Chapter 6
Massive Gravity: A Primer

E.A. Bergshoeff, M. Kovacevic, J. Rosseel, and Y. Yin

Abstract We show that the recently constructed $3D$ higher-derivative "New Massive Gravity theory" is the result of a general procedure that allows one to construct, in the free case, higher-derivative gauge theories for a wide class of "spins" in diverse dimensions. We specify the criterium that the "spin" and dimension need to satisfy in order for the construction to apply. To clarify the general procedure we present examples of higher-derivative gauge theories for the special cases of spin 1 in $D = 3$, 5 and 7 dimensions. We next apply the procedure to spin 2 in $D = 3$ dimensions and show how the New Massive Gravity and Topological Massive Gravity theories are constructed. Both theories allow interactions. We indicate how and under which conditions the $3D$ New Massive Gravity theory can be extended to $D = 4$ dimensions and the $3D$ Topological Massive Gravity theory can be extended to $D = 7$ dimensions. We discuss the issue of interactions of these two theories.

6.1 Introduction

These lectures deal with higher-derivative theories of gravity. Consider first Einstein's theory of gravity as a theory of interacting massless spin 2 particles around a Minkowski space-time background. The dynamics of this theory is described by the Einstein-Hilbert action which is second-order in the derivatives. As is well-known, Einstein's theory of gravity is perturbative non-renormalizable when expanded around a flat Minkowski spacetime. One way to try to cure this problem is

E.A. Bergshoeff (✉) · M. Kovacevic · J. Rosseel · Y. Yin
Centre for Theoretical Physics, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands
e-mail: E.A.Bergshoeff@rug.nl

M. Kovacevic
e-mail: maikovacevic@gmail.com

J. Rosseel
e-mail: J.Rosseel@rug.nl

Y. Yin
e-mail: Y.Yin@rug.nl

G. Calcagni et al. (eds.), *Quantum Gravity and Quantum Cosmology*,
Lecture Notes in Physics 863, DOI 10.1007/978-3-642-33036-0_6,

by adding higher-derivative terms to the Einstein-Hilbert action in order to obtain better behaving propagators that could lead to a perturbative renormalizable theory.

Already in the seventies of the previous century a systematic investigation of the effect of adding fourth-order derivative terms to the Einstein-Hilbert action was undertaken by Stelle [1, 2]. He considered the most general such terms:

$$\mathscr{L} \sim R + a\big(R_{\mu\nu}{}^{ab}\big)^2 + b(R_{\mu\nu})^2 + cR^2. \tag{6.1}$$

Here $R_{\mu\nu}{}^{ab}$, $R_{\mu\nu}$, R are the Riemann tensor, Ricci tensor, Ricci scalar, respectively, and a, b and c are generic coefficients with dimension of one over mass squared. The outcome of his studies was that for generic coefficients the theory is renormalizable[1] but not unitary. It is easy to understand why this is the case. In the above Lagrangian the fourth order derivative terms act like the kinetic terms and the Einstein-Hilbert term as the mass term. Since the kinetic terms are fourth-order in derivatives they can generically be written as the product of two second-order operators. It turns out that one operator corresponds to a massless graviton and the other one to a *massive* graviton. Unfortunately, it turns out that the signs of the two kinetic terms are opposite and that is why ghosts cannot be avoided.

The occurrence of a massive and massless graviton with opposite signs is a generic feature of any dimension. For each dimension this would imply a breakdown of unitarity except for three dimensions since in three dimensions there is no massless graviton! This implies that one is only left with the massive graviton only whose kinetic term can always be given the correct sign by adjusting the over-all sign of the Lagrangian. This is the reason that unitary higher-derivative theories of gravity do exist in three dimensions. There is one more special situation that is less obvious. It turns out that when expanding around an AdS vacuum solution instead of a Minkowski space-time the coefficient in front of the linearized Einstein-Hilbert term gets shifted with a term involving the cosmological term Λ. The value of Λ can be chosen such that the coefficient in front of this term vanishes which has the effect that there is no massive graviton! This special so-called "critical" point in parameter space leads to the so-called "critical" gravity theories. Note that these critical gravity theories are not limited to three dimensions. They will be shortly discussed later in these lectures.

It turns out that in three dimensions there are not one but two unitary higher-derivate gravity theories. They are called Topological Massive Gravity (TMG) [3] and New Massive Gravity (NMG) [4, 5]. An important difference between the two theories is that only one of them (NMG) is parity-invariant. In these lectures we will discuss the general procedure for constructing these TMG and NMG theories. This also shows the way of how to extend these constructions, at least at the linearized level, to higher than three dimensions.

[1]This is not the case for special choices of the coefficients. In particular, scalar gravity, with $a = b = 0$ and Weyl gravity, in which case a, b and c are chosen such that the Weyl tensor squared combination is obtained, are *not* perturbative renormalizable.

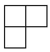
The organization of these lectures is as follows. In Sect. 6.2 we will discuss the general procedure of constructing higher-derivative gauge theories mentioned above for general dimensions and general spin. We will do this on hand of Young tableaux thereby avoiding too many explicit (and complicated!) formulae. In Sect. 6.3 we will elucidate this procedure by working out several examples corresponding to "spin 1" fields. By this we mean fields that carry an index structure corresponding to a Young tableaux with one column. Subsequently, in Sect. 6.4 we will discuss the "spin 2" case, i.e. we will discuss fields whose symmetry structure correspond to Young tableaux with two columns. This will include the construction of the $3D$ NMG and $3D$ TMG theories and a discussion of the higher-dimensional generalization (at the linearized level) of these theories. This will lead to the construction of a new $4D$ NMG and $7D$ TMG theory which will be briefly discussed. In the conclusions we will address a few open issues. We have included an Appendix which contains the four exercises that were mentioned during the lectures together with their answers.

6.2 General Spin

In this section we will explain the general procedure of how to construct a higher-derivative gauge theory for a massive field in a pictorial way using Young tableaux. The precise formulae, corresponding to specific examples, will be presented in the following sections. First, we will explain in Sect. 6.2.1 how to "boost up the derivatives" of a given massive theory. Next, in Sect. 6.2.2 we will explain how to "take the square root" of a massive theory. The techniques of the first subsection may then be applied to boost up the derivatives of this "square root" theory.

6.2.1 "Boosting up the Derivatives"

Following [6, 7],[2] the starting point is a field S in D dimensions with indices corresponding to a $GL(D, \mathbb{R})$ Young tableau with s columns. In order to elucidate the general procedure, we consider as an example a $4D$ field with indices corresponding to the following Young tableau with $s = 2$ columns:

$S \sim$  (6.2)

For simplicity, we will restrict in the discussion below to the cases $s = 1$ and $s = 2$ only. Most of the discussion, however, is valid for any s. In order that the field S

[2]For a recent discussion, see also [8].

describes a massless spin[3] corresponding to the same Young tableau but with the indices now referring to the $SO(D-2)$ little group[4] the field S should transform under a set of gauge transformations whose parameters λ correspond to $GL(D,\mathbb{R})$ Young tableaux that are obtained from the original tableau by deleting one box in all possible ways such as to obtain an allowed Young tableau. For our example (6.2) given above this leads to gauge parameters λ_1 and λ_2 corresponding to the following two $GL(D,\mathbb{R})$ Young tableaux

$$\lambda_1 \sim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \qquad \lambda_2 \sim \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \tag{6.3}$$

This corresponds to a generic 2-tensor gauge parameter $\lambda = \lambda_1 + \lambda_2$. The transformation rule of the gauge field S is obtained by hitting the parameters $\lambda_{1,2}$ with a derivative and projecting to the original Young tableau:

$$\delta \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline \ & \partial \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline \partial & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{6.4}$$

or, shortly, $\delta S = \partial\lambda_1 + \partial\lambda_2$.

For a Young tableau with s columns a gauge-invariant curvature is obtained by adding one box, representing a derivative, to each column. This leads to a curvature with s derivatives. Following the $4D$ spin 2 case we will call this curvature the "generalized" Riemann tensor $R(S)$ or, shortly, the Riemann tensor. For our example (6.2) we obtain

$$R(S) \sim \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \partial \\ \hline \partial & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{6.5}$$

That this Riemann tensor is gauge-invariant can be seen from the fact that the substitution of the transformation rule (6.4) into the expression (6.5) always leads to a column with two derivatives and hence a vanishing result since two derivatives commute [8].

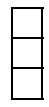
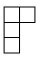
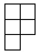
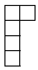
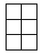
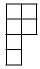
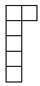
We now construct out of the Riemann tensor $R(S)$ another tensor $G(S)$ by taking the dual of each column. Due to the Bianchi identities of the Riemann tensor this new tensor is divergence-free in *each* of its indices. We now assume that the field S and the tensor $G(S)$ have indices corresponding to the *same* Young tableau. For the example given in Eq. (6.2) this assumption is valid. Assuming this property we can identify $G(S)$ with the "generalized Einstein" tensor for S and write down the following equations of motion for S:

$$G(S) = 0. \tag{6.6}$$

[3] In $3D$ there is no concept of massless spin. In $D=3,4$ a Young tableau with s columns always describes (massless or massive) degrees of freedom of spin s or less. For $D>4$ the specification of spin requires more than one number. For ease of notation we will call in these lectures any field with indices corresponding to a $GL(D,\mathbb{R})$ Young tableau with s columns a "spin-s" field.

[4] To obtain an irreducible $SO(D-2)$ representation from the field S one should first require that all indices only take values in the $(D-2)$ transverse directions and, next, that all traces in any of these transverse directions vanish.

Table 6.1 This table lists, for $s = 1, 2$, all the $GL(D, \mathbb{R})$ representations of S in $3 \le D \le 7$ dimensions for which the massless representation describes zero physical degrees of freedom. The star indicates that the equation of motion of the corresponding field S cannot be integrated to a Lagrangian. The $s = 2$ Young tableaux with a † indicate the family of fields S that are all dual to a symmetric tensor

	$D=3$	$D=4$	$D=5$	$D=6$	$D=7$
$s=1$			⋆		
$s=2$	†	†	 †	 †	  †

Restricting to $s = 1, 2$, we find that for a single-column $s = 1$ Young tableau with p boxes (p odd) and for any two-column $s = 2$ Young tableau these equations of motion can be integrated to the following Lagrangian for S:[5]

$$\mathscr{L} \sim SG(S). \tag{6.7}$$

Making use of the property that the Einstein tensor $G(S)$ is divergence-free in each of its indices one can show that this Lagrangian is invariant under the gauge transformations (6.4). The corresponding Euler-Lagrange equations imply the equations of motion (6.6). To derive these equations we use the fact that the Einstein tensor $G(S)$ defines a rank s self-adjoint differential operator. The special thing about the cases described by the Lagrangian (6.7) is that the vanishing of the Einstein tensor $G(S)$ implies the vanishing of the Riemann tensor $R(S)$ since, by construction, the two are dual to each other. Since the Riemann tensor is zero, the original field S is a pure gauge and, therefore, does not describe any massless physical degrees of freedom. The fact that there are no non-trivial solutions S of the equation $G(S) = 0$ is the crucial property that underlies the construction of the higher-derivative massive gauge theories we are going to describe below.

For a single-column $s = 1$ Young tableau with p boxes the fact that S and $G(S)$ correspond to the same Young tableau implies that the following relation between p and D must hold:

$$p = \frac{1}{2}(D - 1). \tag{6.8}$$

Similarly, for a Young tableau with $s = 2$ columns, of height p and q, we obtain the condition

$$p + q = D - 1, \quad p, q \neq 0. \tag{6.9}$$

Consider now a field S corresponding to a given $GL(D, \mathbb{R})$ Young tableaux. Following [6, 9] we may write down the massive "generalized" Fierz-Pauli (FP) equa-

[5] For $s = 1$ and p even this Lagrangian would be a total derivative.

tions for this field as follows:

$$\left(\Box - m^2\right)S = 0, \qquad S^{\mathrm{tr}} = 0, \quad \partial \cdot S = 0. \tag{6.10}$$

Here S^{tr} indicates the trace of any of the two indices carried by S while $\partial \cdot S$ denotes the divergence taken with respect to any of the indices of S. The effect of the algebraic and differential subsidiary conditions given in Eq. (6.10) is that the massive physical degrees of freedom described by S transform according to a $SO(D-1)$ Young tableau that is equal to the original $GL(D,\mathbb{R})$ Young tableau that corresponds to S. We now assume that the massless representation corresponding to S describes zero degrees of freedom. This requires imposing the restrictions (6.8) and (6.9), for $s = 1$ and $s = 2$, respectively. For $3 \leq D \leq 7$ this leads to the cases listed in Table 6.1. Note that for $s = 2$ we obtain in each dimension a mixed-symmetry tensor that is the massive dual of a symmetric tensor [10]. This family of fields is indicated with a dagger in Table 6.1. They play a special role in the construction of "New Massive Gravity" theories beyond $3D$, see Sect. 6.4.3 [11].

Assuming from now on that we restrict to the cases listed in Table 6.1 we know that the Einstein tensor $G(S)$ is in the same representation as S. We may now exploit this fact and solve the divergence-free condition $\partial \cdot S = 0$ by making the following replacement in the massive equations of motion (6.10):

$$S = G(T), \tag{6.11}$$

for some other field T that is in the same $GL(D,\mathbb{R})$ representation as S. Note that after the replacement (6.11) one ends up with a gauge-theory for T although the starting point (6.10) is not a gauge theory. The important thing is that the equation $G(T) = 0$ does not have any non-trivial solution which is not a pure gauge. Therefore, the replacement (6.11) represents *all* solutions of the equation $\partial \cdot S = 0$. This implies that the degrees of freedom remain the same independent of whether they are described in terms of S or T. The substitution (6.11) therefore leads us to an equivalent higher-derivative gauge theory for the massive field T with the following equations of motion:

$$\left(\Box - m^2\right)G(T) = 0, \qquad G(T)^{\mathrm{tr}} = 0. \tag{6.12}$$

For $s = 2$ the above procedure was first applied to the case of a symmetric tensor in $3D$ in which case it leads to the (linearized) equations of motion of NMG [4, 5].

For Young tableaux with $s = 1$ or $s = 2$ columns one can write down actions corresponding to the equations of motion (6.10) and the boosted up equations of motion (6.12).[6] However, it is not guaranteed that after boosting up the derivatives the action will not contain ghosts. We consider first the $s = 1$ case. It turns out that for a $(2k-1)$-form T in $D = 4k - 1$ dimensions ghosts will occur. The reason for this is that in these dimensions the "helicities" described by the $(2k-1)$-form T split

[6] Starting from $s = 3$ one needs to introduce an extra set of auxiliary fields to write down such actions.

into two groups which are not in the same induced representation of the Poincaré group. They can only be mapped to each other by a parity transformation. Since the replacement (6.11) breaks parity in these cases one does end up with a relative minus sign between the kinetic terms of these two groups of helicities. Therefore, one cannot adapt the overall sign of the action such as to avoid ghosts. On the other hand, for a $2k$-form in $D = 4k + 1$ dimensions the equations of motion cannot be integrated to an action and the issue does not arise. It turns out that for $s = 2$ the issue of ghosts does not arise since the replacement (6.11) never breaks parity for $s = 2$. It has been conjectured that the same is true for any even s [12].

6.2.2 "Taking the Square Root"

The feature described at the end of the previous subsection, namely that the helicities described by a field S, for given s, split into two groups which are only connected by a parity transformation, manifests itself in a factorization of the Klein-Gordon operator acting on that field. To be explicit, for $D = 4k - 1$ one can show that the Klein-Gordon operator $\Box - m^2$, when acting on a field S corresponding to a Young tableaux with s columns of height $2k - 1$ each, that satisfies the massive FP equation (6.10), can be factorized in terms of two first-order matrix operators $\mathscr{D}(\pm m)_{\mu_1\cdots\mu_{2k-1}}{}^{\nu_1\cdots\nu_{2k-1}}$ as follows:[7]

$$\mathscr{D}(m)\mathscr{D}(-m)S = 0, \qquad S^{\mathrm{tr}} = 0, \quad \partial \cdot S = 0, \tag{6.13}$$

where the full index structure of the operator $\mathscr{D}(m)$ is given by

$$\mathscr{D}(m)_{\mu_1\cdots\mu_{2k-1}}{}^{\nu_1\cdots\nu_{2k-1}} = \frac{1}{(2k-1)!}\varepsilon_{\mu_1\cdots\mu_{2k-1}}{}^{\alpha\nu_1\cdots\nu_{2k-1}}\partial_\alpha + m\delta^{\nu_1\cdots\nu_{2k-1}}_{\mu_1\cdots\mu_{2k-1}}. \tag{6.14}$$

It is understood that this operator acts on the first column of the Young tableaux corresponding to S. It is an on-shell projector:

$$\mathscr{D}^2(m)S = \mathscr{D}(m)S \quad \text{if } S \text{ satisfies (6.13)}. \tag{6.15}$$

One can show that the symmetry properties of $\mathscr{D}(m)\mathscr{D}(-m)S$ are the same as that of S itself as a consequence of the algebraic and differential subsidiary conditions.

One could try to write down a similar factorization in $D = 4k + 1$ dimensions for a Klein-Gordon operator when acting on a Young tableau with s columns of height $2k$ each. However, in this case one finds that the Klein-Gordon operator with the "wrong" sign of the mass term factorizes:

$$\left(\Box + m^2\right)S = -\mathscr{D}(m)\mathscr{D}(-m)S = 0, \qquad S^{\mathrm{tr}} = 0, \quad \partial \cdot S = 0. \tag{6.16}$$

[7] We do not indicate indices. In later sections we will give the precise form of the equations in specific examples, including the indices.

The factorization (6.13) of the Klein-Gordon operator $\Box - m^2$ in $D = 4k - 1$ dimensions shows that one can take the "square root" of the generalized FP equations (6.10) and describe the dynamics of only half of the degrees of freedom by the first-order differential equations

$$\mathscr{D}(m)S = 0. \tag{6.17}$$

Note that this equation is not in the same representation as that of S. One can show that it implies the algebraic conditions $S^{\text{tr}} = 0$ and the differential subsidiary conditions $\partial \cdot S = 0$. The other half of the degrees of freedom are described by a similar set of equations but with m replaced by $-m$. Under parity the two equations are mapped into each other. For $s = 1$ these equations reduce to the massive self-duality equations [13, 14]

$$R(S) = \pm m^{\star} S. \tag{6.18}$$

Such massive self-duality equations occur for instance in seven-dimensional gauged supergravity theories where S is a 3-form and m plays the role of the gauge coupling constant [13].

One can play the same trick of "boosting up the derivatives" not only on the generalized FP equations (6.10) but also, in $D = 4k - 1$ dimensions, on the "square root" of these equations, see Eq. (6.17). One thus arrives at the following higher-order derivative equations describing the same degrees of freedom:

$$\mathscr{D}(m)G(T) = 0. \tag{6.19}$$

The integration of these equations of motion to an action in this case does not lead to ghosts since the degrees of freedom are always in the same irreducible induced representation of the Poincaré group. In $D = 3$ dimensions this leads to Topological Massive Electrodynamics (TME) for $s = 1$ [15, 16] and Topological Massive Gravity (TMG) for $s = 2$ [3]. The analogue of Eqs. (6.19) does not exist in $D = 4k + 1$ dimensions since the integration of these equations would lead to a Klein-Gordon equation with the "wrong" sign in front of the mass term.

This ends our discussion of the general procedure of how to obtain out of a generalized massive FP theory for a massive field S, or its "square root", a massive higher-derivative gauge theory for a field T without ghosts. In the next sections we will further explain the general expressions introduced in this section at the hand of the one-column Young tableaux, i.e. $s = 1$.

6.3 Spin 1

In this section we consider the general case of a field S in D dimensions with indices corresponding to a one-column $s = 1$ Young tableau. As explained in footnote 3 we will generically denote this set of fields as "spin-1" fields. In these cases we are dealing with a p-form gauge field $S_{\mu_1 \cdots \mu_p}(x)$ with gauge transformation

$$\delta S_{\mu_1 \cdots \mu_p}(x) = p \partial_{[\mu_1} \lambda_{\mu_2 \cdots \mu_p]}(x). \tag{6.20}$$

The gauge-invariant curvature or "Riemann tensor" of S is given by the curl of this gauge field:

$$R_{\mu_1\cdots\mu_{p+1}}(S) = (p+1)\partial_{[\mu_1} S_{\mu_2\cdots\mu_{p+1}]}. \tag{6.21}$$

In the following we discuss the cases $p=1$, $p=2$ and $p=3$ in more detail.

p = 1 The simplest case that satisfies the condition (6.8) is a vector ($p=1$) in $D=3$ dimensions. In that case the curvature or "Riemann tensor" $R(S)$ and the "Einstein tensor" $G(S)$ are given by

$$R_{\mu\nu}(S) = 2\partial_{[\mu} S_{\nu]}, \qquad G_\mu(S) = \frac{1}{2}\varepsilon_\mu{}^{\nu\rho} R_{\nu\rho}(S). \tag{6.22}$$

The massless Lagrangian (6.7) is now given by

$$\mathscr{L} = \frac{1}{2}\varepsilon^{\mu\nu\rho} S_\mu R_{\nu\rho}(S), \tag{6.23}$$

which indeed does not describe any massless spin 1 degree of freedom.

We next consider the massive Proca equation for a $3D$ massive vector field S_μ:

$$\left(\Box - m^2\right) S_\mu = 0, \qquad \partial^\mu S_\mu = 0. \tag{6.24}$$

These equations are derivable from the Proca Lagrangian

$$\mathscr{L} = \frac{1}{2} G^\mu(S) G_\mu(S) - \frac{1}{2} m^2 S^\mu S_\mu. \tag{6.25}$$

This Lagrangian describes the unitary propagation of two states, one with helicity $+1$ and one with helicity -1, see Exercise 1.[8] The differential subsidiary condition is solved by making the substitution:

$$S_\mu = G_\mu(T) \tag{6.26}$$

in terms of another vector field T_μ. Note that T_μ is a gauge field with gauge transformations $\delta T_\mu = \partial_\mu \lambda$. The substitution (6.26) leads to the following higher-derivative so-called "extended Proca" equation for T_μ:

$$\left(\Box - m^2\right) G_\mu(T) = 0, \tag{6.27}$$

which can be integrated to the following Lagrangian containing the "extended Chern-Simons" term introduced in [17]:

$$\mathscr{L} = -\frac{1}{2} T^\mu G_\mu(T) + \frac{1}{2m^2}\varepsilon^{\mu\nu\rho} G_\mu(T)\partial_\nu G_\rho(T). \tag{6.28}$$

---

[8]The exercises, together with their solutions, are given in the Appendix.

Table 6.2 This table lists all the $s = 1$ cases, with $3 \leq D \leq 7$, where the "boosting up the derivatives" trick works without encountering ghosts. This leads to the $3D$ and $7D$ "Topological Massive Electrodynamics" (TME) theories indicated in the table. The $5D$ "Extended Proca" (EP) theory, indicated by a star in the table, is special in the sense that the equation of motion of this theory cannot be integrated to a Lagrangian

	$D = 3$	$D = 4$	$D = 5$	$D = 6$	$D = 7$
EP			$\begin{array}{\|c\|}\hline \\ \hline \\ \hline \end{array}^{\star}$		
TME	$\square$				$\begin{array}{\|c\|}\hline \\ \hline \\ \hline \\ \hline \end{array}$

A canonical analysis shows that this higher-derivative gauge theory contains ghosts [12, 17]. For a proof of this statement, see Exercise 2.

To avoid ghosts one should first take the "square root" and consider the massive self-duality equations

$$R_{\mu\nu}(S) = m\varepsilon_{\mu\nu}{}^{\rho} S_{\rho}. \tag{6.29}$$

Boosting up the derivatives and integrating the equations of motion leads to the Lagrangian of $3D$ TME [15, 16], see Table 6.2

$$\mathscr{L} = -\frac{1}{4m} R^{\mu\nu}(T) R_{\mu\nu}(T) + \frac{1}{2}\varepsilon^{\mu\nu\rho} T_{\mu}\partial_{\nu} T_{\rho}. \tag{6.30}$$

$\mathbf{p = 2}$ We now move on and consider the next simplest case of a 2-form ($p = 2$) in $5D$. In this case we are dealing with gauge fields S, gauge parameters λ and Riemann tensors $R(S)$ corresponding to the following Young tableaux

$$S \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \qquad \lambda \sim \square \qquad R(S) \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \partial \\ \hline \end{array} \tag{6.31}$$

These expressions correspond to the following formulae:

$$\delta S_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]}, \qquad R_{\mu\nu\rho}(S) = 3\partial_{[\mu} S_{\nu\rho]}, \tag{6.32}$$

while the Einstein tensor $G_{\mu\nu}(S)$ is given by

$$G_{\mu\nu}(S) = \frac{1}{3}\varepsilon_{\mu\nu}{}^{\rho\sigma\tau} R_{\rho\sigma\tau}(S). \tag{6.33}$$

In this case the equation $G_{\mu\nu}(S) = 0$ cannot be integrated to a Lagrangian since the candidate kinetic term $S^{\mu\nu} G_{\mu\nu}(S)$ is a total derivative, see Table 6.2. This is similar to the self-dual 2-form in IIB string theory whose dynamics can be described by an equation of motion without having a Lagrangian.

We next consider the equations of motion for a massive $5D$ two-form $S_{\mu\nu}$:

$$\left(\Box - m^2\right) S_{\mu\nu} = 0, \qquad \partial^{\mu} S_{\mu\nu} = 0. \tag{6.34}$$

These equations are derivable from the following Lagrangian:

$$\mathscr{L} = \frac{1}{8} G^{\mu\nu}(S) G_{\mu\nu}(S) + \frac{1}{2} m^2 S^{\mu\nu} S_{\mu\nu}. \tag{6.35}$$

The differential subsidiary condition given in (6.34) is solved by making the following substitution:

$$S_{\mu\nu} = G_{\mu\nu}(T) \tag{6.36}$$

in terms of another 2-form field $T_{\mu\nu}$. Note that $T_{\mu\nu}$ is a gauge field with gauge transformations $\delta T_{\mu\nu} = 2\partial_{[\mu}\lambda_{\nu]}$. The substitution (6.36) leads to the following higher-derivative equations of motion for T:

$$\left(\Box - m^2\right) G_{\mu\nu}(T) = 0. \tag{6.37}$$

Again, these equations cannot be integrated. Trying a Lagrangian of the form $\mathscr{L} \sim \alpha T^{\mu\nu} G_{\mu\nu}(T) + \beta \varepsilon^{\mu\nu\rho\sigma\tau} G_{\mu\nu}(T) \partial_\rho G_{\sigma\tau}(T)$ one finds that both terms are total derivatives. The dynamics of this case can only be described by a set of equations of motion without having a Lagrangian. Taking the "square root" is not an option in this case since the integrability conditions of the massive self-duality equations would lead to a Klein-Gordon equation with the wrong sign in front of the mass term.

p = 3 Finally, we consider a 3-form ($p = 3$) in $D = 7$ dimensions. We are now dealing with gauge fields S, gauge parameters λ and Riemann tensors $R(S)$ given by the following Young tableaux:

$$S \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \qquad \lambda \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \qquad R(S) \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \partial \\ \hline \end{array} \tag{6.38}$$

These expressions correspond to the following formulae:

$$\delta S_{\mu\nu\rho} = 3\partial_{[\mu}\lambda_{\nu\rho]}, \qquad R_{\mu\nu\rho\sigma}(S) = 4\partial_{[\mu} S_{\nu\rho\sigma]}, \tag{6.39}$$

while the Einstein tensor $G_{\mu\nu\rho}(S)$ is given by

$$G_{\mu\nu\rho}(S) = \frac{1}{4} \varepsilon_{\mu\nu\rho}{}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}(S). \tag{6.40}$$

This leads to the following massless Lagrangian

$$\mathscr{L} = S^{\mu\nu\rho}(S) G_{\mu\nu\rho}(S), \tag{6.41}$$

which does not describe any massless degrees of freedom.

We next consider the massive Proca equation for a $7D$ massive 3-form $S_{\mu\nu\rho}$:

$$\left(\Box - m^2\right) S_{\mu\nu\rho} = 0, \qquad \partial^\mu S_{\mu\nu\rho} = 0. \tag{6.42}$$

These equations are derivable from the Lagrangian

$$\mathscr{L} = G^{\mu\nu\rho}(S)G_{\mu\nu\rho}(S) + \frac{1}{2}m^2 S^{\mu\nu\rho}S_{\mu\nu\rho}. \tag{6.43}$$

The differential subsidiary condition is solved by making the substitution:

$$S_{\mu\nu\rho} = G_{\mu\nu\rho}(T) \tag{6.44}$$

in terms of another 3-form field $T_{\mu\nu\rho}$. This substitution leads to the following higher-derivative equations for $T_{\mu\nu\rho}$:

$$\left(\Box - m^2\right)G_{\mu\nu\rho}(T) = 0, \tag{6.45}$$

which can be integrated to the following Lagrangian

$$\mathscr{L} = \frac{1}{2}T^{\mu\nu\rho}\left(\Box - m^2\right)G_{\mu\nu\rho}(T). \tag{6.46}$$

To see whether the Lagrangian (6.46) describes ghosts or not we perform a canonical analysis. We first fix all gauge degrees of freedom by imposing the following gauge-fixing conditions on the 3-form T and the 2-form gauge parameters λ:

$$\partial^i T_{i\mu\nu} = 0, \qquad \partial^i \lambda_{i\mu} = 0, \quad i = 1, \ldots, 6. \tag{6.47}$$

Using these conditions it follows that $\delta(\partial^i T_{i\mu\nu}) = \nabla^2 \lambda_{\mu\nu}$, which shows that indeed all gauge degrees of freedom in T are fixed.

Taking the gauge-fixing conditions (6.47) into account, we decompose T as follows:

$$T_{0ij} = T_{ij}, \qquad T_{ijk} = \varepsilon_{ijk}{}^{lmn}\partial_l U_{mn}, \tag{6.48}$$

where $T_{ij} = -T_{ji}$, $U_{ij} = -U_{ji}$, $\partial^i T_{ij} = 0$ and $\partial^i U_{ij} = 0$. Therefore, T_{ij} and U_{ij} each describe 10 components.[9]

Using the decomposition (6.48) and dropping all terms with a spatial divergence of T or U, the Lagrangian (6.46) can be rewritten as follows:

$$\mathscr{L} = 36T^{ij}\left(\Box - m^2\right)\nabla^2 U_{ij}.$$

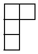
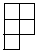
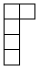
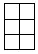
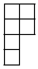
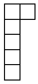
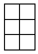
The off-diagonal nature of this expression shows that this Lagrangian describes 20 massive degrees of freedom but that half of them are ghosts.

To avoid ghosts one should first take the "square root" and consider the massive self-duality equations

$$R_{\mu\nu\rho\sigma}(S) = \frac{1}{3!}m\varepsilon_{\mu\nu\rho\sigma}{}^{\alpha\beta\gamma}S_{\alpha\beta\gamma}. \tag{6.49}$$

[9] It is always understood that T_{ij} and U_{ij} are spatially divergenceless. This means that when we apply the variational principle, we should not vary the "divergenceful degrees of freedom".

Table 6.3 This table lists all the $s = 2$ cases where the "boosting up the derivatives" trick works without introducing ghosts. This leads to the different NMG and TMG theories indicated in the table for $3 \leq D \leq 7$. The cases with the sub-indices 1–3 are discussed in Sects. 6.4.1–6.4.3

	$D = 3$	$D = 4$	$D = 5$	$D = 6$	$D = 7$
NMG	 1	 3	 	 	  
TMG	 2				 3

Boosting up the derivatives and integrating the equations of motion leads to the $7D$ higher-derivative TME Lagrangian, see Table 6.2

$$\mathscr{L} = -\frac{3}{4m} R^{\mu\nu\rho\sigma}(T) R_{\mu\nu\rho\sigma}(T) + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma\alpha\beta\gamma} T_{\mu\nu\rho} \partial_\sigma T_{\alpha\beta\gamma} . \tag{6.50}$$

This finishes our discussion of the one-column Young tableaux.

6.4 Spin 2

We now consider fields corresponding to two-column Young tableaux, i.e. $s = 2$. For $3 \leq D \leq 7$ the cases where the "boosting up the derivatives" procedure does not lead to ghosts are indicated in Table 6.3. In the first subsection we will discuss the $3D$ NMG theory [4, 5]. In the next subsection we will review the $3D$ TMG theory [3]. In Sect. 6.4.3 we will briefly discuss the extensions of the $3D$ NMG and TMG theories to higher dimensions. To keep in line with notational conventions we will denote the two-column fields with the letter h instead of S since in specific cases h can be viewed as the linearization of a metric tensor g.

6.4.1 3D New Massive Gravity

It is well-known that the pure Einstein-Hilbert term in three dimensions does not describe any physical degrees of freedom: there are no gravitational waves in three dimensions. For a proof of this, see Exercise 3. This is consistent with our analysis in Sect. 6.2 where we concluded that setting the Einstein tensor corresponding to a $3D$ symmetric tensor to zero implies that there are only gauge degrees of freedom left. In this section we will show that adding a specific combination of higher-derivative terms quadratic in the Riemann tensor has the effect that *massive* gravitons, with helicities $+2$ and -2, start propagating unitarily. The corresponding model is called NMG [4, 5]. The mass parameter is related to the dimension-full parameter in front of the higher-derivative terms. Effectively, the higher-derivative term acts as the kinetic term and the original Einstein-Hilbert term behaves like a mass term.

It is surprising that NMG, given the fact that it contains higher derivatives, does not contain ghosts. The same is not true for similar higher-derivative models in four spacetime dimensions [1, 2]. In general our method of "boosting up the derivatives" does not guarantee that this is the case. However, since in this case the theory is parity-preserving, there are no ghosts to be expected. Below we will give a separate proof that integrating the NMG equations of motion leads to a Lagrangian without ghosts. But first we will describe how NMG is obtained by the boosting up procedure.

Our starting point are the Fierz-Pauli (FP) equations for a symmetric tensor $\tilde{h}_{\mu\nu}$ in D dimensions:

$$\left(\Box - m^2\right)\tilde{h}_{\mu\nu} = 0, \qquad \eta^{\mu\nu}\tilde{h}_{\mu\nu} = 0, \qquad \partial^\mu \tilde{h}_{\mu\nu} = 0. \tag{6.51}$$

The last two of these FP equations are algebraic and differential subsidiary conditions that have to the imposed in order to obtain the correct counting of degrees of freedom. This counting is as follows:

$$\frac{1}{2}D(D+1) - 1 - D = \begin{cases} 5 & \text{for } 4D, \\ 2 & \text{for } 3D. \end{cases} \tag{6.52}$$

In $3D$, the Lagrangian that gives these FP equations is given by

$$\mathscr{L}_{\mathrm{FP}} = \frac{1}{2}\tilde{h}^{\mu\nu} G^{\mathrm{lin}}_{\mu\nu}(\tilde{h}) - \frac{1}{2}m^2\left(\tilde{h}^{\mu\nu}\tilde{h}_{\mu\nu} - \tilde{h}^2\right), \tag{6.53}$$

where we denote $\tilde{h} \equiv \eta^{\mu\nu}\tilde{h}_{\mu\nu}$. The $3D$ linearized Einstein tensor $G^{\mathrm{lin}}_{\mu\nu}(\tilde{h})$ for any symmetric tensor $\tilde{h}_{\mu\nu}$ is defined as

$$G^{\mathrm{lin}}_{\mu\nu}(\tilde{h}) \equiv \varepsilon_\mu{}^{\alpha\beta}\varepsilon_\nu{}^{\gamma\delta}\partial_\alpha\partial_\gamma\tilde{h}_{\beta\delta}. \tag{6.54}$$

We note that the trace $\tilde{h}$ plays the role of an *auxiliary field*: it is needed to write down a Lagrangian but it does not describe a physical degree of freedom. Such auxiliary fields become more and more abundant when one considers fields with spin higher than two. It is instructive to see what goes wrong if one actually tries to write down a FP Lagrangian in terms of a symmetric and traceless tensor $H_{\mu\nu}$ alone. The Klein-Gordon equation and the differential subsidiary condition for $H_{\mu\nu}$ would read:

$$\left(\Box - m^2\right)H_{\mu\nu} = 0, \qquad \partial^\mu H_{\mu\nu} = 0. \tag{6.55}$$

In analogy with the spin-1 case one could try to combine the above equations into the following single equation of motion:

$$\partial^\rho(\partial_\rho H_{\mu\nu} - \partial_\mu H_{\rho\nu}) - m^2 H_{\mu\nu} = 0. \tag{6.56}$$

The nice thing about this equation is that it implies the differential subsidiary condition. However, unlike the spin 1 case, this equation can never serve as the equation of motion for $H_{\mu\nu}$ since, unlike $H_{\mu\nu}$ itself, it is not symmetric in the free indices μ and ν. One could next try to write down the most general symmetric and traceless equation but it turns out that that does not work. The problem is that, in order to derive the differential subsidiary condition $\partial^\mu H_{\mu\nu} = 0$ one needs to make use of the constraint $\partial^\rho \partial^\sigma H_{\rho\sigma} = 0$ first. In order to impose this constraint we must extend the field content and introduce an additional auxiliary scalar H. Making the most general Ansatz in terms of $H_{\mu\nu}$ and H one can indeed arrange things such that the equations of motion imply both the constraint $\partial^\rho \partial^\sigma H_{\rho\sigma} = 0$ as well as $H = 0$. Hence, the Lagrange multiplier H does not introduce a new physical degree of freedom. Having derived the Lagrangian in terms of $H_{\mu\nu}$ and H one can go back to the $\tilde{h}$-basis via the transformation

$$\tilde{h}_{\mu\nu} \equiv H_{\mu\nu} + \frac{1}{3}\eta_{\mu\nu} H \tag{6.57}$$

and recover the FP Lagrangian (6.53).

We now apply our "boosting up the derivatives" procedure as explained in Sect. 6.2.1. We take the standard FP equations in terms of a symmetric tensor $\tilde{h}_{\mu\nu}$. We next solve the differential subsidiary condition $\partial^\mu \tilde{h}_{\mu\nu} = 0$ by expressing $\tilde{h}_{\mu\nu}$ in terms of the Einstein tensor of another symmetric field $h_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = G^{\text{lin}}_{\mu\nu}(h), \tag{6.58}$$

with the linearized Einstein tensor $G^{\text{lin}}_{\mu\nu}(h)$ defined in (6.54). Substituting this solution of the constraint into the original FP equations (6.51) we obtain the following equivalent higher-order equations of motion:

$$\left(\Box - m^2\right) G^{\text{lin}}_{\mu\nu}(h) = 0, \qquad R^{\text{lin}}(h) = 0, \tag{6.59}$$

where $R^{\text{lin}}(h)$ is the linearized Ricci scalar, i.e. the trace of the linearized Ricci tensor

$$R^{\text{lin}}_{\mu\nu}(h) = \Box h_{\mu\nu} - 2\partial_{(\mu}\partial^\rho h_{\nu)\rho} + \partial_\mu \partial_\nu h. \tag{6.60}$$

The linearized Einstein tensor can be written as $G^{\text{lin}}_{\mu\nu}(h) = R^{\text{lin}}_{\mu\nu}(h) - \frac{1}{2} g_{\mu\nu} R^{\text{lin}}$, so $R^{\text{lin}}(h) = 0$ is equivalent to $\eta^{\mu\nu} G^{\text{lin}}_{\mu\nu}(h) = 0$. The equations of motion (6.59) can be integrated to a Lagrangian. At this point there are two surprises. First of all, as we will show below, this Lagrangian does not contain ghosts. Secondly, it turns out that the Lagrangian can be extended to a more general non-linear one with interactions. More precisely, the quadratic (in $h_{\mu\nu}$) Lagrangian corresponding to (6.59) can be viewed as the linearization of a non-linear quadratic curvature Lagrangian where the metric $g_{\mu\nu}$ is expanded around a flat Minkowski spacetime metric $\eta_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{6.61}$$

Upon making this substitution into this quadratic curvature Lagrangian and retaining only the terms quadratic in $h_{\mu\nu}$ one obtains the quadratic Lagrangian that yields the equations of motion (6.59). It turns out that the quadratic curvature Lagrangian in question is the NMG Lagrangian given by:

$$\mathscr{L} = \sqrt{-g}\left[-R - \frac{1}{2m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right)\right]. \tag{6.62}$$

A noteworthy feature of this NMG Lagrangian is that the Einstein Hilbert term has the so-called "wrong" sign in the sense that it is not the sign it should have in four spacetime dimensions. Note that this is possible due to the fact that the Einstein-Hilbert term plays the role of a mass term and not of a kinetic term.

Before we prove that the NMG Lagrangian (6.62) describes unitarily the helicity states $+2$ and -2 it is convenient to introduce the following generalization of this Lagrangian:

$$\mathscr{L} = \sqrt{-g}\left[\sigma R + 4\lambda m^2 - \frac{1}{2m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right)\right]. \tag{6.63}$$

We have introduced here two new parameters: a sign parameter $\sigma = \pm$ and a cosmological parameter λ. The Lagrangian (6.63) is sometimes referred to as "Cosmological New Massive Gravity" (CNMG). Note that the cosmological parameter λ we have introduced is not necessarily equal to the cosmological constant Λ characterizing a maximally symmetric background. This is typical for higher-derivative theories. Substituting the Ansatz

$$G_{\mu\nu} = 2\Lambda g_{\mu\nu} \tag{6.64}$$

into the NMG equations of motion leads to the following quadratic relationship between λ and Λ:

$$4m^4\lambda = \Lambda\big(\Lambda + 4m^2\sigma\big). \tag{6.65}$$

To analyze the modes propagated by the CNMG Lagrangian (6.63) it is convenient to first lower the number of derivatives by introducing a second auxiliary symmetric tensor field $f_{\mu\nu}$. In terms of $g_{\mu\nu}$ and $f_{\mu\nu}$ one can write down the following equivalent Lagrangian:

$$\mathscr{L} = \sqrt{-g}\left[\sigma R + 4\lambda m^2 + f^{\mu\nu}G_{\mu\nu} + \frac{1}{2}m^2\big(f^{\mu\nu}f_{\mu\nu} - f^2\big)\right]. \tag{6.66}$$

The equation of motion of $f_{\mu\nu}$ may be used to solve for $f_{\mu\nu}$ in terms of $G_{\mu\nu}(g)$. Substituting this solution back into the Lagrangian (6.66) one obtains the CNMG Lagrangian (6.63).

We now consider the linearization of (6.66) around a maximally symmetric background with metric $\bar{g}_{\mu\nu}$ with cosmological constant Λ. We first expand the metric $g_{\mu\nu}$ around this background as follows:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{6.67}$$

It turns out to be convenient to expand the auxiliary field $f_{\mu\nu}$ as

$$f_{\mu\nu} = \frac{1}{m^2}\big\{\Lambda[\bar{g}_{\mu\nu} + h_{\mu\nu}] - k_{\mu\nu}\big\}, \tag{6.68}$$

where $k_{\mu\nu}$ is an independent symmetric tensor fluctuation field. Substituting the expansions (6.67) and (6.68) into the CNMG Lagrangian (6.63) one obtains (the details can be found in [4, 5]) the following quadratic Lagrangian in terms of the fluctuations $h_{\mu\nu}$ and $k_{\mu\nu}$:

$$\mathscr{L}_{\text{quadr}} \sim -\frac{1}{2}\bar{\sigma} h^{\mu\nu}\mathscr{G}^{\text{lin}}_{\mu\nu}(h) - \frac{1}{m^2}k^{\mu\nu}\mathscr{G}^{\text{lin}}_{\mu\nu}(h) + \frac{1}{2m^2}\big(k^{\mu\nu}k_{\mu\nu} - k^2\big). \tag{6.69}$$

Here

$$\bar{\sigma} = \sigma - \frac{\Lambda}{2m^2} \tag{6.70}$$

is a shifted σ parameter and $\mathscr{G}^{\text{lin}}_{\mu\nu}(h)$ is the linearized Einstein tensor in the presence of a cosmological constant:

$$\mathscr{G}^{\text{lin}}_{\mu\nu}(h) = \mathscr{R}^{\text{lin}}_{\mu\nu}(h) - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\mathscr{R}^{\text{lin}}_{\rho\sigma}(h) + 4\Lambda h_{\mu\nu} - 2\Lambda\bar{g}_{\mu\nu}h. \tag{6.71}$$

The linearized Ricci tensor $\mathscr{R}^{\text{lin}}_{\mu\nu}$ is given by

$$\mathscr{R}^{\text{lin}}_{\mu\nu}(h) = \Box h_{\mu\nu} - \nabla^\rho\nabla_\mu h_{\rho\nu} - \nabla^\rho\nabla_\nu h_{\rho\mu} + \nabla_\mu\nabla_\nu h. \tag{6.72}$$

Some general properties of the generalized Einstein tensor (6.71) are given in Exercise 4.

One can show that, after an appropriate diagonalization, the Lagrangian (6.69) can be written as the sum of a massless spin 2 Lagrangian and a *massive* spin 2 Lagrangian, with mass

$$M^2 = -m^2\bar{\sigma}. \tag{6.73}$$

This is related to the fact that the kinetic operator, which is of fourth-order in the derivatives, can be written as the product of two second-order derivative operators. One of these operators describes a massless graviton while the other factor describes a massive graviton. In general the Lagrangian (6.69) contains a ghost because the signs of the kinetic terms of the massless and massive graviton turn out to be of opposite sign. There are now two special situations where this does not cause any problem:

$D = 3$ In this case there is no massless graviton but only a massive graviton. This implies that one can always adapt the overall sign of the Lagrangian such that the kinetic term of the massive graviton has the correct sign. This case leads to the $3D$ NMG theory of [4, 5].

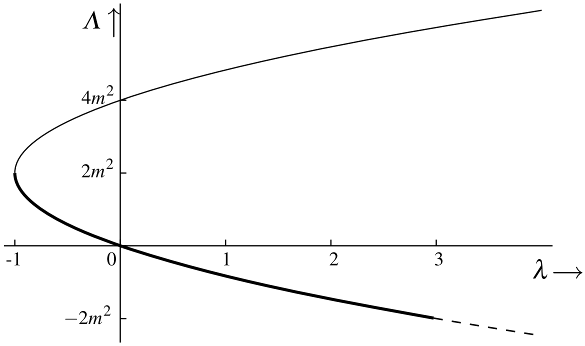


Fig. 6.1 This figure indicates the unitary bulk region (*the boldface line*) for the choice $\sigma = -1$. The boundaries of this unitary region occur for $\lambda = -1$ and $\lambda = 3$ and are discussed in the text

$\Lambda = 2m^2\sigma$ For this special value of the cosmological constant the coefficient $\bar{\sigma}$ in front of the linearized Einstein-Hilbert term vanishes and the massive graviton becomes massless. This special point in the parameter space is more subtle in the sense that it is a degenerate point in the spectrum where one mode, the massive graviton, gets replaced by another, so-called logarithmic mode. This leads to a $3D$ so-called "critical gravity" theory. The interesting thing about this critical point is that it allows a natural generalization to $D > 3$ dimensions [18, 19].

Due to the fact that the sign parameter σ gets shifted to a $\bar{\sigma}$ in a cosmological background one can have unitary bulk models for both signs of σ depending on the value of Λ. There are now several situations. As an example we have given the unitary bulk region in Fig. 6.1 for the choice of $\sigma = -1$. Note that for each choice of the cosmological parameter λ there may be two distinct values of the cosmological constant Λ. The boundary points $\lambda = -1$ and $\lambda = 3$ are special. For $\lambda = -1$ there is an enhanced gauge symmetry leading to a so-called "partial massless" graviton [4, 5] while the $\lambda = 3$ case corresponds to the critical gravity case discussed above.

6.4.2 3D Topological Massive Gravity

In this section we consider the "square root", as described in Sect. 6.2.2, of the $3D$ massive FP equation and show how the procedure of "boosting up the derivatives", as described in Sect. 6.2.1, leads to $3D$ TMG. Our starting point is the massive spin 2 FP equations (6.51). Following the general procedure as described in Sect. 6.2.2 we write the $3D$ Klein-Gordon operator as the product of two first-order matrix operators

$$\left[\mathscr{O}(\pm m)\right]_\mu{}^\rho = \varepsilon_\mu{}^{\tau\rho}\partial_\tau \pm m\delta_\mu^\rho. \tag{6.74}$$

Using such first-order operators, the Klein-Gordon operator acting on a symmetric, traceless and divergenceless rank-2 tensor factorizes as follows:

$$(\Box - m^2)\tilde{h}_{\mu\nu} = \left[\mathscr{O}(m)\right]_\mu{}^\sigma \left[\mathscr{O}(-m)\right]_\sigma{}^\rho \tilde{h}_{\rho\nu}. \tag{6.75}$$

To show this factorization one must use that $\tilde{h}_{\mu\nu}$ satisfies the algebraic and differential subsidiary conditions of the FP equations.

We now take only one of the two first-order operators and consider the $\sqrt{\text{FP}}$ equation $[\mathscr{O}(-m)]_\mu{}^\rho \tilde{h}_{\rho\nu} = 0$:

$$m\tilde{h}_{\mu\nu} = \varepsilon_\mu{}^{\rho\sigma}\partial_\rho \tilde{h}_{\sigma\nu}. \tag{6.76}$$

One can easily prove from this equation that the symmetric tensor $\tilde{h}_{\mu\nu}$ satisfies the FP subsidiary conditions of tracelessness and divergencefreeness. This equation can be integrated to the following first-order action [20]

$$S = \frac{1}{2}\int d^3x \left\{\varepsilon^{\mu\nu\rho}\tilde{h}_\mu{}^\sigma \partial_\nu \tilde{h}_{\rho\sigma} - m\left(\tilde{h}^{\nu\mu}\tilde{h}_{\mu\nu} - \tilde{h}^2\right)\right\}, \tag{6.77}$$

which contains a *non-symmetric* tensor $\tilde{h}_{\mu\nu} \neq \tilde{h}_{\nu\mu}$. The tensor $\tilde{h}_{\mu\nu}$ can be proven to be symmetric after applying the variational principle and then manipulating its equations of motion, but being a fundamental field in the action, it's not symmetric. Its anti-symmetric part behaves like the kind of auxiliary fields we discussed in the case of NMG, see Sect. 6.4.1.

We now apply the "boosting up" procedure and consider the $\sqrt{\text{FP}}$ equations (6.76) in terms of a symmetric tensor $\tilde{h}_{\mu\nu}$. We next solve for the divergence-less condition by expressing the tensor $\tilde{h}_{\mu\nu}$ in terms of a linearized second-order Einstein operator acting on another symmetric tensor $h_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = G^{\text{lin}}_{\mu\nu}(h). \tag{6.78}$$

Substituting this solution of the differential subsidiary condition into the original $\sqrt{\text{FP}}$ equations (6.76) one obtains the following equivalent set of higher-order equations of motion:

$$mG^{\text{lin}}_{\mu\nu}(h) = \varepsilon_\mu{}^{\rho\sigma}\partial_\rho G^{\text{lin}}_{\sigma\nu}(h). \tag{6.79}$$

These equations can be integrated to a Lagrangian that can be viewed as the linearization of the Lagrangian of TMG [3] around a Minkowski spacetime. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ the TMG Lagrangian in terms of $g_{\mu\nu}$ is given by [3]

$$\mathscr{L} = -\sqrt{-g}R + \frac{1}{m}\mathscr{L}_{\text{LCS}}, \tag{6.80}$$

where the last term represents a Lorentz Chern Simons term:

$$\mathscr{L}_{\text{LCS}} = -\varepsilon^{\mu\nu\rho}\left[\Gamma^\alpha_{\mu\beta}\partial_\nu \Gamma^\beta_{\rho\alpha} + \frac{2}{3}\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta}\Gamma^\beta_{\rho\alpha}\right]. \tag{6.81}$$

Here Γ is the usual Levi-Civita connection for the spacetime metric g:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{6.82}$$

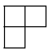
The Riemann curvature tensor is determined, in $3D$, by the Ricci tensor, which is

$$R_{\mu\nu} \equiv R_{\rho\mu}{}^{\rho}{}_{\nu} = -2\big(\partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu}\big). \tag{6.83}$$

Note that, like in the NMG case, the Einstein-Hilbert term in the TMG Lagrangian has the "wrong" sign.

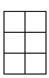
6.4.3 Extensions

It turns out that, using our general procedure described in Sect. 6.2, both the $3D$ NMG as well as the $3D$ TMG theories constructed in the previous two subsections allow, at least at the linearized level, a natural extension to $D > 3$ dimensions. The case of NMG has recently been discussed in [11]. Since this result was published only after the Naxos lectures we will be rather brief here. The basic idea, needed to extend NMG beyond three dimensions, is to use exotic representations to describe the massive spin 2 states. Only in three dimensions the usual symmetric tensor description suffices. In $D > 3$ dimensions one should use, instead, a massive dual representation. These are the mixed-symmetry representations indicated by a dagger in Table 6.1. A common feature of all these representations is that the corresponding Einstein tensor does not describe any massless degrees of freedom. Starting from the generalized FP equations of these fields one can therefore use our "boosting up the derivatives" procedure and construct an equivalent higher-order in derivatives Lagrangian that describes, unitarily, the same massive degrees of freedom as the original massive spin 2 FP equation. The $4D$ NMG Lagrangian makes use of the following exotic representation

$h \sim$  $4D$ NMG (6.84)

For the actual construction of the $4D$ NMG Lagrangian and for more details we refer to [11].

The $3D$ TMG theory can also be extended, at the linearized level, to $D > 3$ dimensions but it requires the use of different exotic representations of the massive spin 2 states. Instead of using the massive dual of the symmetric tensor representation one should use a *self-dual* representation. Only in three dimensions these two representations coincide and that is why both the $3D$ NMG and the $3D$ TMG theories can be formulated in terms of the symmetric tensor representation. Such massive self-dual representations exist in odd dimensions only and only in $D = 4k - 1$ dimensions, with k integer, do the integrability conditions that follow from the corresponding self-duality equations yield the desired Klein-Gordon operators with the correct sign in front of the m^2 term. The first dimensions beyond $3D$ where this occurs is the $7D$ case. In that case the h field corresponds to the following self-dual representation:

$h \sim$  $7D$ TMG (6.85)

The details of the construction of the corresponding (linearized) $7D$ TMG theory will be discussed elsewhere [21].

6.5 Conclusions

In these lecture we have indicated the general procedure that can be applied to construct higher-derivative theories of gravity. The method is based on the assumption that the field involved does not describe any degrees of freedom as a massless representation. This requires the property that setting its Einstein tensor to zero implies that the field in question is a pure gauge. We derived the general criterium that needs to be satisfied in order for this to be true. We exemplified our procedure by first working out several cases involving "spin" 1 fields. Next, we applied the procedure to the "spin 2" case. We first reviewed the constructions of the $3D$ NMG and the $3D$ TMG theories and, subsequently, showed how the procedure can also be applied to construct higher-dimensional generalizations of these theories, at least at the linearized level. The lowest-dimensional examples beyond $3D$ that we discussed were the $4D$ NMG and the $7D$ TMG theories.

It is an open question whether interactions can be introduced for the $D > 3$ NMG and TMG theories. An example of a $4D$ non-linear theory that makes use of an exotic representation is the Eddington-Schrödinger theory which is equivalent to general relativity with a cosmological constant, see [11]. This is some encouragement that it might be possible to introduce interactions for the case of $4D$ NMG. It might necessitate that we need to consider an AdS background instead of a flat Minkowski spacetime. It would be interesting to see whether the $7D$ TMG theory can be reformulated as a Chern-Simons theory like the $3D$ case. This could facilitate the introduction of interactions in that case. Clearly, at the time of writing these lectures the issue of interactions is unresolved and requires a further investigation.

Acknowledgements We would like to thank our collaborators Olaf Hohm and Paul Townsend, with whom we wrote most of the papers upon which these lectures are based, for the many fruitful interactions we had. These proceedings are based on two lectures given by one of us (E.B.). He wishes to thank the organizers of the 6th Aegean Summer School for their kind invitation to deliver these lectures and for the stimulating atmosphere provided by the School.

Appendix: Exercises

During the lectures several exercises were given. They are repeated here together with their solutions.

Exercise 1 Show that the kinetic terms of the two degrees of freedom described by the $3D$ Lagrangian (6.25) have the same sign. Hint: Use the following decomposition:

$$S_0 = \frac{1}{\sqrt{-\nabla^2}}(\phi_0 + \dot{\lambda}), \qquad S_i = \frac{1}{\sqrt{-\nabla^2}}\big(\varepsilon^{ij}\partial_j\phi_1 + \partial_i\lambda\big), \quad i = 1, 2. \tag{6.86}$$

Solution The $3D$ Lagrangian is given by:

$$\mathscr{L} = \frac{1}{2} G^\mu(S) G_\mu(S) - \frac{1}{2} m^2 S^\mu S_\mu, \tag{6.87}$$

where $G_\mu(S) = \frac{1}{2}\varepsilon_\mu{}^{\nu\rho} R_{\nu\rho}(S)$ and $R_{\mu\nu}(S) = 2\partial_{[\mu} S_{\nu]}$. Now use the decomposition (6.86) to calculate both terms in the Lagrangian (6.87). For the mass term we obtain

$$S^\mu S_\mu = S^0 S_0 + S^i S_i = -S_0 S_0 + S_i S_i \tag{6.88}$$

$$= (\phi_0 + \dot\lambda) \frac{1}{\nabla^2} (\phi_0 + \dot\lambda) - (\hat\partial_i \phi_1 + \partial_i \lambda) \frac{1}{\nabla^2} (\hat\partial_i \phi_1 + \partial_i \lambda), \tag{6.89}$$

where $\hat\partial_i \equiv \varepsilon^{ij} \partial_j$. This can be rewritten as follows:

$$S^\mu S_\mu = \phi_0 \frac{1}{\nabla^2} \phi_0 + \dot\lambda \frac{1}{\nabla^2} \dot\lambda + 2\phi_0 \frac{1}{\nabla^2} \dot\lambda + \phi_1^2 + \lambda^2. \tag{6.90}$$

Similarly, the first term in (6.87) reduces to:

$$G^\mu(S) G_\mu(S) = -\frac{1}{2} R_{\mu\nu} R^{\mu\nu} = -2 R_{0i} R_{0i} + R_{ij} R_{ij} = -2\big(\phi_1 \Box \phi_1 + \phi_0^2\big). \tag{6.91}$$

Substituting the expressions

$$R_{0i} = \frac{1}{\sqrt{-\nabla^2}} (\hat\partial_i \dot\phi_1 - \partial_i \phi_0), \qquad R_{ij} = \frac{1}{\sqrt{-\nabla^2}} (\partial_i \hat\partial_j \phi_1 - \partial_j \hat\partial_i \phi_1) \tag{6.92}$$

we obtain

$$G^\mu(S) G_\mu(S) = -2\big(\phi_1 \Box \phi_1 + \phi_0^2\big). \tag{6.93}$$

Putting everything together, the Lagrangian becomes:

$$\mathscr{L} = \frac{1}{2} \left(\phi_0^2 - m^2 \phi_0 \frac{1}{\nabla^2} \phi_0 - 2m^2 \phi_0 \frac{1}{\nabla^2} \dot\lambda - m^2 \lambda^2 - m^2 \dot\lambda \frac{1}{\nabla^2} \dot\lambda + \phi_1 \Box \phi_1 - m^2 \phi_1^2 \right). \tag{6.94}$$

From this Lagrangian we obtain the EOM for the field ϕ_0:

$$\phi_0 - \frac{m^2}{\nabla^2} \phi_0 - \frac{m^2}{\nabla^2} \dot\lambda = 0 \quad \text{or} \quad \phi_0 = \frac{m^2}{\nabla^2 - m^2} \dot\lambda. \tag{6.95}$$

Substituting this back into the Lagrangian and using that

$$\phi_0^2 = -\dot\lambda^2 + 2\dot\lambda \frac{1}{1 - \frac{\nabla^2}{m^2}} \dot\lambda = -\dot\lambda^2 - 2\dot\lambda \phi_0, \tag{6.96}$$

$$\frac{1}{\nabla^2} \phi_0 = -\frac{1}{\nabla^2} \dot\lambda + \frac{1}{m^2} \phi_0 \tag{6.97}$$

we obtain the following expression for the Lagrangian:

$$\mathscr{L} = \frac{1}{2}\left(\phi_1 \Box \phi_1 - m^2 \phi_1^2 - \dot{\lambda} \frac{m^2}{\nabla^2 - m^2} \dot{\lambda} - m^2 \lambda^2\right). \tag{6.98}$$

We next redefine λ in terms of a $\tilde{\lambda}$

$$\lambda = \sqrt{-\nabla^2 + m^2}\tilde{\lambda} \tag{6.99}$$

to obtain the simpler expression

$$\mathscr{L} = \frac{1}{2}\left(\phi_1 \Box \phi_1 - m^2 \phi_1^2 - m^2 \tilde{\lambda} \partial_0 \partial_0 \tilde{\lambda} + m^2 \tilde{\lambda} \nabla^2 \tilde{\lambda} - m^4 \tilde{\lambda}^2\right), \tag{6.100}$$

which reduces to

$$\mathscr{L} = \frac{1}{2}\phi_1\left(\Box - m^2\right)\phi_1 + \frac{1}{2}\lambda_1\left(\Box - m^2\right)\lambda_1 \tag{6.101}$$

in terms of $\lambda_1 = m^2\tilde{\lambda}$.

We deduce that the Lagrangian (6.101) describes two degrees of freedom, where both of them have the same sign in front of the kinetic terms, and therefore, there are no ghosts.

Exercise 2 We have seen that "boosting up the derivatives" in the $3D$ Lagrangian (6.25) leads to the Lagrangian (6.28). Show that this Lagrangian describes two propagating degrees of freedom, one of which is a ghost. Hint: Work in the transverse gauge $\partial_i T_i = 0$ and use the following decomposition:

$$T_0 = \frac{1}{\sqrt{-\nabla^2}}\phi_0, \qquad T_i = \frac{1}{\sqrt{-\nabla^2}}\varepsilon^{ij}\partial_j \phi_1. \tag{6.102}$$

Solution We consider the Lagrangian:

$$\mathscr{L} = -\frac{1}{2}T^\mu G_\mu(T) + \frac{1}{2m^2}\varepsilon^{\mu\nu\rho} G_\mu(T)\partial_\nu G_\rho(T), \tag{6.103}$$

where

$$G_\mu(T) = \frac{1}{2}\varepsilon_\mu{}^{\nu\rho} R_{\nu\rho}(T), \qquad R_{\mu\nu}(T) = 2\partial_{[\mu} T_{\nu]}. \tag{6.104}$$

Using the decomposition (6.102) the first term in the Lagrangian becomes:

$$\begin{aligned} T^\mu G_\mu(T) &= -\varepsilon_0{}^{jk} T_0 \partial_j T_k + \varepsilon^{oij} T_j \partial_0 T_i + \varepsilon^{ij0} T_i \partial_j T_0 + \varepsilon^{ijk} T_i \partial_j T_k \\ &= -\left(\frac{1}{\sqrt{-\nabla^2}}\phi_0\right)\hat{\partial}^k\left(\frac{1}{\sqrt{-\nabla^2}}\hat{\partial}_k \phi_1\right) + \left(\frac{1}{\sqrt{-\nabla^2}}\hat{\partial}_i \phi_1\right)\left(\frac{1}{\sqrt{-\nabla^2}}\hat{\partial}^i \phi_0\right) \\ &= 2\phi_0\phi_1. \end{aligned} \tag{6.105}$$

Similarly, the second term in the Lagrangian (6.103) becomes

$$\begin{aligned}\varepsilon^{\mu\nu\rho}G_\mu(T)\partial_\nu G_\rho(T) &= \varepsilon_0{}^{ij}T_j\Box\partial_i T_0 - \varepsilon_i{}^{0j}T_j\partial_0 T^i - \varepsilon_i{}^{j0}T_0\Box\partial_j T^i - \varepsilon_i{}^{jk}T_k\Box\partial_j T^i \\ &= 2\varepsilon_0{}^{ij}T_j\Box\partial_i T_0 = 2T_j\Box\hat{\partial}^j T_0 \\ &= 2\left(\frac{1}{\sqrt{-\nabla^2}}\partial^j\phi_1\right)\Box\hat{\partial}^j\left(\frac{1}{\sqrt{-\nabla^2}}\phi_0\right) = 2\phi_1\Box\phi_0. \end{aligned} \tag{6.106}$$

Substituting the calculated terms into the original Lagrangian we obtain:

$$\mathscr{L} = \frac{1}{m^2}\phi_1\left(\Box - m^2\right)\phi_0. \tag{6.107}$$

Writing

$$\phi_0 = \eta - \psi, \qquad \phi_1 = \eta + \psi \tag{6.108}$$

the Lagrangian reads

$$\mathscr{L} = \frac{1}{m^2}\left[\eta\left(\Box - m^2\right)\eta - \psi\left(\Box - m^2\right)\psi\right]. \tag{6.109}$$

The relative minus sign between the two kinetic terms shows that there are two propagating degrees of freedom, one of which is a ghost.

Exercise 3 Consider the $3D$ linearized Einstein-Hilbert Lagrangian when linearized around a Minkowski spacetime:

$$\mathscr{L} = \frac{1}{2}S^{\mu\nu}G_{\mu\nu}(S), \tag{6.110}$$

with the Einstein tensor $G_{\mu\nu}(S)$ defined by

$$G_{\mu\nu}(S) = \varepsilon_\mu{}^{\alpha\beta}\varepsilon_\nu{}^{\gamma\delta}\partial_\alpha\partial_\gamma S_{\beta\delta}. \tag{6.111}$$

Show that this Lagrangian does not describe any physical degrees of freedom. Hint: use the following decomposition:

$$S_{00} = -\frac{1}{\nabla^2}\phi_0, \qquad S_{0i} = -\frac{1}{\nabla^2}\hat{\partial}_i\phi_1, \qquad S_{ij} = -\frac{1}{\nabla^2}\hat{\partial}_i\hat{\partial}_j\phi_2, \tag{6.112}$$

with $\hat{\partial}_i \equiv \varepsilon^{ij}\partial_j$.

Solution We first rewrite the Lagrangian (6.110) as follows

$$\mathscr{L} = \frac{1}{2}\left(S^{00}G_{00} + 2S^{0i}G_{0i} + S^{ij}G_{ij}\right) = \frac{1}{2}\left(S_{00}G_{00} - 2S_{0i}G_{0i} + S_{ij}G_{ij}\right). \tag{6.113}$$

Using the definition (6.111) of the Einstein tensor and the decomposition (6.112) of $S_{\mu\nu}$ we obtain:

$$\begin{aligned} G_{00}(S) &= \hat{\partial}^i \hat{\partial}^j S_{ij} = -\nabla^2 \phi_2, \\ G_{0i}(S) &= -\partial_i \dot{\phi}_2 - \hat{\partial}_i \phi_1, \\ G_{ij}(S) &= -\frac{1}{\nabla^2}(\hat{\partial}_i \hat{\partial}_j \phi_0 + \hat{\partial}_i \partial_j \dot{\phi}_1 + \partial_i \hat{\partial}_j \dot{\phi}_1 + \partial_i \partial_j \ddot{\phi}_2) \end{aligned} \tag{6.114}$$

and hence

$$\begin{aligned} S_{00} G_{00}(S) &= \left(-\frac{1}{\nabla^2}\phi_0\right)\left(-\nabla^2 \phi_2\right) = \phi_0 \phi_2, \\ S_{0i} G_{0i}(S) &= \left(-\phi_1 \frac{1}{\nabla^2} \hat{\partial}_i \partial_i \dot{\phi}_2 - \phi_1 \frac{1}{\nabla^2} \hat{\partial}_i \hat{\partial}_i \phi_1\right) = -\phi_1^2, \\ S_{ij} G_{ij}(S) &= \phi_2 \frac{1}{\nabla^2} \frac{1}{\nabla^2} \nabla^2 \nabla^2 \phi_0 = \phi_0 \phi_2. \end{aligned} \tag{6.115}$$

Using all these expressions the Lagrangian as given in Eq. (6.113) reduces to

$$\mathscr{L} = \frac{1}{2}\left(2\phi_0\phi_2 + 2\phi_1^2\right) = \phi_0\phi_2 + \phi_1^2. \tag{6.116}$$

This Lagrangian does not describe any propagating degrees of freedom.

