Preservation under Substructures modulo Bounded Cores

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Abstract. We investigate a model-theoretic property that generalizes the classical notion of preservation under substructures. We call this property *preservation under substructures modulo bounded cores*, and present a syntactic characterization via Σ_2^0 sentences for properties of arbitrary structures definable by FO sentences. Towards a sharper characterization, we conjecture that the count of existential quantifiers in the Σ_2^0 sentence equals the size of the smallest bounded core. We show that this conjecture holds for special fragments of FO and also over special classes of structures. We present a (not FO-definable) class of finite structures for which the conjecture fails, but for which the classical Łoś-Tarski preservation theorem holds. As a fallout of our studies, we obtain combinatorial proofs of the Łoś-Tarski theorem for some of the aforementioned cases.

Keywords: Model theory, First Order logic, Łoś-Tarski preservation theorem.

1 Introduction

Preservation theorems have traditionally been an important area of study in model theory. These theorems provide syntactic characterizations of semantic properties that are preserved under model-theoretic operations. One of the earliest preservation theorems is the Łoś-Tarski theorem, which states that over arbitrary structures, a First Order (FO) sentence is preserved under taking substructures iff it is equivalent to a Π_1^0 sentence [5]. Subsequently many other preservation theorems were studied, e.g. preservation under unions of chains, homomorphisms, direct products, etc. With the advent of finite model theory, the question of whether these theorems hold over finite structures became interesting. It turned out that several preservation theorems fail in the finite [1,7,9]. This inspired research on preservation theorems over special classes of finite structures, e.g. those with bounded degree, bounded tree-width etc. These efforts eventually led to some preservation theorems being "recovered" [2,3]. Among the theorems whose status over the class of all finite structures was open for long was the homomorphism preservation theorem. This was recently resolved in [10], which showed that the theorem survives in the finite.

In this paper, we look at a generalization of the preservation under substructures property that we call *preservation under substructures modulo bounded cores*. In Section 2, we show that for FO sentences, this property has a syntactic characterization in terms of Σ_2^0 sentences over arbitrary structures. Towards a sharper characterization,

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we conjecture that for core sizes bounded by a number B, there is a syntactic characterization in terms of Σ_2^0 sentences that use at most B existential quantifiers. In Section 3, we discuss how the notion of *relativization* can be used to prove the conjecture in special cases. We present our studies of the conjecture for special classes of FO and over special classes of structures in Sections 4 and 5. As a fallout of our studies, we obtain combinatorial proofs of the classical Łoś-Tarski theorem for some of the aforesaid special cases, and also obtain semantic characterizations of natural subclasses of the Δ_2^0 fragment of FO. We conclude with questions for future work in Section 6.

We assume that the reader is familiar with standard notation and terminology used in the syntax and semantics of FO (see [8]). A *vocabulary* τ is a set of predicate, function and constant symbols. In this paper, we will restrict ourselves to finite vocabularies only. A *relational vocabulary* has only predicate and constant symbols, and a *purely relational vocabulary* has only predicate symbols. We denote by $FO(\tau)$, the set of all FO formulae over vocabulary τ. A sequence $(x_1, \ldots, x_k)$ of variables is denoted by $\bar{x}$. We will abbreviate a block of quantifiers of the form $Qx_1 \ldots Qx_k$ by $Q\bar{x}$, where $Q \in \{\forall, \exists\}$. By Σ_k^0 (resp. Π_k^0), we mean FO sentences in Prenex Normal Form (PNF) over an arbitrary vocabulary, whose quantifier prefix begins with a $\exists$ (resp. $\forall$) and consists of $k-1$ alternations of quantifiers. We use the standard notions of τ-structures, substructures and extensions, as in [8]. Given $\tau-$structures M and N, we denote by $M \subseteq N$ that M is a substructure of N (or N is an extension of M). Given M and a subset S (resp. a tuple $\bar{a}$ of elements) of its universe, we denote by $M(S)$ (resp. $M(\bar{a})$) the smallest substructure (under set inclusion ordering of the universe) of M containing S (resp. underlying set of $\bar{a}$) and call it the substructure of M induced by S (resp. underlying set of $\bar{a}$). Finally, by *size* of M, we mean the cardinality of its universe and denote it by $|M|$. As a final note of convention, whenever we talk of FO definability in the paper, we mean definability via FO sentences (as opposed to theories), unless stated otherwise.

2 Preservation under Substructures Modulo Cores

We denote by $\mathbb{PS}$ the collection of all classes of structures, in any vocabulary, that are closed under taking substructures. This includes classes that are not definable in any logic. We let PS denote the collection of FO definable classes in $\mathbb{PS}$. We identify classes in PS with their defining FO sentences and will henceforth treat PS as a set of sentences. We now consider a natural generalization of $\mathbb{PS}$. Our discussion will concern arbitrary (finite) vocabularies and arbitrary structures over them.

2.1 The Case of Finite Cores

Definition 1 ***(Preservation under substructures modulo finite cores)***
A class of structures S is said to be preserved under substructures modulo a finite core *(denoted $S \in \mathbb{PSC}_f$), if for every structure $M \in S$, there exists a finite subset C of elements of M such that if $M_1 \subseteq M$ and M_1 contains C, then $M_1 \in S$. The set C is called a* core of M w.r.t. S. *If S is clear from context, we will call C as a* core of M.

Note that any finite subset of the universe of M containing a core is also a core of M. Also, there can be multiple cores of M having the same size. A *minimal* core of M is a core, no subset of which is a core of M.

We will use $\mathbb{PSC}_f$ to denote the collection of all classes preserved under substructures modulo a finite core. Similarly, we will use PSC_f to denote the collection of FO definable classes in $\mathbb{PSC}_f$. We identify classes in PSC_f with their defining FO sentences, and will henceforth treat PSC_f as a set of sentences.

Example 1: Let S be the class of all graphs containing cycles. For any graph in S, the vertices of any cycle is a core of the graph. Thus $S \in \mathbb{PSC}_f$.

Note that $\mathbb{PS} \subseteq \mathbb{PSC}_f$ since for any class in $\mathbb{PS}$ and for any structure in the class, any element is a core. However it is easy to check that S in above example is not in $\mathbb{PS}$; so $\mathbb{PSC}_f$ strictly generalizes $\mathbb{PS}$. Further, the FO inexpressibility of S shows that $\mathbb{PSC}_f$ contains classes not definable in FO.

Example 2: Consider $\phi = \exists x \forall y E(x, y)$. In any graph satisfying ϕ, any witness for x is a core of the graph. Thus $\phi \in PSC_f$. In fact, one can put a uniform bound of 1 on the minimal core size for all models of ϕ.

Again it is easy to see that $PS \subsetneq PSC_f$. Specifically, the sentence ϕ in Example 2 is not in PS. This is because a directed graph with exactly two nodes a and b, and having all directed edges except the self loop on a models ϕ but the subgraph induced by a does not model ϕ. Hence $PS \subsetneq PSC_f$. Extending the example above, one can show that for any sentence φ in Σ^0_2, in any model of φ, any witness for the $\exists$ quantifiers in φ forms a core of the model. Hence $\Sigma^0_2 \subseteq PSC_f$. In fact, for any sentence in Σ^0_2, the number of $\exists$ quantifiers serves as a uniform bound on the minimal core size for all models. Surprisingly, even for an arbitrary $\phi \in PSC_f$, it is possible to bound the minimal core size for all models!

Towards the result, we use the notions of *chain* and *union of chain* from the literature. The reader is referred to [5] for the definitions. We denote a chain as $M_1 \subseteq M_2 \subseteq \ldots$ and its union as $\bigcup_{i \geq 0} M_i$. We say that a sentence ϕ is *preserved under unions of chains* if for every chain of models of ϕ, the union of the chain is also a model of ϕ. We now recall the following characterization theorem from the '60s [5].

Theorem 1. *(Chang-Łoś-Suszko) A sentence ϕ is preserved under unions of chains iff it is equivalent to a Π^0_2 sentence.*

Now we have the following theorem.

Theorem 2. *A sentence $\phi \in PSC_f$ iff ϕ is equivalent to a Σ^0_2 sentence.*

Proof: We infer from Theorem 1 the following equivalences.
ϕ is equivalent to a Σ^2_0 sentence iff
$\neg\phi$ is equivalent to a Π^2_0 sentence iff
$\forall M_1, M_2, \ldots ((M_1 \subseteq M_2 \subseteq \ldots) \wedge (M = \bigcup_{i \geq 1} M_i) \wedge \forall i (M_i \models \neg\phi)) \rightarrow M \models \neg\phi$
iff
$\forall M_1, M_2, \ldots ((M_1 \subseteq M_2 \subseteq \ldots) \wedge (M = \bigcup_{i \geq 1} M_i) \wedge (M \models \phi)) \rightarrow \exists i (M_i \models \phi)$

Assume $\phi \in PSC_f$. Suppose $M_1 \subseteq M_2 \subseteq \ldots$ is a chain, $M = \bigcup_{i \geq 0} M_i$ and $M \models \phi$. Then, there exists a finite core C of M. For any $a \in C$, there exists an ordinal i_a s.t. $a \in M_{i_a}$ (else a would not be in the union M). Since C is finite, let $i = \mathbf{max}(i_a | \, a \in C)$. Since $i_a \leq i$, we have $M_{i_a} \subseteq M_i$; hence $a \in M_i$ for all $a \in C$. Thus M_i contains C. Since C is a core of M and $M_i \subseteq M$, $M_i \models \phi$ by definition of PSC_f. By the equivalences shown above, ϕ is equivalent to a Σ_2^0 sentence. We have seen earlier that $\Sigma_2^0 \subseteq PSC_f$. ■

Corollary 1. *If $\phi \in PSC_f$, there exists $B \in \mathbb{N}$ such that every model of ϕ has a core of size at most B.*

Proof: Take B to be the number of $\exists$ quantifiers in the equivalent Σ_2^0 sentence. ■

Given Corollary 1, it is natural to ask if B is computable. In this context, the following recent (unpublished) result by Rossman [11] is relevant. Let $|\phi|$ denote the size of ϕ.

Theorem 3. *(Rossman) There is no recursive function $f : \mathbb{N} \to \mathbb{N}$ such that if $\phi \in PS$, then there is an equivalent Π_1^0 sentence of size at most $f(|\phi|)$. The result holds even for relational vocabularies and further even if PS is replaced with $PS \cap \Sigma_2^0$.*

Corollary 2. *There is no recursive function $f : \mathbb{N} \to \mathbb{N}$ such that if $\phi \in PS$, then there is an equivalent Π_1^0 sentence with at most $f(|\phi|)$ universal variables. The result holds even for relational vocabularies and further even if PS is replaced with $PS \cap \Sigma_2^0$.*

Proof: Let $\varphi = \forall^n \bar{z} \psi(\bar{z})$ be a Π_1^0 sentence equivalent to ϕ where $n = f(|\phi|)$. Let k be the number of atomic formulae in ψ. Since ϕ and ψ have the same vocabulary, $k \in O(|\phi| \cdot n^{|\phi|})$. The size of the Disjunctive Normal Form of ψ is therefore bounded above by $O(k \cdot n \cdot 2^k)$. Hence $|\varphi|$ is a recursive function of $|\phi|$ if f is recursive. ■

Theorem 3 strengthens the non-elementary lower bound given in [6]. Corollary 2 gives us the following.

Lemma 1. *There is no recursive function $f : \mathbb{N} \to \mathbb{N}$ s.t. if $\phi \in PSC_f$, then every model of ϕ has a core of size at most $f(|\phi|)$.*

Proof: Consider such a function f. For any sentence ϕ in a relational vocabulary τ s.t. $\phi \in PS$, $\neg\phi$ is equivalent to a Σ_1^0 sentence by Łoś-Tarski theorem. Hence $\neg\phi \in PSC_f$. By assumption about f, the size of minimal models of $\neg\phi$ is bounded above by $n = f(|\phi|) + k$, where k is the number of constants in τ. Therefore, $\neg\phi$ is equivalent to an $\exists^n$ sentence and hence ϕ is equivalent to a $\forall^n$ sentence. Corollary 2 now forbids n, and hence f, from being recursive. It is easy to see that the result extends to vocabularies with functions too (by using functions in a trivial way). ■

Corollary 1 motivates us to consider sentences with bounded cores since all sentences in PSC_f have bounded cores.

2.2 The Case of Bounded Cores

We first give a more general definition.

Definition 2 ***(Preservation under substructures modulo a bounded core)****. A class of structures S is said to be* preserved under substructures modulo a bounded core *(denoted $S \in \mathbb{PSC}$), if $S \in \mathbb{PSC}_f$ and there exists a natural number B dependent only on S such that every structure in S has a core of size at most B.*

The collection of all such classes is denoted by $\mathbb{PSC}$. Let $\mathbb{PSC}(B)$ be the sub-collection of $\mathbb{PSC}$ in which each class has minimal core sizes bounded by B. Then $\mathbb{PSC} = \bigcup_{B \geq 0} \mathbb{PSC}(B)$. An easy observation is that $\mathbb{PSC}(i) \subseteq \mathbb{PSC}(j)$ for $i \leq j$. As before, $\mathbb{PSC}$ and each $\mathbb{PSC}(B)$ contain non-FO definable classes. As an example, the class of forests is in $\mathbb{PSC}(0)$. Let PSC (resp. $PSC(B)$) be the FO definable classes in $\mathbb{PSC}$ (resp. $\mathbb{PSC}(B)$). *Observe that $PSC(0)$ is exactly PS and $PSC = \bigcup_{B \geq 0} PSC(B)$.* Therefore, PSC generalizes PS. Further, the hierarchy in PSC is strict. Consider $\phi \in PSC(k)$ given by $\phi = \exists x_1 \ldots \exists x_k \bigwedge_{1 \leq i < j \leq k} \neg(x_i = x_j)$. Then $\phi \notin PSC(l)$ for $l < k$. From Corollary 1, we have

Lemma 2. $PSC = PSC_f$.

As noted earlier, a Σ^0_2 sentence ϕ with B existential quantifiers is in PSC_f with minimal core size bounded by B. Hence $\phi \in PSC(B)$. In the other direction, Theorem 2 and Lemma 2 together imply that for a sentence $\phi \in PSC(B)$, there is an equivalent Σ^0_2 sentence. We can then ask the following sharper question: For $\phi \in PSC(B)$, is there an equivalent Σ^0_2 sentence having B existential quantifiers?

The remainder of the paper is an account of our studies for a number of special cases of the above question. Since the answer in all of these cases in which arbitrary structures were considered turned out positive, we put forth the following conjecture[1].

Conjecture 1. A sentence $\phi \in PSC(B)$ iff it is equivalent to a Σ^0_2 sentence with B existential quantifiers.

3 Revisiting Relativization

For purposes of our discussion in this and in the remaining sections of the paper, we will assume relational vocabularies (only predicates and constants).

A notion that has proved immensely helpful in proving most of our positive cases for the conjecture is that of *relativization*. Informally speaking, given a sentence ϕ, we would like to define a formula (with free variables $\bar{x}$) which asserts that ϕ is true in the submodel induced by $\bar{x}$. The following lemma shows the existence of such a formula.

Lemma 3. *If τ is a relational vocabulary, for every $FO(\tau)$ sentence ϕ and variables $\bar{x} = (x_1, \ldots, x_k)$, there exists a* quantifier-free *formula $\phi|_{\bar{x}}$ with free variables $\bar{x}$ such*

[1] Post submission of this paper, we have obtained a proof of the conjecture, over arbitary structures, using non-combinatorial model-theoretic arguments. However, this has not benefited from the scrutiny of the anonymous reviewers. Details of our proof may be found in [12].

that the following holds: Let M be a model and $\bar{a} = (a_1, \ldots, a_k)$ be a sequence of elements of M. Then

$$(M, a_1, \ldots, a_k) \models \phi|_{\bar{x}} \textit{ iff } M(\{a_1, \ldots, a_k\}) \models \phi$$

Proof: Let $X = \{x_1, \ldots, x_k\}$ and C be the set of constants in τ. First, replace every $\forall$ quantifier in ϕ by $\neg\exists$. Then, replace every subformula of ϕ of the form $\exists x \chi(x, y_1, \ldots, y_k)$ by $\bigvee_{z \in X \cup C} \chi(z, y_1, \ldots, y_k)$. ■

We refer to $\phi|_{\bar{x}}$ as 'ϕ relativized to $\bar{x}$'. For clarity of exposition, we will abuse notation and use $\phi|_{\{x_1,\ldots,x_k\}}$ to denote $\phi|_{\bar{x}}$ (although $\bar{x}$ is a sequence and $\{x_1, \ldots, x_k\}$ is a set), whenever convenient.

We begin with the following observation.

Lemma 4. *Over any given class $\mathcal{C}$ of structures in $\mathbb{PS}$, if $\phi \leftrightarrow \forall z_1 \ldots \forall z_n \varphi$ where φ is quantifier-free, then $\phi \leftrightarrow \psi$ where $\psi = \forall z_1 \ldots \forall z_n \phi|_{\{z_1,\ldots,z_n\}}$.*

Proof: It is easy to see that $\phi \to \psi$. Let $M \in \mathcal{C}$ be s.t. $M \models \psi$. Let $\bar{a}$ be an $n-$tuple from M. Then, by Lemma 3, $M(\bar{a}) \models \phi$. Since $\mathcal{C} \in \mathbb{PS}$, $M(\bar{a}) \in \mathcal{C}$ so that $M(\bar{a}) \models \forall z_1 \ldots \forall z_n \varphi$. Then $M(\bar{a}) \models \varphi(\bar{a})$ and hence $M \models \varphi(\bar{a})$. Then $M \models \forall z_1 \ldots \forall z_n \varphi$ and hence $M \models \phi$. ■

Using Łoś-Tarski theorem and the above lemma, it follows that a sentence ϕ in PS has an equivalent universal sentence whose matrix is ϕ itself relativized to the universal variables. However we give a proof of this latter fact directly using relativization, and hence an alternate proof of the Łoś-Tarski theorem. We emphasize that our proof works only for relational vocabularies (Łoś-Tarski is known to hold for arbitrary vocabularies). This would show that relativization helps us resolve the conjecture for the case of $B = 0$.

3.1 A Proof of Łoś-Tarski Theorem Using Relativization

We first introduce some notation. Given a $\tau-$structure M, we denote by τ_M, the vocabulary obtained by expanding τ with as many constant symbols as the elements of M - one constant per element. We denote by $\mathcal{M}$ the τ_M structure whose $\tau-$reduct is M and in which each constant in τ_M is interpreted as the element of M corresponding to the constant. It is clear that M uniquely determines $\mathcal{M}$. Finally, $\mathcal{D}(M)$ denotes the *diagram* of M - the collection of quantifier free τ_M-sentences true in $\mathcal{M}$.

Theorem 4. *(Łoś-Tarski) A FO sentence ϕ is in PS iff there exists an $n \in \mathbb{N}$ such that ϕ is equivalent to $\forall z_1 \ldots \forall z_n \phi|_{\{z_1,\ldots,z_n\}}$.*

Proof:
Consider a set of sentences $\Gamma = \{\xi_k \mid k \in \mathbb{N},\ \xi_k = \forall z_1 \ldots \forall z_k \phi|_{\{z_1,\ldots,z_k\}}\}$. Observe that $\xi_{k+1} \to \xi_k$ so that a finite collection of ξ_ks will be equivalent to ξ_{k^*} where k^* is the highest index k appearing in the collection. We will show that $\phi \leftrightarrow \Gamma$. Once we show this, by compactness theorem, $\phi \leftrightarrow \Gamma_1$ for some finite subset Γ_1 of Γ and by the preceding observation, ϕ is equivalent to $\xi_n \in \Gamma_1$ for some n.

If $M \models \phi$, then since $\phi \in PS$, every substructure of it models ϕ - in particular, the substructure induced by any k-elements of M. Then $M \models \xi_k$ for every k and hence $M \models \Gamma$.

Conversely, suppose $M \models \Gamma$. Then every finite substructure of M models ϕ. Let $\mathcal{M}$ be the τ_M structure corresponding to M. Consider any finite subset S of the diagram $\mathcal{D}(M)$ of M. Let C be the finite set of constants referred to in S. Clearly $\mathcal{M}|_{\tau \cup C}$, namely the $(\tau \cup C)$-reduct of $\mathcal{M}$ models S since $\mathcal{M} \models \mathcal{D}(M)$. Then consider the substructure $\mathcal{M}_1$ of $\mathcal{M}|_{\tau \cup C}$ induced by the interpretations of the constants of C - this satisfies S. Now since C is finite, so is $\mathcal{M}_1$. Then the $\tau-$reduct of $\mathcal{M}_1$ - a finite substructure of M models ϕ.

Thus $S \cup \{\phi\}$ is satisfiable by $\mathcal{M}_1$. Since S was arbitrary, every finite subset of $\mathcal{D}(M) \cup \{\phi\}$ is satisfiable so that by compactness, $\mathcal{D}(M) \cup \{\phi\}$ is satisfiable by some structure say $\mathcal{N}$. Then the $\tau-$reduct N of $\mathcal{N}$ is s.t. (i) M is embeddable in N and (ii) $N \models \phi$. Since $\phi \in PS$, the embedding of M in N models ϕ and hence $M \models \phi$. ∎

The above proof shows that for $\phi \in PS$, there is an equivalent universal sentence whose matrix is ϕ itself, relativized to the universal variables. In fact, by Lemma 4, there is an optimal (in terms of the number of universal variables) such sentence.

An observation from the proof of Theorem 4 is that, the Łoś-Tarski theorem is true over any class of structures satisfying compactness - hence in particular the class of structures definable by a FO theory (indeed this result is known). But there are classes of structures which are not definable by FO theories but still satisfy compactness: Consider any FO theory having infinite models and consider the class of models of this theory whose cardinality is not equal to a given infinite cardinal. This class satisfies compactness but cannot be definable by any FO theory due to Löwenheim-Skolem theorem. Yet Łoś-Tarski theorem would hold over this class.

Having seen the usefulness of relativization in proving Conjecture 1 when B equals 0, it is natural to ask if this technique works for higher values of B too. We answer this negatively.

3.2 Limitations of Relativization

We show by a concrete example that relativization cannot be used to prove the conjecture in general. This motivates us to derive necessary and sufficient conditions for relativization to work.

Example 3: Consider $\phi = \exists x \forall y E(x, y)$ over $\tau = \{E\}$. Note that ϕ is in $PSC(1)$. Suppose ϕ is equivalent to $\psi = \exists x \forall^n \bar{y} \phi|_{x\bar{y}}$ for some n. Consider the structure $M = (\mathbb{Z}, \leq)$ namely the integers with usual $\leq$ linear order. Any finite substructure of M satisfies ϕ since it has a minimum element (under the linear order). Then taking x to be any integer, we see that $M \models \psi$. However $M \not\models \phi$ since M has no minimum element - a contradiction. The same argument can be used to show that ϕ cannot be equivalent to any sentence of the form $\exists^n \bar{x}\ \forall^m \bar{y}\ \phi|_{\bar{x}\bar{y}}$.

We now give necessary and sufficient conditions for relativization to work. Towards this, we introduce the following notion. Consider $\phi \in FO(\tau)$ s.t. $\phi \in PSC(B)$.

Consider a vocabulary τ_B obtained by expanding τ with B fresh constants. Consider the class S_ϕ^{all} of τ_B-structures with the following properties:

1. For each $(M, a_1, \ldots, a_B) \in S_\phi^{\text{all}}$ where M is a $\tau-$structure and $a_1, \ldots, a_B \in M$, $M \models \phi$ and $\{a_1, \ldots, a_B\}$ forms a core of M w.r.t. ϕ.
2. For each model M of ϕ, for each core $C = \{a_1, \ldots, a_l\}$ of M w.r.t. ϕ s.t. $l \leq B$ and for each function $p : \{1, \ldots, B\} \to C$ with range C, it must be that $(M, p(1), \ldots, p(B)) \in S_\phi^{\text{all}}$.

We now have the following.

Theorem 5. *Given $\phi \in PSC(B)$, the following are equivalent.*

1. *S_ϕ^{all} is finitely axiomatizable.*
2. *ϕ is equivalent to $\exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$ for some $n \in \mathbb{N}$.*
3. *ϕ is equivalent to a $\exists^B \forall^*$ sentence ψ such that in any model M of ψ and ϕ, the following hold:*
   (a) The underlying set of any witness for ψ is a core of M w.r.t. ϕ.
   (b) Conversely, if C is a core of M w.r.t. ϕ, $x_1, \ldots, x_B$ are the $\exists$ variables of ψ and $f : \{x_1, \ldots, x_B\} \to C$ is any function with range C, then $(f(x_1), \ldots, f(x_B))$ is witness for ψ in M.

Proof:
$\underline{(1) \to (2)}$: Let S_ϕ^{all} be finitely axiomatizable. Check that $S_\phi^{\text{all}} \in \mathbb{PS}$ so that by Łoś-Tarski theorem, it is axiomatizable by a Π_1^0 $FO(\tau_B)$-sentence ψ having say n $\forall$ quantifiers. Further, by Lemma 4, ψ is equivalent to $\gamma = \forall^n \bar{z} \psi|_{\bar{z}}$. Now consider $\varphi = \exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$. Firstly, from Lemma 5, $\phi \to \varphi$. Conversely, suppose $M \models \varphi$. Let $a_1, \ldots, a_B$ be witnesses and consider the τ_B-structure $M_B = (M, a_1, \ldots, a_B)$. Now $M_B \models \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$. We will show that $M_B \models \gamma$. Consider $b_1, \ldots, b_n \in M$ and let $M_1 = M_B(\{b_1, \ldots, b_n\})$. Then $M_1 \models \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$. Check that the $\tau-$reduct of M_1 (i) models ϕ and (ii) contains $\{a_1, \ldots, a_B\}$ as a core. Then $M_1 \in S_\phi^{\text{all}}$ and hence $M_1 \models \psi$. Since $b_1, \ldots, b_n$ were arbitrary, $M_B \models \gamma$. Since $\gamma \leftrightarrow \psi$ and ψ axiomatizes S_ϕ^{all}, the $\tau-$reduct of M_B, namely M, models ϕ.

$\underline{(2) \to (3)}$: Take ψ to be $\exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$. Consider a model M of ϕ and ψ. The set C of elements of any witness for ψ forms a core of M w.r.t. ψ. Then since $\phi \leftrightarrow \psi$, C is also a core of M w.r.t. ϕ. Conversely, consider a core C of M w.r.t. ϕ. Then any substructure of M containing C satisfies ϕ. Then check that elements of C form a witness for ψ.

$\underline{(3) \to (1)}$: Let $\phi \leftrightarrow \psi$ where $\psi = \exists^B \bar{x}\, \forall^n \bar{y} \beta(\bar{x}, \bar{y})$ where β is quantifier free and ψ satisfies the conditions mentioned in (3). Consider $\varphi = \forall^n \bar{y}\, \beta[x_1 \mapsto c_1, \ldots, x_B \mapsto c_B]$ where $c_1, \ldots, c_B$ are B fresh constants and $x_i \mapsto c_i$ means replacement of x_i by c_i. If $M_B = (M, a_1, \ldots, a_B) \models \varphi$, then $M \models \psi$ and hence $M \models \phi$. Since $a_1, \ldots, a_B$ are witnesses for ψ in M, they form a core of M w.r.t. ϕ by assumption, so that $M_B \in S_\phi^{\text{all}}$. Conversely, if $M_B = (M, a_1, \ldots, a_B) \in S_\phi^{\text{all}}$, then $M \models \phi$ and $a_1, \ldots, a_B$ form a core in M. Then by assumption, $M \models \psi$ and $a_1, \ldots, a_B$ are witnesses for ψ. Then $M_B \models \varphi$. To sum up, φ axiomatizes S_ϕ^{all}. ■

Consider ϕ and M in Example 3 above. Take any finite substructure M_1 of M - it models ϕ. There is exactly one witness for ϕ in M_1, namely the least element under $\leq$. However every element in M_1 serves as a core. The above theorem shows that no $\exists\forall^*$ sentence will be able to capture exactly all the cores through its $\exists$ variable.

In the following sections, we shall study the conjecture for several special classes of FO and over special structures. Interestingly, in most of the cases in which the conjecture turns out true, relativization works! However we also show a case for the conjecture in which relativization does not work, yet the conjecture is true.

4 Positive Cases for the Conjecture

4.1 The Conjecture Holds for Special Fragments of FO

Unless otherwise stated, we consider relational vocabularies throughout the section. The following lemma will be repeatedly used in the subsequent results.

Lemma 5. *Let $\phi \in PSC(B)$. For every $n \in \mathbb{N}$, ϕ implies $\exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$.*

Proof: Suppose $M \models \phi$. Since $\phi \in PSC(B)$, there is a core C of M of size at most B. Interpret $\bar{x}$ to include all the elements of C (in any which way). Since C is a core, for any n-tuple $\bar{d}$ of elements of M, having underlying set D, the substructure of M induced by $C \cup D$ models ϕ. Then $(M, \bar{a}, \bar{d}) \models \phi|_{\bar{x}\bar{y}}$ for all $\bar{d}$ from M. ■

Lemma 6. *Let τ be a monadic vocabulary containing k unary predicates. Let $\phi \in FO(\tau)$ be a sentence of rank r s.t. $\phi \in PSC(B)$. Then ϕ is equivalent to ψ where $\psi = \exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$ where $n = r \times 2^k$. For $B = 0$, n is optimal i.e. there is an FO sentence in $PSC(0)$ for which any equivalent Π^0_1 sentence has at least n quantifiers.*

Proof: That ϕ implies ψ follows from Lemma 5. For the converse, suppose $M \models \psi$ where $n = r \times 2^k$. By an Ehrenfeucht-Fräissé game argument, we can show that M contains a substructure M_S such that (i) $M \equiv_r M_S$, with $|M_S| \leq n$ and (ii) for any extension M' of M_S in M, $M' \equiv_r M_S$. The substructure M_S is obtained by taking up to r elements of each colour $c \in 2^\tau$ present in M. An element a in structure M is said to have colour c if for every predicate $P \in \Sigma$, $M \models P(a)$ iff $P \in c$. Since $M \models \psi$, there exists witnesses $\bar{a}$ for ψ in M. Choose $\bar{b}$ to be an n-tuple which includes the elements of M_S. This is possible because $|M_S| \leq n$. Then we have, $(M, \bar{a}, \bar{b}) \models \phi|_{\bar{x}\bar{y}}$ so that $M(\bar{a}\bar{b}) \models \phi$. But $M_S \subseteq M(\bar{a}\bar{b}) \subseteq M$ so that $M(\bar{a}\bar{b}) \equiv_r M$. Then $M \models \phi$.

To see the optimality of n for $B = 0$, consider the sentence ϕ which states that there exists at least one colour $c \in 2^\tau$ such that there exist at most $r - 1$ elements with colour c. The sentence ϕ can be written as a formula with rank r, as the disjunction over all colours, of sentences of the form, $\exists x_1 \exists x_2 \cdots \exists x_{r-1} \forall x_r (\bigwedge_{i=1}^{r-1} x_r \neq x_i) \rightarrow \neg C(x_r)$. From the preceding paragraph, $\phi \leftrightarrow \forall^n \bar{y}\, \phi|_{\bar{y}}$ where $n = r \times 2^k$. Suppose ϕ is equivalent to a $\forall^s$ sentence for some $s < n$. Then by Lemma 4, $\phi \leftrightarrow \varphi$ where $\varphi = \forall^s \bar{y}\, \phi|_{\bar{y}}$. Then consider the structure M, which has r elements of each colour. Clearly, $M \not\models \phi$. However check that every s-sized substructure of M models ϕ. Then $M \models \varphi$ and hence $M \models \phi$ - a contradiction. ■

Lemma 7. *Let $S \in \mathbb{PSC}(B)$ be a finite collection of $\tau-$structures so that S is definable by a Σ_2^0 sentence $\phi \in PSC(B)$. Then S is definable by the sentence ψ where $\psi = \exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$ for some $n \in \mathbb{N}$.*

Proof: Check that all structures in S must be of finite size so that ϕ exists. Let the size of the largest structure in S be at most n. Consider ψ. Lemma 5 shows that $\phi \to \psi$. Conversely, suppose $M \models \psi$. Then there exists a witness $\bar{a}$ s.t. any extension of $M(\bar{a})$ within M with at most n additional elements models ϕ. Since M is of size at most n, taking the extension M of $M(\bar{a})$, we have $M \models \phi$. Since ϕ defines S so does ψ. ■

Lemma 8. *Consider $\phi \in \Pi_2^0$ given by $\phi = \forall^n \bar{x}\, \exists^m \bar{y}\, \beta(\bar{x}, \bar{y})$ where β is quantifier free. If $\phi \in PSC(B)$, then ϕ is equivalent to ψ where $\psi = \exists^B \bar{u}\, \forall^n \bar{v}\, \phi|_{\bar{u}\bar{v}}$.*

Proof: From Lemma 5, $\phi \to \psi$. For the converse, let $M \models \psi$ and let $\bar{a}$ be a witness. Consider an $n-$tuple $\bar{b}$ from M. Then $M_1 = M(\bar{a}\bar{b})$ is s.t. $M_1 \models \phi$. Then for $\bar{x} = \bar{b}$, there exists $\bar{y} = \bar{d}$ s.t. $\bar{d}$ is an $m-$tuple from M_1 and $M_1 \models \beta(\bar{b}, \bar{d})$. Then $M \models \beta(\bar{b}, \bar{d})$ since $M_1 \subseteq M$. Hence $M \models \phi$. ■

Lemma 9. *Suppose $\phi \in PSC(B)$ and $\neg\phi \in PSC(B')$. Then ϕ is equivalent to ψ where $\psi = \exists^B \bar{x}\, \forall^{B'} \bar{y}\, \phi|_{\bar{x}\bar{y}}$.*

Proof: From Lemma 5, ϕ implies ψ. For the converse, suppose $M \models \psi$. Then there is a witness $\bar{a}$ for ψ s.t. for any B'-tuple $\bar{b}$, the substructure induced by $\bar{a}\bar{b}$ i.e. $M(\bar{a}\bar{b})$ models ϕ. Suppose $M \not\models \phi$. Then $M \models \neg\phi$ so that there is a core C of M w.r.t. $\neg\phi$, of size at most B'. Let $\bar{d}$ be a B'-tuple which includes all the elements of C. Then $M(\bar{a}\bar{d}) \models \phi$. But $M(\bar{a}\bar{d}) \subseteq M$ contains C so that $M(\bar{a}\bar{d}) \models \neg\phi$ – a contradiction. ■

Observe that for the special case of $B = 0$, we get combinatorial proofs of Łoś-Tarski theorem for the fragments mentioned above. Moreover all of these proofs and hence the results hold in the finite. We mention that the result of Lemma 8 holding in the finite was proved by Compton too (see [7]). We were unaware of this until recently and have independently arrived at the same result. The reader is referred to [12] for our studies on more *positive* cases of Łoś-Tarski in the finite.

Interestingly, Lemma 9 has implications for the Δ_2^0 fragment of FO. Define $\Delta_2^0(k,l) \subseteq \Delta_2^0$ to be the class of sentences which have a $\exists^k\forall^*$ and a $\forall^l\exists^*$ equivalent. Note that $\Delta_2^0 = \bigcup_{l,k \geq 0} \Delta_2^0(k,l)$. Lemma 9 gives us the following right away.

Theorem 6. *The following are equivalent:*

1. *$\phi \in PSC(k)$ and $\neg\phi \in PSC(l)$.*
2. *ϕ is equivalent to a $\exists^k\forall^l$ and a $\forall^l\exists^k$ sentence.*
3. *$\phi \in \Delta_2^0(k,l)$.*

As a corollary, we see that $\Delta_2^0(k,l)$ is a finite class up to equivalence. We are not aware of any other semantic characterization of these natural fragments of Δ_2^0. This highlights the importance of the notion of cores and the sizes thereof.

4.2 The Conjecture over Special Classes of Structures

We first look at the conjecture over finite words. These are finite structures in the vocabulary containing one binary predicate $\leq$ (always interpreted as a linear order) and a finite number of unary predicates (which form a partition of the universe). Interestingly, we obtain something stronger than the conjecture. Towards this, we note that the idea of relativization can be naturally extended to MSO. Given ϕ in MSO and a set of variables $Z = \{z_1, \ldots, z_n\}$, $\phi|_Z$ is obtained by first converting all $\forall X$ to $\neg\exists X$ and then replacing every subformula $\exists X \chi(X, \ldots)$ with $\bigvee_{Y \subseteq Z}((\bigwedge_{z \in Y} X(z) \wedge \bigwedge_{z \in Z \setminus Y} \neg X(z)) \wedge \chi(X, \ldots))$. The resulting *FO* formula is then relativized to Z and simplified to eliminate the (original) SO variables. As before, abusing notation, we use $\phi|_Z$ and $\phi|_{\bar{z}}$ interchangeably.

Theorem 7. *Over words, a MSO sentence ϕ is in $\mathbb{PSC}(B)$ iff it is equivalent to ψ where $\psi = \exists^B \bar{x} \forall^k \bar{y} \phi|_{\bar{x}\bar{y}}$ for some $k \in \mathbb{N}$.*

Proof sketch: We use the fact that over words, by the Büchi-Elgot-Trakhtenbrot theorem [4], MSO sentences define regular languages. The 'If' direction is easy. For the 'Only if' direction, let the regular language L defined by ϕ be recognized by an n state automaton, say $\mathcal{M}$. If there is no word of length $> N = (B+1) \times n$ in L, then L is a finite language of finite words and hence from Lemma 7, we are done. Else suppose there is a word of length $> N$ in L. Then consider ψ above for $k = N$. It is easy to observe that ϕ implies ψ. In the other direction, suppose $w \models \psi$ for some word w. Then there exists a set A of elements $i_1, \ldots, i_m$ s.t. (i) $m \leq B$ and $i_1 < i_2 \cdots < i_m$ and (ii) every substructure of w of size at most $N + m$ containing A models ϕ. We claim (proof sketched below) that there exists a substructure w_1 of w containing A such that (i) $|w_1| \leq N$ and (ii) $w_1 \in L$ iff $w \in L$. Then w_1 models ϕ and hence $w \models \phi$. Thus ψ implies ϕ and hence is equivalent to ϕ.

The proof of the claim used in the argument above proceeds as follows. Let q_j be the state reached by automaton $\mathcal{M}$ upon reading the subword $w[1 \ldots i_j]$. The subword $w[(i_j+1), \ldots i_{j+1}]$ takes $\mathcal{M}$ from q_j to q_{j+1} through a sequence S of states. Since $\mathcal{M}$ has only n states, if $w[(i_j+1), \ldots i_{j+1}]$ is long, then S will contain at least one loop. Then getting rid of the subwords that give rise to loops, we will be able to obtain a subword of $w[(i_j+1), \ldots i_{j+1}]$ that takes $\mathcal{M}$ from q_j to q_{j+1} without causing $\mathcal{M}$ to loop in between. It follows that this subword must be of length at most n. Collecting such subwords of $w[(i_j+1), \ldots i_{j+1}]$ for each j and concatenating them, we get a subword of w of length at most N containing set A that takes $\mathcal{M}$ from the initial state to the same state as w. Details can be found in [12]. ■

For the special case of $B = 0$, we obtain Łoś-Tarski theorem for words and also give a bound for the number of $\forall$s in the equivalent Π^0_1 sentence in terms of the number of states of the automaton for ϕ (A simpler proof of Łoś-Tarski using Higman's lemma can be found in [12] though this does not tell anything about the number of $\forall$s). We have not encountered this result in our literature survey.

So far, relativization has worked in all the cases we have seen. We now give an example of a class of structures over which relativization fails, yet the conjecture is true.

Consider a subclass $\mathcal{C}$ of bounded degree graphs in which each graph is a collection (finite or infinite) of *oriented* paths (finite or infinite). For clarity, by oriented path we mean a graph isomorphic to a connected induced subgraph of the graph (V, E) where $V = \mathbb{Z}$ and $E = \{(i, i+1) \mid i \in \mathbb{Z}\}$. Observe that $\mathcal{C}$ can be axiomatized by a theory $\mathcal{T}$ which asserts that every node has in-degree at most 1 and out-degree at most 1 and that there is no directed cycle of length k for each $k \geq 0$. We first show the following.

Lemma 10. *For each $B \geq 1$, there is a sentence $\phi \in PSC(B)$ which is not equivalent, over $\mathcal{C}$, to any ψ of the form $\exists^B \bar{x}\, \forall^n \bar{y}\, \phi|_{\bar{x}\bar{y}}$.*

Proof: Consider ϕ which asserts that there are at least B elements of *total* degree at most 1 where total degree is the sum of in-degree and out-degree. Clearly $\phi \in PSC(B)$ since it is expressible as a $\exists^B \forall^*$ sentence. Suppose ϕ is equivalent to ψ of the form above for some $n \in \mathbb{N}$. Consider $M \in \mathcal{C}$ which is a both-ways infinite path so that every node in M has total degree 2 - then $M \not\models \phi$. Consider B distinct points on this path at a distance of at least $2n$ from each other and form a $B-$tuple say $\bar{a}$ with them. Let $\bar{b}$ be any $n-$tuple from M. Now observe that $M(\bar{a}\bar{b})$ is a finite structure which has at least B distinct paths (0-sized paths included). Then $M(\bar{a}\bar{b}) \models \phi$ so that $(M, \bar{a}, \bar{b}) \models \phi|_{\bar{x}\bar{y}}$. Since $\bar{b}$ was arbitrary, $M \models \psi$ so that $M \models \phi$. Contradiction. ∎

However the conjecture holds over $\mathcal{C}$! The proof is currently lengthy so we provide only a sketch and refer the reader to [12] for details.

Theorem 8. *Over the class $\mathcal{C}$ of graphs defined above, $\phi \in PSC(B)$ iff ϕ is equivalent to a $\exists^B \forall^*$ sentence.*

Proof Sketch: If $\tau = \{E\}$ is the vocabulary of ϕ, let τ_B be a vocabulary obtained by adding B fresh constants to τ. Given a class $\mathcal{S}$ of $\tau-$structures, define $\mathcal{S}_B$ to be the class of *all* τ_B-structures s.t. the $\tau-$reduct of each structure in $\mathcal{S}_B$ is in $\mathcal{S}$. Then the proof can be divided into two main steps. Below $\equiv$ denotes elementary equivalence.

Step 1: Given ϕ, define class $\mathcal{C}' \subseteq \mathcal{C}$ such that for every structure $A \in \mathcal{C}_B$, there exists a structure $D \in \mathcal{C}'_B$ such that $A \equiv D$ (Property I). