Discrete-Time Nonautonomous Dynamical Systems

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Abstract These notes present and discuss various aspects of the recent theory for time-dependent difference equations giving rise to nonautonomous dynamical systems on general metric spaces:
First, basic concepts of autonomous difference equations and discrete-time (semi-) dynamical systems are reviewed for later contrast in the nonautonomous case. Then time-dependent difference equations or discrete-time nonautonomous dynamical systems are formulated as processes and as skew products. Their attractors including invariants sets, entire solutions, as well as the concepts of pullback attraction and pullback absorbing sets are introduced for both formulations. In particular, the limitations of pullback attractors for processes is highlighted. Beyond that Lyapunov functions for pullback attractors are discussed.
Two bifurcation concepts for nonautonomous difference equations will be introduced, namely attractor and solution bifurcations.
Finally, random difference equations and discrete-time random dynamical systems are investigated using random attractors and invariant measures.

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1 Introduction

The qualitative theory of dynamical systems has seen an enormous development since the groundbreaking contributions of Poincaré and Lyapunov over a century ago. Meanwhile it provides a successful framework to describe and understand a large variety of phenomena in areas as diverse as physics, life science, engineering or sociology.

Such a success benefits, in part, from the fact that the law of evolution in various problems from the above areas is static and does not change with time (or chance). Thus a description with autonomous evolutionary equations is appropriate. Nevertheless, many real world problems involve time-dependent parameters and, furthermore, one wants to understand control, modulation or other effects. In doing so, periodically or almost periodically driven systems are special cases, but, in principle, a theory for arbitrary time-dependence is desirable. This led to the observation that many of the meanwhile well-established concepts, methods and results for autonomous systems are not applicable and require an appropriate extension—the theory of *nonautonomous dynamical systems*.

The goal of these notes is to give a solid foundation to describe the long-term behaviour of nonautonomous evolutionary equations. Here we restrict to the discrete-time case in form of nonautonomous difference equations. This has the didactical advantage to feature many aspects of infinite-dimensional continuous-time problems (namely nonexistence and uniqueness of backward solutions) without an involved theory to guarantee the existence of a semiflow. Moreover, even in low dimensions, discrete dynamics can be quite complex.

Beyond that a time-discrete theory is strongly motivated from applications e.g., in population biology. In addition, it serves as a basic tool to understand numerical temporal discretization and is often essential for the analysis of continuous-time problems thorough concepts like time-1- or Poincaré mappings.

The focus of our presentation is on two formulations of time-discrete nonautonomous dynamical systems, namely processes (two-parameter semigroups) and skew-product systems. For both we construct, discuss and compare the so-called pullback attractor in Chaps. 4–6. A pullback attractor serves as nonautonomous counterpart to the global attractor, i.e., the object capturing the essential dynamics of a system. Furthermore, in Chap. 7 we sketch two approaches to a bifurcation theory for time-dependent problems to illuminate a current field of research. The final Chap. 8 on random dynamical systems emphasises similarities to the corresponding nonautonomous theory and provides results on random Markov chains and the approximation of invariant measures.

To conclude this introduction we point out that a significantly more comprehensive approach is given in the up-coming monograph [25] (see also the lecture notes [38, 44]). In particular, we neglect various contributions to the discrete-time nonautonomous theory: An appropriate spectral notion for linear difference equations (cf. [6, 35, 46, 47]) substitutes the dynamical role of eigenvalues from the autonomous special case. Gaps in this spectrum enable to construct nonautonomous invariant manifolds (so-called invariant fiber bundles, see [5, 42]). As special case they include centre fiber bundles and therefore allow one to deduce a time-dependent version of Pliss's reduction principle [33, 41]. The pullback attractors constructed in these notes are, generally, only upper semi-continuous in parameters. Thus, for approximation purposes it might be advantageous to embed them into a more robust dynamical object, namely a discrete counterpart to an inertial manifold [34]. Topological linearization of nonautonomous difference equations has been addressed in [7, 8], while a smooth linearization theory via normal forms was developed in [50].

2 Autonomous Difference Equations

A difference equation of the form

$$x_{n+1} = f(x_n), \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$, is called a first-order *autonomous difference equation* on the state space $\mathbb{R}^d$. There is no loss of generality in the restriction to first-order difference equations (1), since higher-order difference equations can be reformulated as (1) by the use of an appropriate higher dimensional state space.

Successive iteration of an autonomous difference equation (1) generates the forwards solution mapping $\pi : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$x_n = \pi(n, x_0) = f^n(x_0) := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0),$$

which satisfies the *initial condition* $\pi(0, x_0) = x_0$ and the *semigroup property*

$$\begin{aligned} \pi(n, \pi(m, x_0)) &= f^n(\pi(m, x_0)) = f^n \circ f^m(x_0) = f^{n+m}(x_0) \\ &= \pi(n+m, x_0) \quad \text{for all } n, m \in \mathbb{Z}^+,\ x_0 \in \mathbb{R}^d. \end{aligned} \tag{2}$$

Here, and later,

$$\mathbb{Z}^+ := \{0, 1, 2, 3, \ldots\}, \qquad \mathbb{Z}^- := \{\ldots, -3, -2, -1, 0\}$$

denote the nonnegative and nonpositive integers, respectively, and a *discrete interval* is the intersection of a real interval with the set of integers $\mathbb{Z}$.

Property (2) says that the solution mapping π forms a semigroup under composition; it is typically only a semigroup rather than a group since the mapping f need not be invertible. It will be assumed here that the mapping f in the difference

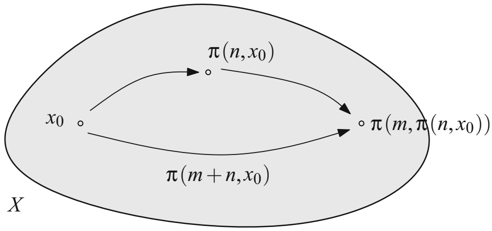


Fig. 1 Semigroup property (ii) of a discrete-time semidynamical system $\pi : \mathbb{Z}^+ \times X \to X$

equation (1) is at least continuous, from which it follows that the mappings $\pi(n, \cdot)$ are continuous for every $n \in \mathbb{Z}^+$. The solution mapping π then generates a discrete-time *semidynamical system* on $\mathbb{R}^d$.

More generally, the state space could be a metric space (X, d).

Definition 2.1. A mapping $\pi : \mathbb{Z}^+ \times X \to X$ satisfying

(i) $\pi(0, x_0) = x_0$ for all $x_0 \in X$,
(ii) $\pi(m + n, x_0) = \pi(m, \pi(n, x_0))$ for all $m, n \in \mathbb{Z}^+$ and $x_0 \in X$,
(iii) The mapping $x_0 \mapsto \pi(n, x_0)$ is continuous for each $n \in \mathbb{Z}^+$,

is called a (discrete-time) *autonomous semidynamical system* or a *semigroup* on the *state space* X.

The semigroup property (ii) is illustrated in Fig. 1 below. Note that such an autonomous semidynamical system π on X is equivalent to a first-order autonomous difference equation on X with the right-hand side f defined by $f(x) := \pi(1, x)$ for all $x \in X$.

If $\mathbb{Z}^+$ in Definition 2.1 is replaced by $\mathbb{Z}$, then π is called a (discrete-time) autonomous *dynamical system* or *group* on the state space X. See [10, 49]

Autonomous dynamical systems need not be generated by autonomous difference equations as above.

Example 2.2. Consider the space $X = \{1, \cdots, r\}^{\mathbb{Z}}$ of bi-infinite sequences $x = \{k_n\}_{n\in\mathbb{Z}}$ with $k_n \in \{1, \cdots, r\}$ w.r.t. the group of left shift operators $\theta_n := \theta^n$ for $n \in \mathbb{Z}$, where the mapping $\theta : X \to X$ is defined by $\theta(\{k_n\}_{n\in\mathbb{Z}}) = \{k_{n+1}\}_{n\in\mathbb{Z}}$. This forms an autonomous dynamical system on X, which is a compact metric space with the metric

$$d\left(x, x'\right) = \sum_{n\in\mathbb{Z}} (r+1)^{-|n|} \left|k_n - k_n'\right|.$$

The proximity and convergence of sets is given in terms of the *Hausdorff separation* $\mathrm{dist}_X(A, B)$ of nonempty compact subsets $A, B \subseteq X$ as

$$\mathrm{dist}_X(A, B) := \max_{a\in A} \mathrm{dist}(a, B) = \max_{a\in A} \min_{b\in B} d(a, b)$$

and the *Hausdorff metric* $H_X(A, B) = \max\{\mathrm{dist}_X(A, B), \mathrm{dist}_X(B, A)\}$ on the space $\mathscr{H}(X)$ of nonempty compact subsets of X. In absence of possible confusion we simply write dist or H for the Hausdorff separation resp. metric.

2.1 Autonomous Semidynamical Systems

The dynamical behaviour of a semidynamical system π on a state space X is characterised by its invariant sets and what happens in neighbourhoods of such sets. A nonempty subset A of X is called *invariant* under π, or π-invariant, if

$$\pi(n, A) = A \quad \text{for all } n \in \mathbb{Z}^+ \tag{3}$$

or, equivalently, if $f(A) = \pi(1, A) = A$.

Simple examples are equilibria (steady state solutions) and periodic solutions; in the first case A consists of a single point, which must thus be a fixed point of the mapping f, whereas for a solution with period r it consists of a finite set of r distinct points $\{p_1, \ldots, p_r\}$ which are fixed point of the composite mapping f^r (but not for an f^j with j smaller than r).

Invariant sets can also be much more complicated, for example fractal sets. Many are the ω-limit sets of some trajectory, i.e., defined by

$$\omega^+(x_0) = \left\{y \in X \ : \ \exists n_j \to \infty, \ \ \pi(n_j, x_0) \to y\right\},$$

which is nonempty, compact and π-invariant when the forwards trajectory $\{\pi(n, x_0);\ n \in \mathbb{Z}^+\}$ is a precompact subset of X and the metric space (X, d) is complete. However, $\omega(x_0)$ needs not to be connected.

The asymptotic behaviour of a semidynamical system is characterised by its ω-limit sets, in general, and by its attractors and their associated absorbing sets, in particular. An *attractor* is a nonempty π-invariant compact set A^* that attracts all trajectories starting in some neighbourhood $\mathcal{U}$ of A^*, that is with $\omega^+(x_0) \subset A^*$ for all $x_0 \in \mathcal{U}$ or, equivalently, with

$$\lim_{n\to\infty} \operatorname{dist}\left(\pi(n, x_0), A^*\right) = 0 \quad \text{for all } x_0 \in \mathcal{U}.$$

A^* is called a maximal or *global attractor* when $\mathcal{U}$ is the entire state space X. Note that a global attractor, if it exists, must be unique. For later comparison the formal definition follow.

Definition 2.3. A nonempty compact subset A^* of X is a *global attractor* of the semidynamical system π on X if it is π-invariant and attracts bounded sets, i.e.,

$$\lim_{n\to\infty} \operatorname{dist}\left(\pi\,(n, D)\,, A^*\right) = 0 \quad \text{for any bounded subset } D \subset X. \tag{4}$$

As simple example consider the autonomous difference equation (1) on $X = \mathbb{R}$ with the map $f(x) := \max\{0, 4x(1-x)\}$ for $x \in \mathbb{R}$. Then $A^* = [0, 1]$ is invariant and $f(x_0) \in A^*$ for all $x_0 \in \mathbb{R}$, so A^* is the maximal attractor. The dynamics are very simple outside of the attractor, but chaotic within it.

The existence and approximate location of a global attractor follow from that of more easily found absorbing sets, which typically have a convenient simpler shape such as a ball or ellipsoid.

Definition 2.4. A nonempty compact subset B of X is called an *absorbing set* of a semidynamical system π on X if for every bounded subset D of X there exists a $N_D \in \mathbb{Z}^+$ such that $\pi(n, D) \subset B$ for all $n \geq N_D$ in $\mathbb{Z}^+$.

Absorbing sets are often called attracting sets when they are also *positively invariant* in the sense that $\pi(n, B) \subseteq B$ holds for all $n \in \mathbb{Z}^+$, i.e., if one has the inclusion $f(B) = \pi(1, B) \subseteq B$. Attractors differ from attracting sets in that they consist entirely of limit points of the system and are thus strictly invariant in the sense of (3).

Theorem 2.5 (Existence of global attractors). *Suppose that a semidynamical system π on X has an absorbing set B. Then π has a unique global attractor $A^* \subset B$ given by*

$$A^* = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \pi(n, B)}, \tag{5}$$

or simply by $A^* = \bigcap_{m \geq 0} \pi(n, B)$ when B is positively invariant.

For a proof we refer to the more general situation of Theorem 4.11.

Similar results hold if the absorbing set is assumed to be only closed and bounded and the mapping π to be compact or asymptotically compact.

For later comparison note that, in view of the invariance of A^*, the attraction (4) can be written equivalently as the *forwards convergence*

$$\operatorname{dist}\left(\pi\left(n, D\right), \pi\left(n, A^*\right)\right) \to 0 \qquad \text{as} \quad n \to \infty. \tag{6}$$

A global attractor is, in fact, uniformly Lyapunov asymptotically stable. The asymptotic stability of attractors and that of attracting sets in general can be characterised by Lyapunov functions. Such Lyapunov functions can be used to establish the existence of an absorbing set and hence that of a nearby global attractor in a perturbed system.

2.2 Lyapunov Functions for Autonomous Attractors

Consider an autonomous semidynamical system π on a compact metric space (X, d) which is generated by an autonomous difference equation

$$x_{n+1} = f(x_n), \tag{7}$$

where $f : X \to X$ is globally Lipschitz continuous with Lipschitz constant $L > 0$, i.e.,

$$d(f(x), f(y)) \leq L d(x, y), \quad \text{for all } x, y \in X.$$

Definition 2.6. A nonempty compact subset $A \subsetneq X$ is called *globally uniformly asymptotically stable* if it is both

(i) *Lyapunov stable*, i.e., for all $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ with

$$\operatorname{dist}(x, A) < \delta \Rightarrow \operatorname{dist}(f^n(x), A) < \epsilon \quad \text{for all } n \in \mathbb{Z}^+ , \tag{8}$$

(ii) *Globally uniformly attracting*, i.e., for all $\epsilon > 0$, there exists an integer $N = N(\epsilon) > 1$ such that

$$\operatorname{dist}(f^n(x), A) < \epsilon \quad \text{for all } x \in X,\ n \geq N . \tag{9}$$

Note that such a set A is the global attractor for the semidynamical system generated by an autonomous difference equation (7). In particular, it is invariant, i.e., $f(A) = A$.

Global uniform asymptotical stability is characterized in terms of a Lyapunov function by the following necessary and sufficient conditions. The following theorem is taken from Diamond and Kloeden [13]. See also [53].

Theorem 2.7. *Let $f : X \to X$ be globally Lipschitz continuous, and let A be a nonempty compact subset of X. Then A is globally uniformly asymptotically stable w.r.t. the dynamical system generated by* (7) *if and only if there exist*

(i) A Lyapunov function $V : X \to \mathbb{R}^+$,
(ii) Monotone increasing continuous functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = \beta(0) = 0$ and $0 < \alpha(r) < \beta(r)$ for all $r > 0$, and
(iii) Constants $K > 0$, $0 \leq q < 1$ such that for all $x, y \in X$, it holds that

1. $|V(x) - V(y)| \leq K d(x, y)$,
2. $\alpha(\operatorname{dist}(x, A)) \leq V(x) \leq \beta(\operatorname{dist}(x, A))$ and
3. $V(f(x)) \leq qV(x)$.

Proof. Sufficiency. Let V be a Lyapunov function as described in the theorem. Choose $\epsilon > 0$ arbitrarily and define $\delta := \beta^{-1}(\alpha(\epsilon)/q)$, which means that $\alpha(\epsilon) = q\beta(\delta)$. This implies that

$$\alpha(\operatorname{dist}(f^n(x), A)) \leq V(f^n(x)) \leq q^n V(x) \leq qV(x) \leq q\beta(\operatorname{dist}(x, A)) ,$$

so that

$$\operatorname{dist}(f^n(x), A) \leq \alpha^{-1}\left(q\beta(\operatorname{dist}(x, A))\right) \leq \alpha^{-1}(\alpha(\epsilon)) \leq \epsilon \quad \text{for all } n \in \mathbb{N},$$

when $\operatorname{dist}(x, A) < \delta$. Thus, A is Lyapunov stable. Now define

$$N := \max\left\{1, 1 + \left\lfloor \frac{\ln(\alpha(\epsilon)/V_0)}{\ln q} \right\rfloor\right\} ,$$

where $V_0 := \max_{x \in X} V(x)$ is finite by continuity of V and compactness of X. For $n \geq N$, one has $q^n \leq q^N$, since $0 \leq q < 1$. Since, from above,

$$\alpha(\text{dist}(f^n(x), A)) \leq q^n V(x) \leq q^n V_0 \leq q^N V_0 \leq \alpha(\epsilon) \quad \text{for all } n \geq N,$$

one has $\text{dist}(f^n(x), A) < \epsilon$ for $n \geq N$, $x \in X$. This means that A is globally uniformly attracting and hence globally uniformly asymptotically stable.

Necessity. This will just be sketched here; the details can be found in [13]. Let A be globally uniformly asymptotically stable, i.e., for given $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that (8) holds, and for given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that (9) holds. Define $G_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ for $k \in \mathbb{N}$ by

$$G_k(r) := \begin{cases} r - \frac{1}{k} : r \geq \frac{1}{k}, \\ 0 \quad : 0 \leq r < \frac{1}{k}, \end{cases} \qquad \text{for all } r \geq 0.$$

Then

$$|G_k(r) - G_k(s)| \leq |r - s| \quad \text{for all } r, s \geq 0.$$

Now choose q so that $0 < q < \min\{1, L\}$, where L is the Lipschitz constant of the mapping f, and define

$$g_k := \left(\frac{q}{L}\right)^{N(1/k)} \qquad \text{for all } k \in \mathbb{N}$$

and

$$V_k(x) = g_k \sup_{n \in \mathbb{Z}^+} q^{-n} G_k(\text{dist}(f^n(x), A)) \quad \text{for all } k \in \mathbb{N}.$$

Then

(i) $V_k(x) = 0$ if and only if $\text{dist}(x, A) < \delta(1/k)$, due to Lyapunov stability.
(ii) Since $|\text{dist}(x, A) - \text{dist}(y, A)| \leq d(x, y)$ and

$$d(f^n(x), f^n(y)) \leq Ld\left(f^{n-1}(x), f^{n-1}(y)\right) \leq \cdots \leq L^n d(x, y),$$

it follows that

$$\begin{aligned} &|V_k(x) - V_k(y)| \\ &\leq g_k \sup_{n \geq 0} q^{-n} \left|G_k(\text{dist}(f^n(x), A)) - G_k(\text{dist}(f^n(y), A))\right| \\ &\leq g_k \sup_{0 \leq n \leq N(1/k)} q^{-n} \left|G_k(\text{dist}(f^n(x), A)) - G_k(\text{dist}(f^n(y), A))\right| \\ &\leq g_k \sup_{0 \leq n \leq N(1/k)} q^{-n} d\left(f^n(x), f^n(y)\right) \\ &\leq g_k \sup_{0 \leq n \leq N(1/k)} q^{-n} L^n d(x, y) = d(x, y). \end{aligned}$$

(iii) From above, it holds that $V_k(x) \leq V_k(y) + d(x, y)$. For all $y \in A$, one obtains that $V_k(y) = 0$ and $V_k(x) \leq d(x, y)$, and since A is compact, the minimum over all $y \in A$ is attained and $V_k(x) \leq \operatorname{dist}(x, A)$.

(iv) $V_k(f(x)) \leq qV_k(x)$, since

$$\begin{aligned} V_k(f(x)) &\leq g_k \sup_{n\geq 0} q^{-n} G_k(\operatorname{dist}(f^n(f(x)), A)) \\ &= qg_k \sup_{n\geq 0} q^{-n-1} G_k(\operatorname{dist}(f^{n+1}(x), A)) \\ &= qg_k \sup_{n\geq 1} q^{-n} G_k(\operatorname{dist}(f^n(x), A)) \\ &\leq qg_k \sup_{n\geq 0} q^{-n} G_k(\operatorname{dist}(f^n(x), A)) = qV_k(x) \end{aligned}$$

Finally, define

$$V(x) = \sum_{k=1}^{\infty} 2^{-k} V_k(x).$$

The main difficulty is to show the existence of the lower bound function α. This is systematically built up via the component functions V_k, which vanish successively on a closed $\frac{1}{k}$-neighbourhood of the set A. □

Remarks. For a more comprehensive introduction to discrete dynamical systems and their attractors we refer to e.g. [32, 51]. In particular, for the case of infinite-dimensional state spaces see [14] and [48, Chap. 2], where also connectedness issues of attractors or compactness properties for the semigroup π are addressed.

3 Nonautonomous Difference Equations

Difference equations on $\mathbb{R}^d$ of the form

$$x_{n+1} = f_n(x_n), \tag{Δ}$$

in which continuous mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$ on the right-hand side are allowed to vary with the time n, are called *nonautonomous difference equations*.

Such nonautonomous difference equations arise quite naturally in many different ways. The mappings f_n in (Δ) may of course vary completely arbitrarily, but often there is some relationship between them or some regularity in the way in which they are given.

For example, the mappings may all be the same as in the very special autonomous subcase (1) or they may vary periodically within, or be chosen irregularly from, a finite family $\{g_1, \cdots, g_r\}$, in which case (Δ) can be rewritten as

$$x_{n+1} = g_{k_n}(x_n), \tag{10}$$

with the $k_n \in \{1, \ldots, r\}$ and $f_n = g_{k_n}$.

As another example, the difference equation (Δ) may represent a variable time-step discretization method for a differential equation $\dot{x} = f(x)$, the simplest of which being the Euler method with a variable time-step $h_n > 0$,

$$x_{n+1} = x_n + h_n f(x_n), \tag{11}$$

in which case $f_n(x) = x + h_n f(x)$. More generally, a difference equation may involve a parameter $\lambda \in \Lambda$ which varies in time by choice or randomly, giving rise to the nonautonomous difference equation

$$x_{n+1} = g(x_n, \lambda_n), \tag{12}$$

so $f_n(x) = g(x, \lambda_n)$ here for the prescribed choice of $\lambda_n \in \Lambda$.

The nonautonomous difference equation (Δ) generates a solution mapping $\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$, where

$$\mathbb{Z}^2_{\geq} := \{(n, n_0) \in \mathbb{Z}^2 \ : \ n \geq n_0\},$$

through iteration, i.e.,

$$\phi(n_0, n_0, x_0) := x_0, \qquad \phi(n, n_0, x_0) := f_{n-1} \circ \cdots \circ f_{n_0}(x_0) \quad \text{for all } n > n_0\,,$$

$n_0 \in \mathbb{Z}$, and each $x_0 \in \mathbb{R}^d$. This solution mapping satisfies the *two-parameter semigroup* property

$$\phi(m, n_0, x_0) = \phi(m, n, \phi(n, n_0, x_0))$$

for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$, $(m, n) \in \mathbb{Z}^2_{\geq}$ and $x_0 \in \mathbb{R}^d$. In this sense, ϕ is called *general solution* of (Δ). In particular, as composition of continuous functions the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ is continuous for $(n, n_0) \in \mathbb{Z}^2_{\geq}$.

The general nonautonomous case differs crucially from the autonomous in that the starting time n_0 is just as important as the time that has elapsed since starting, i.e., $n - n_0$, and hence many of the concepts that have been developed and extensively investigated for autonomous dynamical systems in general and autonomous difference equations in particular are either too restrictive or no longer valid or meaningful.

3.1 Processes

Solution mappings of nonautonomous difference equations (Δ) are one of the main motivations for the process formulation of an abstract nonautonomous dynamical system on a metric state space (X, d) and time set $\mathbb{Z}$.

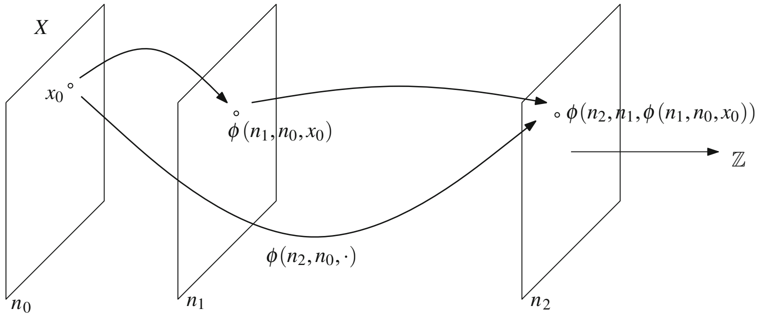


Fig. 2 Property (ii) of a discrete-time process $\phi : \mathbb{Z}^2_{\geq} \times X \to X$

The following definition originates from Dafermos [12] and Hale [14].

Definition 3.1. A (discrete-time) *process* on a state space X is a mapping $\phi : \mathbb{Z}^2_{\geq} \times X \to X$, which satisfies the initial value, two-parameter evolution and continuity properties:

(i) $\phi(n_0, n_0, x_0) = x_0$ for all $n_0 \in \mathbb{Z}$ and $x_0 \in X$,
(ii) $\phi(n_2, n_0, x_0) = \phi\,(n_2, n_1, \phi(n_1, n_0, x_0))$ for all $n_0 \leq n_1 \leq n_2$ in $\mathbb{Z}$ and $x_0 \in X$,
(iii) the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ of X into itself is continuous for all $n_0 \leq n$ in $\mathbb{Z}$.

The evolution property (ii) is illustrated in Fig. 2. Given a process ϕ on X there is an associated nonautonomous difference equation like (Δ) on X with mappings defined by $f_n(x) := \phi(n+1, n, x)$ for all $x \in X$ and $n \in \mathbb{Z}$.

A process is often called a *two-parameter semigroup* on X in contrast with the one-parameter semigroup of an autonomous semidynamical system since it depends on both the initial time n_0 and the actual time n rather than just the elapsed time $n - n_0$. This abstract formalism of a nonautonomous dynamical system is a natural and intuitive generalization of autonomous systems to nonautonomous systems.

3.2 Skew-Product Systems

The skew-product formalism of a nonautonomous dynamical system is somewhat less intuitive than the process formalism. It represents the nonautonomous system as an autonomous system on the cartesian product of the original state space and some other space such as a function or sequence space on which an autonomous dynamical system called the driving system acts. This driving system is the source of nonautonomity in the dynamics on the original state space.

Let (P, d_P) be a metric space with metric d_P and let $\theta = \{\theta_n : n \in \mathbb{Z}\}$ be a group of continuous mappings from P onto itself. Essentially, θ is an autonomous dynamical system on P that models the driving mechanism for the change in the mappings f_n on the right-hand side of a nonautonomous difference equation like (Δ), that will now be written as

$$x_{n+1} = f\left(\theta_n(p), x_n\right) \tag{13}$$

for $n \in \mathbb{Z}^+$, where $f : P \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. The corresponding solution mapping $\varphi : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ is now defined by

$$\varphi(0, p, x) := x, \qquad \varphi(n, p, x) := f\left(\theta_{n-1}(p), \cdot\right) \circ \cdots \circ f\left(p, x\right) \quad \text{for all } n \in \mathbb{N}$$

and $p \in P$, $x \in \mathbb{R}^d$. The mapping φ satisfies the *cocycle property* w.r.t. the driving system θ on P, i.e.,

$$\varphi(0, p, x) := x, \qquad \varphi(m + n, p, x) := \varphi\left(m, \theta_n(p), \varphi\left(n, p, x\right)\right) \tag{14}$$

for all $m, n \in \mathbb{Z}^+$, $p \in P$ and $x \in \mathbb{R}^d$.

3.2.1 Definition

Consider now a state space X instead of $\mathbb{R}^d$, where (X, d) is a metric space with metric d. The above considerations lead to the following definition of a skew-product system, which is an alternative abstract formulation of a discrete nonautonomous dynamical system on the state space X.

Definition 3.2. A (discrete-time) *skew-product system* (θ, ϕ) is defined in terms of a *cocycle mapping* φ on a state space X, driven by an autonomous dynamical system θ acting on a base space P.

Specifically, the *driving system* θ on P is a group of homeomorphisms $\{\theta_n : n \in \mathbb{Z}\}$ under composition on P with the properties

(i) $\theta_0(p) = p$ for all $p \in P$,
(ii) $\theta_{m+n}(p) = \theta_m(\theta_n(p))$ for all $m, n \in \mathbb{Z}$ and $p \in P$,
(iii) The mapping $p \mapsto \theta_n(p)$ is continuous for each $n \in \mathbb{Z}$,

and the *cocycle mapping* $\phi : \mathbb{Z}^+ \times P \times X \to X$ satisfies

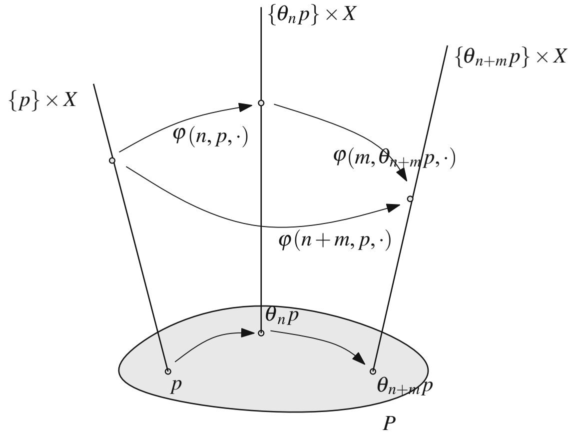
(I) $\varphi(0, p, x) = x$ for all $p \in P$ and $x \in X$,
(II) $\varphi(m + n, p, x) = \varphi(m, \theta_n(p), \varphi(n, p, x))$ for all $m, n \in \mathbb{Z}^+$, $p \in P$, $x \in X$,
(III) The mapping $(p, x) \mapsto \phi(n, p, x)$ is continuous for each $n \in \mathbb{Z}$.

For an illustration we refer to the subsequent Fig. 3. A difference equation of the form (13) can be obtained from a skew-product system by defining $f(p, x) := \varphi(1, p, x)$ for all $p \in P$ and $x \in X$.

A process ϕ admits a formulation as a skew-product system with $P = \mathbb{Z}$, the time shift $\theta_n(n_0) := n + n_0$ and the cocycle mapping

$$\varphi(n, n_0, x) := \phi(n + n_0, n_0, x) \quad \text{for all } n \in \mathbb{Z}^+,\ x \in X.$$

Fig. 3 A discrete-time skew-product system (θ, φ) over the base space P



The real advantage of the somewhat more complicated skew-product system formulation of nonautonomous dynamical systems occurs when P is compact. This never happens for a process reformulated as a skew-product system as above since the parameter space P then is $\mathbb{Z}$, which is only locally compact and not compact.

3.2.2 Examples

The examples above can be reformulated as skew-product systems with appropriate choices of parameter space P and the driving system θ.

Example 3.3. A nonautonomous difference equation (Δ) with continuous right-hand sides $f_n : \mathbb{R}^d \to \mathbb{R}^d$ generates a cocycle mapping φ over the parameter set $P = \mathbb{Z}$ w.r.t. the group of left shift mappings $\theta_j := \theta^j$ for $j \in \mathbb{Z}$, where $\theta(n) := n + 1$ for $n \in \mathbb{Z}$. Here φ is defined by

$$\varphi(0, n, x) := x \quad \text{and} \quad \varphi(j, n, x) := f_{n+j-1} \circ \cdots \circ f_n(x) \quad \text{for all } j \in \mathbb{N}$$

and $n \in \mathbb{Z}$, $x \in \mathbb{R}^d$. The mappings $\varphi(j, n, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are all continuous.

Example 3.4. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping used in an autonomous difference equation (1). The solution mapping φ defined by

$$\varphi(0, x) := x \quad \text{and} \quad \varphi(j, x) = f^j(x) := \underbrace{f \circ \cdots \circ f}_{j \text{ times}}(x) \quad \text{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^d$ generates a semigroup on $\mathbb{R}^d$. It can be considered as a cocycle mapping w.r.t. a singleton parameter set $P = \{p_0\}$ and the singleton group consisting only of identity mapping $\theta := \mathrm{id}_P$ on P. Since the driving system just sits at p_0, the dependence on the parameter in φ can be suppressed.

While the integers $\mathbb{Z}$ appears to be the natural choice for the parameter set in Example 3.3 and the choice is trivial in the autonomous case of Example 3.4, in the remaining examples the use of sequence spaces is more advantageous because such spaces are often compact.

Example 3.5. The nonautonomous difference equation (10) with continuous mappings $g_k : \mathbb{R}^d \to \mathbb{R}^d$ for $k \in \{1, \cdots, r\}$ generates a cocycle mapping over the parameter set $P = \{1, \cdots, r\}^{\mathbb{Z}}$ of bi-infinite sequences $p = \{k_n\}_{n\in\mathbb{Z}}$ with $k_n \in \{1, \cdots, r\}$ w.r.t. the group of left shift operators $\theta_n := \theta^n$ for $n \in \mathbb{Z}$, where $\theta(\{k_n\}_{n\in\mathbb{Z}}) = \{k_{n+1}\}_{n\in\mathbb{Z}}$. The mapping φ is defined by

$$\varphi(0, p, x) := x \quad \text{and} \quad \varphi(j, p, x) := g_{k_{j-1}} \circ \cdots \circ g_{k_0}(x) \quad \text{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^d$, where $p = \{k_n\}_{n\in\mathbb{Z}}$, is a cocycle mapping. Note that the parameter space $\{1, \cdots, r\}^{\mathbb{Z}}$ here is a compact metric space with the metric

$$d\left(p, p'\right) = \sum_{n\in\mathbb{Z}} (r+1)^{-|n|} \left|k_n - k_n'\right|.$$

In addition, $\theta_n : P \to P$ and $\varphi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ are all continuous.

We omit a reformulation of the numerical scheme (11) as it is similar to the next example, but with a bi-infinite sequence $p = \{h_n\}_{n\in\mathbb{Z}}$ of stepsizes satisfying a constraint such as $\frac{1}{2}\delta \le h_n \le \delta$ for $n \in \mathbb{Z}$ with appropriate $\delta > 0$.

Example 3.6. As an example of a parametrically perturbed difference equation (12), consider the mapping $g : \mathbb{R}^1 \times \left[\frac{1}{2}, 1\right] \mapsto \mathbb{R}^1$ defined by

$$g(x, \lambda) = \frac{|x| + \lambda^2}{1 + \lambda},$$

which is continuous in $x \in \mathbb{R}^1$ and $\lambda \in \left[\frac{1}{2}, 1\right]$. Let $P = \left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$ be the space of bi-infinite sequences $p = \{\lambda_n\}_{n\in\mathbb{Z}}$ taking values in $\left[\frac{1}{2}, 1\right]$, which is a compact metric space with the metric

$$d\left(p, p'\right) = \sum_{n\in\mathbb{Z}} 2^{-|n|} \left|\lambda_n - \lambda_n'\right|,$$

and let $\{\theta_n, n \in \mathbb{Z}\}$ be the group generated by the left shift operator θ on this sequence space (analogously to Example 3.5). The mapping φ is defined by

$$\varphi(0, p, x) := x \quad \text{and} \quad \varphi(j, p, x) := g(q_{j-1}, \cdot) \circ \cdots \circ g(q_0, x) \quad \text{for all } j \in \mathbb{N}$$

and $x \in \mathbb{R}^1$, where $p = \{\lambda_n\}_{n\in\mathbb{Z}}$, is a cocycle mapping on $\mathbb{R}^1$ with parameter space $\left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$ and the above shift operators θ_n. The mappings $\theta_n : P \to P$ and $\varphi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ are all continuous here.

3.2.3 Skew-Product Systems as Autonomous Semidynamical Systems

A skew-product system (θ, φ) can be reformulated as autonomous semidynamical system on the *extended state space* $\mathbb{X} := P \times X$. Define a mapping $\pi : \mathbb{Z}^+ \times \mathbb{X} \to \mathbb{X}$ by

$$\pi\,(n, (p, x_0)) := \big(\theta_n(p), \phi(n, p, x_0)\big) \quad \text{for all } n \in \mathbb{Z}^+, (p, x_0) \in \mathbb{X}\,.$$

Note that the variable n in $\pi\,(n, (p, x_0))$ is the time that has elapsed since starting at state (p, x_0).

Theorem 3.7. *π is an autonomous semidynamical system on $\mathbb{X}$.*

Proof. It is obvious that $\pi(n, \cdot)$ is continuous in its variables (p, x_0) for every $n \in \mathbb{Z}^+$ and satisfies the initial condition

$$\pi(0, (p, x_0)) = (p, \varphi(0, p, x_0)) = (p, x_0) \quad \text{for all } p \in P, x_0 \in X.$$

It also satisfies the one-parameter semigroup property

$$\pi(m + n, (p, x_0)) = \pi\,(m, \pi(n, (p, x_0))) \quad \text{for all } m, n \in \mathbb{Z}^+,\ p \in P,\ x_0 \in X$$

since, by the group property of the driving system and the cocycle property of the skew-product,

$$\begin{aligned}
\pi(m + n, (p, x_0)) &= (\theta_{m+n}(p), \varphi(m + n, p, x_0)) \\
&= \big(\theta_m\,(\theta_n(p))\,, \varphi(m, \theta_n(p), \varphi(n, p, x_0))\big) \\
&= \pi\big(m, (\theta_n(p), \varphi(n, p, x_0))\big) = \pi\big(m, \pi(n, (p, x_0))\big)\,.
\end{aligned}$$

□

As seen in Example 3.3, a process ϕ on the state space X is also a skew-product on X with the shift operator θ on $P := \mathbb{Z}$ and thus generates an autonomous semidynamical system π on the extended state space $\mathbb{X} := \mathbb{Z} \times X$. This semidynamical system has some unusual properties. In particular, π has no nonempty ω-limit sets and, indeed, no compact subset of $\mathbb{X}$ can be π-invariant. This is a direct consequence of the fact that the initial time is a component of the extended state space.

Remarks. An early reference to the description of nonautonomous discrete dynamics via processes or skew-product flows, is given in [32, pp. 45–56, Chap. 4].

4 Nonautonomous Invariant Sets and Attractors of Processes

Invariant sets and attractors are important regions of state space that characterize the long-term behaviour of a dynamical system.

Let $\phi : \mathbb{Z}^2_{\geq} \times X \to X$ be a process on a metric state space (X, d). This generates a solution $x_n = \phi(n, n_0, x_0)$ to (Δ) that depends on the starting time n_0 as well as the current time n and not just on the time $n - n_0$ that has elapsed since starting as in an autonomous system. This has some profound consequences in terms of definitions and the interpretation of dynamical behaviour. As pointed out above, many concepts and results from the autonomous case are no longer valid or are too restrictive and exclude many interesting types of possible behaviour.

For example, it is too great a restriction of generality to consider a single subset A of X to be invariant under ϕ in the sense that

$$\phi(n, n_0, A) = A \quad \text{for all } n \geq n_0,\ n_0 \in \mathbb{Z},$$

which is equivalent to $f_n(A) = A$ for every $n \in \mathbb{Z}$, where the f_n are mappings in the corresponding nonautonomous difference equation (Δ). Then, in general, neither the trajectory $\{\chi_n^* : n \in \mathbb{Z}\}$ of a solution χ^* that exists on all of $\mathbb{Z}$ nor a nonautonomous *ω-limit set* defined by

$$\omega^+(n_0, x_0) = \left\{ y \in X \ : \ \exists n_j \to \infty,\ \phi\left(n_j, n_0, x_0\right) \to y \right\},$$

will be invariant in such a sense.

Moreover, such nonautonomous ω-limit sets exist in the infinite future in absolute time rather than in current time like autonomous ω-limit sets, so it is not so clear how useful or even meaningful dynamically they are. Hence, the appropriate formulation of asymptotic behaviour of a nonautonomous dynamical system needs some careful consideration. Lyapunov asymptotical stability of a solution of a nonautonomous system provides a clue. This requires the definition of an entire solution.

Definition 4.1. An *entire solution* of a process ϕ on X is a sequence $\{\chi_k : k \in \mathbb{Z}\}$ in X such that

$$\phi(n, n_0, \chi_{n_0}) = \chi_n \quad \text{for all } n \geq n_0 \text{ and all } n_0 \in \mathbb{Z},$$

or equivalently, $\chi_{n+1} = f_n(\chi_n)$ for all $n \in \mathbb{Z}$ in terms of the nonautonomous difference equation (Δ) corresponding to the process ϕ.

Definition 4.2. An entire solution χ^* of a process ϕ on X is said to be (globally) *Lyapunov asymptotically stable* if it is *Lyapunov stable*, i.e., for every $\epsilon > 0$ and $n_0 \in \mathbb{Z}$ there exists a $\delta = \delta(\epsilon, n_0) > 0$ such that

$$d\left(\phi(n, n_0, x_0), \chi_n^*\right) < \epsilon \quad \text{for all } n \geq n_0 \text{ whenever } d\left(x_0, \chi_{n_0}^*\right) < \delta,$$

and *attracting* in the sense that

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \to 0 \quad \text{as } n \to \infty \tag{15}$$

for all $x_0 \in X$ and $n_0 \in \mathbb{Z}$.

Note, in particular, that the limiting "target" χ_n^* exists for all time and is, in general, also changing in time as the limit is taken.

4.1 Nonautonomous Invariant Sets

Let χ^* be an entire solution of a process ϕ on a metric space (X, d) and consider the family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of singleton subsets $A_n := \{\chi_n^*\}$ of X. Then by the definition of an entire solution it follows that

$$\phi\left(n, n_0, A_{n_0}\right) = A_n \quad \text{for all } n \geq n_0,\ n_0 \in \mathbb{Z}\,.$$

This suggests the following generalization of invariance for nonautonomous dynamical systems.

Definition 4.3. A family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty subsets of X is *invariant* under a process ϕ on X, or *ϕ-invariant*, if

$$\phi\left(n, n_0, A_{n_0}\right) = A_n \qquad \text{for all } n \geq n_0 \text{ and all } n_0 \in \mathbb{Z},$$

or, equivalently, if $f_n(A_n) = A_{n+1}$ for all $n \in \mathbb{Z}$ in terms of the corresponding nonautonomous difference equation (Δ).

A ϕ-invariant family consists of entire solutions. This is essentially due to have in fact a process is onto between the component subsets. The backward solutions, however, need not to be uniquely determined, since the mappings f_n are usually not assumed to be one-to-one.

Proposition 4.4 (Characterization of invariant sets). *A family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ is ϕ-invariant if and only if for every pair $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ there exists an entire solution χ such that $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$.*

Moreover, the entire solution χ is uniquely determined provided the mapping $f_n(\cdot) := \phi(n+1, n, \cdot) : X \to X$ is one-to-one for every $n \in \mathbb{Z}$.

Proof. Sufficiency. Let $\mathscr{A}$ be ϕ-invariant and pick an arbitrary $x_0 \in A_{n_0}$. For $n \geq n_0$ define the sequence $\chi_n := \phi(n, n_0, x_0)$. Then the ϕ-invariance of $\mathscr{A}$ yields $\chi_n \in A_n$. On the other hand, $A_{n_0} = \phi(n_0, n, A_n)$ for $n \leq n_0$, so there exists a sequence $x_n \in A_n$ with $x_0 = \phi(n_0, n, x_n)$ and $x_n = \phi(n, n-1, x_{n-1})$ for all $n < n_0$. Hence define $\chi_n := x_n$ for $n < n_0$ and χ becomes an entire solution with the desired properties. If the mappings f_n are all one-to-one, then the sequence $\{x_n\}$ is uniquely determined.

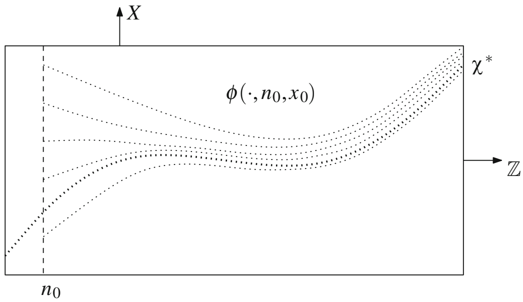


Fig. 4 Forward convergence $n \to \infty$

Necessity. Suppose for an arbitrary $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ that there is an entire solution χ with $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$. Hence $\phi(n, n_0, x_0) = \phi(n, n_0, \chi_{n_0}) = \chi_n \in A_n$ for $n \geq n_0$. From this it follows that $f_n(A_n) \subseteq A_{n+1}$. The remaining inclusion $f_n(A_n) \supseteq A_{n+1}$ follows from the fact that $x_0 = \phi(n_0, n, \chi_n) \in \phi(n_0, n, A_n)$ for $n \leq n_0$. □

4.2 Forwards and Pullback Convergence

The convergence

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \to 0 \quad \text{as } n \to \infty \quad (n_0 \text{ fixed})$$

in the attraction property (15) in the definition of a Lyapunov asymptotically stable entire solution χ^* of a process ϕ will be called *forwards convergence* (cf. Fig. 4) to distinguish it from another kind of convergence that is useful for nonautonomous systems.

Forwards convergence does not, however, provide convergence to a particular point $\chi_{n^*}^*$ for a fixed $n^* \in \mathbb{Z}$, which is important in many practical situations because the actual solution χ^* may not be known and thus needs to be determined. To obtain such convergence one has to start progressively earlier. This leads to the concept of *pullback convergence*, defined by

$$d\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \to 0 \quad \text{as } n_0 \to -\infty \quad (n \text{ fixed})$$

and illustrated in Fig. 5.

In terms of the elapsed time j, forwards convergence can be rewritten as

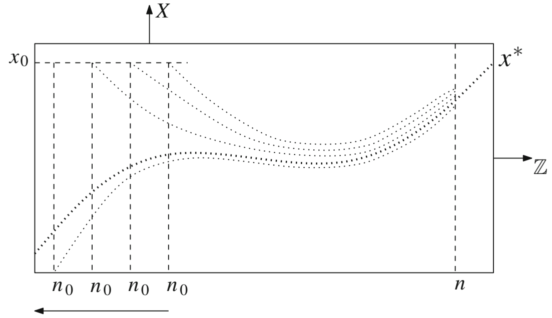
$$d\left(\phi\left(n_0 + j, n_0, x_0\right), \chi_{n_0+j}^*\right) \to 0 \quad \text{as } j \to \infty \tag{16}$$

for all $x_0 \in X$ and $n_0 \in \mathbb{Z}$, while pullback convergence becomes

$$d\left(\phi\left(n, n - j, x_0\right), \chi_n^*\right) \to 0 \quad \text{as } j \to \infty$$

for all $x_0 \in X$ and $n \in \mathbb{Z}$.

Fig. 5 Pullback convergence $n_0 \to -\infty$



Example 4.5. The nonautonomous difference equation $x_{n+1} = \frac{1}{2}x_n + g_n$ on $\mathbb{R}$ has the solution mapping $\phi(j + n_0, n_0, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-j+k} g_{n_0+n}$, for which pullback convergence gives

$$\phi(n_0, n_0 - j, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-k} g_{n_0-k} \to \sum_{k=0}^{\infty} 2^{-k} g_{n_0-k} \quad \text{as } j \to \infty,$$

provided the infinite series here converges. The limiting solution χ^* is given by $\chi^*_{n_0} := \sum_{k=0}^{\infty} 2^{-k} g_{n_0-k}$ for each $n_0 \in \mathbb{Z}$. It is an entire solution of the nonautonomous difference equation.

Pullback convergence makes use of information about the nonautonomous dynamical system from the past, while forwards convergence uses information about the future.

In autonomous dynamical systems, forwards and pullback convergence are equivalent since the elapsed time $n - n_0 \to \infty$ if either $n \to \infty$ with n_0 fixed or $n_0 \to -\infty$ with n fixed. In nonautonomous dynamical systems pullback convergence and forwards convergence do not necessarily imply each other.

Example 4.6. Consider the process ϕ on $\mathbb{R}$ generated $f_n = g_1$ for $n \leq 0$ and $f_n = g_2$ for $n \geq 1$ where the mappings $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by $g_1(x) := \frac{1}{2}x$ and $g_2(x) := \max\{0, 4x(1 - x)\}$ for all $x \in \mathbb{R}$. Then ϕ is pullback convergent to the entire solution χ^* defined by $\chi^*_n \equiv 0$ for $n \in \mathbb{Z}$, but is not forwards convergent to χ^*. In particular, χ^* is not Lyapunov stable.

4.3 Forwards and Pullback Attractors

Forwards and pullback convergence can be used to define two distinct types of nonautonomous attractors for a process ϕ on a state space X. Instead of a family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of singleton subsets $A_n := \{\chi^*_n\}$ for an entire solution χ^* of the process consider a ϕ-invariant family of $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty subsets A_n of X.

In this context forwards convergence generalizes to

$$\operatorname{dist}\big(\phi(n_0 + j, n_0, x_0), A_{n_0+j}\big) \to 0 \quad \text{as } j \to \infty \quad (n_0 \text{ fixed}) \tag{17}$$

and pullback convergence to

$$\operatorname{dist}(\phi(n, n-j, x_0), A_n) \to 0 \quad \text{as } j \to \infty \quad (n \text{ fixed}). \tag{18}$$

More generally, $\mathscr{A}$ is said to forwards (resp. pullback) attract bounded subsets of X if x_0 is replaced by an arbitrary bounded subset D of X in (17) (resp. (18)).

Definition 4.7. A ϕ–invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a *forward attractor* if it forward attracts bounded subsets of X and a *pullback attractor* if it pullback attracts bounded subsets of X.

As a ϕ-invariant family $\mathscr{A}$ of nonempty compact subsets of X, by Proposition 4.4, both pullback and forwards attractors consist of entire solutions.

In fact when the component subsets of a pullback attractor are uniformly bounded, i.e., if there exists a bounded subset B of X such that $A_n \subset B$ for all $n \in \mathbb{Z}$, then pullback attractors are characterized by the bounded entire solutions of the process.

Proposition 4.8 (Dynamical characterization of pullback attractors). *A uniformly bounded pullback attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ admits the dynamical characterization: for each $n_0 \in \mathbb{Z}$*

$$x_0 \in A_{n_0} \quad \Leftrightarrow \quad \textit{there exists a bounded entire solution } \chi \textit{ with } \chi_{n_0} = x_0.$$

Such a pullback attractor is therefore uniquely determined.

Proof. Sufficiency. Pick $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ arbitrarily. Then, due to the ϕ-invariance of the pullback attractor $\mathscr{A}$, by Proposition 4.4 there exists an entire solution χ with $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for each $n \in \mathbb{Z}$. Moreover, χ is bounded since the component sets of the pullback attractor are uniformly bounded.
Necessity. If there exists a bounded entire solution χ of the process ϕ, then the set of points $D_\chi := \{\chi_n : n \in \mathbb{Z}\}$ is bounded in X. Since $\mathscr{A}$ pullback attracts bounded subsets of X, for each $n \in \mathbb{Z}$,

$$0 \le \operatorname{dist}(\chi_n, A_n) \le \lim_{j\to\infty} \operatorname{dist}\big(\phi(n, n-j, D_\chi), A_n\big) = 0,$$

so $\chi_n \in A_n$. □

4.4 Existence of Pullback Attractors

Absorbing sets can also be defined for pullback attraction. Wider applicability can be attained if they are also allowed to depend on time.

Definition 4.9. A family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a *pullback absorbing family* for a process ϕ on X if for each $n \in \mathbb{Z}$ and every bounded subset D of X there exists an $N_{n,D} \in \mathbb{Z}^+$ such that

$$\phi(n, n-j, D) \subseteq B_n \quad \text{for all } j \geq N_{n,D},\ n \in \mathbb{Z}.$$

The existence of a pullback attractor follows from that of a pullback absorbing family in the following generalization of Theorem 2.5 for autonomous global attractors. The proof is simpler if the pullback absorbing family is assumed to be ϕ-positive invariant.

Definition 4.10. A family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is said to be *ϕ-positive invariant* if

$$\phi(n, n_0, B_{n_0}) \subseteq B_n \quad \text{for all } n \geq n_0.$$

Theorem 4.11 (Existence of pullback attractors). *Suppose that a process ϕ on a complete metric space (X, d) has a ϕ-positive invariant pullback absorbing family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$. Then there exists a global pullback attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ with component sets determined by*

$$A_n = \bigcap_{j \geq 0} \phi\left(n, n-j, B_{n-j}\right) \quad \text{for all } n \in \mathbb{Z}. \tag{19}$$

Moreover, if $\mathscr{A}$ is uniformly bounded then it is unique.

Proof. Let $\mathscr{B}$ be a pullback absorbing family and let A_n be defined as in (19). Clearly $A_n \subset B_n$ for each $n \in \mathbb{Z}$.

(i) First, it will be shown for any $n \in \mathbb{Z}$ that

$$\lim_{j \to \infty} \operatorname{dist}\left(\phi(n, n-j, B_{n-j}), A_n\right) = 0\,. \tag{20}$$

Assume to the contrary that there exist sequences $x_{j_k} \in \phi\left(n, n-j_k, B_{n-j_k}\right) \subset B_n$ and $j_k \to \infty$ such that $\operatorname{dist}(x_{j_k}, A_n) > \epsilon$ for all $k \in \mathbb{N}$. The set $\{x_{j_k} : k \in \mathbb{N}\} \subset B_n$ is relatively compact, so there is a point $x_0 \in B_n$ and an index subsequence $k' \to \infty$ such that $x_{j_{k'}} \to x_0$. Now

$$x_{j_{k'}} \in \phi\left(n, n-j_{k'}, B_{n-j_{k'}}\right) \subset \phi(n, n-k, B_{n-k})$$

for all $k_{j'} \geq k$ and each $k \geq 0$. This implies that

$$x_0 \in \phi(n, n-k, B_{n-k}) \quad \text{for all } k \geq 0\,.$$

Hence, $x_0 \in A_n$, which is a contradiction. This proves the assertion (20).

(ii) By (20), for every $\epsilon > 0$, $n \in \mathbb{Z}$, there exists an $N = N_{\epsilon,n} \geq 0$ such that

$$\operatorname{dist}(\phi(n, n-N, B_{n-N}), A_n) < \epsilon .$$

Let D be a bounded subset of X. The fact that $\mathscr{B}$ is a pullback absorbing family implies that $\phi(n, n-j, D) \subset B_n$ for all sufficiently large j. Hence, by the cocycle property,

$$\begin{aligned} \phi(n, n-N-j, D) &= \phi(n, n-N, \phi(n-N, n-N-j, D)) \\ &\subset \phi(n, n-N, B_{n-N}) . \end{aligned}$$

(iii) The ϕ-invariance of the family $\mathscr{A}$ will now be shown. By (19), the set $F_m(n) := \phi(n, n-m, B_{n-m})$ is contained in B_n for every $m \geq 0$, and by definition, $A_{n-j} = \bigcap_{m\geq 0} F_m(n-j)$. First, it will be shown that

$$\phi\left(n, n-j, \bigcap_{m\geq 0} F_m(n-j)\right) = \bigcap_{m\geq 0} \phi(n, n-j, F_m(n-j)) . \tag{21}$$

One sees directly that "$\subset$" holds. To prove "$\supset$", let x be contained in the set on the right side. Then for any $n \geq 0$, there exists an $x^m \in F_m(n-j) \subset B_{n-j}$ such that $x = \phi(n, n-j, x^m)$. Since the sets $F_m(n-j)$ are compact and monotonically decreasing with increasing m, the set $\{x^m : m \geq 0\}$ has a limit point $\hat{x} \in \bigcap_{m\geq 0} F_m(n-j)$. By the continuity of $\phi(n, n-j, \cdot)$, it follows that $x = \phi(n, n-j, \hat{x})$. Thus,

$$x \in \phi\left(n, n-j, \bigcap_{m\geq 0} F_m(n-j)\right) = \phi\left(n, n-j, A_{n-j}\right).$$

Hence, equation (21), the compactness of $F_m(n-j)$ and the continuity of $\phi(n, n-j, \cdot)$ imply that

$$\begin{aligned} \phi\left(n, n-j, A_{n-j}\right) &= \bigcap_{m\geq 0} \phi(n, n-j, F_m(n-j)) \\ &= \bigcap_{m\geq 0} \phi\left(n, n-j, \phi\left(n-j, n-j-m, B_{n-j-m}\right)\right) \\ &= \bigcap_{m\geq 0} \phi\left(n, n-j-m, B_{n-j-m}\right) \\ &= \bigcap_{m\geq j} \phi(n, n-m, B_{n-m}) , \\ &\supset A_n \end{aligned}$$

which means that

$$A_n \subset \phi\left(n, n-j, A_{n-j}\right), \quad j \in \mathbb{Z}^+ \quad \text{for all } n \in \mathbb{Z}. \tag{22}$$

Replacing n by $m - m$ in (22) and using the cocycle property gives

$$\begin{aligned}
\phi\left(n, n-m, A_{n-m}\right) &\subset \phi\left(n, n-m, \phi\left(m-n, n-m-j, A_{n-m-j}\right)\right) \\
&= \phi\left(n, n-j, \phi\left(n-j, n-m-j, A_{n-m-j}\right)\right) \\
&\subset \phi\left(n, n-j, \phi(n-j, n-m-j, B_{n-m-j})\right) \\
&\subset \phi\left(n, n-j, B_{n-j}\right) \subset U_\epsilon(A_n)
\end{aligned}$$

for all ϵ-neighborhoods $U_\epsilon(A_n)$ of A_n, where $\epsilon > 0$, provided that $j = J(\epsilon)$ is sufficiently large. Hence, $\phi\left(n, n-m, A_{n-m}\right) \subset A_n$ For all $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}$. With m replaced by j, this yields with (22) the ϕ-invariance of the family $\{A_n : n \in \mathbb{Z}\}$.

(iv) It remains to observe that if the sets in $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ are uniformly bounded, then the pullback attractor $\mathscr{A}$ is unique by Proposition 4.8. □

Remark 4.12. There is no counterpart of Theorem 4.11 for nonautonomous forwards attractors.

If the pullback absorbing family $\mathscr{B}$ is not ϕ-positive invariant, then the proof is somewhat more complicated and the component subsets of the pullback attractor of $\mathscr{A}$ are given by

$$A_n = \bigcap_{k \geq 0} \overline{\bigcup_{j \geq k} \phi\left(n, n-j, B_{n-j}\right)}.$$

However, the assumption in Theorem 4.11 that ϕ-positively invariant pullback absorbing systems is not a serious restriction.

Proposition 4.13. *If $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ is a pullback absorbing system for a process ϕ fulfilling $B_n \subset C$ for $n \in \mathbb{Z}$, where C is a bounded subset of X, then there exists a ϕ-positively invariant pullback absorbing system $\widehat{\mathscr{B}} = \{\widehat{B}_n : n \in \mathbb{Z}\}$ containing $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ component set-wise.*

Proof. For each $n \in \mathbb{Z}$ define

$$\widehat{B}_n := \overline{\bigcup_{j \geq 0} \phi(n, n-j, B_{n-j})}.$$

Obviously $B_n \subset \widehat{B}_n$ for every $n \in \mathbb{Z}$.

To show positive invariance, the cocycle property is used in what follows.

$$\begin{aligned}\phi(n+1,n,\widehat{B}_n) &= \overline{\bigcup_{j\geq 0}\phi(n+1,n,\phi(n,n-j,B_{n-j}))}\\ &= \overline{\bigcup_{j\geq 0}\phi(n+1,n-j,B_{n-j})}\\ &= \overline{\bigcup_{i\geq 1}\phi(n+1,n+1-i,B_{n+1-i})}\\ &\subseteq \overline{\bigcup_{i\geq 0}\phi(n+1,n+1-i,B_{n+1-i})} \;=\; \widehat{B}_{n+1},\end{aligned}$$

so $\phi(n+1,n,\widehat{B}_n) \subseteq \widehat{B}_{n+1}$. By this and the cocycle property again

$$\begin{aligned}\phi(n+2,n,\widehat{B}_n) &= \phi\left(n+2,n+1,\phi(n+1,n,\widehat{B}_n)\right)\\ &\subseteq \phi(n+2,n+1,\widehat{B}_{n+1}) \subseteq \widehat{B}_{n+2}.\end{aligned}$$

The general positive invariance assertion then follows by induction.

Now by the continuity of $\phi(n,n-j,\cdot)$ and the compactness of B_{n-j}, the set $\phi(n,n-j,B_{n-j})$ is compact for each $j \geq 0$ and $n \in \mathbb{Z}$. Moreover, $B_{n-j} \subset C$ for each $j \geq 0$ and $n \in \mathbb{Z}$, so by the pullback absorbing property of $\mathscr{B}$ there exists an $N = N_{n,C} \in \mathbb{N}$ such that

$$\phi(n,n-j,B_{n-j}) \subset \phi(n,n-j,C) \subset B_n$$

for all $j \geq N$. Hence

$$\begin{aligned}\widehat{B}_n &= \overline{\bigcup_{j\geq 0}\phi(n,n-j,B_{n-j})}\\ &\subseteq B_n \cup \overline{\bigcup_{0\leq j<N}\phi(n,n-j,B_{n-j})}\\ &= \overline{\bigcup_{0\leq j<N}\phi(n,n-j,B_{n-j})},\end{aligned}$$

which is compact as a finite union of compact sets, so $\widehat{B}_n$ is compact.

To see that $\widehat{\mathscr{B}}$ so constructed is pullback absorbing, let D be a bounded subset of X and fix $n \in \mathbb{Z}$. Since $\mathscr{B}$ is pullback absorbing, there exists an $N_{n,D} \in \mathbb{N}$ such that $\phi(n, n-j, D) \subset B_n$ for all $j \geq N_{n,D}$. But $B_n \subset \widehat{B}_n$, so

$$\phi(n, n-j, D) \subset \widehat{B}_n \quad \text{for all } j \geq N_{n,D}.$$

Hence $\widehat{\mathscr{B}}$ is pullback absorbing as required. □

4.5 Limitations of Pullback Attractors

Pullback attractors are based on the behaviour of a nonautonomous system in the past and may not capture the complete dynamics of a system when it is formulated in terms of a process. This was already indicated by Example 4.6 and will be illustrated here through some simpler examples. See [29].

First consider the autonomous scalar difference equation

$$x_{n+1} = \frac{\lambda x_n}{1 + |x_n|} \tag{23}$$

depending on a real parameter $\lambda > 0$. Its zero solution $x^* = 0$ exhibits a pitchfork bifurcation at $\lambda = 1$. Its global dynamical behavior can be summarized as follows (see Fig. 6):

- If $\lambda \leq 1$, then $x^* = 0$ is the only constant solution and is globally asymptotically stable. Thus $\{0\}$ is the global attractor of the autonomous dynamical system generated by the difference equation (23).
- If $\lambda > 1$, then there exist two additional nontrivial constant solutions given by $x_\pm := \pm(\lambda - 1)$. The zero solution $x^* = 0$ is an unstable steady state solution and the symmetric interval $A = [x_-, x_+]$ is the global attractor.

These constant solutions are the fixed points of the mapping $f(x) = \frac{\lambda x}{1+|x|}$.

Piecewise autonomous difference equation: Consider now the piecewise autonomous equation

$$x_{n+1} = \frac{\lambda_n x_n}{1 + |x_n|}, \qquad \lambda_n := \begin{cases} \lambda, & n \geq 0, \\ \lambda^{-1}, & n < 0 \end{cases} \tag{24}$$

for some $\lambda > 1$, which corresponds to a switch between the two autonomous problems (23) at $n = 0$.

The zero solution of the resulting nonautonomous system is the only bounded entire solution, so by Proposition 4.8 the pullback attractor $\mathscr{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$. Note that the zero solution seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$. Moreover the interval $[x_-, x_+]$ is like a global attractor for the whole equation on $\mathbb{Z}$, but it is not really one since it is not invariant or minimal for $n < 0$.

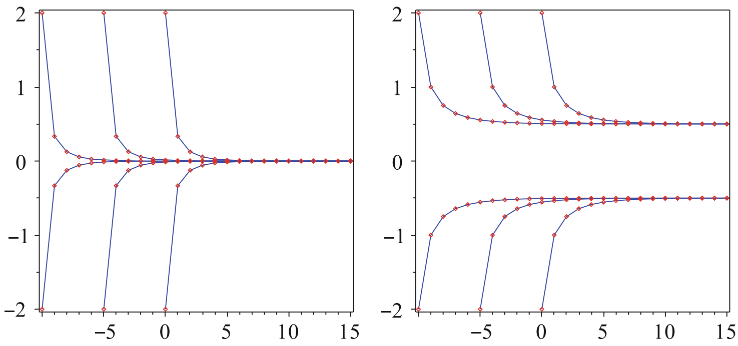


Fig. 6 Trajectories of the autonomous difference equation (23) with $\lambda = 0.5$ (*left*) and $\lambda = 1.5$ (*right*)

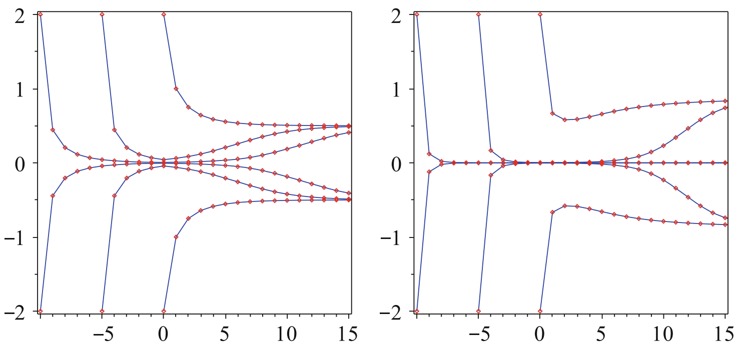


Fig. 7 Trajectories of the piecewise autonomous equation (24) with $\lambda = 1.5$ (*left*) and the asymptotically autonomous equation (25) with $\lambda_k = 1 + \frac{0.9k}{1+|k|}$ (*right*)

The nonautonomous difference equation (24) is asymptotically autonomous in both directions, but the pullback attractor does not reflect the full limiting dynamics (see Fig. 7 (left)), in particular in the forwards time direction.

Fully nonautonomous equation: If the parameters λ_n do not switch from one constant to another as above, but increase monotonically, e.g., such as $\lambda_n = 1 + \frac{0.9n}{1+|n|}$, then the dynamics is similar, although the limiting dynamics is not so obvious from the equation. See Fig. 7 (left).

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a monotonically increasing sequence with $\lim_{k\to\pm\infty} \lambda_n = \bar{\lambda}^{\pm 1}$ for $\bar{\lambda} > 1$. The nonautonomous problem

$$x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|}. \tag{25}$$

is asymptotically autonomous in both directions with the limiting autonomous systems given above.

Its pullback attractor $\mathscr{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution. As above, the zero solution $x^* = 0$ seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$. However, the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, neither of which is a solution at all. In particular, they are not entire solutions, so cannot belong to an attractor, forward or pullback, since these consist of entire solutions. See Fig. 7 (right).

Remark 4.14. Pullback attraction alone does not characterize fully the bounded limiting behaviour of a nonautonomous system formulated as a process. Something in addition like nonautonomous limit sets [25, 38], limiting equations [18] or asymptotically invariant sets [19] and eventual asymptotic stability [20] or a mixture of these ideas is needed to complete the picture. However, this varies from example to example and is somewhat ad hoc. In contrast, this information is built into the skew-product system formulation of a nonautonomous dynamical system, especially when the state space P of the driving system is compact. Essentially, the skew-product system already includes the limiting dynamics and no further ad hoc methods are needed to determine it.

Remarks. Pullback attractors for nonautonomous difference equations were introduced in [23, 24] and a comparison between different attractor types is given in [11] (see also Sect. 5.2).

Without the assumption of being uniformly bounded, pullback attractors of processes need not be unique (see [38, p. 18, Example 1.3.5]). In applications, absorbing sets are frequently not compact and one has to assume ambient compactness properties of a process in order to establish the existence of a pullback attractor (see [38, pp. 12ff]).

5 Nonautonomous Invariant Sets and Attractors: Skew-Product Systems

5.1 Existence of Pullback Attractors

Consider a discrete-time skew-product system (θ, φ) on $P \times X$, where (P, d_P) and (X, d) are metric spaces. There are counterparts for skew-product systems of the concepts of invariance, forwards and pullback convergence and forwards and pullback attractors considered in the previous section for discrete-time processes.

Definition 5.1. A family $\mathscr{A} = \{A_p : p \in P\}$ of nonempty subsets of X is called *φ-invariant* for a skew-product system (θ, φ) on $P \times X$ if

$$\varphi(n, p, A_p) = A_{\theta_n(p)} \quad \text{for all } n \in \mathbb{Z}^+,\ p \in P.$$

It is called *φ-positively invariant* if

$$\varphi(n, p, A_p) \subseteq A_{\theta_n(p)} \quad \text{for all } n \in \mathbb{Z}^+,\ p \in P.$$

Definition 5.2. A family $\mathscr{A} = \{A_p : p \in P\}$ of nonempty compact subsets of X is called *pullback attractor* of a skew-product system (θ, φ) on $P \times X$ if it is φ-invariant and pullback attracts bounded sets, i.e.,

$$\operatorname{dist}\left(\varphi(j, \theta_{-j}(p), D), A_p\right) \to 0 \quad \text{for } j \to \infty \tag{26}$$

for all $p \in P$ and all bounded subsets D of X. It is called a *forwards attractor* if it is φ-invariant and forward attracts bounded sets, i.e.,

$$\operatorname{dist}\left(\varphi(j, p, D), A_{\theta_j(p)}\right) \to 0 \quad \text{for } j \to \infty. \tag{27}$$

As with processes, the existence of a pullback attractor for skew-product systems is ensured by that of a pullback absorbing system.

Definition 5.3. A family $\mathscr{B} = \{B_p : p \in P\}$ of nonempty compact subsets of X is called a *pullback absorbing family* for a skew-product system (θ, φ) on $P \times X$ if for each $p \in P$ and every bounded subset D of X there exists an $N_{p,D} \in \mathbb{Z}^+$ such that

$$\varphi\left(j, \theta_{-j}(p), D\right) \subseteq B_p \quad \text{for all } j \geq N_{p,D},\ p \in P.$$

The following result generalizes Theorem 2.5 for autonomous semidynamical systems and the first half is the counterpart of Theorem 4.11 for processes. The proof is similar in the latter case, essentially with j and $\theta_{-j}(p)$ changed to n_0 and $n_0 - j$, respectively, but additional complications due to the fact that the pullback absorbing family is no longer assumed to be φ-positively invariant. See [26] for details.

Theorem 5.4 (Existence of pullback attractors). *Let (X, d) and (P, d_P) be complete metric spaces and suppose that a skew-product system (θ, φ) has a pullback absorbing set family $\mathscr{B} = \{B_p : p \in P\}$. Then there exists a pullback attractor $\mathscr{A} = \{A_p : p \in P\}$ with component sets determined by*

$$A_p = \bigcap_{n \geq 0} \overline{\bigcup_{j \geq n} \varphi\left(j, \theta_{-j}(p), B_{\theta_{-j}(p)}\right)}; \tag{28}$$

it is unique if its component sets are uniformly bounded.

The pullback attractor of a skew-product system (θ, φ) has some nice properties when its component subsets are contained in a common compact subset or if the state space P of the driving system is compact.

Proposition 5.5 (Upper semi-continuity of pullback attractors). *Suppose that $A(P) := \overline{\bigcup_{p \in P} A_p}$ is compact for a pullback attractor $\mathscr{A} = \{A_p : p \in P\}$.*

Then the set-valued mapping $p \mapsto A_p$ is upper semi-continuous in the sense that

$$\operatorname{dist}\left(A_q, A_p\right) \to 0 \quad \textit{as } q \to p.$$

On the other hand, if P is compact and the set-valued mapping $p \mapsto A_p$ is upper semi-continuous, then $A(P)$ is compact.

Proof. First note that, since $A(P)$ is compact, the pullback attractor is uniformly bounded by a compact set and hence is uniquely determined.

Assume that the set-valued mapping $p \mapsto A_p$ is not upper semi-continuous. Then there exist an $\epsilon_0 > 0$ and a sequence $p_n \to p_0$ in P such that $\operatorname{dist}\left(A_{p_n}, A_{p_0}\right) \geq 3\epsilon_0$ for all $n \in \mathbb{N}$. Since the sets A_{p_n} are compact, there exists an $a_n \in A_{p_n}$ such that

$$\operatorname{dist}\left(a_n, A_{p_0}\right) = \operatorname{dist}\left(A_{p_n}, A_{p_0}\right) \geq 3\epsilon_0 \quad \text{for each } n \in \mathbb{N}\,. \tag{29}$$

By pullback attraction, $\operatorname{dist}\left(\varphi\left(m, \theta_{-m}(p_0), B\right), A_{p_0}\right) \leq \epsilon_0$ for $m \geq M_{B,\epsilon_0}$ for any bounded subset B of X; in particular, below $A(P)$ will be used for the set B. By the φ-invariance of the pullback attractor, there exist $b_n \in A_{\theta_{-m}(p_n)} \subset A(P)$ for $n \in \mathbb{N}$ such that $\varphi\left(m, \theta_{-m}(p_n), b_n\right) = a_n$. Since $A(P)$ is compact, there is a convergent subsequence $b_{n'} \to \bar{b} \in A(P)$. Finally, by the continuity of $\theta_{-m}(\cdot)$ and of the cocycle mapping $\varphi(n, \cdot, \cdot)$,

$$d\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), \varphi(m, \theta_{-m}(p_0), \bar{b})\right) \leq \epsilon_0 \quad \text{for } n' \text{ large enough.}$$

Thus,

$$\begin{aligned}\operatorname{dist}\left(a_{n'}, A_{p_0}\right) &= \operatorname{dist}\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), A_{p_0}\right)\\ &\leq d\left(\varphi(m, \theta_{-m}(p_{n'}), b_{n'}), \varphi(m, \theta_{-m}(p_0), \bar{b})\right)\\ &\quad + \operatorname{dist}\left(\varphi(m, \theta_{-m}(p_0), \bar{b}), A_{p_0}\right) \leq 2\epsilon_0\,,\end{aligned}$$

which contradicts (29). Hence, $p \mapsto A_p$ must be upper semi-continuous.

The remaining assertion follows since the image of a compact subset under an upper semi-continuous compact set-valued mapping is compact (cf. [4]). □

Pullback attractors are in general not forwards attractors. When, however, the state space P of the driving system is compact, then one has the following partial forwards convergence result for the pullback attractor.

Theorem 5.6. *In addition to the assumptions of Theorem 5.4, suppose that P is compact and suppose that the pullback absorbing family $\mathscr{B}$ is uniformly bounded by a compact subset C of X. Then*

$$\lim_{n\to\infty} \sup_{p\in P} \operatorname{dist}\left(\varphi(n, p, D), A(P)\right) = 0 \tag{30}$$

for every bounded subset D of X, where $A(P) := \overline{\bigcup_{p\in P} A_p}$.

Proof. First note that $A(P)$ is compact since the component subsets A_p are all contained in the common compact set C. This means also that the pullback attractor is unique.

Suppose to the contrary that the convergence (30) does not hold. Then there exist an $\epsilon_0 > 0$ and sequences $n_j \to \infty$, $\hat{p}_j \in P$ and $x_j \in C$ such that

$$\operatorname{dist}\left(\varphi(n_j, \hat{p}_j, x_j), A(P)\right) > \epsilon_0 \,. \tag{31}$$

Set $p_j = \theta_{n_j}(\hat{p}_j)$. By the compactness of P, there exists a convergent subsequence $p_{j'} \to p_0 \in P$. From the pullback attraction, there exists an $n > 0$ such that

$$\operatorname{dist}\left(\varphi(n, \theta_{-n}(p_0), C), A_{p_0}\right) < \frac{\epsilon_0}{2} \,.$$

The cocycle property then gives

$$\varphi\left(n_j, \theta_{-n_j}(p_j), x_j\right) = \varphi\left(n, \theta_{-n}(p_j), \varphi\left(n_j - n, \theta_{-n_j}(p_j), x_j\right)\right)$$

for any $n_j > n$. By the pullback absorption of $\mathscr{B}$, it follows that

$$\varphi\left(n_j - n, \theta_{-n_j}(p_j), x_j\right) \subset B_{\theta_{-n}(p_j)} \subset C \,,$$

and since C is compact, there is a further index subsequence j'' of j' (depending on n) such that

$$z_{n_{j''}} := \varphi\left(n_{j''} - n, \theta_{-n_{j''}}(p_{j''}), x_{j''}\right) \to z_0 \in C.$$

The continuity of the skew-product mappings in the p and x variables implies

$$\operatorname{dist}\Big(\varphi(n, \theta_{-n}(p_{j''}), z_{n_{j''}}), \varphi(n, \theta_{-n}(p_0), z_0)\Big) < \frac{\epsilon_0}{2}, \quad \text{when } n_{j''} > n(\epsilon_0) \,.$$

Therefore,

$$\begin{aligned} \epsilon_0 &> \operatorname{dist}\Big(\varphi(n_{j''}, \theta_{-n_{j''}}(p_0), x_{j''}), A_{p_0}\Big) \\ &= \operatorname{dist}\left(\varphi\left(n_{j''}, \hat{p}_{j''}, x_{j''}\right), A_{p_0}\right) \geq \operatorname{dist}\left(\varphi\left(n_{j''}, \hat{p}_{j''}, x_{j''}\right), A(P)\right) \,, \end{aligned}$$

which contradicts (31). Thus, the asserted convergence (30) must hold. □

5.2 Comparison of Nonautonomous Attractors

Recall from Theorem 3.7 that the mapping $\pi : \mathbb{Z}^+ \times \mathbb{X} \to \mathbb{X}$ defined by

$$\pi(n, (p, x)) := (\theta_n(p), \varphi(n, p, x))$$

for all $j \in \mathbb{Z}^+$ and $(p,x) \in \mathbb{X} := P \times X$ forms an autonomous semidynamical system on the extended state space $\mathbb{X}$ with the metric

$$\mathrm{dist}_{\mathbb{X}}\left((p_1,x_1),(p_2,x_2)\right) = d_P(p_1,p_2) + d(x_1,x_2).$$

Proposition 5.7 (Uniform and global attractors). *Suppose that $\mathscr{A}$ is a uniform attractor (i.e., uniformly attracting in both the forward and pullback senses) of a skew-product system (θ,φ) and that $\bigcup_{p\in P} A_p$ is precompact in X. Then the union $\mathbb{A} := \bigcup_{p\in P}\{p\} \times A_p$ is the global attractor of the autonomous semidynamical system π.*

Proof. The π-invariance of $\mathbb{A}$ follows from the φ-invariance of $\mathscr{A}$, and the θ-invariance of P via

$$\pi(n,\mathbb{A}) = \bigcup_{p\in P}\{\theta_n(p)\}\times\varphi(n,p,A_p) = \bigcup_{p\in P}\{\theta_n(p)\}\times A_{\theta_n(p)} = \bigcup_{q\in P}\{q\}\times A_q = \mathbb{A}.$$

Since $\mathscr{A}$ is also a pullback attractor and $\bigcup_{p\in P} A_p$ is precompact in X (and P is compact too), the set-valued mapping $p \mapsto A_p$ is upper semi-continuous, which means that $p \mapsto F(p) := \{p\} \times A_p$ is also upper semi-continuous. Hence, $F(P) = \mathbb{A}$ is a compact subset of $\mathbb{X}$. Moreover, the definition of the metric $\mathrm{dist}_{\mathbb{X}}$ on $\mathbb{X}$ implies that

$$\begin{aligned}
\mathrm{dist}_{\mathbb{X}}\left(\pi(n,(p,x)),\mathbb{A}\right) &= \mathrm{dist}_{\mathbb{X}}\left((\theta_n(p),\varphi(n,p,x)),\mathbb{A}\right)\\
&\le \mathrm{dist}_{\mathbb{X}}\left((\theta_n(p),\varphi(n,p,x)),\{\theta_n(p)\}\times A_{\theta_n(p)}\right)\\
&= \mathrm{dist}_P\left(\theta_n(p),\theta_n(p)\right) + \mathrm{dist}\left(\varphi(n,p,x),A_{\theta_n(p)}\right)\\
&= \mathrm{dist}\left(\varphi(n,p,x),A_{\theta_n(p)}\right),
\end{aligned}$$

where $\pi(n,(p,x)) = (\theta_n(p),\varphi(n,p,x))$. The desired attraction to $\mathbb{A}$ w.r.t. π then follows from the forward attraction of $\mathscr{A}$ w.r.t. φ. □

Without uniform attraction as in Proposition 5.7 a pullback attractor need not give a global attractor, but the following result does hold.

Proposition 5.8. *If $\mathscr{A}$ is a pullback attractor for a skew-product system (θ,φ) and $\bigcup_{p\in P} A_p$ is precompact in X, then $\mathbb{A} := \bigcup_{p\in P}\{p\}\times A_p$ is the maximal invariant compact set of the autonomous semidynamical system π.*

Proof. The compactness and π-invariance of $\mathbb{A}$ are proved in the same way as in first part of the proof of Proposition 5.7. To prove that the compact invariant set $\mathbb{A}$ is maximal, let $\mathbb{C}$ be any other compact invariant set of the autonomous semidynamical system π. Then $\mathbb{A}$ is a compact and φ-invariant family of compact sets, and by pullback attraction,

$$\mathrm{dist}\left(\varphi\left(n,\theta_{-n}(p),C_{\theta_{-n}(p)}\right),A_p\right) \le \mathrm{dist}\left(\varphi\left(n,\theta_{-n}(p),K\right),A_p\right) \to 0$$

as $n \to \infty$, where $K := \overline{\bigcup_{p\in P} C_p}$ is compact. Hence, $C_p \subseteq A_p$ for all $p \in P$, i.e., $\mathbb{C} := \bigcup_{p\in P}\{p\} \times C_p \subseteq \mathbb{A}$, which finally means that $\mathbb{A}$ is a maximal π-invariant set. □

The set $\mathbb{A}$ here need not be the global attractor of π. In the opposite direction, the global attractor of the associated autonomous semidynamical system always forms a pullback attractor of the skew-product system.

Proposition 5.9 (Global and pullback attractors). *If an autonomous semidynamical system π has a global attractor*

$$\mathbb{A} = \bigcup_{p\in P}\{p\} \times A_p,$$

then $\mathscr{A} = \{A_p : p \in P\}$ is a pullback attractor for the skew-product system (θ, φ).

Proof. The sets P and $K := \bigcup_{p\in P} A_p$ are compact by the compactness of $\mathbb{A}$. Moreover, $\mathbb{A} \subset P \times K$, which is a compact set. Now

$$\begin{aligned} \operatorname{dist}(\varphi(n,p,x),K) &= \operatorname{dist}_P(\theta_n(p),P) + \operatorname{dist}(\varphi(n,p,x),K) \\ &= \operatorname{dist}_{\mathbb{X}}((\theta_n(p),\varphi(n,p,x)),P\times K) \\ &\le \operatorname{dist}_{\mathbb{X}}(\pi(n,(p,x)),P\times K) \\ &\le \operatorname{dist}_{\mathbb{X}}(\pi(n,P\times D),\mathbb{A}) \to 0 \text{ as } n\to\infty \end{aligned}$$

for all $(p,x) \in P \times D$ and every arbitrary bounded subset D of X, since $\mathbb{A}$ is the global attractor of π.

Hence, replacing p by $\theta_{-n}(p)$ implies

$$\lim_{n\to\infty} \operatorname{dist}(\varphi(n,\theta_{-n}(p),D),K) = 0\,.$$

Then the system is pullback asymptotic compact (see the definition in Chapter 12 of [25]) and by Theorem 12.12 in [25] this is a sufficient condition for the existence of a pullback attractor $\mathscr{A}' = \{A'_p : p \in P\}$ with $\bigcup_{p\in P} A'_p \subset K$. From Proposition 5.8, $\mathbb{A}' := \bigcup_{p\in P}\{p\} \times A'_p$ is the maximal π-invariant subset of $\mathbb{X}$, but so is the global attractor $\mathbb{A}$. This means that $\mathbb{A}' = \mathbb{A}$. Thus, $\mathscr{A}$ is a pullback attractor of the skew-product system (θ, φ). □

5.3 *Limitations of Pullback Attractors Revisited*

The limitations of pullback attraction for processes were illustrated in Sect. 4.5 through the scalar nonautonomous difference equation

$$x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|}, \tag{32}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is an increasing sequence with $\lim_{n\to\pm\infty} \lambda_n = \bar{\lambda}^{\pm 1}$ for $\bar{\lambda} > 1$.

The pullback attractor $\mathscr{A}$ of the corresponding process has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution. The zero solution $x^* = 0$ seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$. However the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, which both are not solutions at all. In particular, they are not entire solutions.

An elegant way to resolve the problem is to consider the skew-product system formulation of a nonautonomous dynamical system. This includes an autonomous dynamical system as a driving mechanism, which is responsible for the temporal change in the dynamics of the nonautonomous difference equation. It also includes the dynamics of the asymptotically autonomous difference equations above and their limiting autonomous systems.

The nonautonomous difference equation (32) can be formulated as a skew-product system with the diving system defined in terms of the shift operator θ on the space of bi-infinite sequences

$$\Lambda_L = \{\lambda = \{\lambda_n\}_{n\in\mathbb{Z}} : \lambda_n \in [0, L], \quad n \in \mathbb{Z}\}$$

for some $L > \bar{\lambda} > 1$. It yields a compact metric space with the metric

$$d_{\Lambda_L}\left(\lambda, \lambda'\right) := \sum_{n\in\mathbb{Z}} (L+1)^{-|n|} \left|\lambda_n - \lambda'_n\right|.$$

This is coupled with a cocycle mapping with values $x_n = \varphi(n, \lambda, x_0)$ on $\mathbb{R}$ generated by the difference equation (32) with a given coefficient sequence λ.

For the sequence λ from (32), the limit of the shifted sequences $\theta_n(\lambda)$ in the above metric as $n \to \infty$ is the constant sequence λ_+^* equal to $\bar{\lambda}$, while the limit as $n \to -\infty$ is the sequence λ_-^* with all components equal to $\bar{\lambda}^{-1}$.

The pullback attractor of the corresponding skew-product system (θ, φ) on $\Lambda \times \mathbb{R}$ consists of compact subsets A_λ of $\mathbb{R}$ for each $\lambda \in \Lambda_L$. It is easy to see that $A_\lambda = \{0\}$ for any λ with components $\lambda_n < 1$ for $n \leq 0$, which includes the constant sequence λ_-^* as well as the switched sequence in (32). On the other hand, $A_{\lambda_+^*} = [-\bar{\lambda}, \bar{\lambda}]$. Here $\cup_{\lambda\in\Lambda_L} A_\lambda$ is precompact, so contains all future limiting dynamics.

The pullback attractor of the skew-product system includes that of the process for a given bi-infinite coefficient sequence, but also includes its forward asymptotic limits and much more. The coefficient sequence set Λ_L includes all possibilities, in fact, far more than may be of interest in particular situation.

If one is interested in the dynamics of a process corresponding to a specific $\hat{\lambda} \in \Lambda_L$, then it would suffice to consider the skew-product system w.r.t. the driving system on the smaller space $\Lambda_{\hat{\lambda}}$ defined as the hull of this sequence, i.e., the set of accumulation points of the set $\{\theta_n(\hat{\lambda}) : n \in \mathbb{Z}\}$ in the metric space $(\Lambda_L, d_{\Lambda_L})$. In particular, if $\hat{\lambda}$ is the specific sequence in (32), then the union $\cup_{\lambda \in \Lambda_{\hat{\lambda}}} A_\lambda = A_{\lambda_+^*} = [-\bar{\lambda}, \bar{\lambda}]$ contains all future limiting dynamics, i.e.,

$$\lim_{n\to\infty} \text{dist}\left(\varphi(n,\lambda,x), [-\bar{\lambda},\bar{\lambda}]\right) = 0 \quad \text{for all } x \in \mathbb{R}.$$

The example described by nonautonomous difference equation (32) is asymptotically autonomous with $\Lambda_\lambda = \{\lambda_\pm^*\} \cup \{\theta_n(\lambda) : n \in \mathbb{Z}\}$. The forward limit points $\pm(\bar{\lambda} - 1)$ of the process generated by (25), which were not steady states of the process, are now locally asymptotic steady states of the skew product flow with base space $P = \bar{\Lambda}$ consisting of the single constant sequence $\lambda_k \equiv \bar{\lambda}$, when the skew product system is interpreted as an autonomous semidynamical system on the product space $P \times X$. More generally, unlike the process formulation, the skew-product system formulation and its pullback attractor include the forwards limiting dynamics.

5.4 Local Pullback Attractors

Less uniform behaviour such as parameter dependent domains of definition and local pullback attractors can be handled by introducing the concept of a basin of attraction system.

Let $\text{Dom}_p \subset X$ be the domain of definition of $f(p,\cdot)$ in the nonautonomous equation (13), which requires $f(p,\text{Dom}_p) \subset \text{Dom}_{\theta(p)}$. Then the corresponding cocycle mapping φ has the domain of definition $\mathbb{Z}^+ \times \bigcup_{p\in P}\left(\{p\}\times \text{Dom}_p\right)$. Consequently one needs to restrict the admissible families of bounded sets in the pullback convergence to subsets of Dom_p for each $p \in P$.

Definition 5.10. An ensemble $\mathfrak{D}_{ad}$ of families $\mathscr{D} = \{D_p : p \in P\}$ of nonempty subsets X is called *admissible* if

(i) D_p is bounded and $D_p \subset \text{Dom}_p$ for each $p \in P$ and every $\mathscr{D} = \{D_p : p \in P\} \in \mathfrak{D}_{ad}$; and
(ii) $\widehat{D}^{(1)} = \{D_p^{(1)} : p \in P\} \in \mathfrak{D}_{ad}$ whenever $\widehat{D}^{(2)} = \{D_p^{(2)} : p \in P\} \in \mathfrak{D}_{ad}$ and $D_p^{(1)} \subseteq D_p^{(2)}$ for all $p \in P$.

Further restrictions will allow one to consider local or otherwise restricted form of pullback attraction.

Definition 5.11. A φ-invariant family $\mathscr{A} = \{A_p : p \in P\}$ of nonempty compact subsets of X with $A_p \subset \text{Dom}_p$ for each $p \in P$ is called a *pullback attractor* w.r.t. the basin of attraction system $\mathfrak{D}_{att}$ if $\mathfrak{D}_{att}$ is an admissible ensemble of families of subsets such that

$$\lim_{j\to\infty} \text{dist}\left(\varphi(j,\theta_{-j}(p), D_{\theta_{-j}(p)}), A_p\right) = 0 \tag{33}$$

for every $\mathscr{D} = \{D_p : p \in P\} \in \mathfrak{D}_{att}$.