Exercise 4 Show that the linearized generalized Einstein tensor $\mathscr{G}^{\text{lin}}_{\mu\nu}(h)$ defined in (6.71) satisfies the Bianchi identities

$$\nabla^\mu \mathscr{G}^{\text{lin}}_{\mu\nu}(h) = 0. \tag{6.117}$$

Show that the tensor $\mathscr{G}^{\text{lin}}_{\mu\nu}(h)$ is invariant under the linear diffeomorphisms

$$\delta h_{\mu\nu} = \nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu. \tag{6.118}$$

Hint: Use that

$$[\nabla_\mu, \nabla_\nu] V_\rho = \Lambda(\bar{g}_{\mu\rho} V_\nu - \bar{g}_{\nu\rho} V_\mu). \tag{6.119}$$

Solution Taking the divergence ∇^μ of the generalized Einstein tensor gives:

$$\begin{aligned} \nabla^\mu \mathscr{G}^{\text{lin}}_{\mu\nu}(h) &= \nabla^\mu \mathscr{R}^{\text{lin}}_{\mu\nu} - \frac{1}{2}\nabla_\nu \bar{g}^{\rho\sigma} \mathscr{R}^{\text{lin}}_{\rho\sigma} + 4\Lambda \nabla^\mu h_{\mu\nu} - 2\Lambda \nabla_\nu h \\ &= \nabla^\mu \Box h_{\mu\nu} - \nabla^\mu \nabla^\rho \nabla_\mu h_{\rho\nu} - \nabla^\mu \nabla^\rho \nabla_\nu h_{\rho\mu} + \Box \nabla_\nu h \\ &\quad - \nabla_\nu\left(\Box h - \nabla^\alpha \nabla^\sigma h_{\alpha\sigma}\right) + 4\Lambda \nabla^\mu h_{\mu\nu} - 2\Lambda \nabla_\nu h. \end{aligned} \tag{6.120}$$

Using the property $[\nabla_\mu, \nabla_\nu] V_\rho = \Lambda(\bar{g}_{\mu\rho} V_\nu - \bar{g}_{\nu\rho} V_\mu)$ together with

$$[\nabla_\mu, \nabla_\nu] V_{\rho\sigma} = 2\Lambda(\bar{g}_{\rho[\mu} V_{\nu]\sigma} + \bar{g}_{\sigma[\mu} V_{\nu]\rho}), \tag{6.121}$$

the previous relation reduces to:

$$\begin{aligned}\nabla^{\mu}\mathscr{G}^{\rm lin}_{\mu\nu}(h) &= -\Lambda\nabla^{\mu}h_{\mu\nu} + \Lambda\nabla_{\nu}h - \bar{g}^{\mu\alpha}\bar{g}^{\rho\beta}[\nabla_{\alpha},\nabla_{\beta}]\nabla_{\mu}h_{\rho\nu} \\ &= -\Lambda\nabla^{\mu}h_{\mu\nu} + \Lambda\nabla_{\nu}h - \Lambda\big(3\nabla^{\rho}h_{\rho\nu} - \nabla^{\rho}h_{\rho\nu} + \nabla_{\nu}h\big) \\ &\quad - \Lambda\big(-\nabla^{\alpha}h_{\nu\alpha} + \nabla^{\rho}h_{\rho\nu} - 3\nabla^{\alpha}h_{\alpha\nu}\big) \\ &= -\Lambda\nabla^{\mu}h_{\mu\nu} + \Lambda\nabla^{\alpha}h_{\alpha\nu} + \Lambda\nabla_{\nu}h - \Lambda\nabla_{\nu}h \\ &= 0, \end{aligned} \tag{6.122}$$

where we used:

$$[\nabla_{\alpha},\nabla_{\beta}]\nabla_{\mu}h_{\rho\nu} = 2\Lambda(\bar{g}_{\mu[\alpha}\nabla_{\beta]}h_{\rho\nu} + \bar{g}_{\nu[\alpha}\nabla_{|\mu}h_{\rho|\beta]} + \bar{g}_{\rho[\alpha}\nabla_{|\mu|}h_{\beta]\nu}). \tag{6.123}$$

We now calculate the variation of the generalized Einstein tensor $\mathscr{G}^{\rm lin}_{\mu\nu}(h)$ under the linearized diffeomorphisms (6.118). We first calculate $\delta\mathscr{R}^{\rm lin}_{\mu\nu}$:

$$\begin{aligned}\delta\mathscr{R}^{\rm lin}_{\mu\nu} &= \Box\delta h_{\mu\nu} - \nabla^{\rho}\nabla_{\mu}\delta h_{\rho\nu} - \nabla^{\rho}\nabla_{\nu}\delta h_{\rho\mu} + \nabla_{\mu}\nabla_{\nu}\bar{g}^{\alpha\beta}\delta h_{\alpha\beta} \\ &= -4\Lambda\nabla_{\mu}\varepsilon_{\nu} - 4\Lambda\nabla_{\nu}\varepsilon_{\mu}, \end{aligned} \tag{6.124}$$

where we used (6.118) and the following relation:

$$\big[\nabla^{\rho},\nabla_{\mu}\big]\nabla_{\nu}\varepsilon_{\rho} = 3\Lambda\nabla_{\mu}\varepsilon_{\nu} - \Lambda\bar{g}_{\mu\nu}\nabla^{\rho}\varepsilon_{\rho}. \tag{6.125}$$

It then follows that:

$$\begin{aligned}\delta\mathscr{G}^{\rm lin}_{\mu\nu}(h) &= -4\Lambda(\nabla_{\mu}\varepsilon_{\nu} + \nabla_{\nu}\varepsilon_{\mu}) + \Lambda\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}(2\nabla_{\rho}\varepsilon_{\sigma} + 2\nabla_{\sigma}\varepsilon_{\rho}) \\ &\quad + 4\Lambda(\nabla_{\mu}\varepsilon_{\nu} - 2\nabla_{\nu}\varepsilon_{\mu}) - 2\Lambda\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}(\nabla_{\rho}\varepsilon_{\sigma} + \nabla_{\sigma}\varepsilon_{\rho}) \\ &= -4\Lambda\bar{g}_{\mu\nu}\nabla^{\rho}\varepsilon_{\rho} + 4\Lambda\bar{g}_{\mu\nu}\nabla^{\rho}\varepsilon_{\rho} = 0, \end{aligned} \tag{6.126}$$

which is what we wanted to proof.

References

1. K.S. Stelle, Renormalization of higher derivative quantum gravity. Phys. Rev. D **16**, 953 (1977)
2. K.S. Stelle, Classical gravity with higher derivatives. Gen. Relativ. Gravit. **9**, 353 (1978)
3. S. Deser, R. Jackiw, S. Templeton, Topologically massive gauge theories. Ann. Phys. **140**, 372 (1982) [Erratum-ibid. **185**, 406 (1988), Ann. Phys. **281**, 409 (2000)]
4. E.A. Bergshoeff, O. Hohm, P.K. Townsend, Massive gravity in three dimensions. Phys. Rev. Lett. **102**, 201301 (2009). arXiv:0901.1766 [hep-th]
5. E.A. Bergshoeff, O. Hohm, P.K. Townsend, More on massive 3D gravity. Phys. Rev. D **79**, 124042 (2009). arXiv:0905.1259 [hep-th]
6. T. Curtright, Generalized gauge fields. Phys. Lett. B **165**, 304 (1985)

7. J.M.F. Labastida, Massless particles in arbitrary representations of the Lorentz group. Nucl. Phys. B **322**, 185 (1989)
8. M. Henneaux, A. Kleinschmidt, H. Nicolai, Higher spin gauge fields and extended Kac-Moody symmetries. arXiv:1110.4460 [hep-th]
9. T.L. Curtright, P.G.O. Freund, Massive dual fields. Nucl. Phys. B **172**, 413 (1980)
10. B. Gonzalez, A. Khoudeir, R. Montemayor, L.F. Urrutia, Duality for massive spin two theories in arbitrary dimensions. J. High Energy Phys. **0809**, 058 (2008). arXiv:0806.3200 [hep-th]
11. E.A. Bergshoeff, J.J. Fernandez-Melgarejo, J. Rosseel, P.K. Townsend, On 'new massive' $4D$ gravity. arXiv:1202.1501 [hep-th]
12. E.A. Bergshoeff, O. Hohm, P.K. Townsend, On higher derivatives in $3D$ gravity and higher spin gauge theories. Ann. Phys. **325**, 1118 (2010). arXiv:0911.3061 [hep-th]
13. P.K. Townsend, K. Pilch, P. van Nieuwenhuizen, Selfduality in odd dimensions. Phys. Lett. B **136**, 38 (1984). Addendum-ibid. B **137**, 443 (1984)
14. S. Deser, R. Jackiw, 'Selfduality' of topologically massive gauge theories. Phys. Lett. B **139**, 371 (1984)
15. J.F. Schonfeld, A mass term for three-dimensional gauge fields. Nucl. Phys. B **185**, 157 (1981)
16. S. Deser, R. Jackiw, S. Templeton, Three-dimensional massive gauge theories. Phys. Rev. Lett. **48**, 975 (1982)
17. S. Deser, R. Jackiw, Higher derivative Chern-Simons extensions. Phys. Lett. B **451**, 73 (1999). hep-th/9901125
18. H. Lu, C.N. Pope, Critical gravity in four dimensions. Phys. Rev. Lett. **106**, 181302 (2011). arXiv:1101.1971 [hep-th]
19. S. Deser, H. Liu, H. Lu, C.N. Pope, T.C. Şişman, B. Tekin, Critical points of D-dimensional extended gravities. Phys. Rev. D **83**, 061502 (2011). arXiv:1101.4009 [hep-th]
20. C. Aragone, A. Khoudeir, Selfdual massive gravity. Phys. Lett. B **173**, 141 (1986)
21. E.A. Bergshoeff, M. Kovacevic, J. Rosseel, Y. Yin, On topologically massive spin-2 gauge theories beyond three dimensions. J. High Energy Phys. **1210**, 055 (2012). arXiv:1207.0192 [hep-th]