Chapter 4
Precession and Nutation of the Earth

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Abstract Precession and nutation of the Earth originate in the tidal forces exerted by the Moon, the Sun, and the planets on the equatorial bulge of the Earth. Discovered respectively in the 2nd century B.C. by Hipparcus and in the 18th century by Bradley, their existence and characteristics were deduced theoretically by Newton for the precession and by d'Alembert for the nutation. After a historical review we explain, both in an intuitive manner and by simple calculations, the gravitational origin and the main characteristics of the precession-nutation. Then we describe in detail two fundamental theories, one using the Lagrangian formalism, the other the Hamiltonian one. A large final part is devoted to successive improvements of the precession-nutation theory in the last decades, both when considering the Earth as a rigid body and when taking into account the small effects of non-rigidity.

4.1 Introduction

Among various astronomical phenomena that have their origin in the lunar and solar tides, the precession of the equinoxes exhibit a very small effect on the time-scale of a human life. Yet it was discovered as early as the 2nd century B.C. by Hipparcus, who was comparing the positions of the stars in his era to those recorded by his predecessor Timocharis, about 150 years earlier. In Chap. 2 of volume III of *Almagest*, Claudius Ptolemy reports the work of Hipparchus on the length of the year[1]: the most surprising fact for him was that when the return of the Sun at an equinox is measured, 1 year amounted to a little less than $365 + 1/4$ days, whereas when this

[1] No writing by Hipparcus survives. According to O. Neugbauer in *A History of Ancient Mathematical Astronomy*, he lived between 190 and 120 B.C., whereas Ptolemy lived between 100 and 170. The observations attributed to Timocharis seem to date back to a period between 300 and 270 B.C.

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return is compared to the fixed stars, he found it a little longer than this value. From this observation, Hipparcus deduced that the celestial sphere itself (with the stars fixed on it) was undergoing a slow motion with respect to the equinoxes, and vice versa.

A physical explanation of the precession had to wait for more than eighteen centuries, until Newton (1642–1727). In *Philosophiae Naturalis Principia Mathematica* published in 1687, he understood for the first time that the Earth was flattened oblately and that the precession was caused by the gravitational torque exerted by an external body (the Moon or the Sun), owing to this flattened asymmetry with respect to the direction of the external body. Newton tackles the problem of the motion of the axis of rotation of a spheroid in Corallaries 18, 20, 21, 22 of Proposition 66 of *Principia*, as an application of the three-body problem. First, he studies the motion of a satellite revolving around a planet and perturbed by the Sun. Second, he replaces the satellite by a fluid ring, composed of infinitely many independent particles. Third, he replaces the fluid ring by a rigid one, fixed to a homogeneous sphere. The rigid ring, which represents the equatorial bulge, imparts to the sphere its own motion. Newton shows that the nodal line of the equatorial plane of the sphere (containing the ring) with respect to the orbital plane of the planet around the Sun undergoes a retrograde motion, due to the gravitational perturbation of the Sun. Thus precession was explained for the first time. Later on, Newton observed that, for the Earth, the precessional motion contains two components, due to the Moon and to the Sun, and that the ratio of the amplitudes of the two components is the same as the ratio of the forces they exert in the phenomena of tides. He evaluates this ratio to be 4.5.

More than half a century later, in 1747, James Bradley (1693–1762), after observing for a period of roughly 20 years the transit of zenithal stars at Kew and Wansted, remarked that the polar axis traces, in addition to the by then well-known precession, a small loop of 18.6-year period, with an amplitude close to $9''$, that is to say far beyond the capacity of detection with the naked eye.[2]

Just after this discovery, d'Alembert (1717–1783) realized that the 18.6-year period corresponds exactly to the period of retrogradation of the nodes of the lunar orbit with respect to the ecliptic. Then he elaborated in barely a year a complete theory of the Earth's rotation, improving Newton's calculations of precession by correcting a substantial number of errors and approximations, and by making full use of the new mathematical tools of calculus. He succeeded in proving the existence and the nature of the nutation loop, showing that it originates from exactly the same cause as the precession. His book, entitled *Recherches sur la Précession des Equinoxes et sur le Nutation de l'Axe de la Terre dans le Système Newtonien* and published in 1749, must be considered as the first treatise dealing with a complete theory of the precession-nutation of the Earth. It opens the path to a new era of

[2]The discovery of nutation, following that of aberration, by Bradley, is officially recorded in a memoir as a letter to Lord Macclesfield, his protector and friend, later President of the Royal Society 1752–1764. Bradley's memoir, dated 31 December 1747, was read at the Royal Society on 14 February 1748, and published later in the Philosophical Transactions.

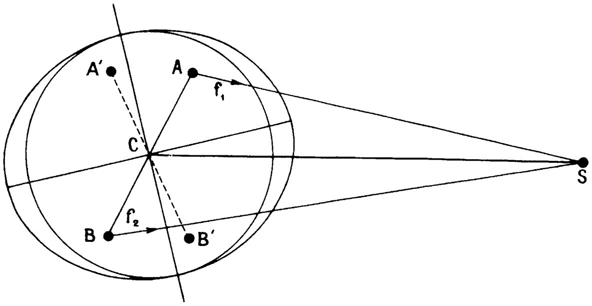

Fig. 4.1 Geometric proof of the cancellation of the torque exerted by an external body S over the sphere tangential to the ellipsoid (from [16])

increasingly refined theories of the Earth's rotation, which we will describe in the following.[3]

4.2 A Simple Geometric Explanation

In this section we show how the precession mechanism can be explained in a simple geometric way, following the exposition in Danjon [16], from Newton's calculations. We suppose here (Fig. 4.1) that the Earth is an ellipsoid of revolution, homogeneous and flattened at the poles. What is the effect of the solar attraction on the orientation of its axis of rotation in space? It is easy to show that the effect involves both a force, responsible for the orbital motion of the Earth around the Sun, and also a torque, which tends to make the equator coincide with the ecliptic.

4.2.1 Precession-Nutation due to the Sun

First we can show that the attraction of the Sun on the mass included in the sphere centered in C, and internally tangential to the ellipsoid is reduced to a force. For this let us figure out that we cut the sphere by a plan containing the center S of the Sun and the axis of the Earth. A and B represent two ranges of material, identical, normal to the plan of figure, and symmetrical with respect to the center C (Fig. 4.1). Of course the two gravitational forces $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$ exerted by the Sun on these ranges are not equivalent, neither in amplitude, nor in direction. Their effects can be replaced by a force positioned at the center C and a torque perpendicular to the plan of figure. Now we can consider two other ranges of material A′ and B′, symmetrical respectively to the ranges A and B with respect to the plan crossing C and perpendicular to the direction CS. If we reduce in the same manner as above the gravitational attraction exerted by the Sun on A′ and B′, we observe that the resulting force is equal to that of the ranges AB, whereas the resulting torque is exactly the opposite. As the

[3]Notice that Newton in his *Principia* predicted the semi-annual and semi-monthly nutations which their small amplitude rendered undetectable at his time.

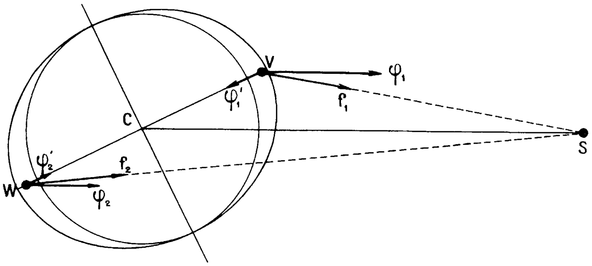

Fig. 4.2 Geometric proof of the existence of a torque exerted by an external body S on an ellipsoid (from [16])

total mass of the sphere can be completely decomposed into such groups as A, B, A′, B′, we conclude that the resulting torque on the sphere is zero.

In contrast, this is not the case of the torque exerted on the part of the ellipsoid exterior to the sphere, generally called the *equatorial bulge*. Here the symmetry which ensured the annihilation of the torques does not generally exist. A simple geometric argument proves that fact. Let us consider (Fig. 4.2) two point like elements of material V and W located at the surface of the Earth on the equator itself, symmetrical with respect to C. Then it is possible to decompose the gravitational force $\mathbf{f}_1$ exerted by the Sun on V into two other ones: $\boldsymbol{\varphi}_1$, parallel to CS and $\boldsymbol{\varphi}'_1$ towards the direction of the center CV. In the same way it is possible to decompose the force $\mathbf{f}_2$ into two sub-components $\boldsymbol{\varphi}_2$ and $\boldsymbol{\varphi}'_2$. The forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are obviously inverse to the square of the distances CV and CW to the Sun. As a consequence it is also easy to prove that the forces $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$, parallel one to each other, are proportional to the inverse of the cube of these distances. Therefore the torque due to the action of the Sun on V has an amplitude larger than the torque due to this same action on W. As a conclusion the resulting torque is represented by a vector perpendicular to the plane of figure and oriented backward. It tends to make a rotation of the segment VW clockwise, in such a way that the obliquity, i.e. the angle between the ecliptic and the equator planes, decreases.

We can remark that for two particular cases, the resulting torque exerted in the elements of material V and W is equal to zero: when the line CS is along the equatorial plane, or when it is perpendicular to that plane. The first case occurs during the equinoxes, and the second one never occurs. Indeed these two cases correspond respectively to a declination of the Sun $\delta_{Sun} = 0°$ and $\delta_{Sun} = 90°$, whereas the declination of the Sun varies in the range $\pm 23°27'$. Following the fact that the resulting torque vanishes for the two values of the declination above, it follows that the torque reaches its maximum for a value included between these two extrema. In summary we can conclude that the Sun exerts on the Earth considered as an homogeneous ellipsoid a torque varying with its declination. This torque is zero during the equinoxes, maximum during the solstices. The moment of this torque is located along the equator, and tends to decrease the obliquity.

It is easy to construct a vector which verifies those properties: let us consider (Fig. 4.3) a constant vector $\mathbf{M}_1$ in the equatorial plane directed towards the equinox γ, and a second one $\mathbf{M}_2$ with the same amplitude and symmetric of $\mathbf{M}_1$ with respect to the line CU, itself perpendicular to the solar hour circle (the great circle

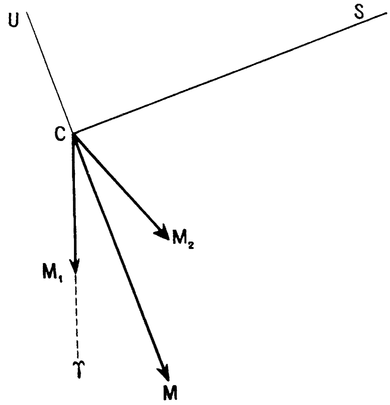

Fig. 4.3 Decomposition of the external torque M into two components: a fixed $\mathbf{M}_1$ and a moving $\mathbf{M}_2$ with the same amplitude and symmetric with respect to the CU line (from [16])

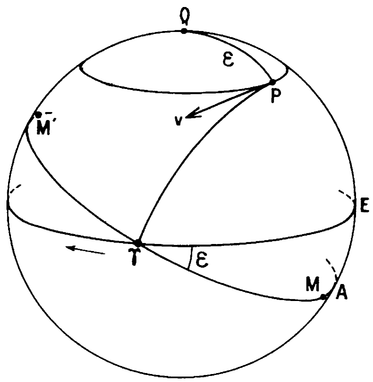

Fig. 4.4 Motion of the celestial pole P under the effect of the solar torque (from [16])

containing the poles and passing through the line CS). Then the resultant vector $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ satisfies the properties above: its amplitude is zero during the equinoxes, maximum during the solstices and it corresponds to the moment of a torque leading to a decrease of the obliquity. As a consequence it is interesting to study the action of the solar torque by analyzing independently the effects of $\mathbf{M}_1$ and $\mathbf{M}_2$.

4.2.1.1 Effect of $\mathbf{M}_1$: Precessional Motion

The first moment $\mathbf{M}_1$ tends to impart a rotation around Cγ. In the same time, the Earth undergoes a rotation around its instantaneous axis of rotation (which at first approximation can be considered as coinciding with the axis of figure). This angular velocity is represented by a vector **v** oriented towards the pole P (Fig. 4.4). Thus the perturbation caused by the solar attraction due to the first component $\mathbf{M}_1$ tends to move P closer to the equinox γ. In other words; the instantaneous pole of rotation is moving in such a way that its velocity **v** remains tangent to the hour circle of the equinox γ. At the same time, this hour circle remains tangent at P to a small circle centered in Q, the ecliptic pole, in such a way that the angle $\widehat{\text{CQP}} = \varepsilon$, where ε is the obliquity. In summary **v** is oriented towards the tangent common to the two

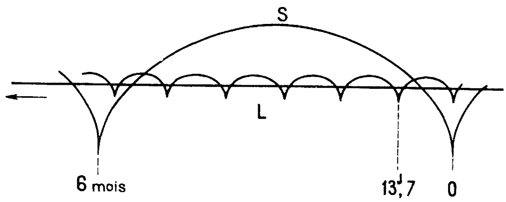

Fig. 4.5 Loops traced by the celestial pole under the effect of the precession added to the semi-annual and fortnightly nutation components. The leading component with period of 18.6 years is not included in this simplified model (from [16])

circles. As a result P describes the small circle above, the obliquity ε remaining constant. The motion of both P and γ is uniform and retrograde. In the same time, the celestial equator undergoes a rotation around the diameter MM′ perpendicular to the hour circle Pγ, and the point γ retrogrades along the ecliptic with an angular velocity of $15''.8/\text{year}$. In parallel P undergoes its circular motion with an angular velocity of $15''.5 \times \sin\varepsilon = 6''.3/\text{year}$.

4.2.1.2 Effect of $\mathbf{M}_2$: Semi-annual Nutational Motion

The instantaneous effect of the torque with moment $\mathbf{M}_2$ is the following one: if the torque acts alone, the velocity of P on the celestial sphere should be included in the hour circle of the fictitious point towards the moment $\mathbf{M}_2$. This point moves on the equator and it accomplishes two complete tropical revolutions during one tropical year. As can be verified when referring to Fig. 4.3, it is opposed to γ during the equinoxes and coincides with it during the solstices. As a result, the velocity vector of P undergoing the effect of $\mathbf{M}_2$ moves around P in the plane tangent to the celestial sphere in P, accomplishing two revolutions per year. In conclusion:

- the displacement of the pole P under the effect of $\mathbf{M}_2$ is characterized by a periodic orbit, very close to a circle, with radius $0''.55$, with a 6-month period;
- this nutation of P is naturally accompanied both by a periodic displacement of γ along the ecliptic, with amplitude $1''.3$, and a periodic variation of the obliquity, with amplitude $0''.55$. This obliquity is maximum during the equinoxes and minimum during the solstices.

4.2.1.3 Combined Precession-Nutation Motion

Now that we have characterized individually the solar precession caused by $\mathbf{M}_1$ and the solar nutation caused by $\mathbf{M}_2$ we can combine the two effects: they result in a cycloidal motion of the pole P in the celestial sphere, represented in Fig. 4.5. The turning back points correspond to the equinoxes, when the amplitude of the two torques $\mathbf{M}_1$ and $\mathbf{M}_2$ added together is zero.

4.2.2 Precession-Nutation due to the Moon

All that has been demonstrated above for the Sun can be repeated by analogy for the Moon. We know that the perturbing forces involved are proportional both to the mass of the perturbing body and to the inverse of the cube of its distance. Knowing the ratios between the masses and the distances of the Sun and the Moon, we can conclude that the amplitude of the lunar torque is roughly 2.2 times that of the Sun. As a consequence, the lunar precession in longitude, i.e. the linear retrogradation of the point γ along the ecliptic due to the sole action of the Moon, reaches roughly 2.2 times that due to the Sun, that is to say $2.2 \times 15''.8 = 34''.6$ per year.

Concerning the nutation, by analogy with the semi-annual nutation coming from the moment $\mathbf{M}_2$ due to the Sun, the corresponding moment due to the Moon gives birth to a nutation with period half that of the tropical revolution of the Moon (27.5 days). The amplitude of this semi-monthly (also called *fortnightly*) nutation is much smaller than the semi-annual one due to the Sun. The point γ oscillates around its mean (precession) motion with a $0''.2$ amplitude (instead of $1''.3$) and in the same time its obliquity can be increased or decreased by $0''.09$ (instead of $0''.55$). The corresponding displacement of the pole P in the celestial sphere is a small circle with amplitude $0''.09$.

Therefore, taking into account the effect of the Moon alone on the combined precession and nutation leads, as it is the case for the Sun, to a cycloidal motion of the pole (Fig. 4.5). In that case, this motion is characterized by 27 arches per year (27 is the number of lunar half-period cycles during one year). In fact this basic representation of the lunar nutation has been established with the implicit and simplified idea that the Moon is moving along the ecliptic, which is not the case, for its orbit presents an inclination of roughly 5° with respect to this last plane. Moreover the Moon's orbit is not fixed: its ascending node with respect to the ecliptic is precessing in the retrograde direction, with a 18.6 years period. The effect of this retrogradation is to create another nutation component, much larger than the semi-monthly one explained above. This component is often called the *principal nutation*. It is characterized by an elliptical loop (close to a circle) with $9''$ amplitude described by the pole of rotation with respect to the celestial sphere, in this same 18.6 years period. As a direct consequence, the obliquity is varying with this same amplitude of $9''$ and the γ point is undergoing an oscillation with an amplitude of roughly $18''$ alternatively in the prograde and retrograde direction along the ecliptic.

4.2.3 Global Motion of the Pole of Rotation in Space

Following all the leading effects described above, the main components of the combined gravitational torque exerted by the Moon and the Sun on the equatorial bulge of the Earth are:

- a linear retrograde displacement of the γ point (the ascending node of the ecliptic on the celestial equator) which was traditionally called the *luni-solar precession in longitude* with amplitude $34''.6 + 15''.8 = 50''.4$ per year

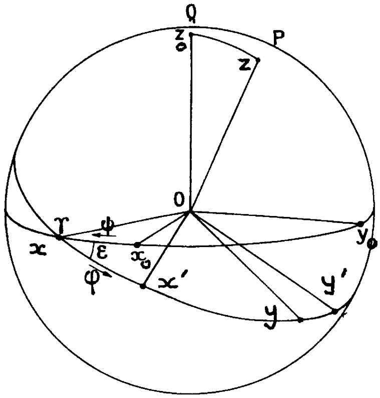

Fig. 4.6 Parametrization for the study of the precession-nutation motion of the Earth. $\Re_0 = (O, x_0, y_0, z_0)$ is a fixed inertial coordinate system; $\Re' = (O, x', y', z')$ is fixed to the Earth. $\Re = (O, x, y, z)$ is an intermediate coordinate system (from [16])

- an elliptical loop of lunar origin with amplitude roughly 9″ described in a prograde sense in 18.6 years.
- a quasi-circular loop of solar origin with amplitude 0″.55 described in a prograde sense in a half year period.

To these motions we must add the semi-monthly nutation loop already described above and a series of smaller nutational oscillations coming from secondary components in the perturbing function due in particular to the irregularities in the relative orbital motion of the Moon and the Sun around the Earth; the number of these components depends of course on the truncature limit for their amplitudes.

4.3 A Basic Mathematical Proof of the Precession-Nutation Phenomena

As we saw in the previous section, the lunisolar precession and nutation are due to the action, on the oblate Earth, of the torque exerted both by the Moon and the Sun on the Earth assimilated to an ellipsoid flattened at the equator. Since Euler's pioneering work, we know how to deal with the motion of a body around its center of gravity subject to a torque. Following the exposition in Danjon [16] we give the expression for the lunisolar torque acting on the Earth as a function of the celestial coordinates of the Sun and the Moon. Then we present the classic Euler equations for the rotational motion of our planet. The integration of the equations will furnish both the lunisolar precession and the main nutation components.

4.3.1 Reference Frames and Parametrization

In this subsection we define the rectangular reference systems which enable one to characterize the rotation of our planet, O being its center (Fig. 4.6):

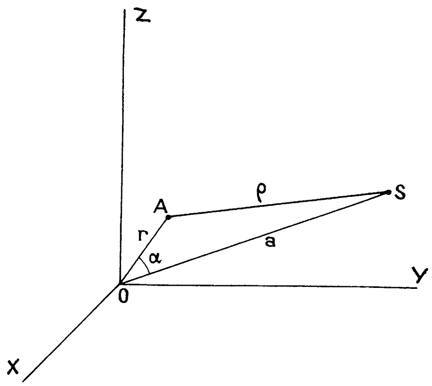

Fig. 4.7 Parameters involved in the calculations. O is the Earth barycenter. A is a point in the Earth. r is the distance OA. S is the Sun barycenter. ρ and a are respectively the distances AS and OS (from [16])

– $\Re_0 = (O, x_0, y_0, z_0)$ is constructed in such a way that (O, x_0) and (O, y_0) are two fixed directions in the ecliptic, considered as fixed in first approximation, (O, z_0) being directed towards the ecliptic pole.
– $\Re' = (O, x', y', z')$ is fixed with respect to the Earth, in such a way that the axes (O, x') and (O, y') are located in the equator, and (O, z') is oriented towards the pole of figure. We choose these axes in such a way that they coincide with the principal axes of inertia of the Earth.
– $\Re = (O, x, y, z)$ is a third rectangular coordinate system insuring the link between the two previous reference frames $\Re_0$ and $\Re'$: the (O, x) axis is oriented toward the ascending node of the ecliptic with respect to the equator, $(O, z) = (O, z')$ has been defined previously, and (O, y) completes the triad. We call A, B, C respectively the moments of inertia of the Earth with respect to these three rectangular axes.

Thus the rotation of the Earth is defined by a set of 3 angles, called Euler's angles (Fig. 4.6):

$$\psi = \widehat{x_0 x}$$
$$\varphi = \widehat{x x'}$$
$$\varepsilon = \widehat{z_0 z}$$

4.3.2 Expression of the Tidal Torque

Now let us name x, y, z the rectangular coordinates in $\Re$ of an element A of the Earth with infinitesimal mass dm, and X_S, Y_S, Z_S the coordinates of the Sun in $\Re$.

The distances OS, OA, AS are respectively noted a, r, ρ (Fig. 4.7). Let us call $M_\odot$ the mass of the Sun, and G the constant of gravitation. Therefore, the gravitational force exerted by the Sun per mass is

$$f = \frac{GM_\odot}{\rho^2} \tag{4.1}$$

By using the vectorial notation this force is expressed as:

$$\frac{GM_\odot}{\rho^3}\overrightarrow{\mathrm{AS}} = \frac{GM_\odot}{\rho^3}\overrightarrow{\mathrm{AO}} + \frac{GM_\odot}{\rho^3}\overrightarrow{\mathrm{OS}} \tag{4.2}$$

Here we can already notice that the torque with respect to O due to the first component at the right-hand side of this equation is zero. Therefore we can let it aside. The amplitude of the second component can be rewritten

$$\frac{GM_\odot a}{\rho^3} = \frac{GM_\odot}{a^2} + GM_\odot a\left[\frac{1}{\rho^3} - \frac{1}{a^3}\right] \tag{4.3}$$

The first component at the right hand side of Eq. (4.3) corresponds to an acceleration independent on the position of A on the Earth. Integrated over the whole Earth, it is the acceleration of the orbital motion of the Earth. The second component has a small amplitude, because of the relatively very close values of a and ρ. We can write

$$\frac{1}{\rho^3} - \frac{1}{a^3} = \frac{a^3-\rho^3}{a^3\rho^3} = \frac{(a-\rho)(a^2+a\rho+\rho^2)}{\rho^3 a^3} \approx \frac{3(a-\rho)}{a^4} \tag{4.4}$$

and

$$GM_\odot a\left[\frac{1}{\rho^3} - \frac{1}{a^3}\right] \approx -\frac{3GM_\odot}{a^3}(\rho - a) \tag{4.5}$$

In first approximation, we can regard a as constant, which means that the orbital motion of the Earth is a circle with radius a. Moreover, according to Kepler's third law,

$$GM_\odot = \frac{4\pi^2 a^3}{T^2} = n^2 a^3 \tag{4.6}$$

Then the combination of (4.5) and (4.6) gives

$$GM_\odot\left[\frac{1}{\rho^3} - \frac{1}{a^3}\right] \approx -3n^2(\rho - a) \tag{4.7}$$

Still in first approximation, we can consider that ρ and a are parallel (Fig. 4.7). Thus we have

$$\rho - a = -r\cos\alpha = -\frac{xX_S + yY_S + zZ_S}{a} \tag{4.8}$$

α standing for the angle $\widehat{\mathrm{AOS}}$, x, y, z and X_S, Y_S, Z_S representing respectively rectangular coordinates of A and S in $\Re$ Finally, the elementary perturbing force applied to the element of mass dm is given by

$$\mathrm{d}F = 3\frac{n^2}{a}(xX_S + yY_S + zZ_S)\,\mathrm{d}m \tag{4.9}$$

This force is parallel to OS. Its components along the three equatorial axes are respectively $(X_S/a)\,\mathrm{d}F$, $(Y_S/a)\,\mathrm{d}F$ and $(Z_S/a)\,\mathrm{d}F$.

4.3.3 Expression of the Solar Torque

Now we can express the moment of the torque coming from the elementary force $\mathrm{d}F$ on A:

$$\mathrm{d}\vec{M} = \begin{pmatrix} \mathrm{d}L_S \\ \mathrm{d}M_S \\ \mathrm{d}N_S \end{pmatrix} = \frac{\mathrm{d}F}{a} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} \tag{4.10}$$

which gives, after projection on the axes (O, x), (O, y) and (O, z):

$$\begin{aligned} \mathrm{d}L_S &= \frac{\mathrm{d}F}{a}(yZ_S - zY_S), \qquad \mathrm{d}M_S = \frac{\mathrm{d}F}{a}(zX_S - xZ_S) \\ \mathrm{d}N_S &= \frac{\mathrm{d}F}{a}(xY_S - yX_S) \end{aligned} \tag{4.11}$$

By combining (4.9) and (4.11), and after integration, we deduce the total torque exerted on the whole Earth:

$$L_S = \frac{3n^2}{a^2} \int (xX_S + yY_S + zZ_S)(yZ_S - zY_S)\,\mathrm{d}m \tag{4.12.1}$$

$$M_S = \frac{3n^2}{a^2} \int (xX_S + yY_S + zZ_S)(zX_S - xZ_S)\,\mathrm{d}m \tag{4.12.2}$$

$$N_S = \frac{3n^2}{a^2} \int (xX_S + yY_S + zZ_S)(xY_S - yX_S)\,\mathrm{d}m \tag{4.12.3}$$

The expansion of the expressions at the right hand side of these three equations is considerably simplified when using the following properties:

$$\int x\,\mathrm{d}m = \int y\,\mathrm{d}m = \int z\,\mathrm{d}m = \int xy\,\mathrm{d}m = \int xz\,\mathrm{d}m = \int yz\,\mathrm{d}m = 0 \tag{4.13}$$

$$\int \left(y^2 + z^2\right)\mathrm{d}m = A, \qquad \int \left(x^2 + z^2\right)\mathrm{d}m = B, \qquad \int \left(x^2 + y^2\right)\mathrm{d}m = C \tag{4.14}$$

$$\begin{aligned} &\int \left(y^2 - z^2\right)\mathrm{d}m = C - B, \qquad \int \left(z^2 - x^2\right)\mathrm{d}m = A - C \\ &\int \left(x^2 - y^2\right)\mathrm{d}m = B - A \end{aligned} \tag{4.15}$$

where A, B, C are the moments of inertia of the Earth with respect to the space-fixed axes (O, x), (O, y), (O, z). By taking into account all these equations, we finally get the following simplified expressions for the three components:

$$\begin{aligned} L_S &= 3n^2 \frac{Y_S Z_S}{a^2}(C - B), \qquad M_S = -3n^2 \frac{Z_S X_S}{a^2}(C - A) \\ N_S &= 3n^2 \frac{X_S Y_S}{a^2}(B - A) \end{aligned} \tag{4.16}$$

Here we still make an approximation consisting in assuming that A = B, which means implicitly that the Earth is rigorously axisymmetric with respect to the (0, z) axis. Thus the equations above become

$$L_S = 3n^2 \frac{Y_S Z_S}{a^2}(C - A), \qquad M_S = -3n^2 \frac{Z_S X_S}{a^2}(C - A), \qquad N_S = 0 \tag{4.17}$$

The Sun is moving along the ecliptic plane (O, x_0, y_0). Then it is possible to replace the celestial rectangular coordinates X_S, Y_S, Z_S of the Sun by their expression as function of its distance $a =$ OS from the Earth, of its ecliptic longitude $\lambda_\odot$ counted from γ (Fig. 4.6) and the obliquity ε, taking into account the rotation with angle ε from the ecliptic to the equatorial frames:

$$X_S = a\cos\lambda_\odot, \qquad Y_S = a\cos\varepsilon\sin\lambda_\odot, \qquad Z_S = a\sin\varepsilon\sin\lambda_\odot \tag{4.18}$$

Inserting these expressions in (4.17) we get

$$L_S = 3n^2(C - A)\cos\varepsilon\sin\varepsilon\sin^2\lambda_\odot = \frac{3n^2}{4}(C - A)\sin 2\varepsilon(1 - \cos 2\lambda_\odot) \tag{4.19}$$

$$M_S = -3n^2(C - A)\sin\varepsilon\sin\lambda_\odot\cos\lambda_\odot = -\frac{3n^2}{2}(C - A)\sin\varepsilon\sin 2\lambda_\odot \tag{4.20}$$

$$N_S = 0 \tag{4.21}$$

4.3.4 Expression for the Lunar Torque

By analogy with the calculations carried out above for the Sun, we can start from Eqs. (4.16) to calculate the components of the moment of the torque exerted by the Moon on the Earth by

- replacing X_S, Y_S and Z_S by the rectangular coordinates X_M, Y_M, Z_M of the Moon
- replacing the scaling factor n^2 by the corresponding one n'^2 related to the orbital motion of the Moon around the Earth

Indeed, calling $M_\circ$ the mass of the Moon and a' its semi-major axis, we have

$$GM_\odot = n^2 a^3, \qquad GM_\circ = n'^2 a'^3 \tag{4.22}$$

which gives

$$n'^2 = n^2 \left(\frac{a}{a'}\right)^3 \left(\frac{M_\circ}{M_\odot}\right) \tag{4.23}$$

Still here, we consider at first approximation that the orbital motion of the perturbing body (the Moon) is circular, with radius a', and $a/a' = 388.93$. Moreover:

$$\frac{M_\circ}{M_\odot} = \frac{M_\circ}{M_\oplus}\frac{M_\oplus}{M_\odot} = \left(\frac{1}{81.3}\right) \times \left(\frac{1}{332\,946}\right) \approx \frac{1}{27\,068\,500} \tag{4.24}$$

where $M_\oplus$ stands for the mass of the Earth. Finally this gives $n'^2 = kn^2$ with $k = 2.174$. The next step consists in expressing the rectangular coordinates X_M, Y_M, Z_M of the Moon as a function of its distance a' from the Earth's barycenter, the ecliptic longitude of the Moon λ_M, the longitude Ω_M of the Moon's ascending node of its orbit with respect to the ecliptic, the inclination of its orbit i_M with respect to the ecliptic, and the obliquity ε. These expressions are more complex for the Moon than for the Sun, because by contrast to the Sun, which is moving along the ecliptic, our satellite is moving along an orbit inclined with $i_M \approx 5°$ with respect to this plane. Through classical geometrical transformations we get easily:

$$X_M = a' \cos \lambda_M \tag{4.25.1}$$

$$Y_M = a' \sin \lambda_M \cos \varepsilon - a' \sin \varepsilon \sin i_M \sin(\lambda_M - \Omega_M) \tag{4.25.2}$$

$$Z_M = a' \sin \lambda_M \sin \varepsilon + a' \cos \varepsilon \sin i_M \sin(\lambda_M - \Omega_M) \tag{4.25.3}$$

Finally, by substituting these coordinates of the Moon to that of the Sun in Eq. (4.17) and by neglecting the components in i_M^2, we find:

$$L_M = \frac{3}{4} n'^2 (C - A) \times \left[\sin 2\varepsilon (1 - \cos 2\lambda_M) + 2 \sin i_M \cos 2\varepsilon \big(\cos \Omega_M - \cos(2\lambda_M - \Omega_M)\big)\right] \tag{4.26.1}$$

$$M_M = -\frac{3}{2} n'^2 (C - A) \times \left[\sin \varepsilon \sin 2\lambda_M - \sin i_M \cos \varepsilon \big(\sin \Omega_M - \sin(2\lambda_M - \Omega_M)\big)\right] \tag{4.26.2}$$

$$N_M = 0 \tag{4.26.3}$$

Then the total lunisolar torque with components L, M, N is obtained by combination of the components: $\mathbf{L} = (L_S + L_M, M_S + M_M, N_S + N_M)$. We note that the component along (O, x) contains a constant term, with amplitude

$$L_{const.} = \frac{3}{4}(C - A)\big(n^2 + n'^2\big) \sin 2\varepsilon = \frac{3}{4}(C - A)(1 + k) n^2 \sin 2\varepsilon \tag{4.27}$$

4.3.5 *Equations for the Rotational Motion of the Earth*

Now we have an explicit formulation of the lunisolar torque exerted on the Earth, we apply the fundamental equation related to the angular momentum $\boldsymbol{\sigma}$:

$$\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\Re_0} = (\mathbf{L})_{\Re_0} \tag{4.28}$$

In the moving body-fixed frame $\Re'$, this equation becomes

$$\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}\right)_{\Re'} + (\boldsymbol{\omega} \wedge \boldsymbol{\sigma})_{\Re'} = (\mathbf{L})_{\Re'} \tag{4.29}$$

Let us define the Earth's rotation vector $\boldsymbol{\omega}$ with coordinates $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in $\Re'$. In matrix notation, we have

$$\boldsymbol{\sigma} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{pmatrix} \tag{4.30}$$

Thus the combination of (4.29) and (4.30) leads to

$$\begin{pmatrix} A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} \\ B\frac{\mathrm{d}\omega_2}{\mathrm{d}t} \\ C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \wedge \begin{pmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{pmatrix} = \begin{pmatrix} L' \\ M' \\ N' \end{pmatrix} \tag{4.31}$$

where L', M' and N' are the components of $\mathbf{L}$ in $\Re'$.

Projected along the axes of the body-fixed frame $\Re'$ corresponding to the principal axes of inertia, these equations give, still considering that $A = B$:

$$A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (C - A)\omega_2\omega_3 = L' \tag{4.32}$$

$$A\frac{\mathrm{d}\omega_2}{\mathrm{d}t} - (C - A)\omega_1\omega_3 = M' \tag{4.33}$$

$$C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = N' \tag{4.34}$$

Moreover the components L', M' and N' of the lunisolar torque in $\Re'$ are related to the corresponding ones L, M and N in $\Re$ through a rotation with angle φ around $(O, z) = (O, z')$ (cf. Fig. 4.6):

$$L = L'\cos\varphi - M'\sin\varphi \tag{4.35}$$

$$M = L'\sin\varphi + M'\cos\varphi \tag{4.36}$$

A combination of Eqs. (4.32), (4.33), (4.35) and (4.36) gives immediately the following set of differential equations:

$$A\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}\cos\varphi - \frac{\mathrm{d}\omega_2}{\mathrm{d}t}\sin\varphi\right) + (C - A)\omega_3(\omega_1\sin\varphi + \omega_2\cos\varphi) = L \tag{4.37}$$

$$A\left(\frac{\mathrm{d}\omega_1}{\mathrm{d}t}\sin\varphi + \frac{\mathrm{d}\omega_2}{\mathrm{d}t}\cos\varphi\right) - (C - A)\omega_3(\omega_1\cos\varphi - \omega_2\sin\varphi) = M \tag{4.38}$$

$$C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = N \tag{4.39}$$

Now we consider the components ω_x, ω_y and ω_z of the rotation vector with respect to the equatorial non rotating frame $\Re$. They can be expressed by the following elementary rotations:

$$\omega_x = -\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}, \qquad \omega_y = \frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\varepsilon, \qquad \omega_z = \frac{\mathrm{d}\varphi}{\mathrm{d}t} - \frac{\mathrm{d}\psi}{\mathrm{d}t}\cos\varepsilon \tag{4.40}$$

At the same time, these components are also deduced from ω_1, ω_2 and ω_3 in $\Re'$ by the rotation with angle φ:

$$\omega_x = \omega_1 \cos\varphi - \omega_2 \sin\varphi, \qquad \omega_y = \omega_1 \sin\varphi + \omega_2 \cos\varphi \tag{4.41}$$

After derivation of the equations above we get easily:

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t}\cos\varphi - \frac{\mathrm{d}\omega_2}{\mathrm{d}t}\sin\varphi = \frac{\mathrm{d}\omega_x}{\mathrm{d}t} + \omega_y\frac{\mathrm{d}\varphi}{\mathrm{d}t} \tag{4.42}$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t}\sin\varphi + \frac{\mathrm{d}\omega_2}{\mathrm{d}t}\cos\varphi = \frac{\mathrm{d}\omega_y}{\mathrm{d}t} - \omega_x\frac{\mathrm{d}\varphi}{\mathrm{d}t} \tag{4.43}$$

Substituting the transformations of Eqs. (4.42) and (4.43) in Eqs. (4.37) and (4.38) leads to the following equations:

$$A\left[\frac{\mathrm{d}\omega_x}{\mathrm{d}t} + \omega_y\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] + (C-A)\omega_z\omega_y = L \tag{4.44}$$

$$A\left[\frac{\mathrm{d}\omega_y}{\mathrm{d}t} - \omega_x\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] - (C-A)\omega_z\omega_x = M \tag{4.45}$$

$$C\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = N = 0 \tag{4.46}$$

Then replacing ω_x, ω_y and ω_z by their values in function of ε, φ and ψ thanks to (4.40), and after combination of terms,

$$-A\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} - (C-A)\sin\varepsilon\cos\varepsilon\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 + C\sin\varepsilon\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\psi}{\mathrm{d}t} = L \tag{4.47}$$

$$A\sin\varepsilon\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} - (C-2A)\cos\varepsilon\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right) + C\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = M \tag{4.48}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\cos\varepsilon + \frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\sin\varepsilon = 0 \tag{4.49}$$

The third equation is equivalent to $\mathrm{d}\omega_z/\mathrm{d}t = 0$, which means that the component of the vector rotation along the figure axis (O, z) is constant. Moreover we know that the rates $\mathrm{d}\psi/\mathrm{d}t$ and $\mathrm{d}\varepsilon/\mathrm{d}t$ are very small in comparison to $\mathrm{d}\varphi/\mathrm{d}t$, and also that the expressions $\mathrm{d}^2\psi/\mathrm{d}t^2$ $\mathrm{d}^2\psi/\mathrm{d}t^2$ and $\mathrm{d}\psi/\mathrm{d}t \times \mathrm{d}\varepsilon/\mathrm{d}t$ are negligible at first approximation. Therefore the system of Eqs. (4.47) to (4.49) can be simplified to

$$C\sin\varepsilon\frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = L \tag{4.50}$$

$$C\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = M \tag{4.51}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = 0 \tag{4.52}$$

where L (respectively M) is the sum of a solar contribution L_S (respectively M_S) and a lunar one L_M (respectively M_M), given by Eqs. (4.19) and (4.26.1) (respectively (4.20) and (4.26.2)). The integration of both Eqs. (4.50) and (4.51) will naturally give the theoretical expression of the precession-nutation in longitude ψ and in obliquity ε. The lunisolar precession in longitude, which is the linear variation $\psi_1 t$ of ψ comes from the constant part L^0 of L:

$$\begin{aligned} L^0 = L_S^0 + L_M^0 &= \frac{3}{4}n^2(C-A)\sin 2\varepsilon + \frac{3}{4}n'^2(C-A)\sin 2\varepsilon \\ &= \frac{3}{4}\left[n^2(1+k)\right](C-A)\sin 2\varepsilon \end{aligned} \tag{4.53}$$

Inserting this value in Eq. (4.50) we get

$$\psi_1 = \frac{3}{2}\frac{n^2(1+k)}{\dot{\varphi}}\frac{(C-A)}{C}\cos\varepsilon \tag{4.54}$$

The physical parameter $(C-A)/C$ characterizes the relative difference between the moments of inertia C along the figure axis, and A perpendicular to it: it is called the *dynamical ellipticity*, in the case $A = B$ considered here. A priori all the quantities at the right hand side of this equation are known with very good accuracy, excepted this last parameter. n is the mean motion of the Earth: $n = 2\pi rd./y$. Moreover $n/\dot{\varphi} = T_{s.d.}/T_y = 1/366.24$ where $T_{s.d.}$ is the period of a sidereal day and T_y is the period of the sidereal revolution of the Earth. The obliquity ε can be considered as constant when not taking into account the small variations $\Delta\varepsilon$ calculated in the following and due to the nutation $\varepsilon \approx 23°26'21''$, which gives $\cos\varepsilon \approx 0.9174$. On the other side, the lunisolar precession ψ_1 is well known and determined with high precision from observational data. Its value is set at $\psi_1 = 50''.37/y$. Therefore a remarkable fact is that from (4.54) we can deduce theoretically the value of the dynamical flattening $(C-A)/C$ from the value of ψ_1 determined observationally.

$$\begin{aligned} \frac{C-A}{C} &= \frac{2}{3}\frac{\psi_1}{n}\frac{\dot{\phi}}{n}\frac{1}{1+k}\frac{1}{\cos\varepsilon} \\ &= \frac{2}{3} \times \frac{50.37}{360 \times 3600} \times \frac{366.24}{1+2.74} \times \frac{1}{0.9174} = \frac{1}{306.8} \end{aligned} \tag{4.55}$$

Noting $\Delta\psi$ and $\Delta\varepsilon$ the nutations respectively in longitude and in obliquity, that is to say the periodic components of ψ and ε, we have, from (4.50) and (4.51),

$$C\sin\varepsilon\dot{\varphi}\frac{\mathrm{d}\Delta\psi}{\mathrm{d}t} = L_S^{per.} + L_M^{per.} \tag{4.56}$$

$$C\dot{\varphi}\frac{\mathrm{d}\Delta\varepsilon}{\mathrm{d}t} = M_S^{per.} + M_M^{per.} \tag{4.57}$$

where the symbol *per.* stands fore the periodic part of each corresponding component of the torque.

By substituting to $L_S^{per.}$, $L_M^{per.}$, $M_S^{per.}$, $M_M^{per.}$ their expressions in Eqs. (4.19), (4.26.1) (4.20) and (4.26.2) we find

$$\begin{aligned} \frac{\mathrm{d}\Delta\psi}{\mathrm{d}t} = &-\frac{3}{2}\frac{n^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\cos\varepsilon\cos 2\lambda_\odot - \frac{3}{2}\frac{kn^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\cos\varepsilon\cos 2\lambda_M \\ &+ \frac{3}{2}\frac{kn^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\frac{\cos 2\varepsilon}{\cos\varepsilon}\sin i_M\left[\cos\Omega_M - \cos(2\lambda_M - \Omega_M)\right] \end{aligned} \tag{4.58}$$

$$\frac{\mathrm{d}\Delta\varepsilon}{\mathrm{d}t} = -\frac{3}{2}\frac{n^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\sin\varepsilon\sin 2\lambda_\odot - \frac{3}{2}\frac{kn^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)$$
$$\times\left[\sin\varepsilon\sin 2\lambda_M - \sin i_M\cos\varepsilon\big(\sin\Omega_M - \sin(2\lambda_M - \Omega_M)\big)\right] \quad (4.59)$$

Finally we can separate the nutations in longitude coming from the solar and lunar parts:

$$\Delta\psi = \Delta\psi^{Sun} + \Delta\psi^{Moon}, \qquad \Delta\varepsilon = \Delta\varepsilon^{Sun} + \Delta\varepsilon^{Moon} \quad (4.60)$$

with

$$\Delta\psi^{Sun} = -\frac{3}{4}\frac{n^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\cos\varepsilon\frac{\sin 2\lambda_\odot}{n} \quad (4.61)$$

$$\Delta\psi^{Moon} = \frac{3}{2}\frac{kn^2}{\dot{\varphi}}\frac{(C-A)}{C}\left[\frac{\sin i_M\cos 2\varepsilon}{\sin\varepsilon}\frac{\sin\Omega_M}{\dot{\Omega}_M} - \frac{\cos\varepsilon\sin 2\lambda_M}{2n'}\right.$$
$$\left. - \frac{\sin i_M\cos 2\varepsilon}{\sin\varepsilon}\frac{\sin(2\lambda_M - \Omega_M)}{2n' - \dot{\Omega}_M}\right] \quad (4.62)$$

And for the nutation in obliquity

$$\Delta\varepsilon = \Delta\varepsilon^{Sun} + \Delta\varepsilon^{Moon} \quad (4.63)$$

with

$$\Delta\varepsilon^{Sun} = -\frac{3}{4}\frac{n^2}{\dot{\varphi}}\left(\frac{C-A}{C}\right)\sin\varepsilon\frac{\cos 2\lambda_\odot}{n} \quad (4.64)$$

$$\Delta\varepsilon^{Moon} = -\frac{3}{2}\frac{kn^2}{\dot{\varphi}}\frac{(C-A)}{C}\left[\sin i_M\cos\varepsilon\frac{\cos\Omega_M}{\dot{\Omega}_M} - \frac{\sin\varepsilon\cos 2\lambda_M}{2n'}\right.$$
$$\left. - \sin i_M\cos\varepsilon\frac{\cos(2\lambda_M - \Omega_M)}{2n' - \dot{\Omega}_M}\right] \quad (4.65)$$

Numerically this gives

$$\Delta\psi = -17''.16\sin\Omega_M - 1''.263\sin 2\lambda_\odot - 0''.205\sin 2\lambda_M$$
$$- 0''.034\sin(2\lambda_M - \Omega_M) \quad (4.66)$$
$$\Delta\varepsilon = 9''.17\cos\Omega_M + 0''.548\cos 2\lambda_\odot + 0''.089\cos 2\lambda_M$$
$$+ 0''.018\cos(2\lambda_M - \Omega_M) \quad (4.67)$$

In this section we have explained in a simplified manner, following Danjon [16] the various steps which lead to the theoretical expressions of the combined precession-nutation of the Earth in space, that is to say the motion of its figure axis in space, when undergoing the lunisolar torque. For that purpose we have made several approximations:

- We neglected the variations of distance of the perturbing bodies (Moon and Sun) considering that their relative motion is circular.
- We assumed that the Earth is axisymmetric ($A = B$), whereas in reality the Earth is triaxial, although the relative difference of moments of inertia $(A - B)/C$ is much smaller than the difference $(C - A)/C$.

– We considered here the Earth as a rigid body whereas in reality we must take into account several effects of non-rigidity: those due to the elastic mantle with a fluid outer core and a solid inner core as well as the influence of the oceans and the atmosphere, which can no more be neglected in comparison with the accuracy of modern observations.
– We neglected 2nd-order terms in Eqs. (4.47), (4.48) and (4.49).
– We did not take into account the gravitational effects of the planets which, although being considerably much smaller than the lunisolar ones, are not negligible when compared with up-to-date observational accuracy.
– The problem has been considered in the Newtonian framework while we must take into account the *geodetic precession* and *geodetic nutation* which are a time-dependent rotation of the geocentric celestial reference system (GCRS) with respect to the barycentric celestial reference system (BCRS) due to General relativity.
– All our calculations were done with a small precision, to 3 significant digits.

It is clear that all these simplifications, although allowing a straightforward and clear demonstration, are not satisfactory as soon as a good accuracy is requested. In the following we describe how the theory of the precession-nutation was pushed to a remarkable precision thanks to recent developments, and in particular by taking into account all the corrections mentioned above.

4.4 Alternative Theories of Precession-Nutation for a Rigid Earth Model

Best modeling the precession-nutation of the real Earth supposes at first step a very accurate determination of this motion when considering the simplified case of a rigid Earth. This will serve as a basis for a more complete and accurate theory including geophysical, atmospheric and oceanic contributions. After pioneering works done by Woolard [79] and Kinoshita [37, 38] to elaborate a very complete theory for rigid Earth precession-nutation, the drastic improvement of observational techniques such as VLBI, reaching the sub-milliarcsecond accuracy during the 1980's required new investigations to develop theories available up to the same level of precision. Competitive works appeared in the 1990's to accomplish this challenge. They consisted essentially in an improvement of the already well established theories mentioned above: Kinoshita's theory based on Hamiltonian formalism with the help of canonical variables [39, 65–67]. Woolard's theory based on the equivalent principles of the theorem of angular momentum and Lagrangian equations [4, 5, 54]. In the following we describe the theoretical basis of these two theoretical ways of calculation.

The latter approach was used also by Capitaine et al. [12] with a new parameterization replacing the traditional Euler angles (see Sect. 4.4.1.1); this refers to the CIP (Celestial intermediate pole) and the CIO (Celestial intermediate origin) as introduced by the IAU 2000 Resolutions (see Sect. 4.4.1.6).

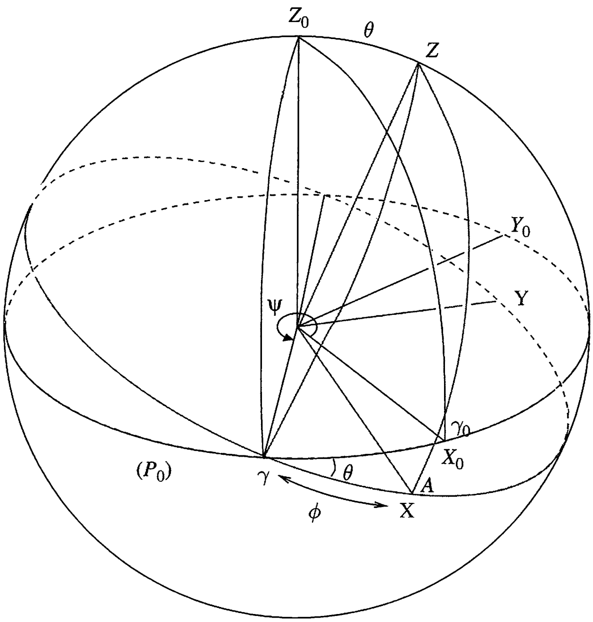

Fig. 4.8 Parametrization of the rotation of a rigid body with Eulerian variables. $\Re_0 = (O, X_0, Y_0, Z_0)$ is a fixed inertial reference frame and $\Re = (O, X, Y, Z)$ is the body fixed moving reference frame. The obliquity θ is the angle between the axes (O, Z_0) and (O, Z). The precession ψ enables one to determine the position of the nodal line (O, γ) between the body fixed equatorial plane (O, X, Y) and the fixed reference plane (O, X_0, Y_0). The angle of proper rotation ϕ enables one to determine the position of the prime meridian (O, X, Z) with respect to the nodal axis (O, γ) (from [79])

4.4.1 Dynamical Equations of the Rotation of the Rigid Earth with Lagrangian Formalism

All the calculations in this chapter are taken from Woolard [79]. The most classical way to represent the rotation of the rigid Earth with respect to a fixed reference frame is done through the Eulerian angles (Fig. 4.8). The two reference frames necessary for the calculations are an inertial one $\Re_0 = (O, X_0, Y_0, Z_0)$ where O stands for the center of mass of the Earth, and a body-fixed one $\Re = (O, X, Y, Z)$ in such a way that (O, Z) is directed towards the axis of maximum moment of inertia C of the Earth, also defined as the figure axis. (O, X) is oriented towards a point on the equator of figure and (O, Z) completes the triad. The three axes (O, X), (O, Y), and (O, Z) coincide respectively with the principal axes of inertia of the Earth, namely A, B and C, with $A < B < C$.

4.4.1.1 Eulerian Parametrization

The Eulerian angles can be defined as follows (Fig. 4.8):

- θ, the *obliquity angle* (or simply the obliquity) represents the inclination of the equator of figure with respect to the fixed plane $(P_0) = (O, X_0, Y_0)$. It is generally reckoned positively.
- ψ, the *precession angle*, is defined in the fixed plane (O, X_0, Y_0) between a reference point γ_0 on (P_0) and the line (O, γ) along which the equator of figure

(O, X, Y) crosses the plane (P_0). For the sake of simplicity we can make γ_0 coinciding with (O, X_0). Therefore we can write $\psi = \gamma_0\gamma$.
- ϕ, sometimes called the *proper rotation*, is the angle between the axes (O, γ) and (O, X).

Thus the position of the axis of figure of the Earth in space is given by the set of the three angles (ψ, θ, ϕ).

4.4.1.2 Euler Kinematical Equations

The rotational motion of the Earth, considered as a rigid body, about its center of mass with respect to $\Re_0$ is the combination of three independent rotations:

– a rotation at rate $\dot{\psi}$ around Z_0.
– a rotation at rate $\dot{\theta}$ around the moving line of the node $(0, \gamma)$ of the equator of figure on the fixed plane (P_0).
– a rotation at rate $\dot{\phi}$ around the axis of figure.

These three individual rotations compound into a resultant rotation vector $\boldsymbol{\omega}$ around the *instantaneous axis of rotation* passing through the center of mass O. Its amplitude ω is the angular velocity around this axis. Therefore the position of the axis of rotation with respect to the Earth-fixed coordinate system $\Re$ is given at any instant by the coordinates $(\omega_1, \omega_2, \omega_3)$ of the rotation vector $\boldsymbol{\omega}$, which are linked to the derivatives of the Eulerian angles through a system of equations called the *Euler's kinematical equations*. These are

$$\omega_1 = -\dot{\theta}\cos\phi - \dot{\psi}\sin\theta\sin\phi \tag{4.68}$$

$$\omega_2 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \tag{4.69}$$

$$\omega_3 = \dot{\psi}\cos\theta + \dot{\phi} \tag{4.70}$$

And reciprocally, this can be written:

$$\dot{\psi}\sin\theta = -\omega_1\sin\phi - \omega_2\cos\phi \tag{4.71}$$

$$\dot{\theta} = -\omega_1\cos\phi + \omega_2\sin\phi \tag{4.72}$$

$$\dot{\phi} = \omega_3 + \cot\theta(\omega_1\sin\phi + \omega_2\cos\phi) \tag{4.73}$$

4.4.1.3 Lagrange Formalism

A straightforward way to determine the equations of the rotational motion consists in applying Lagrange's equations. The kinetic energy of the Earth is expressed in the following classical form:

$$T = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) \tag{4.74}$$

Now we choose the Eulerian angles ψ, θ and ϕ as the generalized coordinates q_i $(i = 1, 2, 3)$. Then the Lagrangian function is

$$L = T + U = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + U \tag{4.75}$$

where U is the force function (potential) representing the lunisolar perturbing potential, which will be explicated in Sect. 4.4.3. The system can be considered as conservative, so that Lagrange equations can be applied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{4.76}$$

which gives, after expansion [79]:

$$A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (C - B)\omega_2\omega_3 = \frac{\sin\phi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial \phi} - \frac{\partial U}{\partial \psi}\right) - \cos\phi\frac{\partial U}{\partial \theta} \tag{4.77}$$

$$B\frac{\mathrm{d}\omega_2}{\mathrm{d}t} - (C - A)\omega_1\omega_3 = \frac{\cos\phi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial \phi} - \frac{\partial U}{\partial \psi}\right) + \sin\phi\frac{\partial U}{\partial \theta} \tag{4.78}$$

$$C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} + (B - A)\omega_1\omega_2 = \frac{\partial U}{\partial \phi} \tag{4.79}$$

These equations are generally called the *Euler's dynamical equations*

4.4.1.4 Method of Variation of Parameters

The external forces that act to affect the rotational motion of the Earth are so comparatively small that the equations of motion may be integrated efficiently by the method of variation of parameters: in a first step we determine a simplified solution that would occur were the external forces to vanish ($U = 0$). Then the solution is approximatively modified (through the parameters involved) to get the motion in actual conditions. As we have already mentioned we consider that

$$\frac{A - B}{C} \ll \frac{C - A}{C} \tag{4.80}$$

so that, at first approximation, the set of Eqs. (4.77) to (4.79) becomes, taking into account that $\frac{\partial U}{\partial \phi} = 0$, due to the symmetry:

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \left(\frac{C - A}{A}\right)\omega_2\omega_3 = -\frac{\sin\phi}{A\sin\theta}\frac{\partial U}{\partial \psi} - \frac{\cos\phi}{A}\frac{\partial U}{\partial \theta} \tag{4.81.1}$$

$$\frac{\mathrm{d}\omega_2}{\mathrm{d}t} - \left(\frac{C - A}{A}\right)\omega_1\omega_3 = -\frac{\cos\phi}{A\sin\theta}\frac{\partial U}{\partial \psi} + \frac{\sin\phi}{A}\frac{\partial U}{\partial \theta} \tag{4.81.2}$$

$$\omega_3 = cte. \tag{4.81.3}$$

According to the method of variation of parameters, we first consider that $U = 0$. Therefore the right-hand side of Eqs. (4.81.1) and (4.81.2) reduces to zero and putting

$$\mu = \frac{C - A}{A}\omega_3 \tag{4.82}$$

leads to the trivial equations:

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \mu\omega_2\omega_3 = 0 \tag{4.83.1}$$

$$\frac{\mathrm{d}\omega_2}{\mathrm{d}t} - \mu\omega_1\omega_2 = 0 \tag{4.83.2}$$

$$\omega_3 = cte. \tag{4.83.3}$$

with the obvious solutions

$$\omega_1 = f_0 \cos\mu t + g_0 \sin\mu t \tag{4.84.1}$$

$$\omega_2 = f_0 \sin\mu t - g_0 \cos\mu t \tag{4.84.2}$$

where f_0 and g_0 are constants of integration. Thus, we show from these equations that when external forces vanish ($U = 0$), and by accepting the condition of axisymmetry ($A = B$), the axis of rotation of the Earth describes with respect to the Earth-fixed reference frame $\Re$ a motion circular and uniform around the axis of figure, represented by the axis (O, Z). This motion, called the *free polar motion*, is described with a frequency $\mu = (C - A/C)\omega_3$.

The second step in the method of variation of parameters consists in adopting the same kind of formalism as in (4.84.1) and (4.84.2) for ω_1 and ω_2 but by replacing the constants f_0 and g_0 by functions f and g depending on time:

$$\omega_1 = f \cos\mu t + g \sin\mu t \tag{4.85.1}$$

$$\omega_2 = f \sin\mu t - g \cos\mu t \tag{4.85.2}$$

Inserting these expressions in Eqs. (4.81.1) and (4.81.2) enables to determine f and g by quadrature:

$$f = f_0 - \int \left(\frac{\sin(\phi + \mu t)}{A \sin\theta} \frac{\partial U}{\partial \psi} + \frac{\cos(\phi + \mu t)}{A} \frac{\partial U}{\partial \theta} \right) \mathrm{d}t \tag{4.86.1}$$

$$g = g_0 + \int \left(\frac{\cos(\phi + \mu t)}{A \sin\theta} \frac{\partial U}{\partial \psi} - \frac{\sin(\phi + \mu t)}{A} \frac{\partial U}{\partial \theta} \right) \mathrm{d}t \tag{4.86.2}$$

Then the combination of Eqs. (4.71), (4.72) and (4.73) with (4.86.1) and (4.86.2) gives the variations of the Eulerian angles which determine the position of the axis of figure in space:

$$\sin\theta \frac{\mathrm{d}\psi}{\mathrm{d}t} = -f \sin(\phi + \mu t) + g \cos(\phi + \mu t) \tag{4.87.1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -f \cos(\phi + \mu t) - g \sin(\phi + \mu t) \tag{4.87.2}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_3 - \cos\theta \left(\frac{\mathrm{d}\psi}{\mathrm{d}t} \right) \tag{4.87.3}$$

The last equation can be rewritten, using Eq. (4.82), as

$$\frac{\mathrm{d}(\phi + \mu t)}{\mathrm{d}t} = \frac{C}{A}\omega_3 - \cos\theta \left(\frac{\mathrm{d}\psi}{\mathrm{dt}} \right) \tag{4.88}$$

4.4.1.5 The Motion of the Axis of Figure in Space

Equations (4.87.1) and (4.87.2) can be transformed into a more advantageous form, by substituting the expressions of f and g given by Eqs. (4.86.1) and (4.86.2) in the right-hand side, and after differentiating both members with respect to t. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) = (\dot{\phi}+\mu)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{A\sin\theta}\frac{\partial U}{\partial\psi} \tag{4.89}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = -(\dot{\phi}+\mu)\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}t} + \frac{1}{A}\frac{\partial U}{\partial\theta} \tag{4.90}$$

Using Eq. (4.88) and after re-combination, we finally obtain the derivative of the two precession-nutation angles:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{C\omega_3\sin\theta}\frac{\partial U}{\partial\psi} + \frac{A}{C\omega_3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) + \frac{A}{C\omega_3}\cos\theta\frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{4.91}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{C\omega_3\sin\theta}\frac{\partial U}{\partial\theta} - \frac{A}{C\omega_3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) + \frac{A}{C\omega_3}\cos\theta\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 \tag{4.92}$$

4.4.1.6 Modern Parametrization Based on IAU 2000 Resolutions

The Euler dynamical equations and the method of variation of parameters described in the previous sections have been used by Capitaine et al. [12] for a modern semi-analytical resolution (analytical representation with numerical coefficients) of the precession-nutation equations based on the CIO based parameters.

The celestial and terrestrial intermediate origins (CIO and TIO respectively), have been defined as origins on the equator of the CIP, based on the concept of the "non-rotating origin" [25] when the CIP moves in space and in the Earth, respectively. Their kinematical property provides a very straightforward definition of the Earth's diurnal rotation based on the *Earth Rotation Angle* (ERA) along the equator of the CIP (Celestial Intermediate Pole) between those two origins, which is linearly related to UT1. The ERA replaces the third Euler angle ϕ from the nodal axis (O, γ).

The CIP is defined as being the intermediate pole, in the transformation between the celestial and terrestrial systems, separating nutation from polar motion by a specific convention in the frequency domain. That convention is such that (i) the GCRS (Geocentric Celestial Reference System) CIP motion includes all the terms with periods greater than 2 days in the GCRS (i.e. frequencies between -0.5 cycles per sidereal day (cpsd) and $+0.5$ cpsd); (ii) the ITRS (International Terrestrial Reference System) CIP motion, includes all the terms outside the retrograde diurnal band in the ITRS (i.e. frequencies less than -1.5 cpsd or greater than -0.5 cpsd).

The CIO based precession-nutation parameters consist in the GCRS coordinates of the CIP unit vector, either in their polar form, E and d, or their rectangular form, $X = \sin d\cos E$, $Y = \sin d\sin E$ (see Fig. 4.9); they contain precession and nutation of the CIP, frame bias between the equator and equinox frame at J2000.0 and

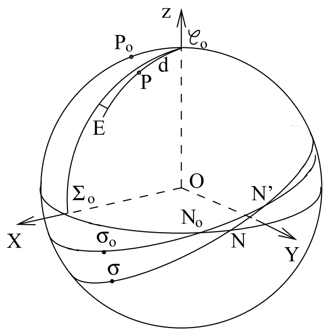

Fig. 4.9 Parametrization of the precession-nutation of the equator using the CIO (σ) based parameters: the coordinates of the CIP unit vector (either E and d, or $X = \sin d \cos E$, $Y = \sin d \sin E$)

the GCRS, plus the cross terms between precession and nutation [9, 10]. Y and X replace respectively the first and second Euler angles, θ and ψ.

The CIO (Celestial Intermediate Origin) based precession-nutation equations for a rigid axially symmetric Earth are as follows [12]:

$$-\ddot{Y} + (C/A)\Omega\dot{X} = L_\Sigma/A + F'' \quad (4.93.1)$$

$$\ddot{X} + (C/A)\Omega\dot{Y} = M_\Sigma/A + G'' \quad (4.93.2)$$

Ω being the mean Earth's angular velocity, L_Σ and M_Σ the equatorial components of the torque referred to Σ (such that $\Sigma N = \Sigma_0 N$), and F'', G'' functions of X, Y and of their first and second time derivatives.

4.4.2 Dynamical Equations of the Rotation of the Rigid Earth with Hamiltonian Formalism

An alternative way to construct a theory of the rotation for a rigid Earth model consists in starting from Andoyer variables [1] instead of Eulerian angles. One of the advantages of such a choice comes from the fact that Andoyer variables are canonical. Thus it is rather easy to apply a perturbation theory based on canonical transformations to the rotational motion, and to separate precessional from nutational motion, or, in other words, secular perturbations from periodic ones [38]. Moreover it is easy to treat separately the motion of the figure, rotation and angular momentum axes.

4.4.2.1 Andoyer Angles Parametrization

We start from the two reference frames, the inertial one $\Re_0 = (O, X_0, Y_0, Z_0)$ and the Earth-fixed one $\Re = (O, X, Y, Z)$ as defined previously. We call $\mathbf{L}$ the angular momentum vector of the Earth. The plane (P_0) has the same meaning as in the previous section, whereas (P_L) stands for the plane perpendicular to $\mathbf{L}$. The Andoyer variables consist in 3 action variables (L, G, H) and 3 angle variables (l, g, h) defined as follows (Fig. 4.10):

Action variables

They are defined with respect to the angular momentum vector.

- G is the amplitude of the angular momentum vector $\mathbf{L}$
- L is the component of $\mathbf{L}$ along the axis (O, Z)
- H is the component of $\mathbf{L}$ along the axis (O, Z_0)

From these definitions we can already introduce two angles which play a fundamental role in the theory: I is the inclination of $\mathbf{L}$ with respect to (O, Z_0) and J its inclination with respect to (O, Z) in such a way that

$$L = G\cos J, \qquad H = G\cos I \tag{4.94}$$

Notice that I represents the obliquity of the axis of angular momentum with respect to the inertial axis (O, Z) and must not be confused with the classical obliquity, i.e. the angle between the axis of figure and the basic plane (P_0) (generally the ecliptic).

Angle variables

As in the case of the Eulerian angles they enable one to give the orientation of $\Re$ with respect to $\Re_0$, but with the intermediary of the plane (P_L) which is not involved in this first case.

- h is the angle measured along the reference plane (P_0) between the fixed point γ_0 and the node Q of (P_L) with respect to (P_0). Notice that it represents the angle of precession but for the equator of angular momentum (P_L) instead of the precession ψ of the equator of figure.
- g is the angle along the equator of angular momentum (P_L) between Q defined previously and the ascending node P of the equator of figure with respect to (P_L).
- l is the angle along the equator of figure between the ascending node P and the Earth-fixed origin axis (O, X).

4.4.2.2 Relationships Between Eulerian and Andoyer Variables

In Fig. 4.10 we represent the reference planes and axes defined together with the Eulerian angles and Andoyer action and angles. The relationships between these two set of variables can be derived from the spherical triangle (P, Q, γ) defined previously. These are

$$\cos\theta = \cos I \cos J - \sin I \sin J \cos g \tag{4.95.1}$$

$$\frac{\sin(\psi - h)}{\sin J} = \frac{\sin(\phi - l)}{\sin I} = \frac{\sin g}{\sin\theta} \tag{4.95.2}$$

In the case of the Earth, the angle J is very small. Its amplitude does not exceed $1''$. This means that the angular momentum axis is nearly coinciding with the

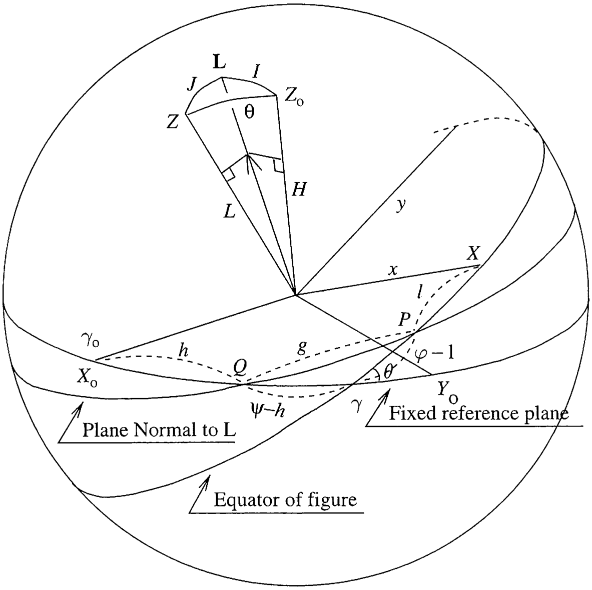

Fig. 4.10 Parametrization of the rotation of a rigid body with Andoyer variables with respect to the fixed inertial reference frame $\Re_0 = (O, X_0, Y_0, Z_0)$. The three angle variables l, g, h enable one to determine the body fixed reference frame $\Re = (O, X, Y, Z)$ with respect to $\Re_0$. The three action variables are L and H, respectively projections of the vector angular momentum $\mathbf{L}$ on the body fixed axis (O, Z) and the inertial axis (O, Z_0), and G, the norm of $\mathbf{L}$ (from [38])

figure axis. Therefore at the first order in J we can write from the two equations above

$$\psi = h + \frac{J}{\sin I} \sin g + O\left(J^2\right) \tag{4.96.1}$$

$$\theta = I + J \cos g + O\left(J^2\right) \tag{4.96.2}$$

$$\phi = l + g - J \cot I \sin g + O\left(J^2\right) \tag{4.96.3}$$

4.4.2.3 Andoyer Variables Referred to the Fixed and Moving Ecliptics

In practical case, the inertial reference frame $\Re_0$ is defined in such a way that the (O, X_0) axis is directed towards the fixed mean equinox of a reference epoch T_0 (for instance J2000.0), the plane $(P_0) = (O, X_0, Y_0)$ coinciding with the mean ecliptic of T_0. For the study of the rotation of the Earth, it is convenient to adopt instead of $\Re_0$ a moving reference frame $\bar{\Re} = (O, \bar{X}, \bar{Y}, \bar{Z})$ in such a way that the plane $(\bar{P}) = (O, \bar{X}, \bar{Y})$ coincides to the moving ecliptic of the date, $(O, \bar{Z})$ being directed towards the axis of this ecliptic, and $(O, \bar{X})$ towards the *departure point* as defined by Kinoshita [38]. Now we can define the motions of the moving ecliptic of date with respect to the fixed ecliptic of epoch through the set of two variables π_e and Π_e: they stand respectively for the inclination and the longitude of the node of $(\bar{P})$ with respect to (P_0).

Kinoshita [38] showed that it is possible to transform the variables h, I, and g referred to the fixed plane (P_0) into a corresponding set of variables h', I' and g' referred to a slightly moving reference plane $(\bar{P})$, in such a way that the old set of canonical variables (l, g, h, L, G, H) is transformed in a new canonical one (l', g', h', L', G', H'). If $\mathscr{H}$ designates the Hamiltonian of the system of the rotational motion of the rigid Earth, the new Hamiltonian, called $\mathscr{K}$, satisfies

$$G\,\mathrm{d}g + H\,\mathrm{d}h - \mathscr{H}\,\mathrm{d}t = G'\,\mathrm{d}g' + H'\,\mathrm{d}h' - \mathscr{K}\,\mathrm{d}t \tag{4.97}$$

with

$$G' = G, \qquad H' = G\cos I' \tag{4.98}$$

We can remark that the variables l and L do not depend on the reference frame: $l' = l$, and $L' = L$ The new Hamiltonian becomes

$$\mathscr{K} = \mathscr{H} + E \tag{4.99}$$

with

$$\begin{aligned} E = {} & H'(1 - \cos\pi_e)\frac{\mathrm{d}\Pi_e}{\mathrm{d}t} \\ & + G\sin I'\left[\frac{\mathrm{d}\Pi_e}{\mathrm{d}t}\sin\pi_e\cos\left(h' - \Pi_e\right) - \frac{\mathrm{d}\pi_e}{\mathrm{d}t}\sin\left(h' - \Pi_e\right)\right] \end{aligned} \tag{4.100}$$

4.4.2.4 Hamiltonian of the System and Canonical Equations of the Rotational Motion

According to Kinoshita [37, 38], the Hamiltonian $\mathscr{K}$ for the rotational motion of the rigid Earth referred to the slightly moving reference frame $\bar{\mathfrak{R}}$ defined in the previous section is

$$\mathscr{K} = F_0 + E + U \tag{4.101}$$

- F_0 represents the kinetic energy of the rotational motion. It is written

$$F_0 = \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)\left(G^2 - L^2\right) + \frac{1}{2C}L^2 \tag{4.102}$$

- E is the complementary Hamiltonian component due to the change of reference frame as explained in the previous section.

$$\begin{aligned} E = {} & H'(1 - \cos\pi_e)\frac{\mathrm{d}\Pi_e}{\mathrm{d}t} \\ & + G\sin I'\left[\frac{\mathrm{d}\Pi_e}{\mathrm{d}t}\sin\pi_e\cos\left(h' - \Pi_e\right) - \frac{\mathrm{d}\pi_e}{\mathrm{d}t}\sin\left(h' - \Pi_e\right)\right] \end{aligned} \tag{4.103}$$

- U is the disturbing potential for the rotational motion due to the external bodies (Moon, Sun and planets). An exhaustive study of the determination of U will be given in the next section.

4.4.2.5 Equations of Motion

The equations of motion of the rotation of the rigid Earth are directly derived from the property of canonicity of the adopted Andoyer variables (l, g', h', L, G, H'). For the sake of simplicity, the prime symbols are removed from the variables, and in the moving reference frame $\bar{\mathfrak{R}}$, the canonical equations can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(L, G, H) = -\frac{\partial \mathscr{K}}{\partial (l, g, h)} \tag{4.104.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(l, g, h) = \frac{\partial \mathscr{K}}{\partial (L, G, H)} \tag{4.104.2}$$

4.4.2.6 Precession-Nutation of the Axis of Angular Momentum

In the scope of this chapter dealing with the precession-nutation motion, we essentially focus on the two variables h and I which represent respectively the precession angle and the obliquity of the axis of the plane perpendicular to the angular momentum vector. We have already explained that I is defined starting from the canonical variables through the relationship $H = G\cos I$. In addition we call Δh and ΔI the periodic variations h and I. By writing W the determining function defined by

$$W = \int \left(U_1^{per} + U_2^{per}\right) \mathrm{d}t \tag{4.105}$$

where U_1^{per} and U_2^{per} stand for the periodic parts of respectively U_1 and U_2, we have

$$\Delta h = -\frac{\partial W}{H} = \frac{\partial W}{\partial I}\frac{\partial I}{\partial H} = -\frac{1}{G\sin I}\frac{\partial W}{\partial I} \tag{4.106}$$

$$\Delta I = \frac{1}{G}\left(-\frac{1}{\sin I}\Delta H + \cot I\,\Delta G\right) = \frac{1}{G}\left(\frac{1}{\sin I}\frac{\partial W}{\partial h} - \cot I\frac{\partial W}{\partial g}\right) \tag{4.107}$$

4.4.2.7 Precession-Nutation of the Figure Axis

The nutation of the figure axis $(\Delta\psi_f, \Delta\varepsilon_f)$ are directly deduced from the geometrical relationships (4.96.1) and (4.96.2) linking the axis of angular momentum and the axis of figure[4]:

$$\Delta\psi_f = \Delta h + \Delta\left[\frac{J\sin g}{\sin I}\right] + O\left(J^2\right) \tag{4.108}$$

$$\Delta\varepsilon_f = \Delta\theta = \Delta I + \Delta[J\cos g] + O\left(J^2\right) \tag{4.109}$$

[4]The reader must pay attention to the sign conventions: here ψ and h as well as ε and I have the same signs. This follows the classical geometrical rules of a positive sign in the trigonometric (counter clockwise) sense, whereas for conventional astronomical rules ψ and ε are counted positively in the clockwise sense.

with

$$\Delta\left(\frac{J\sin g}{\sin I}\right) = \frac{1}{\sin I}(\sin g\Delta J + J\cos g\Delta g) - \frac{J\sin g}{\sin^2 I}\Delta I$$
$$= \frac{1}{G\sin I}\left[\frac{\partial W}{\partial g}\sin g - W\cos g\right] \tag{4.110}$$

and

$$\Delta(J\cos g) = \frac{1}{G}\left[\frac{\partial W}{\partial g}\cos g + W\sin g\right] \tag{4.111}$$

These two last expressions are called the *Oppolzer terms*.

4.4.3 The Determination of the Disturbing Potential U

As seen in Sects. 4.4.1 and 4.4.2, whatever be the theory used to determine the rotational motion, and more precisely the precession-nutation of the Earth, it necessitates the precise calculation of the disturbing potential U exerted by the external body (Moon, Sun, planet). This disturbing potential can be represented by expansion in spherical harmonics of first order (U_1) and second one (U_2) [69]

$$U = U_1 + U_2 \tag{4.112}$$

with

$$U_1 = \frac{GM}{r^3}\left[\frac{2C-A-B}{2}P_2(\sin\delta) + \frac{A-B}{4}P_2^2(\sin\delta)\cos 2\alpha_E\right] \tag{4.113}$$

$$U_2 = \sum_{n=3}^{\infty}\frac{GMM_Ea_E^n}{r^{n+1}} \times \left[J_nP_n(\sin\delta) - \sum_{m=1}^{n}P_n^m(\sin\delta)(C_{n,m}\cos m\alpha_E + S_{nm}\sin m\alpha_E)\right] \tag{4.114}$$

where α_E stands for the geocentric longitude of the perturbing body as measured from a prime meridian on the Earth[5] and δ its declination. M is the mass of the perturbing body, r its distance from the center of the Earth. J_n, C_{nm} and S_{nm} are the coefficients of the geopotential, which characterize the repartition of mass inside the Earth. P_i^j are the Legendre polynomials of degree i and order j.

4.4.3.1 Use of Ecliptic Coordinates

The reference plane to measure the precession-nutation motion being the ecliptic of the date, it is convenient to express the Legendre polynomials $P_2(\sin\delta)$ and

[5] α_E must not be confused with the classical right ascension α measured from an equinox.

$P_2^2(\sin\delta)\cos 2\alpha$ as a function of the ecliptic longitude λ and latitude β of the perturbing body. Kinoshita et al. [40] showed how this step is possible trough the intermediary of the modified Jacobi polynomials. After expansion, we find [38]:

$$\begin{aligned}
P_2(\sin\delta) = {} & \frac{1}{2}\left(3\cos^2 J - 1\right)\left[\frac{1}{2}\left(3\cos^2 I - 1\right)P_2(\sin\beta)\right. \\
& \left. - \frac{1}{2}\sin 2I\, P_2^1(\sin\beta)\sin(\lambda - h) - \frac{1}{4}\sin^2 I\, P_2^2(\sin\beta)\cos 2(\lambda - h)\right] \\
& + \sin 2J\left[-\frac{3}{4}\sin 2I\, P_2(\sin\beta)\cos g\right. \\
& - \frac{1}{4}\sum_{\varepsilon=\pm 1}(1 + \varepsilon\cos I)(-1 + 2\varepsilon\cos I)P_2^1(\sin\beta)\sin(\lambda - h - \varepsilon g) \\
& \left. - \sum_{\varepsilon=\pm 1}\frac{1}{8}\varepsilon\sin I(1 + \varepsilon\cos I)P_2^2(\sin\beta)\cos(2\lambda - 2h - \varepsilon g)\right] \\
& + \sin^2 J\left[\frac{3}{4}\sin^2 I\, P_2(\sin\beta)\cos 2g\right. \\
& + \frac{1}{4}\sum_{\varepsilon=\pm 1}\varepsilon\sin I(1 + \varepsilon\cos I)P_2^1(\sin\beta)\sin(\lambda - h - 2\varepsilon g) \\
& \left. - \frac{1}{16}\sum_{\varepsilon=\pm 1}(1 + \varepsilon\cos I)^2 P_2^2(\sin\beta)\cos 2(\lambda - h - \varepsilon g)\right]
\end{aligned} \quad (4.115)$$

The same kind of expansion as a function of I, J, the coordinates λ, β and the Andoyer variables l, g, h is done for the expression $P_2^2(\sin\delta)\cos 2\alpha_E$ which takes place in the part of the potential, depending on the triaxiality in Eq. (4.113).

Analytically, Woolard [79] and Kinoshita [38] showed that it is possible for the Moon as well as for the Sun to express the functions $P_2(\sin\beta)$, $P_2^1(\sin\beta)\sin(\lambda - h)$ and $P_2^2(\sin\beta)\cos 2(\lambda - h)$ etc. as Fourier series with arguments Θ_ν themselves combination of the five Delaunay arguments, which are

- l the mean anomaly of the Moon
- l' the mean anomaly of the Sun
- Ω the longitude of the node
- $F = \lambda_M - \Omega$, where λ_M is the mean longitude of the Moon
- $D = \lambda_M - \lambda_S$, where λ_S is the mean longitude of the Sun.

Kinoshita and Souchay [39] as well as Souchay et al. [67] generalized this kind of formulation by introducing the mean longitudes of the planets (excepted Neptune whose influence is negligible) when including the direct and indirect planetary perturbations, as well as a the general precession on longitude p_A.

In these works, Θ_ν are written

$$\begin{aligned}
\Theta_\nu = {} & i_1 l + i_2 l' + i_3 F + i_4 D + i_5\Omega + i_6\lambda_{Me} + i_7\lambda_{Ve} + i_8\lambda_{Ea} \\
& + i_9\lambda_{Ma} + i_{10}\lambda_{Ju} + i_{11}\lambda_{Sa} + i_{12}\lambda_{Ur} + i_{13}p_A
\end{aligned} \quad (4.116)$$

Thus we can adopt a generic formula for the expansions used:

$$\frac{1}{2}\left(\frac{a}{r}\right)^3\left(1-3\sin^2\beta\right)=\sum_\nu A_\nu^0\cos\Theta_\nu \tag{4.117.1}$$

$$\left(\frac{a}{r}\right)^3\sin\beta\cos\beta\sin\lambda=\sum_\nu A_\nu^1\cos\Theta_\nu \tag{4.117.2}$$

$$\left(\frac{a}{r}\right)^3\sin\beta\cos\beta\cos\lambda=-\sum_\nu A_\nu^1\sin\Theta_\nu \tag{4.117.3}$$

$$\left(\frac{a}{r}\right)^3\cos^2\beta\cos 2\lambda=\sum_\nu A_\nu^2\cos\Theta_\nu \tag{4.117.4}$$

$$\left(\frac{a}{r}\right)^3\cos^2\beta\sin 2\lambda=-\sum_\nu A_\nu^2\sin\Theta_\nu \tag{4.117.5}$$

4.4.3.2 Generic Formula for the Expressions of the Potential *U*

According to (4.113) the determination of U presupposes the knowledge of $(\frac{a}{r})^3 P_2(\sin\delta)$ and $(\frac{a}{r})^3 P_2^2(\sin\delta)\cos 2\alpha_E$. Kinoshita [38] has shown that these expressions can be conveniently expanded in the following form:

$$\begin{aligned}\left(\frac{a}{r}\right)^3 P_2(\sin\delta)=&\ \frac{3}{2}\left(3\cos^2 J-1\right)\sum_\nu B_\nu\cos\Theta_\nu\\&-\frac{3}{2}\sin 2J\sum_{\varepsilon=\pm1}\sum_\nu C_\nu(\varepsilon)\cos(g-\varepsilon\Theta_\nu)\\&+\frac{3}{4}\sin^2 J\sum_{\varepsilon=\pm1}\sum_\nu D_\nu(\varepsilon)\cos(2g-\varepsilon\Theta_\nu)\end{aligned} \tag{4.118}$$

and

$$\begin{aligned}&\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\cos 2\alpha_E\\&=-\frac{9}{2}\sin^2 J\sum_\nu B_\nu\cos(2l-\varepsilon\Theta_\nu)\\&\quad-3\sum_{\rho=\pm1}\sin J(1+\rho\cos J)\sum_{\varepsilon=\pm1}\sum_\nu C_\nu(\varepsilon)\cos(g+2\rho l-\varepsilon\Theta_\nu)\\&\quad-\frac{3}{4}\sum_{\varepsilon=\pm1}\sum_{\rho=\pm1}(1+\rho\cos J)^2\sum_\nu D_\nu(\varepsilon)\cos(2g+2\rho l-\varepsilon\Theta_\nu)\end{aligned} \tag{4.119}$$

in which

$$B_\nu=-\frac{1}{6}\left(3\cos^2 I-1\right)A_\nu^0-\frac{1}{2}\sin 2I A_\nu^1-\frac{1}{4}\sin^2 I A_\nu^2 \tag{4.120}$$

$$C_\nu(\varepsilon) = -\frac{1}{4}\sin 2I A_\nu^0 + \frac{1}{2}(1+\varepsilon\cos I)(-1+2\varepsilon\cos I)A_\nu^1 + \frac{1}{4}\varepsilon\sin I(1+\varepsilon\cos I)A_\nu^2 \quad (4.121)$$

$$D_\nu(\varepsilon) = -\frac{1}{2}\sin^2 I A_\nu^0 + \varepsilon\sin I(1+\varepsilon\cos I)A_\nu^1 - \frac{1}{4}(1+\varepsilon\cos I)^2 A_\nu^2 \quad (4.122)$$

4.4.4 Generic Formula for the Expressions of the Nutations $\Delta\psi$ and $\Delta\varepsilon$

Once the potential U has been expressed as a Fourier series the nutations are determined in a straightforward manner by simple integration and partial derivatives with respect to I and h, following Eqs. (4.105), (4.106) and (4.107) [38]

• Nutation of the angular momentum axis.

For the angular momentum the nutations in longitude $\Delta\psi_{AM}$ and in obliquity $\Delta\varepsilon_{AM}$ are given by

$$\Delta\psi_{AM} = \Delta h = -\frac{1}{G\sin I}\left(\frac{\partial W}{\partial I}\right) + O(J) = k\sum\frac{E_\nu}{N_\nu}\sin\Theta_\nu \quad (4.123)$$

with

$$E_\nu = \left[A_\nu^0 - \frac{1}{2}A_\nu^2\right]\cos I - \frac{\cos 2I}{\sin I}A_\nu^1 \quad (4.124)$$

and

$$\Delta\varepsilon_{AM} = \Delta I = \frac{1}{G\sin I}\left(\frac{\partial W}{\partial h}\right) + O(J) = \frac{k}{\sin I}\sum i_5\frac{B_\nu}{N_\nu}\cos\Theta_\nu \quad (4.125)$$

where i_5 is the coefficient of Ω in the argument Θ_ν (see Eq. (4.116)) and $N_\nu = \dot{\Theta}_\nu$.

k is a scaling factor given by

$$k = 3\frac{GM}{a^3\omega_E}\frac{2C-A-B}{2C} = 3\frac{GM}{a^3\omega_E}H_d \quad (4.126)$$

where M is the mass of the perturbing body, a the semi-major axis of its orbit (in the case of the solar potential, M is the mass of the Sun and a the semi-major axis of the Earth), and ω_E the sidereal angular velocity of the Earth. $H_d = (2C-A-B)/2C$ is called the *dynamical ellipticity* of the Earth.[6] We will discuss below how it is determined from the precession deduced from observations.

• Nutations of the figure axis.

The nutations in longitude $\Delta\psi_f$ and in obliquity $\Delta\varepsilon_f$ of the figure axis of the Earth are deduced from the Oppolzer terms whose expressions have already been

[6]In the case of axisymmetry, $H_d = (C-A)/C$.

given in Eqs. (4.110) and (4.111). We have, at the first order in J [38]:

$$\Delta\psi_f = \Delta\psi_{AM} + \frac{1}{G\sin I}\left(\frac{\partial W}{\partial g}\sin g - W\cos g\right) \tag{4.127}$$

$$= \Delta\psi_{AM} + \frac{k}{\sin I}\sum_{\nu}\sum_{\varepsilon=\pm 1}\frac{\varepsilon C_\nu(\varepsilon)}{n_g - \varepsilon N_\nu}\sin\Theta_\nu \tag{4.128}$$

$$\Delta\varepsilon_f = \Delta\varepsilon_{AM} - \frac{1}{G}\left(\frac{\partial W}{\partial g}\cos g + W\sin g\right) \tag{4.129}$$

$$= \Delta\varepsilon_{AM} + k\sum_{\nu}\sum_{\varepsilon=\pm 1}\frac{C_\nu(\varepsilon)}{n_g - \varepsilon N_\nu}\cos\Theta_\nu \tag{4.130}$$

4.5 Modern Precession-Nutation Theories for a Rigid Earth Model

Face to the tremendous improvement by an order 2 or 3 of the precision in the determination of the coefficients of nutation thanks to the VLBI technique, it became necessary, at the end of the 1980's, to construct a new rigid Earth nutation model with an extreme accuracy, at the level of the sub-milliarcsecond (mas). During the 1990's several groups undertook this work, which was reckoned as fundamental. These efforts lead to the a definitive publication of the three different models called SMART97 [5], RDAN97 [54] and REN2000 [67]. These three models based on different theoretical foundations give very close results for the nutation coefficients when each of them is compared to the others [64]. The level of truncature for each coefficient of the related series of nutation in the three works was set at least to 0.1 µas instead of 0.1 mas, that is a factor 1000, with respect to the previous series constructed by Kinoshita [38] about two decades earlier. This new truncature level required to take into account more than one thousand components of nutation instead of the 106 ones in this last paper. It forced also the authors above to include new kinds of contribution, which although being very small, cannot be ignored. After giving a brief review of the way of construction of the three kinds of series of nutation above, we present in detail each of these second-order contributions

The three nutation series SMART97, RDAN97, and REN2000 differ by the methodology used for their construction. Nevertheless they give all very close results for precession-nutation of the three axes concerned: the axis of angular momentum, the axis of rotation and the axis of figure. All the related theories use the analytical solution VSOP87 [3] for the motion of the Sun and the planets, and the analytical solution ELP2000 [15] for the orbital motion of the Moon.

- For the construction of SMART97, Bretagnon et al. [5] used an iterative analytical method based on Eulerian dynamical equations, already described in Sect. 4.4.1.3. They also used a numerical integration to test the validity of the analytical developments, finding a remarkable agreement, of 16 µas for ψ and 8 µas for ε.

- For RDAN97, Roosbeek and Dehant [54] used the torque approach. The Lagrangian equations expressing the rigid Earth response to the torque induced by the external bodies, as seen in Sect. 4.4.1, are solved analytically. In order to validate and further test their analytical model, they have also computed a benchmark series called RDNN97 built from the DE403/LE403 ephemerids [68] and from a numerical integration. Their comparison between RDAN97 and RDNN97 shows that, in the time domain, the maximum difference is 62 µas for $\Delta\psi$ and 29 µas for $\Delta\varepsilon$, whereas in the frequency domain they are respectively 6 µas and 4 µas.
- For REN2000, Souchay et al. [67] up-dated the theory set up by Kinoshita [38] based on Hamiltonian equations and described in details in Sect. 4.4.3. Souchay [64] compared also the analytical nutation given by this series with numerical integration. The r.m.s. of the residuals do not exceed 5 µas both for $\Delta\psi \sin\varepsilon$ and $\Delta\varepsilon$.

The three independent models of nutation for a rigid Earth model mentioned above show a remarkable agreement both between themselves (at the level of 1 µas for the amplitude of each individual coefficient) as with numerical integration of the equations of motion. We can conclude that any of these models is well suited to serve as a basis for a more sophisticated theory of nutation involving a real Earth with non rigid aspects.

Note that an iterative semi-analytical method based on Eulerian dynamical equations similar to that of Bretagnon (1997) was proposed [12] for integrating the equations directly as functions of the coordinates of the CIP in the GCRS.

4.5.1 The Construction of a Highly Accurate Rigid Earth Precession-Nutation Model

The construction of a highly accurate rigid Earth precession-nutation model requires a very accurate determination of the dynamical ellipticity of the Earth as well as the investigation of second-order contributions which cannot be neglected anymore, as given the level of truncature (0.1 µas) of the Fourier series of nutation. They can be enumerated as:

- the direct planetary effects
- the indirect planetary effects
- the effects of the triaxiality of the Earth
- the contributions due to second-order geopotential (J_3, J_4)
- the crossed-nutation effects
- the J_2 and planetary tilt effects
- the geodetic precession

4.5.1.1 The Observed Precession and the Determination of the Dynamical Ellipticity of the Earth

The fit between the observed value of the lunisolar precession in longitude and its theoretical formula allows the determination of the dynamical ellipticity of the Earth $H_d = (2C - A - B)/2C$ which is the fundamental parameter for the calculation of the potential U in Eq. (4.113), and as a consequence for the calculation of the amplitude of all the nutation coefficients. The general precession in longitude p_A up-dated by Williams [78] is $5028''.7700$/cy, which represents a $-0''.3266$/cy correction with respect to a previous value adopted by the IAU 1976 General Assembly [41]. As it was explained by Kinoshita and Souchay [65], this very accurate value coming from modern observations, in particular from VLBI data, and established with respect to the moving ecliptic of the date, includes not only the lunisolar precession, but also a combination of second-order effects. These are a spin-orbit coupling effect in the Earth-Moon system ($0''.380$/cy), the effects due to the J_4 geopotential ($-0''.0026$/cy), to the direct planetary gravitational influence ($-0''.0321$/cy), to the *geodetic precession* as given by Barker and O'Connell [2]. This geodetic precession, also called the *De Sitter precession* is the relativistic rotation of the geocentric fixed celestial system with respect to the barycentric one. Its amplitude is $1''.9194$/cy [78].

Moreover, the difference due to the adoption of a fixed ecliptic or a moving ecliptic, called *planetary precession* amounts to $11''.8745$/cy. All these contributions must be removed from the observed value of the general precession in longitude p_A, to isolate the sole lunisolar contribution ψ_A with respect to a fixed ecliptic. We find $\psi_A = 5040.6445''$/cy. Now ψ_A is also given by the following formula [65]:

$$\begin{aligned}\psi_A &= \psi_A^{Moon} + \psi_A^{Sun}\\ &= 3H_d\left(\left(\frac{M_\circ}{M_\circ + M_\oplus}\right)\left(\frac{n_M^2}{\Omega}M_0 + \left(\frac{M_\odot}{M_\odot + M_\circ + m_\oplus}\right)\left(\frac{n_E^2}{\Omega}\right)S_0\right)\right)\cos\varepsilon_A\end{aligned} \tag{4.131}$$

where $M_\odot$, $M_\oplus$ and $M_\circ$ are respectively the masses of the Sun, the Earth and the Moon, n_E and n_M the mean motions of the Earth-Moon barycenter and of the Moon, and Ω the sidereal angular rate of rotation of the Earth. M_0 and S_0 are quantities coming directly from computations of the lunisolar potential. They are respectively the constant terms in the expressions $\frac{1}{2}(\frac{a}{r})^3(1 - 3\sin^2\beta)$ for the Moon and for the Sun, where λ, β and r are the ecliptic coordinates of the perturbing body (Sun or Moon) with respect to the moving equinox and ecliptic of the date (for the Sun $\beta \approx 0$).

The correspondence between ψ_A as deduced from the observational value of p_A, at the left hand side of Eq. (4.131), and its theoretical expression at the right hand side enables to determine H_d given its status of sole unknown parameter. In fact each of the nutation theories above is associated by its own estimation of H_d. Bretagnon et al. [4] in SMART97 find $H_d = 0.0032737668$ while Souchay and Kinoshita [66] in REN2000 have $H_d = 0.0032737548$ and Roosbeek and Dehant in RDAN97 have $H_d = 0.0032737674$.

4.5.1.2 The Main Lunar Terms

The major contribution to the nutation, as considering the importance of the effect and the number of coefficients, comes from the Main Problem of the Moon, that is to say from the three-body problem involving the Moon orbiting around the Earth in a quasi-Keplerian motion greatly perturbed by the Sun [15]. As a consequence the influence of the planetary perturbations on the Moon's orbit are treated independently and will be considered later. Thus the only arguments entering in the expansions of the angles Θ_ν are the Delaunay's arguments l, l', F, D and Ω. The leading nutation components are those with arguments Ω and 2Ω and respective periods 18.6 y and 9.3 y [67]: $\Delta\psi = -17''.2805921 \sin\Omega + 0''.2090296 \sin 2\Omega$ and $\Delta\varepsilon = 9''.22289220 \cos\Omega - 0''.0903611 \sin 2\Omega$. For the nutation figure axis of the rigid Earth, Souchay et al. [67] find 583 coefficients in longitude and 486 in obliquity, when adopting a truncature level of 0.1 μas.

4.5.1.3 The Main Solar Terms

The main solar terms are those due to the quasi-Keplerian motion of the Earth. In other words, it comes from the expansions of the geocentric ecliptic coordinates of the Sun $\lambda_\odot$, $\beta_\odot \approx 0$ and $r_\odot$ involved in the expression of the solar potential given by Eqs. (4.113) and (4.115) following classical expansions of $a/r_\odot$ and $\lambda_\odot$ as a function of the eccentricity of the Earth. These terms, with arguments Θ_ν linear combinations only of the five Delaunay's arguments in Eq. (4.116), must be separated from those with arguments Θ_ν including also the mean longitude of the planets, which are coming from the perturbations of the planets and are called indirect planetary effects (see Sect. 4.5.1.5). The larger term in the category of the main solar terms is the semi-annual one, with period 182.621 days (see Eqs. (4.66) and (4.67)); its amplitude is $\Delta\psi_{s.a.} = -1''.317090 \sin(2F - 2D + 2\Omega)$ in longitude and $\Delta\varepsilon_{s.a.} = 0''.573034 \cos(2F - 2D + 2\Omega)$ in obliquity.

4.5.1.4 The Direct Planetary Effects

Like the Moon and the Sun, the planets induce also nutations of the Earth's axes. Vondrak [70] calculated for the first time these direct influences of the planets on the nutation, showing that they could reach the 0.1 mas level for individual components. Independently of the three theories considered here, Williams [78] calculated all the coefficients related to the direct influence of the planets, up to 0.5 μas, both for $\Delta\psi \cos\varepsilon$ and for $\Delta\varepsilon$. At this level of truncature he found 1, 103, 26, 22, 5 and 1 terms respectively for Mercury, Venus, Mars, Jupiter, Saturn and Uranus, the influence of Neptune being negligible. Exhaustive tables of the direct influences of the planets included in REN2000 can be found in Souchay and Kinoshita [66] which show a perfect agreement with Williams [78]. The argument of each component is a linear combination of the longitude of the Earth λ_{Ea}, of the longitude of the perturbing planet considered and of the general precession in longitude p_A. The leading terms in longitude and obliquity are by far due to Venus and Jupiter. They are, in µas:

$$\begin{aligned}\Delta\psi = {} & 215.0\sin(3\lambda_{Ve}-5\lambda_{Ea}-2p_A)+84.6\sin(\lambda_{Ve}-\lambda_{Ea})\\ & -50.4\sin(4\lambda_{Ve}-6\lambda_{Ea}-2p_A)+34.9\sin(2\lambda_{Ve}-4\lambda_{Ea}-2p_A)\\ & +35.0\sin(2\lambda_{Ve}-2\lambda_{Ea})-106.2\sin(2\lambda_{Ju}+2p_A)+33.4\sin\lambda_{Ju}\end{aligned} \tag{4.132}$$

$$\begin{aligned}\Delta\varepsilon = {} & 93.2\cos(3\lambda_{Ve}-5\lambda_{Ea}-2p_A)-21.9\sin(4\lambda_{Ve}-6\lambda_{Ea}-2p_A)\\ & +15.1\cos(2\lambda_{Ve}-4\lambda_{Ea}-2p_A)+46.0\cos(2\lambda_{Ju}+2p_A)\end{aligned} \tag{4.133}$$

4.5.1.5 The Indirect Planetary Effects

The indirect planetary effects, first pointed out and estimated by Vondrak [71, 72] originate from the small perturbations of the planets on the orbital motion of the Moon around the Earth and of the Earth around the Sun. These perturbations affect the relative ecliptic coordinates λ and β of the body causing the nutation (the Moon or the Sun). In their turn, these little changes cause a change in the perturbing potential exerted by the body. In REN2000 [67] the corresponding terms of nutation can be recognized easily by the nature of their arguments, as a linear combination of the Delaunay variables l, l', F, D and Ω, of the general precession in longitude p_A and of the mean longitudes of the planets λ_{Me}, λ_{Ve} etc. At high frequency the indirect planetary effects due to Moon are dominated by two components with arguments $-l_M+2F+2\Omega+18\lambda_{Ve}-16\lambda_{Ea}$ and $l_M+2F+2\Omega-18\lambda_{Ve}+16\lambda_{Ea}$ and the same amplitude of 14.1 µas in $\Delta\psi$. Moreover for a 100 y time interval, the peak-to-peak amplitude of these planetary effects are of the order of 1 mas both for the Moon's and the Sun's parts.

4.5.1.6 The Crossed Nutation Effects

When computing the coefficients of nutation at first order, through the intermediary of $P_2(\sin\delta)$ as expressed in Eq. (4.115), the obliquity angle I as well as the longitude λ of the perturbing body are determined without taking into account the nutations. In fact, they must be replaced respectively by $I+\Delta I$ and $\lambda-\Delta h$. In other words, the nutation itself causes a slight modification of the position of the equator which in its turn provokes a slight modification of the potential U exerted by the perturbing body and in the determining function W given by Eq. (4.105), which results in *crossed nutation effects* when applying Eqs. (4.106) and (4.107). They concern 68 components for $\Delta\psi$ and 40 components for $\Delta\varepsilon$, up to 0.1 µas [67]. The leading contribution concerns the component with argument 2Ω resulting from the crossed nutations of the leading term with argument Ω. It amounts to 1.220 mas for $\Delta\psi$ and -0.238 mas for $\Delta\varepsilon$.

4.5.1.7 The J_2 Tilt Effects

The J_2 tilt effect comes from the particular perturbation on the motion of the Moon around the Earth due to the shape of the Earth, i.e. its equatorial bulge. These perturbations involving J_2 modify in their turn the potential exerted by the Moon on the Earth. This change in the potential generates some additional second-order contributions which affect in a significant manner the leading nutation coefficients of lunar origin, with arguments Ω and 2Ω.

4.5.1.8 The Planetary Tilt Effect

This effect was pointed out by Williams [78]: the orbit planes of the planets have small inclinations with respect to the ecliptic plane. As a consequence of the planetary attractions, the ecliptic planes moves. The Moon's mean plane of orbital precession follows the moving ecliptic closely, but not perfectly. This motion causes a $1''.4$ tilt of the plane of orbital precession to the ecliptic. This result in an additional torque on the oblate Earth.

4.5.1.9 Effects due to the Triaxiality of the Earth

The triaxiality of the Earth is characterized by the relative difference $(B - A)/C$ between the moments of inertia along the principal axes perpendicular to the figure axis. Were the Earth perfectly axisymmetric, the triaxiality is zero. For the real Earth we have $(B - A)/(2C - A - B) = 0.0033536$. The triaxiality takes part in the perturbing potential U_2 through the expression $(A - B/4) \times P_2^2(\sin\delta)\cos 2\alpha_E$ in Eq. (4.113). The presence of the component $\cos 2\alpha_E$ with semi-diurnal period combined with long periodic components in $P_2^2(\sin\delta)$ results after integration in quasi-semi-diurnal terms of nutation. They are listed in Souchay and Kinoshita [66] up to 0.1 µas. Two reasons lead to the relatively small values of the nutations due to the triaxiality: first the smallness of the ratio above; second the fact that when carrying out the integration according to Eq. (4.105) a large frequency value appears at the denominator, due to the very high semi-diurnal frequency. This contribution is dominated by 3 coefficients at periods 0.518 d, 0.500 d, 0.499 d with respective arguments $2\Phi - 2F - 2\Omega$, $2\Phi - 2F + 2D - 2\Omega$ and 2Φ, where Φ is the angle of sidereal rotation of the Earth. The respective amplitudes are 27.1 µas, 12.5 µas, −37.8 µas for $\Delta\psi$ and 11.0 µas, 4.7 µas and 15.0 µas for $\Delta\varepsilon$. Note that the largest coefficient originates both from the influence of the Moon and of the Sun. At last the combination of these sinusoidal terms with very close frequencies leads to a beating.

4.5.1.10 Effects due to Second-Order Potential J_3

The second order geopotential coefficient J_3 acts on the second order potential exerted by the Moon with the intermediary of the component $J_3 P_3(\sin\delta)$ in

Eq. (4.114). Because of the scaling factor $(J_3/J_2) \times (a_E/a_M)$ which characterizes the amplitudes of the corresponding nutations with respect to the first order ones depending on J_2 , these amplitudes are comparatively much smaller. 17 coefficients both for $\Delta\psi$ and $\Delta\varepsilon$ are found larger than 0.1 μas [27, 66]. This contribution is characterized by a very large set of frequencies, the smallest period being 6.8 d, and the largest one 20935 y. The leading coefficient, in mas, is $-0.105\sin(-l_M + F + \Omega)$ for $\Delta\psi$ and $-0.1089\cos(-l_M + F + \Omega)$ for $\Delta\varepsilon$.

4.6 Modern Nutation Theory for a Non-rigid Earth Model

The nutation is almost entirely due to the torques resulting from the gravitational action of celestial bodies on the equatorial bulge of the Earth. At first approximation, one can use as a proxy, the so called rigid Earth nutation series representing the action of the torques on a hypothetical rigid Earth, having the same moments of inertia and high order moments as the real Earth. This rigid Earth nutation has been largely discussed in the previous sections. Nevertheless, with the appearance of modern observational techniques, and particularly the VLBI (Very Long Baseline Interferometry) in the early 1980's the precision of estimates obtainable for the Earth orientation parameters, among which the nutation $(\Delta\psi, \Delta\varepsilon)$, has increased greatly. This fact, combined with the increasing volume and longer time span of the data sets available, has made it possible to estimate the amplitudes of a significantly larger number of nutation components and a much accurate value of the precession rate. Face with these developments, the rigid-Earth nutation theory, starting from the early 1980's, could no more match the accuracy of observations. In other words, the various effects due to the non rigidity of the Earth, such as changes in matter distribution, atmospheric pressure variations, oceanic motions, frictions between the core and the mantle, etc., although remaining all relatively small, could no more be neglected in view of the quality of observational data.

4.6.1 Definition of Prograde and Retrograde Circular Nutations

In order to deal with non-rigid Earth nutations, experts in this topic, as geophysicians, introduce the concept of prograde and retrograde circular components of nutation. In the following, we present their definition. Generally astronomers are concerned with the lunisolar nutations in longitude $\Delta\psi$ and in obliquity $\Delta\varepsilon$ as a Fourier series in the form

$$\Delta\psi = \sum_{\nu} \Delta\psi_{\nu I} \sin \nu\Omega_0 t + \Delta\psi_{\nu O} \cos \nu\Omega_0 t \tag{4.134}$$

$$\Delta\varepsilon = \sum_{\nu} \Delta\varepsilon_{\nu I} \cos \nu\Omega_0 t + \Delta\varepsilon_{\nu O} \sin \nu\Omega_0 t \tag{4.135}$$

where $\Delta\psi_{\nu I}$ and $\Delta\psi_{\nu O}$ are, respectively, the in-phase and out-of-phase coefficients of the nutation in longitude, and $\Delta\varepsilon_{\nu I}$ and $\Delta\varepsilon_{\nu O}$ in obliquity. Here $\Omega_0 = 7.292115 \times 10^{-5}$ rad/s stands for the mean sidereal angular rotation rate. Notice that the out-of-phase components are generally very small with respect to the in-phase ones, for they characterize dissipative processes. Moreover we have shown in Eq. (4.116) that the frequency $\Theta_\nu = \nu\Omega_0$ is a combination of fundamental astronomical arguments.

Once the formula above have been established, the in-phase parts of the components of $\Delta\psi$ and $\Delta\varepsilon$ for any particular frequency $\Theta_\nu = \nu\Omega_0$ constitutes an elliptical nutation, whereas the out-of-phase parts constitutes another one. In their turn, these paired elliptical nutations can be resolved into two circular components, one representing a uniform rotation of the figure axis around an inertial (space-fixed) axis in the prograde sense, and the other, in the retrograde sense. The combination of the two prograde and retrograde circular nutations results respectively in the complex components $\eta^{(pro)}$ and $\eta^{(ret)}$. This is materialized by the following relationships [45]:

$$\eta^{(pro)} = -\frac{1}{2}\left(\Delta\varepsilon_{\nu I} - \frac{\nu}{|\nu|}\Delta\psi_{\nu I}\sin\varepsilon_A\right) + \frac{i}{2}\frac{\nu}{|\nu|}\left(\Delta\varepsilon_{\nu O} + \frac{\nu}{|\nu|}\Delta\psi_{\nu O}\sin\varepsilon_A\right) \tag{4.136}$$

$$\eta^{(ret)} = -\frac{1}{2}\left(\Delta\varepsilon_{\nu I} + \frac{\nu}{|\nu|}\Delta\psi_{\nu I}\sin\varepsilon_A\right) - \frac{i}{2}\frac{\nu}{|\nu|}\left(\Delta\varepsilon_{\nu O} - \frac{\nu}{|\nu|}\Delta\psi_{\nu O}\sin\varepsilon_A\right) \tag{4.137}$$

where ε_A is the mean obliquity.

4.6.2 Early Non-rigid Earth Nutation Theories

The earliest nutation theories started from the Earth as a rigid ellipsoid [38, 79]. We have seen previously that in this approximation only the Earth's principal moments of inertia and the amplitude and frequency of the tidal force are important. In pioneer works dealing with non-rigidity, Jeffreys and Vicente [35, 36] and Molodensky [50] greatly extended these results by including the effects of a fluid core (already introduced by Poincaré [53]) and of the elasticity within the mantle. They found differences from rigid Earth results of as much as $0.02''$ for both the principal nutation with period 18.6 y and the leading nutation of solar origin, with semi-annual period. In comparison with the precision of the observations in the 1970's these effects could no more be neglected. These analytical theories take into account a simplified model of core and mantle deformation computed from a spherical non-rotating shell. Shen and Manshina [59], in a numerical way, as well as Sasao et al. [55], analytically, started from these previous works to include more complete dynamical and structural models of fluid core.

4.6.3 The Nutation Series of Wahr

The nutation series of Wahr [74] has been the standard of reference for roughly 20 years. It was adopted by the International Astronomical Union (IAU) as the basic nutation series named the IAU 1980 nutation [58]. It was computed by solving the equations for the field of displacements produced by the action of the tide generating potential (TGP) throughout the Earth, as applied to an oceanless elastic, ellipsoidal Earth model derived on the assumption of a hydrostatic equilibrium. Wahr theory can be considered as a further extension of the previous investigations mentioned above, with accounting more completely for the Earth's ellipticity and rotation. For that purpose he used techniques developed by Smith [62] and Wahr [73]. The first author described the linearization of infinitesimal motion for a rotating, slightly elliptical, self-gravitating, elastic, hydrostatically prestressed and oceanless Earth. Wahr [73] demonstrated that the forced motion of a rotating Earth could be expanded as a decoupled sum of normal modes of the Earth. On the opposite of what was done previously, elliptical and rotational effects were considered by Wahr [74, 75] to compute the rotational motion.

Wahr adopted a model of Earth interior called the model 1066A of Gilbert and Dziewonski [24], based on the assumption of a hydrostatic equilibrium. In order to compute semi-analytically the nutation coefficients, Wahr [73] established a formula expressing the transfer function between a given coefficient for the rigid Earth model, with frequency ω, and the corresponding non rigid Earth coefficient. The ratio between these two coefficients is given by $\eta(a,\omega)/\eta_r(\omega)$ so that

$$\begin{aligned}\frac{\eta(a,\omega)}{\eta_r(a,\omega)} - 1 &= \left[B_0 + (\omega - 0.927\Omega)\left[\frac{B_1}{\omega_1 - \omega} + \frac{B_2}{\omega_2 - \omega} + \frac{1.06}{\omega + 3.28\times 10^{-3}\Omega}\right]\right] \\ &\quad \times \left[\frac{\Omega - \omega}{\Omega}\right]\left[\frac{\omega}{\Omega} + 3.28\times 10^{-3}\right] \end{aligned} \tag{4.138}$$

where B_0, B_1 and B_2 are frequency-independent constants. ω_1 and ω_2 are the eigenfrequencies of the Chandler Wobble (CW) and the Free Core Nutation (FCN) respectively. Ω is the frequency of the sidereal rotation of the Earth.

Soon after its establishment, the predictions of the Wahr theory have been found to differ from VLBI observational data by much more than the uncertainties in the data itself. Face to this unsatisfactory result, an empirical series, called IERS96 series, constructed on the basis of some corrections to leading nutation coefficients from O–C discrepancies, was established, giving close agreement to the data [28, 49]. This series was still improved further by Shirai and Fukushima [60, 61], noticeably by introducing an estimated exponentially decaying free core nutation amplitude. As in Wahr [74] these two empirical series express the nutation amplitudes in terms of a resonant formula for the transfer function.

4.6.4 Further Improvements

An exhaustive review of all the improvements done in the fields of non rigid Earth nutation was done by Mathews et al. [48]. An important step towards a better geophysical accounting of nutation was taken successively by Gwinn et al. [26] and Herring et al. [29, 30], finding that a value higher than approximatively 5 % than that scheduled by the hydrostatic equilibrium state is needed for the dynamical ellipticity of the fluid core to close a gap of approximatively 2 mas (milliarcseconds) found between the observed nutation and the IAU 1980 values. This concerns the in phase part of the amplitude of the annual retrograde nutation. In parallel some studies were devoted to the computation of the effects of the ocean tides [57, 76] as well as those coming from the mantle anelasticity [77].

Alternative theoretical investigations start from the torque equations for the ellipsoidally stratified deformable Earth and its core. First developed by Molodensky [50] they were improved by Sasao et al. [55] and generalized by Mathews et al. [46, 47]. They are well suited for taking into account the dynamics of the inner core. In the last work, the torque equations and an accompanying kinematical equation reduce to a set of simultaneous linear algebraic equations. Such formula are very efficient to take into account the nonhydrostatic ellipticity and the use of an electromagnetic coupling at the core mantle boundary explaining the residuals of approximatively 0.4 mas remaining in the out-of-phase part of the retrograde annual component after taking into account anelasticity and ocean tides effects.

An independent approach was developed abundantly and exhaustively by Getino and Ferrandiz [21–23], starting from the same canonical equations as Kinoshita [38]. These authors introduced modified canonical variables to apply the theory to a non rigid Earth model taking into account an elastic mantle, a FOC (fluid outer core), a SIC (solid inner core) and a delay in the elastic response of the Earth with oceanic corrections. Although a final model with observations based on this work would have offered a better fit with the observations, as the IERS96 nutation series did previously, Mathews et al. [48] underlined the lack of explicit information concerning the fit of several parameters and their physical interpretation.

4.6.5 The Normal Modes of the Rotation of the Earth

To determine an accurate non rigid Earth nutation theory, it looks fundamental to know the normal modes of free rotational motions of the Earth as well as the eigenfrequencies σ_α which are associated with these normal modes. We will see later in Sect. 4.7.1 that those eigenfrequencies play a leading role in the transfer function from rigid Earth to non rigid Earth nutations. The principal normal modes acting as resonance modes in the transfer function are enumerated below. The list is not exhaustive: a quasi-infinite list of other normal modes exist, as the elastic vibration modes, which should not bring wobble components.

4.6.5.1 The Chandler Wobble (CW)

The only normal mode concerning the rigid Earth (in the approximation of the axisymmetric case) has already been described in Sect. 4.4.1.4. It is called the *free polar motion* or *Eulerian free wobble*. If the Earth were perfectly rigid, the frequency of this free wobble should be $(C - A/C)\Omega$ (where Ω already defined in Sect. 4.6.3 is the mean sidereal rotation rate), and its period $C/C - A = 305$ d. Chandler identified for the first time this free wobble from observational data in 1891, fixing its period to 14 months. The difference was soon interpreted as due to the deformability of the Earth [42], as well as the existence of a fluid core [34, 53] Much more recently, Smith and Dahlen [63] showed that the pole tide produced by the oceans should bring also a significant contribution. The amplitude of the Chandler wobble is variable, never exceeding $1''$.

4.6.5.2 The Retrograde Free Core Nutation (RFCN)

The free core nutation (FCN) is a normal mode of the Earth, associated with the existence of a rotating ellipsoidal fluid core inside a rotating elastic mantle. It occurs due to the excitement of a mis-alignment of the instantaneous rotation axes of the core and the mantle. More precisely the non spherical shape of the core-mantle boundary (CMB) has the consequence that any rotational motion of the fluid core relative to the mantle, with the core's rotation axis inclined to the symmetry axis of the CMB, causes imbalance of fluid pressure on the boundary, and a resultant torque which tends to bring the two axes into alignment. For the PREM model of the Earth [19], the theoretical FCN period, computed for an Earth in hydrostatic equilibrium, is 458 sidereal days in the retrograde direction. By analyzing the nutation amplitudes determined from VLBI observations, it has been shown that the FCN period is around 432 sidereal days [17, 26]. Moreover, analyses of gravimetric recordings in the diurnal frequency band also give a FCN period around 430 sidereal days [51]. The significant difference between the theoretical and observational values above can be explained by the increase of about 5 % of the core flattening with respect to the hydrostatic equilibrium value [26] One characteristic of the FCN is its variable amplitude and phase.

4.6.5.3 The Prograde Free Core Nutation (PFCN)

In the early 90's, several authors have shown that the presence of a solid inner core (SIC) gives rise to another diurnal wobble mode which corresponds to a prograde nutation [46, 47], De Vries and Wahr [18]. These last authors refer to this new mode as the *free inner core nutation* (FICN), whereas it is also sometimes called the *prograde free core nutation* (PFCN) Mathews and Shapiro [45]. Mathews et al. [47] found that the relation between the wobble motion of the fluid outer core and the mantle, in this mode, are very close to that in the already well known retrograde FCN.

4.6.5.4 The Inner Core Wobble (ICW)

The last free motion taking part in the transfer function is the inner core wobble (ICW). First mentioned by Mathews et al. [46, 47], it is predominantly a rotation of the figure axis of the inner core relative to the mantle. Moreover it would reduce to the free wobble of the solid inner core (SIC) if the forces between the SIC and the rest of the Earth could vanish.

4.7 The IAU 2006/2000 Precession Nutation

The IAU 2006/2000 Precession-nutation [10, 13] is composed of the IAU 2000 nutation and the IAU 2006 precession that replaced the precession component of the IAU 2000 precession-nutation. That component consisted only in corrections, $\delta\psi_A = -0.29965''/\text{century}$ and $\delta\omega_A = -0.02524''/\text{century}$, to the precession rates (in longitude and obliquity referred to the J2000.0 ecliptic), of the IAU 1976 precession and hence did not correspond to a dynamical theory [11].

4.7.1 The IAU 2000 (MHB2000) Nutation

The present conventional model of nutation adopted by the International Astronomical Union in 2000, called MHB2000, has been developed by Mathews et al. [48]. It is based on the REN2000 rigid Earth nutation series [67] of the axis of figure. The rigid Earth nutation was transformed to the non rigid Earth nutation by applying the MHB2000 transfer function to the full REN2000 series of the corresponding prograde and retrograde nutations and then converting back into elliptical nutation components. This transfer function is based on the solution of the linearized dynamical equations of the wobble-nutation problem and makes use of estimated values of seven of the parameters appearing in the theory called the BEP (Basic Earth Parameters).

The BEP were preliminary defined by Mathews et al. [46]. They consist of ellipticities e, e_F and e_S, and the mean equatorial moments of inertia A, A_F, and A_S of the Earth, the fluid outer core (FOC), and the solid inner core (SIC). Other BEP are compliance parameters $\kappa, \gamma, \zeta, \beta \ldots$ which represent the deformabilities of the Earth and of its core regions under different kinds of forcing. Another BEP is the density ρ_F of the FOC at the inner core boundary (ICB). Others BEP characterize the gravitational coupling between the SIC and the rest of the Earth. They are obtained from a least-squares fit of the theory to an up-to-date precession-nutation VLBI data set Herring et al. [32]. The MHB2000 model improves the IAU 1980 theory of nutation by taking into account the effects of mantle unelasticity, ocean tides, electromagnetic couplings produced between the FOC and the mantle as well as between the SIC and the FOC [8]. Moreover it takes in consideration non linear terms which have hitherto been ignored in previous formulations. The axis of reference, often called *axis of figure* is the axis of maximum moment of inertia of the Earth in steady rotation (ignoring time dependent deformations).

4.7.1.1 Analytical Formulation in MHB2000

At the basis of the formulation, consider a forced or free nutation having an angular frequency of τ cycles per sidereal day (cpsd) in space, 1 cpsd corresponding to the mean sidereal rotation period with angular velocity Ω. This nutation itself corresponds to a wobble of the Earth's mantle, which is a circular motion of its rotation axis around its geometric axis, with frequency σ cpsd with respect to an Earth-fixed frame. Thus we have $\sigma = \tau - 1$. The amplitude $\bar{m}(\sigma)$ of this wobble, the amplitudes $\bar{m}_F(\sigma)$ and $\bar{m}_S(\sigma)$ of accompanying wobbles relative to the mantle of the FOC and the SIC, as well as the amplitudes $\bar{n}_S(\sigma)$ of the effect of the polar axis of the SIC from that of the mantle are the dynamical variables of the wobble-nutation problem on the frequency domain. According to Mathews et al. [46, 47] the amplitude $\bar{\eta}(\sigma)$ of the nutation associated with the wobble of frequency σ cpsd is related to $\bar{m}(\sigma)$ by

$$\bar{\eta}(\sigma) = -\frac{\bar{m}(\sigma)}{1+\sigma} \tag{4.139}$$

It follows that the transfer function $T(\sigma, e)$ from the amplitude for the rigid Earth to that for the non rigid Earth is the same for the wobble and the corresponding nutation. It is presented as a resonance expansion of the form

$$T(\sigma, e) = R + R'(1+\sigma) + \sum_{\alpha} \frac{R_\alpha}{\sigma - \sigma_\alpha} \tag{4.140}$$

where the resonance frequencies σ_α are associated with four normal modes described in Sect. 4.6.5: the Chandler wobble (CW) the retrograde free core nutation (RFCN), the prograde free core nutation (PFCN) due to the presence of an elliptical solid inner core, and a free wobble of the inner core (ICW). Mathews et al. [48] modified in some extent this formula and adopted a generalized transfer function expressed in the form

$$T(\sigma, e/e_R) = \frac{e_R - \sigma}{e_R - 1} . N_O \big(1 + (1+\sigma)\big) \sum_{k=1}^{4} \frac{N_\alpha}{\sigma - \sigma_\alpha} \tag{4.141}$$

with

$$N_0 = \frac{H_d}{H_{dR}} = \frac{e/1+e}{e_R/1+e_R} \tag{4.142}$$

where H_d is the dynamical ellipticity of the Earth, already defined earlier, and H_{dR} its value in the rigid case.

4.7.1.2 Mantle Anelasticity Effects

Mantle anelasticity causes a small frequency-dependent phase lag in the Earth's response to periodic forcing besides altering the magnitude of the response. This anelasticity is characterized by the presence of a complex and frequency dependent shear and bulk moduli for each point of the mantle. The compliances appearing in nutation theory are computed initially for an elastic Earth model, such as the Preliminary Reference Earth Model (PREM) of Dziewonski and Anderson [19] by integrating the equations of tidal deformation, together with small contributions as the compliances coming from the Earth's ellipticity, the Coriolis force due to the Earth rotation, the differential rotations of the FOC and the SIC with respect to the mantle [7]. The anelasticity contributions to the compliances at a given excitation frequency are then computed from the same deformation equations, by evaluation of the changes in deformations resulting from the variations of the shear modulus $\mu(r)$ as studied by Wahr and Bergen [77].

4.7.1.3 Electromagnetic Coupling

The presence of an internal magnetic field influences the Earth's nutation through the effects of electromagnetic torques at the boundaries of the fluid core. Electromagnetic coupling is a consequence of the Lorentz force, which represents the force experienced by current-carrying matter in the presence of a magnetic field. The interaction between a magnetic field that crosses the outer core boundaries and the motion of conducting matter on either side of these boundaries induces an electric current, which locally perturbs the magnetic field. An increase of the Lorentz force opposes relative motion across the outer core boundaries, thereby coupling the motion of the inner core, outer core and mantle. Buffett et al. [8] calculated these effects on nutation by combining a solution for full hydrodynamic response of the fluid core. The coupling of the fluid outer core (FOC) to the mantle and the solid inner core (SIC) is described by two complex constants K^{CMB} and K^{ICB} that characterize the electromagnetic torques at the core-mantle boundary (CMB) and the inner core boundary (ICB). Predictions for K^{CMB} and K^{ICB} are compared with estimates inferred from observations of the Earth's nutation. The estimate of K^{CMB} can be explained by the presence of a thin conducting layer at the base of the mantle whose conductance has been estimated. The value of K^{ICB} can be explained with a mixture of dipole and non dipole components.

4.7.1.4 Ocean Tides Effects

Ocean tides affect nutations through changes in the inertia tensors of the Earth as well as its core regions due to the loading of the crust. Another cause is the contribution to the global angular momentum of the Earth. Ocean tidal motions in the diurnal band of frequencies are influenced by the FCN resonance. Wahr and Sasao

Table 4.1 Principal terms of nutation $\Delta\psi$ and $\Delta\varepsilon$ in the theory MHB2000 [48]

Argument	Period day	$\Delta\psi$ sin($''$)	$\Delta\psi$ cos($''$)	$\Delta\varepsilon$ sin($''$)	$\Delta\varepsilon$ cos($''$)
Ω	6798.384	−17.206416	0.0033338	0.001537	9.205233
2Ω	3399.192	0.207455	−0.001369	−0.000029	−0.089749
l'	365.260	0.147587	0.001181	−0.000192	0.007387
$2F-2D+2\Omega$	182.621	−1.317090	−0.001369	−0.000458	0.573034
$l'+2F-2D+2\Omega$	121.749	−0.051682	−0.000052	−0.000017	0.022438
$2F+2\Omega$	13.661	−0.227641	0.000279	0.000137	0.097846
$2F+\Omega$	13.633	−0.038730	0.000038	0.000032	0.020073
$l+2F+2\Omega$	9.133	−0.030146	0.000082	0.000037	0.012902

[76] have first made theoretical estimates of the contributions from ocean tides to nutation amplitudes, using as inputs the retrograde FCN eigenfrequency from Wahr [75] and tide heights from ad hoc models [52, 56]. To evaluate the role of the angular momentum $\bar{h}$ carried by the ocean tidal current it is enough to introduce it in the dynamical equations of the angular momentum. In MHB2000, values of $\bar{h}$ are taken from Chao et al. [14]. In fact accurate computation of the ocean angular momentum from ocean tide maps is difficult because of large contributions coming from small areas where the ocean is very deep.

4.7.2 The MHB2000 Nutation Series

The MHB2000 series of nutation includes 678 lunisolar terms and 687 planetary terms which are expressed as 'in-phase' and 'out-of-phase' components, together with their time-variations. That model is expected to guarantee an accuracy of about 10 μas for most of his terms. In Table 4.1 we show the principal terms of nutation $\Delta\psi$ and $\Delta\varepsilon$, with their argument, their period, their in-phase and out-of-phase amplitudes. As already calculated roughly in Eqs. (4.66) and (4.67) of Sect. 4.3.5, the largest terms have a 18.6 y period for the lunar contribution and a semi-annual period for the solar contribution, with respective arguments the longitude of the node of the Moon Ω and $2\lambda_{Earth} = 2F-2D+2\Omega$. The respective in-phase amplitudes are $17.206''$ and $1.317''$ for $\Delta\psi$, $9.205''$ and $0.573''$ for $\Delta\varepsilon$.

In Table 4.2 we represent the differences between the amplitudes of the principal terms nutation for a non rigid Earth model, taken from MHB2000 [48], and for a rigid Earth model, taken from REN2000 [67]. These differences reach 74 mas in $\Delta\psi$ and 23 mas in $\Delta\varepsilon$, for the leading term with argument Ω. They are also very important (40 mas and 20 mas) for the semi-annual component with argument $2F-2D+2\Omega$.

Table 4.2 Differences between the amplitudes of the principal terms in $\Delta\psi$ and $\Delta\varepsilon$ obtained from the non-rigid Earth theory MHB2000 [48] and the rigid Earth theory REN2000 [67]

Argument	Period day	$\Delta\psi$ sin(mas)	$\Delta\psi$ cos(mas)	$\Delta\varepsilon$ sin(mas)	$\Delta\varepsilon$ cos(mas)
Ω	6798.384	74.1760	2.9560	1.5408	−22.6769
2Ω	3399.192	−1.5742	−0.0757	−0.0323	0.5877
l'	365.260	22.0841	1.1817	−0.1924	7.5220
$2F-2D+2\Omega$	182.621	−39.6154	−1.3696	−0.4587	19.6988
$l'+2F-2D+2\Omega$	121.749	−1.7443	−0.0524	−0.0174	0.8189
$2F+2\Omega$	13.661	−6.1301	0.2796	0.1374	2.9250
$2F+\Omega$	13.633	−0.8772	0.0380	0.0318	0.6603
$l+2F+2\Omega$	9.133	−0.5633	0.0816	0.0367	0.2928

4.7.3 The IAU 2006 (P03) Precession

The IAU 2006 precession [10, 33] provides improved polynomial expressions up to the 5th degree in time t, both for the precession of the ecliptic (previously named "planetary precession") and the precession of the equator (previously named "luni-solar precession").

The precession of the equator was derived from the dynamical equations expressing the motion of the mean pole about the ecliptic pole. The convention for separating precession from nutation, as well as the integration constants used in solving the equations, has been chosen in order to be consistent with the IAU 2000A nutation. This includes corrections for the perturbing effects in the observed quantities.

In particular, the IAU 2006 value for the precession rate in longitude is such that the corresponding Earth's dynamical flattening is consistent with the MHB value for that parameter. This required applying a multiplying factor to the IAU 2000 precession rate of $\sin\varepsilon_{\mathrm{IAU2000}}/\sin\varepsilon_{\mathrm{IAU2006}} = 1.000000470$ in order to compensate for the change (by 42 mas) of the J2000 mean obliquity of the IAU 2006 model with respect to the IAU 2000 value (i.e. the IAU 1976 value). Moreover, the IAU 2006 precession includes the Earth's J_2 rate effect (*i.e.* $\dot{J}_2 = -3 \times 10^{-9}$/century), mostly due to the post-glacial rebound, which was not taken into account in the IAU precession models previously.

The contributions to the IAU 2006 precession rates for the 2nd order effects, the J_3 and J_4 effects of the luni-solar torque, the J_2 and planetary tilt effects, as well as the tidal effects are from Williams [78], and the non-linear terms are from MHB2000.

The geodetic precession is from Brumberg et al. [6], i.e. $p_g = 1.919883''$/cy. It is important to note that including the geodetic precession and geodetic nutation in the precession-nutation model ensure that the GCRS (Geocentric celestial reference system) is without any time-dependent rotation with respect to the BCRS (Barycentric celestial reference system).

4.7.4 The Agreement of the IAU 2006/2000 Precession-Nutation with Highly Accurate VLBI Observations

The accuracy with which Earth orientation in general and precession-nutation in particular can be determined as a function of time has increased tremendously over the three past decades, due to the advances in the VLBI technology, in techniques of data analysis, and also to the expanding volume of data over a lengthening time span. VLBI (Very Long Baseline Interferometry) can be considered as the most powerful technique to measure the Earth Orientation Parameters (EOP) [31]. These parameters are related to the changes of the position of the Earth's rotation axis or more precisely the axis of the *celestial intermediate pole* (CIP) with respect to its crust, so-called polar motion, and with respect to inertial space, i.e. the precession-nutation motion. One additional EOP is related to changes in the rotation rate of the Earth, and is usually expressed as the difference between UT1 and the time standard UTC (Universal Time Coordinate).

The VLBI technique measures the differential arrival times of radio-signals from extragalactic radio-sources, which provide in particular the most stable definition of inertial system currently available, as it is materialized by the successive updates of the ICRS [43, 44]. A classical VLBI session uses a set of four to eight radio telescopes, with separations of several thousands of kilometers, which make a large amount of measurements of time delays and delay rates from usually 20 to 40 extragalactic radio-sources. Of the various factors which limit the accuracy of the determination, one of the most important is the atmospheric contribution to group delays, especially the part due to water vapor which is the most difficult to estimate reliably [20].

Nevertheless with the basic and well reckoned assumption that rigid Earth nutation is modeled with an optimal accuracy (at the level of 1 µas), the VLBI observations allow a very accurate determination of the non rigid effects of the Earth on the largest nutation coefficients. Herring et al. [32] showed that the analysis of over 20 years of VLBI data yields estimates of the nutation amplitudes with standard deviations of ≈5 µas for the nutations with periods smaller than 400 days. They show that at this level of uncertainty, the estimated amplitudes are consistent with the IAU 2006/2000 precession-nutation model which has been described previously.

Figure 4.11 shows the differences O–C (observed-calculated) between the overall nutations components dX and dY determined from combined VLBI sessions and the same nutations calculated from the series MHB2000. The remarkable agreement at the level of a few 0.1 mas (a few 100 µas) is clearly shown after 1995, whereas the residuals are much larger before that date. This is clearly due to a drastic improvements of the quality VLBI observations around this date. Notice a very dominant systematic oscillation in the residuals: it is interpreted as the retrograde Free Core Nutation (RFCN) whose origin has been explained in Sect. 4.6.5. In Fig. 4.12 we show the residuals after eliminating empirically this systematic oscillation, taking into account its changes in amplitude and phase. The very flat residuals enable to conclude that the general agreement between the theoretical and observational data is remarkable.

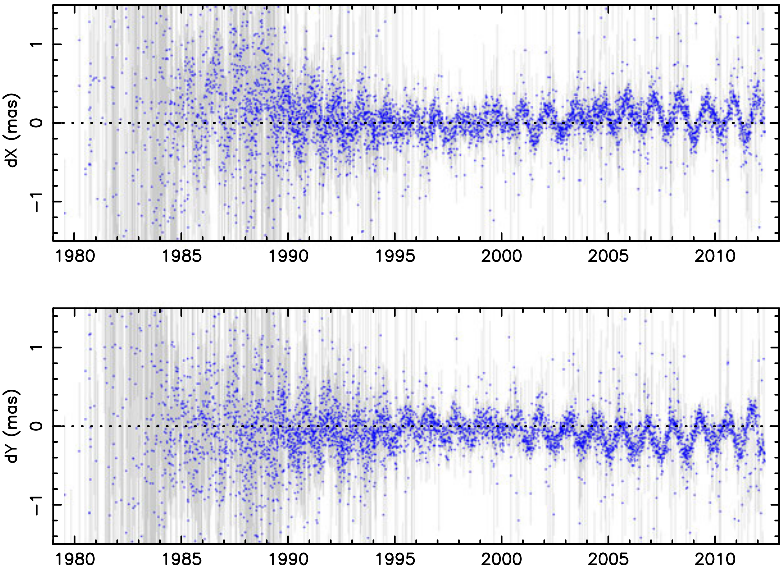

Fig. 4.11 O–C difference between the celestial motion of the pole dX and dY observed from VLBI sessions and the theoretical motion calculated from the IAU 2006/2000 precession-nutation model (credit: IVS OPA Analysis Center, Observatoire de Paris)

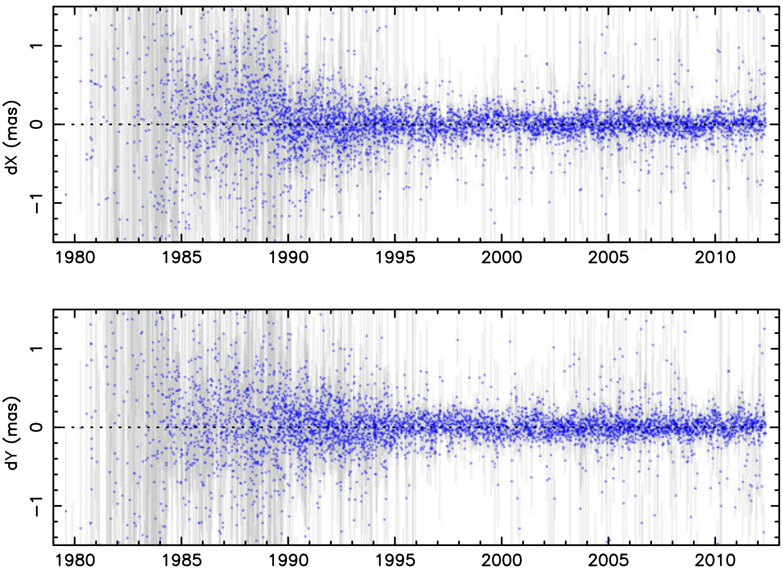

Fig. 4.12 O–C difference between the celestial motion of the pole dX and dY observed from VLBI sessions and the theoretical motion calculated from the IAU 2006/2000 precession-nutation model. The curves correspond to those in Fig. 4.10 after the FCN signal has been empirically subtracted (credit: IVS OPA Analysis Center, Observatoire de Paris)

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