In this case a pullback absorbing set system $\mathscr{B} = \{B_p : p \in P\}$ should also satisfy $\mathscr{B} \in \mathfrak{D}_{att}$ and the pullback absorbing property should be modified to

$$\varphi\left(j, \theta_{-j}(p), D_{\theta_{-j}(p)}\right) \subseteq B_p$$

for all $j \geq N_{p,\mathscr{D}}$, $p \in P$ and $\mathscr{D} = \{D_p;\ p \in P\} \in \mathfrak{D}_{att}$.

A counterpart of Theorem 5.4 then holds here. In this case the pullback attractor is unique within the basin of attraction system, but the skew-product system may have other pullback attractors within other basin of attraction systems, which may be either disjoint from or a proper sub-ensemble of the original basin of attraction system.

Example 5.12. Consider the scalar nonautonomous difference equation

$$x_{n+1} = f_n(x_n) := x_n + \gamma_n x_n \left(1 - x_n^2\right) \tag{34}$$

for given parameters $\gamma_n > 0$, $n \in \mathbb{Z}$.

First let $\gamma_n \equiv \bar{\gamma}$ for all $n \in \mathbb{Z}$, so the system is autonomous. It has the attractor $A^* = [-1, 1]$ for the maximal basin of attraction $\left(-1 - \bar{\gamma}^{-1}, 1 + \bar{\gamma}^{-1}\right)$, but if one restricts attention further to the basin of attraction $\left(0, 1 + \bar{\gamma}^{-1}\right)$ then the attractor is only $A^{**} = \{1\}$.

Now let γ_n be variable with $\gamma_n \in \left[\frac{1}{2}\bar{\gamma}, \bar{\gamma}\right]$ for each $n \in \mathbb{Z}$, so the system is now nonautonomous and representable as a skew-product on the state space $\mathbb{X} = \mathbb{Z} \times \mathbb{R}$ with the parameter set $P = \mathbb{Z}$. Then $\mathscr{A}^* = \{A_n^* : n \in \mathbb{Z}\}$ with $A_n^* = [-1, 1]$ for all $n \in \mathbb{Z}$ is the pullback attractor for the basin of attraction system $\mathfrak{D}_{att}$ consisting of all families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ satisfying $D_n \subset \left(-1 - \bar{\gamma}^{-1}, 1 + \bar{\gamma}^{-1}\right)$, whereas $\mathscr{A}^{**} = \{A_n^{**} : n \in \mathbb{Z}\}$ with $A_n^{**} = \{1\}$ for all $n \in \mathbb{Z}$ is the pullback attractor for the basin of attraction system $\mathfrak{D}_{att}$ consisting of all families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ with $D_n \subset \left(0, 1 + \bar{\gamma}^{-1}\right)$.

6 Lyapunov Functions for Pullback Attractors

A Lyapunov function characterizing pullback attraction and pullback attractors for a discrete-time process in $\mathbb{R}^d$ will be constructed here. Consider a nonautonomous difference equation

$$x_{n+1} = f_n(x_n) \tag{Δ}$$

on $\mathbb{R}^d$, where the $f_n : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous mappings. This generates a process $\phi : \mathbb{Z}_{\geq}^2 \times \mathbb{R}^d \to \mathbb{R}^d$ through iteration by

$$\phi(n, n_0, x_0) = f_{n-1} \circ \cdots \circ f_{n_0}(x_0) \quad \text{for all } n \geq n_0$$

and each $x_0 \in \mathbb{R}^d$, which in particular satisfies the *continuity property*

$$x_0 \mapsto \phi(n, n_0, x_0) \quad \text{is Lipschitz continuous for all } n \geq n_0.$$

The pullback attraction is taken w.r.t. a basin of attraction system, which is defined as follows for a process.

Definition 6.1. A *basin of attraction system* $\mathfrak{D}_{att}$ consists of families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of nonempty bounded subsets of $\mathbb{R}^d$ with the property that $\mathscr{D}^{(1)} = \{D_n^{(1)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ if $\mathscr{D}^{(2)} = \{D_n^{(2)} : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ and $D_n^{(1)} \subseteq D_n^{(2)}$ for all $n \in \mathbb{Z}$.

Although somewhat complicated, the use of such a basin of attraction system allows both nonuniform and local attraction regions, which are typical in nonautonomous systems, to be handled.

Definition 6.2. A ϕ-invariant family of nonempty compact subsets $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ is called a *pullback attractor* w.r.t. a basin of attraction system $\mathfrak{D}_{att}$ if it is pullback attracting

$$\lim_{j \to \infty} \operatorname{dist}\left(\phi(n, n-j, D_{n-j}), A_n\right) = 0 \tag{35}$$

for all $n \in \mathbb{Z}$ and all $\mathscr{D} = \{D_n : n \in \mathbb{Z}\} \in \mathfrak{D}_{att}$.

Obviously $\mathscr{A} \in \mathfrak{D}_{att}$.

The construction of the Lyapunov function requires the existence of a pullback absorbing neighbourhood family.

6.1 Existence of a Pullback Absorbing Neighbourhood System

The following lemma shows that there always exists such a pullback absorbing neighbourhood system for any given pullback attractor. This will be required for the construction of the Lyapunov function for the proof of Theorem 6.4. The proof is very similar to that of Proposition 4.13.

Lemma 6.3. *If $\mathscr{A}$ is a pullback attractor with a basin of attraction system $\mathfrak{D}_{att}$ for a process ϕ, then there exists a pullback absorbing neighbourhood system $\mathscr{B} \subset \mathfrak{D}_{att}$ of $\mathscr{A}$ w.r.t. ϕ. Moreover, $\mathscr{B}$ is ϕ-positive invariant.*

Proof. For each $n_0 \in \mathbb{Z}$ pick $\delta_{n_0} > 0$ such that

$$B[A_{n_0}; \delta_{n_0}] := \{x \in \mathbb{R}^d \,:\, \operatorname{dist}(x, A_{n_0}) \leq \delta_{n_0}\}$$

satisfies $\{B[A_{n_0}; \delta_{n_0}] : n_0 \in \mathbb{Z}\} \in \mathfrak{D}_{att}$ and define

$$B_{n_0} := \overline{\bigcup_{j \geq 0} \phi(n_0, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}])}.$$

Obviously $A_{n_0} \subset \operatorname{int} B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$. To show positive invariance the two-parameter semigroup property will be used in what follows.

$$\phi(n_0+1,n_0,B_{n_0}) = \overline{\bigcup_{j\ge 0}\phi(n_0+1,n_0,\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}]))}$$

$$= \overline{\bigcup_{j\ge 0}\phi(n_0+1,n_0-j,B[A_{n_0-j};\delta_{n_0-j}])}$$

$$= \overline{\bigcup_{i\ge 1}\phi(n_0+1,n_0+1-i,B[A_{n_0+1-i};\delta_{n_0+1-i}])}$$

$$\subseteq \overline{\bigcup_{i\ge 0}\phi(n_0+1,n_0+1-i,B[A_{n_0+1-i};\delta_{n_0+1-i}])} = B_{n_0+1},$$

so $\phi(n_0+1,n_0,B_{n_0}) \subseteq B_{n_0+1}$. This and the two-parameter semigroup property again gives

$$\phi(n_0+2,n_0,B_{n_0}) = \phi(n_0+2,n_0+1,\phi(n_0+1,n_0,B_{n_0})$$

$$\subseteq \phi(n_0+2,n_0+1,B_{n_0+1}) \subseteq B_{n_0+2}.$$

The general positive invariance assertion then follows by induction.

Now referring to the continuity of $\phi(n_0,n_0-j,\cdot)$ and the compactness of $B[A_{n_0-j};\delta_{n_0-j}]$, the set $\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}])$ is compact for each $j\ge 0$ and $n_0\in\mathbb{Z}$. Moreover, by pullback convergence, there exists an $N=N(n_0,\delta_{n_0})\in\mathbb{N}$ such that

$$\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}]) \subseteq B[A_{n_0};\delta_{n_0}] \subset B_{n_0}$$

for all $j\ge N$. Hence

$$B_{n_0} = \overline{\bigcup_{j\ge 0}\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}])}$$

$$\subseteq B[A_{n_0};\delta_{n_0}]\bigcup\overline{\bigcup_{0\le j<N}\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}])}$$

$$= \overline{\bigcup_{0\le j<N}\phi(n_0,n_0-j,B[A_{n_0-j};\delta_{n_0-j}])},$$

which is compact, so B_{n_0} is compact.

To see that $\mathscr{B}$ so constructed is pullback absorbing w.r.t. $\mathfrak{D}_{att}$, let $\mathscr{D}\in\mathfrak{D}_{att}$. Fix $n_0\in\mathbb{Z}$. Since $\mathscr{A}$ is pullback attracting, there exists an $N(\mathscr{D},\delta_{n_0},n_0)\in\mathbb{N}$ such that

$$\mathrm{dist}\left(\phi(n_0, n_0 - j, D_{n_0-j}), A_{n_0}\right) < \delta_{n_0}$$

for all $j \geq N(\mathscr{D}, \delta_{n_0}, n_0)$. But $(\phi(n_0, n_0 - j, D_{n_0-j}) \subset \mathrm{int}B[A_{n_0}; \delta_{n_0}]$ and $B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$, so

$$\phi(n_0, n_0 - j, D_{n_0-j}) \subset \mathrm{int}B_{n_0}$$

for all $j \geq N(\mathscr{D}, \delta_{n_0}, n_0)$. Hence $\mathscr{B}$ is pullback absorbing as required. □

6.2 Necessary and Sufficient Conditions

The main result is the construction of a Lyapunov function that characterizes this pullback attraction. See [21, 22].

Theorem 6.4. *Let the f_n be uniformly Lipschitz continuous on $\mathbb{R}^d$ for each $n \in \mathbb{Z}$ and let ϕ be the process that they generate. In addition, let $\mathscr{A}$ be a ϕ-invariant family of nonempty compact sets that is pullback attracting with respect to ϕ with a basin of attraction system $\mathfrak{D}_{att}$. Then there exists a Lipschitz continuous function $V : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ such that*

Property 1 (upper bound). *For all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$*

$$V(n_0, x_0) \leq \mathrm{dist}(x_0, A_{n_0}); \tag{36}$$

Property 2 (lower bound). *For each $n_0 \in \mathbb{Z}$ there exists a function $a(n_0, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonically increasing in r such that*

$$a(n_0, \mathrm{dist}(x_0, A_{n_0})) \leq V(n_0, x_0) \quad \textit{for all } x_0 \in \mathbb{R}^d; \tag{37}$$

Property 3 (Lipschitz condition). *For all $n_0 \in \mathbb{Z}$ and $x_0,\ y_0 \in \mathbb{R}^d$*

$$|V(n_0, x_0) - V(n_0, y_0)| \leq \|x_0 - y_0\|; \tag{38}$$

Property 4 (pullback convergence). *For all $n_0 \in \mathbb{Z}$ and any $\mathscr{D} \in \mathscr{D}_{att}$*

$$\mathrm{limsup}_{n\to\infty} \sup_{z_{n_0-n} \in D_{n_0-n}} V(n_0, \phi(n_0, n_0 - n, z_{n_0-n})) = 0. \tag{39}$$

In addition,

Property 5 (forwards convergence). *There exists $\mathscr{N} \in \mathfrak{D}_{att}$. which is positively invariant under ϕ and consists of nonempty compact sets N_{n_0} with $A_{n_0} \subset \mathrm{int}N_{n_0}$ for each $n_0 \in \mathbb{Z}$ such that*

$$V(n_0+1, \phi(n_0+1, n_0, x_0)) \le e^{-1} V(n_0, x_0) \tag{40}$$

for all $x_0 \in N_{n_0}$ *and hence*

$$V(n_0+j, \phi(j, n_0, x_0)) \le e^{-j} V(n_0, x_0) \quad \text{for all } x_0 \in N_{n_0},\ j \in \mathbb{N}. \tag{41}$$

Proof. The aim is to construct a Lyapunov function $V(n_0, x_0)$ that characterizes a pullback attractor $\mathscr{A}$ and satisfies properties 1–5 of Theorem 6.4. For this define

$$V(n_0, x_0) := \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0-n, B_{n_0-n}))$$

for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$, where

$$T_{n_0,n} = n + \sum_{j=1}^{n} \alpha^+_{n_0-j}$$

with $T_{n_0,0} = 0$. Here $\alpha_n = \log L_n$, where L_n is the uniform Lipschitz constant of f_n on $\mathbb{R}^d$, and $a^+ = (a+|a|)/2$, i.e., the positive part of a real number a.
Note 4: $T_{n_0,n} \ge n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for $n, m \in \mathbb{N}$, $n_0 \in \mathbb{Z}$.

Proof of property 1

Since $e^{-T_{n_0,n}} \le 1$ for all $n \in \mathbb{N}$ and $\operatorname{dist}(x_0, \phi(n_0, n_0-n, B_{n_0-n}))$ is monotonically increasing from $0 \le \operatorname{dist}(x_0, \phi(n_0, n_0, B_{n_0}))$ at $n = 0$ to $\operatorname{dist}(x_0, A_{n_0})$ as $n \to \infty$,

$$V(n_0, x_0) = \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0-n, B_{n_0-n})) \le 1 \cdot \operatorname{dist}(x_0, A_{n_0}).$$

Proof of property 2

If $x_0 \in A_{n_0}$, then $V(n_0, x_0) = 0$ by Property 1, so assume that $x_0 \in \mathbb{R}^d \setminus A_{n_0}$. Now in

$$V(n_0, x_0) = \sup_{n \ge 0} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \phi(n_0, n_0-n, B_{n_0-n}))$$

the supremum involves the product of an exponentially decreasing quantity bounded below by zero and a bounded increasing function, since the sets $\phi(n_0, n_0-n, B_{n_0-n})$ are a nested family of compact sets decreasing to A_{n_0} with increasing n. In particular,

$$\operatorname{dist}(x_0, A_{n_0}) \ge \operatorname{dist}(x_0, \phi(n_0, n_0-n, B_{n_0-n})) \quad \text{for all } n \in \mathbb{N}.$$

Hence there exists an $N^* = N^*(n_0, x_0) \in \mathbb{N}$ such that

$$\frac{1}{2}\text{dist}(x_0, A_{n_0}) \leq \text{dist}\,(x_0, \phi(n_o, n_0 - n, B_{n_0-n})) \leq \text{dist}(x_0, A_{n_0})$$

for all $n \geq N^*$, but not for $n = N^* - 1$. Then, from above,

$$V(n_0, x_0) \geq e^{-T_{n_0,N^*}}\text{dist}\left(x_0, \phi(n_0, n_0 - N^*, B_{n_0-N^*})\right)$$

$$\geq \frac{1}{2}e^{-T_{n_0,N^*}}\text{dist}\,(x_0, A_{n_0})\,.$$

Define

$$N^*(n_0, r) := \sup\{N^*(n_0, x_0) : \text{dist}\,(x_0, A_{n_0}) = r\}$$

Now $N^*(n_0, r) < \infty$ for $x_0 \notin A_{n_0}$ with $\text{dist}\,(x_0, A_{n_0}) = r$ and $N^*(n_0, r)$ is nondecreasing with $r \to 0$. To see this note that by the triangle rule

$$\text{dist}(x_0, A_{n_0}) \leq \text{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) + \text{dist}(\phi(n_0, n_0 - n, B_{n_0-n}), A_{n_0}).$$

Also by pullback convergence there exists an $N(n_0, r/2)$ such that

$$\text{dist}(\phi(n_0, n_0 - n, B_{n_0-n}), A_{n_0}) < \frac{1}{2}r$$

for all $n \geq N(n_0, r/2)$. Hence for $\text{dist}(x_0, A_{n_0}) = r$ and $n \geq N(n_0, r/2)$,

$$r \leq \text{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})) + \frac{1}{2}r,$$

that is

$$\frac{1}{2}r \leq \text{dist}(x_0, \phi(n_0, n_0 - n, B_{n_0-n})).$$

Obviously $N^*(n_0, r) \leq N^*(n_0, r/2)$.

Finally, define

$$a(n_0, r) := \frac{1}{2}r\; e^{-T_{n_0,N^*(n_0,r)}}. \tag{42}$$

Note that there is no guarantee here (without further assumptions) that $a(n_0, r)$ does not converge to 0 for fixed $r \neq 0$ as $n_0 \to \infty$.

Proof of property 3

$$|V(n_0, x_0) - V(n_0, y_0)|$$

$$= \Bigg|\sup_{n\in\mathbb{N}} e^{-T_{n_0,n}}\text{dist}\,(x_0, \phi(n_0, n_0 - n, B_{n_0-n}))$$

$$- \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}}\text{dist}\,(y_0, \phi(n_0, n_0 - n, B_{n_0-n}))\Bigg|$$

$$\leq \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \left|\text{dist}\left(x_0, \phi(n_0, n_0 - n, B_{n_0-n})\right) - \text{dist}\left(y_0, \phi(n_0, n_0 - n, B_{n_0-n})\right)\right|$$

$$\leq \sup_{n\in\mathbb{N}} e^{-T_{n_0,n}} \|x_0 - y_0\| \leq \|x_0 - y_0\|$$

since

$$|\text{dist}\,(x_0, C) - \text{dist}\,(y_0, C)| \leq \|x_0 - y_0\|$$

for any x_0, $y_0 \in \mathbb{R}^d$ and nonempty compact subset C of $\mathbb{R}^d$.

Proof of property 4

Assume the opposite. Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $\mathbb{N}$ and points $x_j \in \phi(n_0, n_0 - n_j, D_{n_0-n_j})$ such that $V(n_0, x_j) \geq \varepsilon_0$ for all $j \in \mathbb{N}$. Since $\mathscr{D} \in \mathfrak{D}_{att}$ and $\mathscr{B}$ is pullback absorbing, there exists an $N = N(\mathscr{D}, n_0) \in \mathbb{N}$ such that

$$\phi(n_0, n_0 - n_j, D_{n_0-n_j}) \subset B_{n_0} \quad \text{for all } n_j \geq N.$$

Hence, for all j such that $n_j \geq N$, it holds $x_j \in B_{n_0}$, which is a compact set, so there exists a convergent subsequence $x_{j'} \to x^* \in B_{n_0}$. But also

$$x_{j'} \in \overline{\bigcup_{n\geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0-n})}$$

and

$$\bigcap_{n_{j'}} \overline{\bigcup_{n\geq n_{j'}} \phi(n_0, n_0 - n, D_{n_0-n})} \subseteq A_{n_0}$$

by the definition of a pullback attractor. Hence $x^* \in A_{n_0}$ and $V(n_0, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so

$$\varepsilon_0 \leq V(n_0, x_{j'}) = \|V(n_0, x_{j'}) - V(n_0, x^*)\| \leq \|x_{j'} - x^*\|,$$

which contradicts the convergence $x_{j'} \to x^*$. Hence property 4 must hold.

Proof of property 5

Define

$$N_{n_0} := \{x_0 \in B[B_{n_0}; 1] \ : \ \phi(n_0 + 1, n_0, x_0) \in B_{n_0+1}\},$$

where $B[B_{n_0}; 1] = \{x_0 \ : \ \text{dist}(x_0, B_{n_0}) \leq 1\}$ is bounded because B_{n_0} is compact and $\mathbb{R}^d$ is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since $\phi(n_0 + 1, n_0, \cdot)$ is continuous and B_{n_0+1} is compact. Now $A_{n_0} \subset \text{int}B_{n_0}$ and $B_{n_0} \subset N_{n_0}$, so $A_{n_0} \subset \text{int}N_{n_0}$. In addition,

$$\phi(n_0+1, n_0, N_{n_0}) \subset B_{n_0+1} \subset N_{n_0+1},$$

so $\mathscr{N}$ is positive invariant.

It remains to establish the exponential decay inequality (40). This needs the following Lipschitz condition on $\phi(n_0+1, n_0, \cdot) \equiv f_{n_0}(\cdot)$:

$$\|\phi(n_0+1, n_0, x_0) - \phi(n_0+1, n_0, y_0)\| \leq e^{\alpha_{n_0}} \|x_0 - y_0\|$$

for all $x_0, y_0 \in D_{n_0}$. It follows from this that

$$\text{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0+1, n_0, C_{n_0})) \leq e^{\alpha_{n_0}} \text{dist}(x_0, C_{n_0})$$

for any compact subset $C_{n_0} \subset \mathbb{R}^d$. From the definition of V,

$$\begin{aligned} V(n_0+1, \phi(n_0+1, n_0, x_0)) & \\ &= \sup_{n\geq 0} e^{-T_{n_0+1,n}} \text{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \\ &= \sup_{n\geq 1} e^{-T_{n_0+1,n}} \text{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-n, B_{n_0-n})) \end{aligned}$$

since $\phi(n_0+1, n_0, x_0) \in B_{n_0+1}$ when $x_0 \in N_{n_0}$. Hence re-indexing and then using the two-parameter semigroup property and the Lipschitz condition on $\phi(1, n_0, \cdot)$

$$\begin{aligned} & V(n_0+1, \phi(n_0+1, n_0, x_0)) \\ &= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} \text{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0, n_0-j-1, B_{n_0-j-1})) \\ &= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} \text{dist}(\phi(n_0+1, n_0, x_0), \phi(n_0+1, n_0, \phi(n_0, n_0-j, B_{n_0-j}))) \\ &\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}} e^{\alpha_{n_0}} \text{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \end{aligned}$$

Now $T_{n_0+1,j+1} = T_{n_0,j} + 1 - \alpha_{n_0}^+$, so

$$\begin{aligned} & V(n_0+1, \phi(n_0+1, n_0, x_0)) \\ &\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}+\alpha_{n_0}} \text{dist}(x_0, \phi(n_0+j, n_0-j, B_{n_0-j})) \\ &= \sup_{j\geq 0} e^{-T_{n_0,j}-1-\alpha_{n_0}^+ + \alpha_{n_0}} \text{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \\ &\leq e^{-1} \sup_{j\geq 0} e^{-T_{n_0,j}} \text{dist}(x_0, \phi(n_0, n_0-j, B_{n_0-j})) \leq e^{-1} V(n_0, x_0), \end{aligned}$$

which is the desired inequality. Moreover, since $\phi(1, n_0, x_0) \in B_{n_0+1} \subset N_{n_0+1}$, the proof continues inductively to give

$$V(n_0 + j, \phi(n_0 + j, n_0, x_0)) \le e^{-j} V(n_0, x_0) \quad \text{for all } j \in \mathbb{N}.$$

This completes the proof of Theorem 6.4. □

6.2.1 Comments on Theorem 6.4

Note 1: It would be nice to use $\phi(n_0, n_0 - n, x_0)$ for a fixed x_0 in the pullback convergence property (39), but this may not always be possible due to nonuniformity of the attraction region, i.e., there may not be a $\mathscr{D} \in \mathfrak{D}_{att}$ with $x_0 \in D_{n_0-n}$ for all $n \in \mathbb{N}$.

Note 2: The forwards convergence inequality (41) does not imply forwards Lyapunov stability or Lyapunov asymptotical stability. Although

$$a(n_0 + j, \text{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})) \le e^{-j} V(n_0, x_0)$$

there is no guarantee (without additional assumptions) that

$$\inf_{j \ge 0} a(n_0 + j, r) > 0$$

for $r > 0$, so $\text{dist}(\phi(n_0 + j, n_0, x_0), A_{n_0+j})$ need not become small as $j \to \infty$.

As a counterexample consider Example 4.6 of the process ϕ on $\mathbb{R}$ generated by (Δ) with $f_n = g_1$ for $n \le 0$ and $f_n = g_2$ for $n \ge 1$ where the mappings $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by $g_1(x) := \frac{1}{2}x$ and $g_2(x) := \max\{0, 4x(1 - x)\}$ for all $x \in \mathbb{R}$. Then $\mathscr{A}$ with $A_{n_0} = \{0\}$ for all $n_0 \in \mathbb{Z}$ is pullback attracting for ϕ but is not forwards Lyapunov asymptotically stable. (Note one can restrict g_1, g_2 to $[-R, R] \to [-R, R]$ for any fixed $R > 1$ to ensure the required uniform Lipschitz continuity of the f_n).

Note 3: The forwards convergence inequality (41) can be rewritten as

$$V(n_0, \phi(n_0, n_0 - j, x_{n_0-j})) \le e^{-j} V(n_0 - j, x_{n_0-j}) \le e^{-j} \text{dist}(x_{n_0-j}, A_{n_0-j})$$

for all $x_{n_0-j} \in N_{n_0-j}$ and $j \in \mathbb{N}$.

Definition 6.6. A family $\mathscr{D} \in \mathfrak{D}_{att}$ is called *past–tempered* w.r.t. $\mathscr{A}$ if

$$\lim_{j \to \infty} \frac{1}{j} \log^+ \text{dist}(D_{n_0-j}, A_{n_0-j}) = 0 \quad \text{for all } n_0 \in \mathbb{Z},$$

or equivalently if

$$\lim_{j \to \infty} e^{-\gamma j} \text{dist}(D_{n_0-j}, A_{n_0-j}) = 0 \quad \text{for all } n_0 \in \mathbb{Z},\ \gamma > 0.$$

This says that there is at most subexponential growth backwards in time of the starting sets. It is reasonable to restrict attention to such sets.

For a past-tempered family $\mathscr{D} \subset \mathscr{N}$ it follows that

$$V(n_0, \phi(n_0, n_0 - j, x_{n_0 - j})) \le e^{-j} \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$. Hence

$$a\left(n_0, \mathrm{dist}(\phi(n_0, n_0 - j, x_{n_0-j}), A_{n_0})\right) \le e^{-j} \mathrm{dist}(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$. Since n_0 is fixed in the lower expression, this implies the pullback convergence

$$\lim_{j\to\infty} \mathrm{dist}(\phi(n_0, n_0 - j, D_{n_0-j}), A_{n_0}) = 0.$$

A rate of pullback convergence for more general sets $\mathscr{D} \in \mathfrak{D}_{att}$ will be considered in the next subsection.

6.2.2 Rate of Pullback Convergence

Since $\mathscr{B}$ is a pullback absorbing neighbourhood system, then for every $n_0 \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\mathscr{D} \in \mathfrak{D}_{att}$ there exists an $N(\mathscr{D}, n_0, n) \in \mathbb{N}$ such that

$$\phi(n_0 - n, n_0 - n - m, D_{n_0-n-m}) \subseteq B_{n_0-n} \quad \text{for all } m \ge N.$$

Hence, by the two-parameter semigroup property,

$$\begin{aligned} &\phi(n_0, n_0 - n - m, D_{n_0-n-m}) \\ &= \phi(n_0, n_0 - n, \phi(n_0 - n, n_0 - n - m, D_{n_0-n-m})) \\ &\subseteq \phi(n_0, n_0 - n, B_{n_0-n}) \\ &= \phi(n_0, n_0 - i, \phi(n_0 - i, n_0 - n, B_{n_0-n})) \\ &\subseteq \phi(n_0, n_0 - i, B_{n_0-i}) \quad \text{for all } m \ge N,\ 0 \le i \le n, \end{aligned}$$

where the positive invariance of $\mathscr{B}$ was used in the last line. Hence

$$\phi(n_0, n_0 - n - m, D_{n_0-n-m}) \subseteq \phi(n_0, n_0 - i, B_{n_0-i})$$

for all $m \ge N(\mathscr{D}, n_0, n)$ and $0 \le i \le n$, or equivalently

$$\phi(n_0, n_0 - m, D_{n_0-m}) \subseteq \phi(n_0, n_0 - i, B_{n_0-i}) \quad \text{for all } m \ge n + N(\mathscr{D}, n_0, n)$$

and $0 \leq i \leq n$. This means that for any $z_{n_0-m} \in D_{n_0-m}$ the supremum in

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) = \sup_{i\geq 0} e^{-T_{n_0,i}} \operatorname{dist}\left(\phi(n_0, n_0 - m, z_{n_0-m}), \phi(n_0, n_0 - i, B_{n_0-i})\right)$$

need only be considered over $i \geq n$. Hence

$$\begin{aligned} &V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \\ &= \sup_{i\geq n} e^{-T_{n_0,i}} \operatorname{dist}\left(\phi(n_0, n_0 - m, z_{n_0-m}), \phi(n_0, n_0 - i, B_{n_0-i})\right) \\ &\leq e^{-T_{n_0,n}} \sup_{j\geq 0} e^{-T_{n_0-n,j}} \operatorname{dist}\left(\phi(n_0, n_0 - m, z_{n_0-m}), \phi(n_0, n_0 - n - j, B_{n_0-n-j})\right) \\ &\leq e^{-T_{n_0,n}} \operatorname{dist}\left(\phi(n_0, n_0 - m, z_{n_0-m}), A_{n_0}\right) \\ &\leq e^{-T_{n_0,n}} \operatorname{dist}\left(B_{n_0}, A_{n_0}\right) \end{aligned}$$

since $A_{n_0} \subseteq \phi(n_0, n_0 - n - j, B_{n_0-n-j})$ and $\phi(n_0, n_0 - m, z_{n_0-m}) \in B_{n_0}$. Thus

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq e^{-T_{n_0,n}} \operatorname{dist}\left(B_{n_0}, A_{n_0}\right)$$

for all $z_{n_0-m} \in D_{n_0-m}$, $m \geq n + N(\mathscr{D}, n_0, n)$ and $n \geq 0$.

It can be assumed that the mapping $n \mapsto n + N(\mathscr{D}, n_0, n)$ is monotonic increasing in n (by taking a larger $N(\mathscr{D}, n_0, n)$ if necessary), and is hence invertible. Let the inverse of $m = n + N(\mathscr{D}, n_0, n)$ be $n = M(m) = M(\mathscr{D}, n_0, m)$. Then

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq e^{-T_{n_0,M(m)}} \operatorname{dist}\left(B_{n_0}, A_{n_0}\right)$$

for all $m \geq N(\mathscr{D}, n_0, 0) \geq 0$. Usually $N(\mathscr{D}, n_0, 0) > 0$. This expression can be modified to hold for all $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for all $m \geq 0$ and introducing a constant $K_{\mathscr{D},n_0} \geq 1$ to account for the behaviour over the finite time set $0 \leq m < N(\mathscr{D}, n_0, 0)$. For all $m \geq 0$ this gives

$$V(n_0, \phi(n_0, n_0 - m, z_{n_0-m})) \leq K_{\mathscr{D},n_0} e^{-T_{n_0,M^*(m)}} \operatorname{dist}\left(B_{n_0}, A_{n_0}\right).$$

7 Bifurcations

The classical theory of dynamical bifurcation focusses on autonomous difference equations

$$x_{n+1} = g(x_n, \lambda) \tag{43}$$

with a right-hand side $g : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ depending on a parameter λ from some parameter space Λ, which is typically a subset of $\mathbb{R}^n$ (cf., e.g., [30] or [17]). A central question is how stability and multiplicity of invariant sets for (43) changes when the parameter λ is varied. In the simplest, and most often considered situation, these invariant sets are fixed points or periodic solutions to (43).

Given some parameter value λ^*, a fixed point $x^* = g(x^*, \lambda^*)$ of (43) is called *hyperbolic*, if the derivative $D_1 g(x^*, \lambda^*)$ has no eigenvalue on the complex unit circle $\mathbb{S}^1$. Then it is an easy consequence of the implicit function theorem (cf. [31, p. 365, Theorem 2.1]) that x^* allows a unique continuation $x(\lambda) \equiv g(x(\lambda), \lambda)$ in a neighborhood of λ^*. In particular, hyperbolicity rules out bifurcations understood as topological changes in the set $\{x \in \mathbb{R}^d : g(x, \lambda) = x\}$ near (x^*, λ^*) or a stability change of x^*.

On the other hand, eigenvalues on the complex unit circle give rise to various well-understood autonomous bifurcation scenarios. Examples include fold, transcritical or pitchfork bifurcations (eigenvalue 1), flip bifurcations (eigenvalue -1) or the Sacker–Neimark bifurcation (a pair of complex conjugate eigenvalues for $d \geq 2$).

7.1 Hyperbolicity and Simple Examples

Even in the autonomous set-up of (43) one easily encounters intrinsically nonautonomous problems, where the classical methods of, for instance, [17, 30] do not apply:

1. Investigate the behaviour of (43) along an entire reference solution $(\chi_n)_{n \in \mathbb{Z}}$, which is not constant or periodic. This is typically done using the (obviously nonautonomous) *equation of perturbed motion*

$$x_{n+1} = g(x_n + \chi_n, \lambda) - g(\chi_n, \lambda).$$

2. Replace the constant parameter λ in (43) by a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ in Λ, which varies in time. Also the resulting *parametrically perturbed* equation

$$x_{n+1} = g(x_n, \lambda_n)$$

   becomes nonautonomous. This situation is highly relevant from an applied point of view, since parameters in real world problems are typically subject to random perturbations or an intrinsic background noise.

Both the above problems fit into the framework of general nonautonomous difference equations

$$x_{n+1} = f_n(x_n, \lambda) \qquad (\Delta_\lambda)$$

with a sufficiently smooth right-hand side $f_n : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$, $n \in \mathbb{Z}$. In addition, suppose that f_n and its derivatives map bounded subsets of $\mathbb{R}^d \times \Lambda$ into bounded sets uniformly in $n \in \mathbb{Z}$.

Generically, nonautonomous equations (Δ_λ) do not have constant solutions, and the fixed point sequences $x_n^* = f_n(x_n^*, \lambda^*)$ are usually not solutions to (Δ_λ). This gives rise to the following question:

If there are no equilibria, what should bifurcate in a nonautonomous set-up?

Before suggesting an answer, a criterion to exclude bifurcations is proposed. For motivational purposes consider again the autonomous case (43) and the problem of parametric perturbations.

Example 7.1. The autonomous difference equation $x_{n+1} = \frac{1}{2}x_n + \lambda$ has the unique fixed point $x^*(\lambda) = 2\lambda$ for all $\lambda \in \mathbb{R}$. Replace λ by a bounded sequence $(\lambda_n)_{n\in\mathbb{Z}}$ and observe as in Example 4.5 that the nonautonomous counterpart

$$x_{n+1} = \tfrac{1}{2}x_n + \lambda_n$$

has a unique bounded entire solution $\chi_n^* := \sum_{k=-\infty}^{n-1} \left(\frac{1}{2}\right)^{n-k-1} \lambda_k$. For the special case $\lambda_n \equiv \lambda$, this solution reduces to the known fixed point $\chi_n^* \equiv 2\lambda$.

This simple example yields the conjecture that equilibria of autonomous equations persist as bounded entire solutions under parametric perturbations. It will be shown below in Theorem 7.5 (or in [37, Theorem 3.4]) that this conjecture is generically true in the sense that the fixed point of (43) has to be hyperbolic in order to persist under parametric perturbations.

Example 7.2. The linear difference equation $x_{n+1} = x_n + \lambda_n$ has the forward solution $x_n = x_0 + \sum_{k=0}^{n-1} \lambda_n$, whose boundedness requires the assumption that the real sequence $(\lambda_n)_{n\geq 0}$ is summable. Thus, the nonhyperbolic equilibria x^* of $x_{n+1} = x_n$ do not necessarily persist as bounded entire solutions under arbitrary bounded parametric perturbations.

Typical examples of nonautonomous equations having an equilibrium, given by the trivial solution are equations of perturbed motion. Their variational equation along $(\chi_n)_{n\in\mathbb{Z}}$ is given by $x_{n+1} = D_1 g(\chi_n, \lambda)x_n$ and investigating the behaviour of its trivial solution under variation of λ requires an appropriate nonautonomous notion of hyperbolicity.

Suppose that $A_n \in \mathbb{R}^{d\times d}$, $n \in \mathbb{Z}$, is a sequence of invertible matrices, and consider a linear difference equation

$$x_{n+1} = A_n x_n \tag{44}$$

with the *transition matrix*

$$\Phi(n, l) := \begin{cases} A_{n-1} \cdots A_l, & l < n, \\ I, & n = l, \\ A_n^{-1} \cdots A_{l-1}^{-1}, & n < l. \end{cases}$$

Let $\mathbb{I}$ be a discrete interval and define $\mathbb{I}' := \{k \in \mathbb{I} : k + 1 \in \mathbb{I}\}$. An *invariant projector* for (44) is a sequence $P_n \in \mathbb{R}^{d \times d}$, $n \in \mathbb{I}$, of projections $P_n = P_n^2$ such that

$$A_{n+1} P_n = P_n A_n \quad \text{for all } n \in \mathbb{I}'.$$

Definition 7.3. A linear difference equation (44) is said to admit an *exponential dichotomy* on $\mathbb{I}$, if there exist an invariant projector P_n and real numbers $K \geq 0$, $\alpha \in (0, 1)$ such that for all $n, l \in \mathbb{I}$ one has

$$\|\Phi(n,l)P_l\| \leq K\alpha^{n-l} \quad \text{if } l \leq n,$$

$$\|\Phi(n,l)[\mathrm{id} - P_l]\| \leq K\alpha^{l-n} \quad \text{if } n \leq l.$$

Remark 7.4. An autonomous difference equation $x_{n+1} = Ax_n$ has an exponential dichotomy, if and only if the coefficient matrix $A \in \mathbb{R}^{d \times d}$ has no eigenvalues on the complex unit circle.

In terms of this terminology an entire solution $(\chi_n)_{n \in \mathbb{Z}}$ of (Δ_λ) is called *hyperbolic*, if the variational equation

$$x_{n+1} = D_1 f_n(\chi_n, \lambda) x_n \tag{V_λ}$$

has an exponential dichotomy on $\mathbb{Z}$.

Let ℓ^∞ denote the space of bounded sequences in $\mathbb{R}^d$.

Theorem 7.5 (Continuation of bounded entire solutions). *If $\chi^* = (\chi_n^*)_{n \in \mathbb{Z}}$ is an entire bounded and hyperbolic solution of (Δ_{λ^*}), then there exists an open neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^* and a unique function $\chi : \Lambda_0 \to \ell^\infty$ such that*

(i) $\chi(\lambda^*) = \chi^*$,
(ii) Each $\chi(\lambda)$ *is a bounded entire and hyperbolic solution of* (Δ_λ),
(iii) $\chi : \Lambda_0 \to \ell^\infty$ *is as smooth as the functions* f_n.

Proof. The proof is based on the idea to formulate a nonautonomous difference equation (Δ_λ) as an abstract equation $F(\chi, \lambda) = 0$ in the space ℓ^∞. This is solved using the implicit mapping theorem, where the invertibility of the Fréchet derivative $D_1 F(\chi^*, \lambda^*)$ is characterised by the hyperbolicity assumption on χ^*. For details, see [40, Theorem 2.11]. □

Consequently, in order to deduce sufficient conditions for bifurcations, one must violate the hyperbolicity of χ^*. For this purpose, the following characterisation of an exponential dichotomy is useful.

Theorem 7.6 (Characterization of exponential dichotomies). *A variational equation (V_λ) has an exponential dichotomy on $\mathbb{Z}$, if and only if the following conditions are fulfilled:*

(i) (V_λ) *has an exponential dichotomy on* $\mathbb{Z}^+$ *with projector* P_n^+, *as well as an exponential dichotomy on* $\mathbb{Z}^-$ *with projector* P_n^-,

(ii) $R(P_0^+) \oplus N(P_0^-) = \mathbb{R}^d$.

Proof. See [9, Lemma 2.4]. □

The subsequent examples illustrate various scenarios that can arise, if a condition stated in Theorem 7.6 is violated.

Example 7.7 (Pitchfork bifurcation). Consider the difference equation

$$x_{n+1} = f_n(x_n, \lambda), \qquad f_n(x, \lambda) := \frac{\lambda x}{1 + |x|}$$

from Sect. 4.5. It is a prototypical example of a supercritical autonomous pitchfork bifurcation (cf., e.g. [30, pp. 119ff, Sect. 4.4]), where the unique asymptotically stable equilibrium $x^* = 0$ for $\lambda \in (0, 1)$ bifurcates into two asymptotically stable equilibria $x_\pm := \pm(\lambda - 1)$ for $\lambda > 1$.

Along the trivial solution the variational equation $x_{n+1} = \lambda x_n$ becomes nonhyperbolic for $\lambda = 1$. Indeed, criterion (i) of Theorem 7.6 is violated, since the variational equation does not admit a dichotomy on $\mathbb{Z}^+$ or on $\mathbb{Z}^-$. This loss of hyperbolicity causes an attractor bifurcation, since for

- $\lambda \in (0, 1)$, the set $x^* = 0$ is the global attractor
- $\lambda > 1$, the trivial equilibrium $x^* = 0$ becomes unstable and the symmetric interval $A = [x_-, x_+]$ is the global attractor.

Bifurcations of pullback attractors can be observed as nonautonomous versions of pitchfork bifurcations.

Example 7.8 (Pullback attractor bifurcation). Consider for parameter values $\lambda > 0$ the difference equation

$$x_{n+1} = \lambda x_n - \begin{cases} \min\left\{a_n x_n^3, \frac{\lambda}{2} x_n\right\}, & x_n \geq 0, \\ \max\left\{a_n x_n^3, \frac{\lambda}{2} x_n\right\}, & x_n < 0, \end{cases}$$

where $(a_n)_{n\in\mathbb{Z}}$ is a sequence which is both bounded and bounded away from zero. Note that in a neighborhood U of 0, the difference equation is given by $x_{n+1} = \lambda x_n - a_n x_n^3$, and outside of a set $V \supset U$, the difference equation is given by $x_{n+1} = \frac{\lambda}{2} x_n$. Both U and V here can be chosen independently of λ near $\lambda = 1$. Moreover, for fixed $n \in \mathbb{Z}$, the right-hand side of this equation lies between the functions $x \mapsto \frac{\lambda}{2} x$ and $x \mapsto \lambda x$.

It is clear that for $\lambda \in (0, 1)$, the global pullback attractor is given by the trivial solution, which follows from the fact that points are contracted at each time step by the factor λ. For $\lambda > 1$, the trivial solution is no longer attractive, but there exists a (nontrivial) pullback attractor for $\lambda \in (1, 2)$. This follows from Theorem 5.4, because the family $\mathscr{B} = \{V : n \in \mathbb{Z}\}$ is pullback absorbing (the right-hand is given by $x \mapsto \frac{\lambda}{2} x$ outside of V).

At the parameter value $\lambda = 1$, the global pullback attractor changes its dimension. Thus, this difference equation provides an example of a nonautonomous pitchfork bifurcation, which will be treated below in Sect. 7.2.

While these two examples show how (autonomous) bifurcations can be understood as attractor bifurcations, the following scenario is intrinsically nonautonomous (see [36] for a deeper analysis).

Example 7.9 (Shovel bifurcation). Consider a scalar difference equation

$$x_{n+1} = a_n(\lambda)x_n, \qquad a_n(\lambda) := \begin{cases} \frac{1}{2} + \lambda, & n < 0, \\ \lambda, & n \geq 0, \end{cases} \tag{45}$$

with parameters $\lambda > 0$. In order to understand the dynamics of (45), distinguish three cases:

(i) $\lambda \in (0, \frac{1}{2})$: The equation (45) has an exponential dichotomy on $\mathbb{Z}$ with projector $P_n \equiv 1$. The uniquely determined bounded entire solution is the trivial one, which is uniformly asymptotically stable.
(ii) $\lambda > 1$: The equation (45) has an exponential dichotomy on $\mathbb{Z}$ with projector $P_k \equiv 0$. Again, 0 is the unique bounded entire solution, but is now unstable.
(iii) $\lambda \in (\frac{1}{2}, 1)$: In this situation, (45) has an exponential dichotomy on $\mathbb{Z}^+$ with projector $P_n^+ \equiv 1$, as well as an exponential dichotomy on $\mathbb{Z}^-$ with projector $P_n^- = 0$. Thus condition (ii) in Theorem 7.6 is violated and 0 is a nonhyperbolic solution. For this parameter regime, every solution of (45) is bounded. Moreover, (45) is asymptotically stable, but not uniformly asymptotically stable on the whole time axis $\mathbb{Z}$.

The parameter values $\lambda \in \{\frac{1}{2}, 1\}$ are critical. In both situations, the number of bounded entire solutions to the linear difference equation (45) changes drastically. Furthermore, there is a loss of stability in two steps: From uniformly asymptotically stable to asymptotically stable, and finally to unstable, as λ increases through the values $\frac{1}{2}$ and 1. Hence, both values can be considered as bifurcation values, since the number of bounded entire solutions changes as well as their stability properties.

The next example requires the state space to be at least two-dimensional.

Example 7.10 (Fold solution bifurcation). Consider the planar equation

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^2 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{46}$$

with components $x_n = (x_n^1, x_n^2)$, depending on a parameter $\lambda \in \mathbb{R}$ and asymptotically constant sequences

$$b_n := \begin{cases} 2, & n < 0, \\ \frac{1}{2}, & n \geq 0, \end{cases} \qquad c_n := \begin{cases} \frac{1}{2}, & n < 0, \\ 2, & n \geq 0. \end{cases} \tag{47}$$

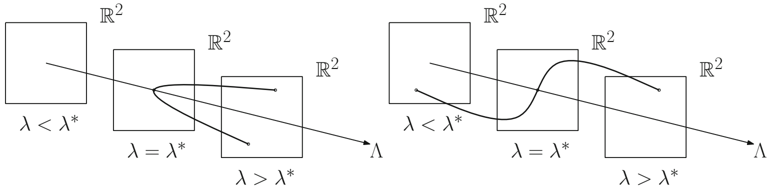


Fig. 8 *Left* (supercritical fold): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\phi_\lambda(\cdot, 0, \eta)$ of (46) for different parameter values λ.
Right (cusp): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\phi_\lambda(\cdot, 0, \eta)$ of (49) for different parameter values λ

The variational equation for (46) corresponding to the trivial solution and the parameter $\lambda^* = 0$ reads as

$$x_{n+1} = D_1 f_n(0, 0)x_n := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n.$$

It admits an exponential dichotomy on $\mathbb{Z}^+$, as well as on $\mathbb{Z}^-$ with corresponding invariant projectors $P_n^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_n^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This yields

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad R(P_0^+) + N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and therefore condition (ii) of Theorem 7.6 is violated. Hence, the trivial solution to (46) for $\lambda = 0$ is not hyperbolic.

Let $\phi_\lambda(\cdot, 0, \eta)$ be the general solution to (46). Its first component ϕ_λ^1 is

$$\phi_\lambda^1(n, 0, \eta) = 2^{-|n|}\eta_1 \quad \text{for all } n \in \mathbb{Z}, \tag{48}$$

while the variation of constants formula (cf. [1, p. 59]) can be used to deduce the asymptotic representation

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n\left(\eta_2 + \frac{4}{7}\eta_1^2 - \lambda\right) + O(1), & n \to \infty, \\ \frac{1}{2^n}\left(\eta_2 - \frac{1}{2}\eta_1^2 + 2\lambda\right) + O(1), & n \to -\infty. \end{cases}$$

Therefore, the sequence $\phi_\lambda(\cdot, 0, \eta)$ is bounded if and only if $\eta_2 = -\frac{4}{7}\eta_1^2 + \lambda$ and $\eta_2 = \frac{1}{2}\eta_1^2 - 2\lambda$ holds, i.e., $\eta_1^2 = \frac{7}{2}\lambda$, $\eta_2 = -\lambda$. From the first relation, one sees that there exist two bounded solutions if $\lambda > 0$, the trivial solution is the unique bounded solution for $\lambda = 0$ and there are no bounded solutions for $\lambda < 0$; see Fig. 8 (left) for an illustration. For this reason, $\lambda = 0$ can be interpreted as bifurcation value, since the number of bounded entire solutions increases from 0 to 2 as λ increases through 0.

The method of explicit solutions can also be applied to the nonlinear equation

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ 0 & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^3 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{49}$$

However, using the variation of constants formula (cf. [1, p. 59]), it is possible to show that the crucial second component of the general solution $\phi_\lambda(\cdot, 0, \eta)$ for (49) fulfills

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n\left(\eta_2 + \frac{8}{15}\eta_1^3 - \lambda\right) + O(1), & n \to \infty, \\ \frac{1}{2^n}\left(\eta_2 - \frac{2}{15}\eta_1^3 + 2\lambda\right) + O(1), & n \to -\infty. \end{cases}$$

Since the first component is given in (48), $\phi_\lambda(\cdot, 0, \eta)$ is bounded if and only if $\eta_2 = -\frac{8}{15}\eta_1^3 + \lambda$ and $\eta_2 = \frac{2}{15}\eta_1^3 - 2\lambda$, which in turn is equivalent to

$$\eta_1 = \sqrt[3]{\frac{9}{2}\lambda}, \qquad \eta_2 = -\frac{7}{5}\lambda.$$

Hence, these particular initial values $\eta \in \mathbb{R}^2$ given by the cusp shaped curve depicted in Fig. 8 (right) lead to bounded entire solutions of (49).

7.2 *Attractor Bifurcation*

An easy example for a bifurcation of a pullback attractor was discussed already in Example 7.8. Now a general bifurcation pattern will be derived, which ensures, under certain conditions on Taylor coefficients, that a pullback attractor changes qualitatively under variation of the parameter. This generalizes the autonomous pitchfork bifurcation pattern. Although the pullback attractor discussed in Example 7.8 is a *global* attractor, the pitchfork bifurcation only yields results for a *local* pullback attractor.

Definition 7.11. Consider a process ϕ on a metric state space X. A ϕ-invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a *local pullback attractor* if there exists an $\eta > 0$ such that

$$\lim_{k\to\infty} \operatorname{dist}\left(\phi(n, n-k, B_\eta(A_{n-k})), A_n\right) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

A local pullback attractor is a special case of a pullback attractor w.r.t. a certain basin of attraction, which was introduced in Definition 5.11. Here, the basin of attraction has to be chosen as a neighborhood of the local pullback attractor.

Suppose now that (Δ_λ) is a scalar equation ($d = 1$) with the trivial solution for all parameters λ from an interval $\Lambda \subseteq \mathbb{R}$. The transition matrix of the corresponding variational equation

$$x_{n+1} = D_1 f_n(0, \lambda) x_n$$

is denoted by $\Phi_\lambda(n, l) \in \mathbb{R}$.

The hyperbolicity condition (i) in Theorem 7.6 will be violated when dealing with attractor bifurcations. This yields a nonautonomous counterpart to the classical pitchfork bifurcation pattern.

Theorem 7.12 (Nonautonomous pitchfork bifurcation). *Suppose that* $f_n(\cdot,\lambda):\mathbb{R}\to\mathbb{R}$ *is invertible and of class* C^4 *with*

$$D_1^2 f_n(0,\lambda)=0 \quad \text{for all } n\in\mathbb{Z} \text{ and } \lambda\in\Lambda.$$

Suppose there exists a $\lambda^*\in\mathbb{R}$ *such that the following hypotheses hold.*

- *Hypothesis on linear part: There exists a* $K\geq 1$ *and functions* $\beta_1,\beta_2:\Lambda\to(0,\infty)$ *which are either both increasing or decreasing with* $\lim_{\lambda\to\lambda^*} b_1(\lambda)=\lim_{\lambda\to\lambda^*} b_2(\lambda)=1$ *and*

$$\Phi_\lambda(n,l)\leq K\beta_1(\lambda)^{n-l} \quad \text{for all } l\leq n,$$
$$\Phi_\lambda(n,l)\leq K\beta_2(\lambda)^{n-l} \quad \text{for all } n\leq l$$

  and all $\lambda\in\Lambda$.
- *Hypothesis on nonlinearity: Assume that if the functions* β_1 *and* β_2 *are increasing, then*

$$-\infty<\liminf_{\lambda\to\lambda^*}\inf_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda)\leq\limsup_{\lambda\to\lambda^*}\sup_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda)<0,$$

  and otherwise (i.e., if the functions β_1 *and* β_2 *are decreasing), then*

$$0<\liminf_{\lambda\to\lambda^*}\inf_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda)\leq\limsup_{\lambda\to\lambda^*}\sup_{n\in\mathbb{Z}} D_1^3 f_n(0,\lambda)<\infty.$$

  In addition, suppose that the remainder satisfies

$$\lim_{x\to 0}\sup_{\lambda\in(\lambda^*-x^2,\lambda^*+x^2)}\sup_{n\leq 0} x\int_0^1(1-t)^3 D^4 f_n(tx,\lambda)\,dt=0,$$

$$\limsup_{\lambda\to\lambda^*}\limsup_{x\to 0}\sup_{n\leq 0}\frac{Kx^3}{1-\min\{\beta_1(\lambda),\beta_2(\lambda)^{-1}\}}\int_0^1(1-t)^3 D^4 f_n(tx,\lambda)\,dt<3.$$

Then there exist $\lambda_-<\lambda^*<\lambda_+$ *so that the following statements hold:*

1. *If the functions* β_1,β_2 *are increasing, the trivial solution is a local pullback attractor for* $\lambda\in(\lambda_-,\lambda^*)$*, which bifurcates to a nontrivial local pullback attractor* $\{A_n^\lambda:n\in\mathbb{Z}\}$*,* $\lambda\in(\lambda^*,\lambda^+)$*, satisfying the limit*

$$\lim_{\lambda\to\lambda^*}\sup_{n\leq 0}\operatorname{dist}(A_n^\lambda,\{0\})=0.$$

2. *If the functions* β_1, β_2 *are decreasing, the trivial solution is a local pullback attractor for* $\lambda \in (\lambda^*, \lambda^+)$*, which bifurcates to a nontrivial local pullback attractor* $\{A_n^\lambda : n \in \mathbb{Z}\}$*,* $\lambda \in (\lambda_-, \lambda^*)$*, satisfying the limit*

$$\lim_{\lambda \to \lambda^*} \sup_{n \leq 0} \operatorname{dist}(A_n^\lambda, \{0\}) = 0.$$

For a proof of this theorem and extensions (to both different time domains and repellers), see [43, 45].

The next example, taken from [15], illustrates the above theorem.

Example 7.13. Consider the nonautonomous difference equation

$$x_{n+1} = \frac{\lambda x_n}{1 + \frac{b_n q}{\lambda} x_n^q}, \tag{50}$$

where $q \in \mathbb{N}$ and the sequence $(b_n)_{n \in \mathbb{N}}$ is positive and both bounded and bounded away from zero. For $q = 1$, this difference equation can be transformed into the well-known Beverton–Holt equation, which describes the density of a population in a fluctuating environment. It was shown in [15] that in this case, the system admits a nonautonomous transcritical bifurcation (the bifurcation pattern of which was derived in [43]).

For $q = 2$, a nonautonomous pitchfork bifurcation occurs. The above theorem can be applied, because the Taylor expansion of the right-hand side of (50) reads as $\lambda x_n + b_n x_n^{q+1} + O(x^{2q+1})$, and the remainder fulfills the conditions of the theorem (see [15] for details). This means that for $\lambda \in (0, 1)$, the trivial solution is a local pullback attractor, which undergoes a transition to a nontrivial local pullback attractor when $\lambda > 1$. Note that the extreme solutions of the nontrivial local pullback attractor for $\lambda > 1$ are also local pullback attractors, which gives the interpretation of this bifurcation as a bifurcation of locally pullback attractive solutions.

7.3 *Solution Bifurcation*

In the previous section on attractor bifurcations, the first hyperbolicity condition (i) in Theorem 7.6, given by exponential dichotomies on both semiaxes, was violated.

The present concept of solution bifurcation is based on the assumption that merely condition (ii) of Theorem 7.6 does not hold. This requires the variational difference equation (V_λ) to be intrinsically nonautonomous. Indeed, if (V_λ) is almost periodic, then an exponential dichotomy on a semiaxis extends to the whole integer axis (cf. [52, Theorem 2]) and the reference solution $\chi = (\chi_n)_{n \in \mathbb{Z}}$ becomes hyperbolic. For this reason the following bifurcation scenarios cannot occur for periodic or autonomous difference equations.

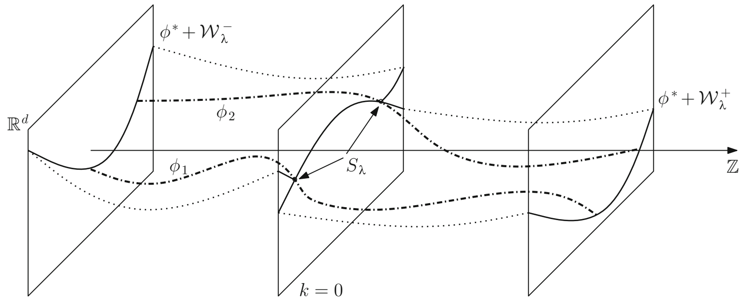


Fig. 9 Intersection $S_\lambda \subseteq \mathbb{R}^d$ of the stable fiber bundle $\phi^* + \mathscr{W}_\lambda^+ \subseteq \mathbb{Z}^+ \times \mathbb{R}^d$ with the unstable fiber bundle $\phi^* + \mathscr{W}_\lambda^- \subseteq \mathbb{Z}^- \times \mathbb{R}^d$ at time $k = 0$ yields two bounded entire solutions ϕ_1, ϕ_2 to (Δ_λ) indicated as *dotted dashed lines*

The crucial and standing assumption is the following:

Hypothesis: *The variational equation* (V_λ) *admits an ED both on* $\mathbb{Z}^+$ *(with projector* P_n^+*) and on* $\mathbb{Z}^-$ *(with projector* P_n^-*) such that there exists nonzero vectors* $\xi_1 \in \mathbb{R}^d$, $\xi_1' \in \mathbb{R}^d$ *satisfying*

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R}\xi_1, \qquad (R(P_0^+) + N(P_0^-))^\perp = \mathbb{R}\xi_1'. \tag{51}$$

Then a *solution bifurcation* is understood as follows: Suppose that for a fixed parameter $\lambda^* \in \Lambda$, the difference equation (Δ_{λ^*}) has an entire bounded reference solution $\chi^* = \chi(\lambda^*)$. One says that (Δ_λ) undergoes a *bifurcation* at $\lambda = \lambda^*$ along χ^*, or χ^* *bifurcates* at λ^*, if there exists a convergent parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with limit λ^* so that (Δ_{λ_n}) has two distinct entire solutions $\chi^1_{\lambda_n}, \chi^2_{\lambda_n} \in \ell^\infty$ both satisfying

$$\lim_{n\to\infty} \chi^1_{\lambda_n} = \lim_{n\to\infty} \chi^2_{\lambda_n} = \chi^*.$$

The above Hypothesis allows a geometrical insight into the following abstract bifurcation results using *invariant fiber bundles*, i.e., nonautonomous counterparts to invariant manifolds: Because (V_λ) has an exponential dichotomy on $\mathbb{Z}^+$, there exists a stable fiber bundle $\chi^* + \mathscr{W}_\lambda^+$ consisting of all solutions to (Δ_λ) approaching χ^* in forward time. Here, $\mathscr{W}_\lambda^+$ is locally a graph over the stable vector bundle $\{R(P_n^+) : n \in \mathbb{Z}^+\}$. Analogously, the dichotomy on $\mathbb{Z}^-$ guarantees an unstable fiber bundle $\chi^* + \mathscr{W}_\lambda^-$ consisting of solutions decaying to χ^* in backward time (cf. [40, Corollary 2.23]). Then the bounded entire solutions to (Δ_λ) are contained in the intersection $(\chi^* + \mathscr{W}_\lambda^+) \cap (\chi^* + \mathscr{W}_\lambda^-)$. In particular, the intersection of the fibers

$$S_\lambda := \left(\chi_0^* + \mathscr{W}_{\lambda,0}^+\right) \cap \left(\chi_0^* + \mathscr{W}_{\lambda,0}^-\right) \subseteq \mathbb{R}^d$$

yields initial values for bounded entire solutions (see Fig. 9).

It can be assumed without loss of generality, using the equation of perturbed motion, that $\chi^* = 0$. In addition suppose that

$$f_n(0, \lambda) \equiv 0 \quad \text{on } \mathbb{Z},$$

which means that (Δ_λ) has the trivial solution for all $\lambda \in \Lambda$. The corresponding variational equation is

$$x_{n+1} = D_1 f_n(0,\lambda)x_n$$

with transition matrix $\Phi_\lambda(n,l) \in \mathbb{R}^{d\times d}$.

Theorem 7.14 (Bifurcation from known solutions). *Let $\Lambda \subseteq \mathbb{R}$ and suppose f_n is of class C^m, $m \geq 2$. If the transversality condition*

$$g_{11} := \sum_{n\in\mathbb{Z}} \langle \Phi_{\lambda^*}(0,n+1)'\xi_1', D_1D_2 f_n(0,\lambda^*)\Phi_{\lambda^*}(n,0)\xi_1\rangle \neq 0 \tag{52}$$

is satisfied, then the trivial solution of a difference equation (Δ_λ) bifurcates at λ^*. In particular, there exists a $\rho > 0$, open convex neighborhoods $U \subseteq \ell^\infty(\Omega)$ of 0, $\Lambda_0 \subseteq \Lambda$ of λ^* and C^{m-1}-functions $\psi : (-\rho,\rho) \to U$, $\lambda : (-\rho,\rho) \to \Lambda_0$ with

1. *$\psi(0) = 0$, $\lambda(0) = \lambda^*$ and $\dot\psi(0) = \Phi_{\lambda^*}(\cdot,0)\xi_1$,*
2. *Each $\psi(s)$ is a nontrivial solution of $(\Delta)_{\lambda(s)}$ homoclinic to 0, i.e.,*

$$\lim_{n\to\pm\infty} \psi(s)_n = 0.$$

Proof. See [39, Theorem 2.14]. □

Corollary 7.15 (Transcritical bifurcation). *Under the additional assumption*

$$g_{20} := \sum_{n\in\mathbb{Z}} \langle \Phi_{\lambda^*}(0,n+1)'\xi_1', D_1^2 f_n(0,\lambda^*)[\Phi_{\lambda^*}(n,0)\xi_1]^2\rangle \neq 0$$

one has $\dot\lambda(0) = -\frac{g_{20}}{2g_{11}}$ and the following holds locally in $U \times \Lambda_0$: The difference equation (Δ_λ) has a unique nontrivial entire bounded solution $\psi(\lambda)$ for $\lambda \neq \lambda^*$ and 0 is the unique entire bounded solution of $(\Delta)_{\lambda^*}$; moreover, $\psi(\lambda)$ is homoclinic to 0.

Proof. See [39, Corollary 2.16]. □

Example 7.16. Consider the nonlinear difference equation

$$x_{n+1} = f_n(x_n,\lambda) := \begin{pmatrix} b_n & 0 \\ \lambda & c_n \end{pmatrix} x_n + \begin{pmatrix} 0 \\ (x_n^1)^2 \end{pmatrix} \tag{53}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and sequences b_n, c_n defined in (47). As in Example 7.10, the assumptions hold with $\lambda^* = 0$ and

$$g_{11} = \frac{4}{3} \neq 0, \qquad g_{20} = \frac{12}{7} \neq 0.$$

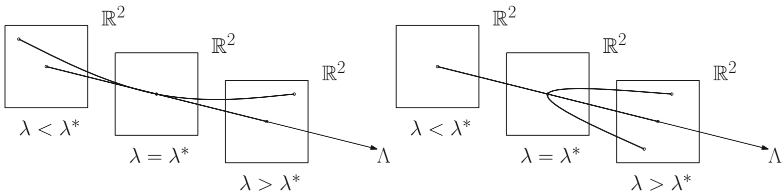


Fig. 10 *Left* (transcritical): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\phi_\lambda(\cdot, 0, \eta)$ of (53) for different parameter values λ.
Right (supercritical pitchfork): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\phi_\lambda(\cdot, 0, \eta)$ of (54) for different parameter values λ

Hence, Corollary 7.15 can be applied in order to see that the trivial solution of (53) has a transcritical bifurcation at $\lambda = 0$. Again, this bifurcation will be described quantitatively. While the first component of the general solution $\phi_\lambda(\cdot, 0, \eta)$ given by (48) is homoclinic, the second component satisfies

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n \left(\eta_2 + \frac{4}{7}\eta_1^2 + \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to \infty, \\ 2^{-n} \left(\eta_2 - \frac{2}{7}\eta_1^2 - \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to -\infty. \end{cases}$$

In conclusion, one sees that $\phi_\lambda(\cdot, 0, \eta)$ is bounded if and only if $\eta = (0, 0)$ or

$$\eta_1 = -\frac{14}{9}\lambda, \qquad \eta_2 = \frac{28}{81}\lambda^2.$$

Hence, besides the zero solution, there is a unique nontrivial entire solution passing through the initial point $\eta = (\eta_1, \eta_2)$ at time $n = 0$ for $\lambda \neq 0$. This means the solution bifurcation pattern sketched in Fig. 10 (left) holds.

Corollary 7.17 (Pitchfork bifurcation). *For $m \geq 3$ and under the additional assumptions*

$$\sum_{n \in \mathbb{Z}} \langle \Phi_{\lambda^*}(0, n+1)'\xi_1', D_1^2 f_n(0, \lambda^*)[\Phi_{\lambda^*}(n, 0)\xi_1]^2 \rangle = 0,$$

$$g_{30} := \sum_{n \in \mathbb{Z}} \langle \Phi_{\lambda^*}(0, n+1)'\xi_1', D_1^3 f_n(0, \lambda^*)[\Phi_{\lambda^*}(n, 0)\xi_1]^3 \rangle \neq 0$$

one has $\dot{\lambda}(0) = 0$, $\ddot{\lambda}(0) = -\frac{g_{30}}{3g_{11}}$ and the following holds locally in $U \times \Lambda_0$:

3. *Subcritical case: If $g_{30}/g_{11} > 0$, then the unique entire bounded solution of (Δ_λ) is the trivial one for $\lambda \geq \lambda^*$ and (Δ_λ) has exactly two nontrivial entire solutions for $\lambda < \lambda^*$; both are homoclinic to 0.*

4. *Supercritical case: If $g_{30}/g_{11} < 0$, then the unique entire bounded solution of (Δ_λ) is the trivial one for $\lambda \le \lambda^*$ and (Δ_λ) has exactly two nontrivial entire solutions for $\lambda > \lambda^*$; both are homoclinic to* 0.

Proof. See [39, Corollary 2.16]. □

Example 7.18. Let δ be a fixed nonzero real number and consider the nonlinear difference equation

$$x_{n+1} = f_n(x_n, \lambda) := \begin{pmatrix} b_n & 0 \\ \lambda & c_n \end{pmatrix} x_n + \delta \begin{pmatrix} 0 \\ (x_n^1)^3 \end{pmatrix} \tag{54}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and the b_n, c_n defined in (47). As in our above Example 7.16, the assumptions of Corollary 7.17 are fulfilled with $\lambda^* = 0$. The transversality condition here reads $g_{11} = \frac{4}{3} \neq 0$. Moreover, $D_1^2 f_n(0,0) \equiv 0$ on $\mathbb{Z}$ implies $g_{20} = 0$, whereas the relation $D_1^3 f_n(0,0)\zeta^3 = \binom{0}{6\delta\zeta_1^3}$ for all $n \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2$ leads to $g_{30} = 4\delta \neq 0$. This gives the crucial quotient $\frac{g_{30}}{g_{11}} = 3\delta$. By Corollary 7.17, the trivial solution to (54) undergoes a subcritical (supercritical) pitchfork bifurcation at $\lambda = 0$ provided $\delta > 0$ (resp. $\delta < 0$). As before one can illustrate this result using the general solution $\phi_\lambda(\cdot, 0, \eta)$ to (54). The first component is given by (48) and helps to show for the second component that

$$\phi_\lambda^2(n, 0, \eta) = \begin{cases} 2^n \left(\eta_2 + \frac{8\delta}{15}\eta_1^3 + \frac{2\lambda}{3}\eta_1\right) + o(1), & n \to \infty, \\ 2^{-n} \left(\eta_2 - \frac{2\delta}{15}\eta_1^3 - \frac{4\lambda}{3}\eta_1\right) + o(1), & n \to -\infty. \end{cases}$$

This asymptotic representation shows that $\phi_\lambda(\cdot, 0, \eta)$ is homoclinic to 0 if and only if $\eta = 0$ or $\eta_1^2 = -\frac{2}{\delta}\lambda$ and $\eta_2 = \frac{4}{15}\frac{(5\delta+16\lambda)}{\delta^2}\lambda^2$. Hence, there is a correspondence to the pitchfork solution bifurcation from in Corollary 7.17. See Fig. 10 (right) for an illustration.

Remarks. In [43, Theorem 5.1] one finds a nonautonomous generalization for transcritical bifurcations.

8 Random Dynamical Systems

Random dynamical systems on a state space X are nonautonomous by the very nature of the driving noise. They can be formulated as skew-product systems with the driving system acting a probability sample space Ω rather than on a topological or metric parameter space P. A major difference is that only measurability and not continuity w.r.t. the parameter can be assumed, which changes the types of results that can be proved. In particular, the skew-product system does not form an autonomous semidynamical system on the product space $\Omega \times X$. Nevertheless, there are many interesting parallels with the theory of deterministic nonautonomous dynamical systems.

For further details see Arnold [2] and, for example, also [27], where the temporal discretization of random differential equations is also considered.

8.1 Random Difference Equations

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\{\xi_n, n \in \mathbb{Z}\}$ be a discrete-time stochastic process taking values in some space Ξ, i.e., a sequence of random variables or, equivalently, $\mathscr{F}$-measurable mappings $\xi_n : \Omega \to \Xi$ for $n \in \mathbb{Z}$. Let (X, d) be a complete metric space and consider a mapping $g : \Xi \times X \to X$.

Then

$$x_{n+1}(\omega) = g\left(\xi_n(\omega), x_n(\omega)\right) \quad \text{for all } n \in \mathbb{Z},\ \omega \in \Omega, \tag{55}$$

is a *random difference equation* on X driven by the stochastic process ξ_n.

Greater generality can be achieved by representing the driving noise process by a metrical (i.e., measure theoretic) dynamical system θ on some canonical sample space Ω, i.e., the group of $\mathscr{F}$-measurable mappings $\{\theta_n, n \in \mathbb{Z}\}$ under composition formed by iterating a measurable mapping $\theta : \Omega \to \Omega$ and its measurable inverse mapping $\theta^{-1} : \Omega \to \Omega$, i.e., with $\theta_0 = \mathrm{id}_\Omega$ and

$$\theta_{n+1} := \theta \circ \theta_n, \qquad \theta_{-n-1} := \theta^{-1} \circ \theta_{-n} \quad \text{for all } n \in \mathbb{N},$$

where $\theta_{-1} := \theta^{-1}$. It is usually assumed that θ generates an ergodic process on Ω.

Let $f : \Omega \times X \to X$ be an $\mathscr{F} \times \mathscr{B}(X)$-measurable mapping, where $\mathscr{B}(X)$ is the Borel σ-algebra on X. Then, in this context, a *random difference equation* has the form

$$x_{n+1}(\omega) = f\left(\theta^n(\omega), x_n(\omega)\right) \quad \text{for all } n \in \mathbb{Z},\ \omega \in \Omega. \tag{56}$$

Define recursively a solution mapping $\varphi : \mathbb{Z}^+ \times \Omega \times X \to X$ for the random difference equation (56) by $\varphi(\omega, 0, x) := x$ and

$$\varphi(\omega, n+1, x) = f(\theta^n(\omega), \phi(\theta^n(\omega), n, x)) \quad \text{for all } n \in \mathbb{N},\ x \in X$$

and $\omega \in \Omega$. Then, φ satisfies the discrete-time *cocycle property* w.r.t. θ, i.e.,

$$\varphi(n+m, \omega, x) = \varphi\left(n, \theta_m(\omega), \varphi(m, \omega, x_0)\right) \quad \text{for all } m, n \in \mathbb{Z}^+,$$

$x \in X$ and $\omega \in \Omega$. The mapping φ is called a cocycle mapping.

In terms of Arnold [2], the random difference equation (56) generates a discrete-time *random dynamical system* (θ, ϕ) on $\Omega \times X$ with the *metric dynamical system* θ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the cocycle mapping φ on the state space X.

Definition 8.1. A (discrete-time) *random dynamical system* (θ, φ) on $\Omega \times X$ consists of a metrical dynamical system θ on Ω, i.e., a group of measure preserving mappings $\theta_n : \Omega \to \Omega$, $n \in \mathbb{Z}$, such that

(i) $\theta_0 = \mathrm{id}_\Omega$ and $\theta_n \circ \theta_m = \theta_{n+m}$ for all $n, m \in \mathbb{Z}$,
(ii) The map $\omega \mapsto \theta_n(\omega)$ is measurable and invariant w.r.t. $\mathbb{P}$ in the sense that $\theta_n(\mathbb{P}) = \mathbb{P}$ for each $n \in \mathbb{Z}$,

and a cocycle mapping $\varphi : \mathbb{Z}^+ \times \Omega \times X \to X$ such that

(a) $\varphi(0, \omega, x_0) = x_0$ for all $x_0 \in X$ and $\omega \in \Omega$,
(b) $\varphi(n + m, \omega, x_0) = \varphi(n, \theta_m(\omega), \varphi(m, \omega, x_0))$ for all n,$m \in \mathbb{Z}^+$, $x_0 \in X$ and $\omega \in \Omega$,
(c) $x_0 \mapsto \varphi(n, \omega, x_0)$ is continuous for each $(n, \omega) \in \mathbb{Z}^+ \times \Omega$,
(d) $\omega \mapsto \varphi(n, \omega, x_0)$ is $\mathscr{F}$-measurable for all $(n, x_0) \in \mathbb{Z}^+ \times X$.

The notation $\theta_n(\mathbb{P}) = \mathbb{P}$ for the measure preserving property of θ_n w.r.t. $\mathbb{P}$ is just a compact way of writing

$$\mathbb{P}(\theta_n(A)) = \mathbb{P}(A) \quad \text{for all } n \in \mathbb{Z}, A \in \mathscr{F} .$$

A systematic treatment of the random dynamical system theory, both continuous and discrete time, is propounded in Arnold [2]. Note that $\pi = (\theta, \phi)$ has a skew-product structure on $\Omega \times X$, but it is not an autonomous semidynamical system on $\Omega \times X$ since no topological structure is assumed on Ω.

8.2 Random Attractors

Unlike a deterministic skew-product system, a random dynamical system (θ, φ) on $\Omega \times X$ is not an autonomous semidynamical system on $\Omega \times X$. Nevertheless, skew-product deterministic systems and random dynamical systems have many analogous properties, and concepts and results for one can often be used with appropriate modifications for the other. The most significant modification concerns measurability and the nonautonomous sets under consideration are random sets.

Let (X, d) be a complete and separable metric space (i.e., a Polish space)

Definition 8.2. A family $\mathscr{D} = \{D_\omega, \omega \in \Omega\}$ of nonempty subsets of X is called a *random set* if the mapping $\omega \mapsto \mathrm{dist}(x, D_\omega)$ is $\mathscr{F}$-measurable for all $x \in X$. A random set $\mathscr{D}$ is called a *random closed set* if D_ω is closed for each $\omega \in \Omega$ and is called a *random compact set* if D_ω is compact for each $\omega \in \Omega$.

Random sets are called *tempered* if their growth w.r.t. the driving system θ is sub-exponential (cf. Definition 6.6).

Definition 8.3. A random set $\mathscr{D} = \{D_\omega, \omega \in \Omega\}$ in X is said to be *tempered* if there exists a $x_0 \in X$ such that

$$D_\omega \subset \{x \in X \ : \ d(x, x_0) \le r(\omega)\} \quad \text{for all } \omega \in \Omega ,$$

where the random variable $r(\omega) > 0$ is tempered, i.e.,

$$\sup_{n\in\mathbb{Z}}\{r(\theta_n(\omega))e^{-\gamma|n|}\} < \infty \quad \text{for all } \omega \in \Omega,\ \gamma > 0.$$

The collection of all tempered random sets in X will be denoted by $\mathfrak{D}$.

A random attractor of a random dynamical system is a random set which is a pullback attractor in the pathwise sense w.r.t. the attracting basin of tempered random sets.

Definition 8.4. A random compact set $\mathscr{A} = (A_\omega)_{\omega\in\Omega}$ from $\mathfrak{D}$ is called a *random attractor* of a random dynamical system (θ, φ) on $\Omega \times X$ in $\mathscr{D}$ if $\mathscr{A}$ is a φ-invariant set, i.e.,

$$\varphi(n, \omega, A_\omega) = A_{\theta_n(\omega)} \quad \text{for all } n \in \mathbb{Z}^+, \omega \in \Omega\ ,$$

and pathwise pullback attracting in $\mathfrak{D}$, i.e.,

$$\lim_{n\to\infty} \operatorname{dist}\big(\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big), A_\omega\big) = 0 \quad \text{for all } \omega \in \Omega,\ \mathscr{D} \in \mathfrak{D}.$$

If the random attractor consists of singleton sets, i.e., $A_\omega = \{Z^*(\omega)\}$ for some random variable Z^* with $Z^*(\omega) \in X$, then $\bar{Z}_n(\omega) := Z^*(\theta_n(\omega))$ is a stationary stochastic process on X.

The existence of a random attractor is ensured by that of a pullback absorbing set. The tempered random set $\mathscr{B} = \{B_\omega, \omega \in \Omega\}$ in the following theorem is called a pullback absorbing random set.

Theorem 8.5 (Existence of random attractors). *Let (θ, φ) be a random dynamical system on $\Omega \times X$ such that $\varphi(n, \omega, \cdot) : X \to X$ is a compact operator for each fixed $n > 0$ and $\omega \in \Omega$. If there exist a tempered random set $\mathscr{B} = \{B_\omega, \omega \in \Omega\}$ with closed and bounded component sets and an $N_{\mathscr{D},\omega} \geq 0$ such that*

$$\varphi\big(n, \theta_{-n}(\omega), D(\theta_{-n}(\omega))\big) \subset B_\omega \quad \textit{for all } n \geq N_{\mathscr{D},\omega}, \tag{57}$$

and every tempered random set $\mathscr{D} = \{D_\omega, \omega \in \Omega\}$, then the random dynamical system (θ, φ) possesses a random pullback attractor $\mathscr{A} = \{A_\omega : \omega \in \Omega\}$ with component sets defined by

$$A_\omega = \bigcap_{m>0} \overline{\bigcup_{n\geq m} \varphi(n, \theta_{-n}(\omega), B(\theta_{-n}(\omega))}\quad \textit{for all } \omega \in \Omega. \tag{58}$$

The proof of Theorem 8.5 is essentially the same as its counterparts for deterministic skew-product systems. The only new feature is that of measurability, i.e., to show that $\mathscr{A} = \{A_\omega), \omega \in \Omega\}$ is a random set. This follows from the fact that the set-valued mappings $\omega \mapsto \varphi\big(n, \theta_{-n}(\omega), B(\theta_{-n}(\omega))\big)$ are measurable for each $n \in \mathbb{Z}^+$.

Arnold and Schmalfuß [3] showed that a random attractor is also a *forward* attractor in the weaker sense of convergence in probability, i.e.,

$$\lim_{n\to\infty}\int_{\Omega}\text{dist}\left(\varphi(n,\omega,D_\omega),A_{\theta_n(\omega)}\right)\,\mathbb{P}(d\omega)=0$$

for all $\mathscr{D}\in\mathfrak{D}$. This allows individual sample paths to have large deviations from the attractor, but for all to converge in this probabilistic sense.

8.3 Random Markov Chains

Discrete-time finite state Markov chains with a tridiagonal structure are common in biological applications. They have a transition matrix $[I_N+\Delta Q]$, where I_N is the $N\times N$ identity matrix and Q is the tridiagonal $N\times N$-matrix

$$Q=\begin{bmatrix} -q_1 & q_2 & & & & \bigcirc \\ q_1 & -(q_2+q_3) & q_4 & & & \\ & \ddots & \ddots\;\ddots & \ddots & \ddots & \\ & & & q_{2N-5} & -(q_{2N-4}+q_{2N-3}) & q_{2N-2} \\ \bigcirc & & & & q_{2N-3} & -q_{2N-2} \end{bmatrix} \tag{59}$$

where the q_j are positive constants.

Such a Markov chain is a first order linear difference equation

$$\mathbf{p}^{(n+1)}=[I_N+\Delta Q]\,\mathbf{p}^{(n)} \tag{60}$$

on the probability simplex Σ_N in $\mathbb{R}^N$ defined by

$$\Sigma_N=\left\{\mathbf{p}=(p_1,\cdots,p_N)^T:\ \sum_{j=1}^{N}p_j=1,\ p_1,\ldots,p_N\in[0,1]\right\}.$$

The Perron-Frobenius theorem applies to the matrix $L_\Delta:=I_N+\Delta Q$ when $\Delta>0$ is chosen sufficiently small. In particular, it has eigenvalue $\lambda=1$ and there is a positive eigenvector $\bar{\mathbf{x}}$, which can be normalized (in the $\|\cdot\|_1$ norm) to give a probability vector $\bar{\mathbf{p}}$, i.e., $[I_N+\Delta Q]\bar{\mathbf{p}}=\bar{\mathbf{p}}$, so $Q\bar{\mathbf{p}}=0$. Specifically, the probability vector

$$\bar{\mathbf{p}}_1=\frac{1}{\|\bar{\mathbf{x}}\|_1},\quad \bar{\mathbf{p}}_{j+1}=\frac{1}{\|\bar{\mathbf{x}}\|_1}\prod_{i=1}^{j}\frac{q_{2i-1}}{q_{2i}}\quad\text{for all } j=1,\ldots,N-1,$$

where

$$\|\bar{\mathbf{x}}\|_1=\sum_{j=1}^{N}\bar{\mathbf{x}}_j=1+\sum_{j=1}^{N-1}\prod_{i=1}^{j}\frac{q_{2i-1}}{q_{2i}}.$$

The following result is well known.

Theorem 8.6. *The probability eigenvector* $\bar{\mathbf{p}}$ *is an asymptotically stable steady state of the difference equation* (60) *on the simplex* Σ_N.

In a random environment, e.g., with randomly varying food supply, the transition probabilities may be random, i.e., the band entries q_i of the matric Q may depend on the sample space parameter $\omega \in \Omega$. Thus, $q_i = q_i(\omega)$ for $i = 1, 2, \ldots, 2N-2$, and these may vary in turn according to some metric dynamical system θ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The following basic assumption will be used.

Assumption 1. *There exist numbers* $0 < \alpha \le \beta < \infty$ *such that the uniform estimates hold*

$$\alpha \le q_i(\omega) \le \beta \quad \text{for all } \omega \in \Omega,\ i = 1, 2, \ldots, 2N-2. \tag{61}$$

Let $\mathscr{L}$ be a set of linear operators $L_\omega : \mathbb{R}^N \to \mathbb{R}^N$ parametrized by the parameter ω taking values in some set Ω and let $\{\theta_n, n \in \mathbb{Z}\}$ be a group of maps of Ω onto itself. The maps $L_\omega x$ serve as the generator of a linear cocycle $F_{\mathscr{L}}(n,\omega)$. Then $(\theta, F_{\mathscr{L}})$ is a random dynamical system on $\Omega \times \Sigma_N$.

Theorem 8.7. *Let* $F_{\mathscr{L}}(n,\omega)x$ *be the linear cocycle*

$$F_{\mathscr{L}}(n,\omega)x = L_{\theta_{n-1}\omega} \cdots L_{\theta_1\omega} L_{\theta_0\omega} x.$$

with matrices $L_\omega := I_N + \Delta Q(\omega)$*, where the tridiagonal matrices* $Q(\omega)$ *are of the form* (59) *with the entries* $q_i = q_i(\omega)$ *satisfying the uniform estimates* (61) *in Assumption 1. In addition, suppose that* $0 < \Delta < \frac{1}{2\beta}$.

Then, the simplex Σ_N *is positively invariant under* $F_{\mathscr{L}}(n,\omega)$*, i.e.,*

$$F_{\mathscr{L}}(n,\omega)\Sigma_N \subseteq \Sigma_N \quad \text{for all } \omega \in \Omega.$$

Moreover, for n large enough, the restriction of $F_{\mathscr{L}}(n,\omega)x$ *to the set* Σ_N *is a uniformly dissipative and uniformly contractive cocycle (w.r.t. the Hilbert metric), which has a random attractor* $\mathscr{A} = \{A_\omega, \omega \in \Omega\}$ *such that each set* A_ω*,* $\omega \in \Omega$*, consists of a single point.*

The proof can be found in [28]. It involves positive matrices and the Hilbert projective metric on positive cones in $\mathbb{R}^N$.

Henceforth write $A_\omega = \{a_\omega\}$ for the singleton component subsets of the random attractor $\mathscr{A}$. Then the random attractor is an entire random sequence $\{a_{\theta_n\omega}, n \in \mathbb{Z}\}$ in $\Sigma_N(\gamma) \subset \overset{\circ}{\Sigma}_N$, where

$$\Sigma_N(\gamma) = \left\{ x = (x_1, x_2, \ldots, x_N) : \sum_{i=1}^N x_i = 1,\ x_1, x_2, \ldots, x_N \ge \gamma^{N-1} \right\}.$$

with $\gamma := \min\{\Delta\alpha, 1 - 2\Delta\beta\} > 0$. It attracts other iterates of the random Markov chain in the pullback sense. Pullback convergence involves starting at earlier initial times with a fixed end time. It is, generally, not the same as forward convergence in the sense usually understood in dynamical systems, but in this case it is the same due to the uniform boundedness of the contractive rate w.r.t. ω.

Corollary 8.8. *For any norm $\|\cdot\|$ on $\mathbb{R}^N$, $\mathbf{p}^{(0)} \in \Sigma_N$ and $\omega \in \Omega$*

$$\left\|\mathbf{p}^{(n)}(\omega) - a_{\theta_n \omega}\right\| \to 0 \quad as \quad n \to \infty.$$

The random attractor is, in fact, asymptotically Lyapunov stable in the conventional forward sense.

8.4 Approximating Invariant Measures

Consider now a compact metric space (X, d). A random difference equation (56) on X driven by the noise process θ generates a random dynamical system (θ, φ). It can be reformulated as a difference equation with a triangular or skew-product structure

$$(\omega, x) \;\mapsto\; F(\omega, x) \;:=\; \begin{pmatrix} \theta(\omega) \\ f(\omega, x) \end{pmatrix}$$

An invariant measure μ of $F = (\theta, \varphi)$ on $\Omega \times X$ defined by $\mu = F^*\mu$ (which is shorthand for an integral expression) can be decomposed as

$$\mu(\omega, B) \;=\; \mu_\omega(B)\,\mathbb{P}(d\omega) \quad \text{for all } B \in \mathscr{B}(X),$$

where the measures μ_ω on X are θ-invariant w.r.t. f, i.e.,

$$\mu_{\theta(\omega)}(B) \;=\; \mu_\omega\left(f^{-1}(\omega, B)\right) \quad \text{for all } B \in \mathscr{B}(X),\ \omega \in \Omega.$$

This decomposition is very important since only the state space X, but not the sample space Ω, can be discretized.

To compute a given invariant measure μ consider a sequence of finite subsets X_N of X given by

$$X_N = \{x_1^{(N)}, \cdots, x_N^{(N)}\} \subset X,$$

for $N \in \mathbb{N}$ with maximal step size

$$h_N = \sup_{x \in X} \mathrm{dist}(x, X_N)$$

such that $h_N \to 0$ as $N \to \infty$.

Then the invariant measure μ will be approximated by a sequence of invariant stochastic vectors associated with *random Markov chains* describing transitions between the states of the discretized state spaces X_N. These involve random $N \times N$ matrices, i.e., measurable mappings

$$P_N : \Omega \to \mathscr{S}_N,$$

where $\mathscr{S}_N$ denotes the set of $N \times N$ (nonrandom) stochastic matrices, satisfying the property

$$P_N^n(\theta^m(\omega))P_N^m(\omega) = P_N^{m+n}(\omega) \quad \text{for all } m, n \in \mathbb{Z}_+. \tag{62}$$

Recall that a stochastic matrix has non-negative entries with the columns summing to 1.

Consider a random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ and a random probability vector $\{p_N(\omega), \omega \in \Omega\}$ on the deterministic grid X_N. Then

$$p_{N,n+1}(\theta^{n+1}(\omega)) \;=\; p_{N,n}(\theta^n(\omega))P_N(\theta^n(\omega))$$

and an equilibrium probability vector is defined by

$$\bar{p}_N(\theta(\omega)) \;=\; \bar{p}_N(\omega)P_N(\omega) \quad \text{for all } \omega \in \Omega.$$

It can be represented trivially as a random measure $\mu_{N,\omega}$ on X.

The distance between random probability measures will be given with the *Prokhorov metric* ρ and the distance of a random Markov chain $P : \Omega \to \mathscr{S}_N$ and the generating mapping f of the random dynamical system is defined by

$$D(P(\omega), f) = \sum_{i,j=1}^{N} \left(p_{i,j}(\omega)\,\mathrm{dist}_{X\times X}((x_i^{(N)}, x_j^{(N)}), \mathrm{Gr} f(\omega,\cdot)\right), \tag{63}$$

where the distance to the random graph is given by

$$\mathrm{dist}_{X\times X}((x, y), \mathrm{Gr} f(\omega,\cdot)) = \inf_{z\in X} \max\{d(x,z), d(y, f(\omega,z))\} \quad \text{for all } x, y \in X.$$

The following necessary and sufficient result holds if θ-semi-invariant rather than θ-invariant families of decomposed probability measures are used.

Definition 8.9. A family of probability measures μ_ω on X is called θ-semi-invariant w.r.t. f, if

$$\mu_{\theta(\omega)}(B) \;\le\; \mu_\omega\left(f^{-1}(\omega, B)\right) \quad \text{for all } B \in \mathscr{B}(X),\ \omega \in \Omega.$$

Such θ-semi-invariant families are, in fact, θ-invariant when the mappings $x \mapsto f(\omega, x)$ are continuous.

Theorem 8.10. *A random probability measure $\{\mu_\omega, \omega \in \Omega\}$ is θ-semi-invariant w.r.t. f on X if and only if it is randomly stochastically approachable, i.e., for each N there exist*

(i) A grid X_N with fineness $h_N \to 0$ as $N \to \infty$
(ii) A random Markov chain $\{P_N(\omega), \omega \in \Omega\}$ on X_N
(iii) Random probability measure $\{\mu_{N,\omega}, \omega \in \Omega\}$ on X corresponding to a random equilibrium probability vector $\{\bar{p}_N(\omega), \omega \in \Omega\}$ of $\{P_N(\omega), \omega \in \Omega\}$ on X_N

with the expected convergences

$$\mathbb{E}D\left(P_N(\omega), f(\omega,\cdot)\right) \to 0, \qquad \mathbb{E}\rho\left(\mu_{N,\omega}, \mu_\omega\right) \to 0 \quad \text{as } n \to \infty.$$

Proof. See Imkeller and Kloeden [16]. □

The double terminology "random stochastic" seems to be an overkill, but just think of a Markov chain for which the transition probabilities are not fixed, but can vary randomly in time.

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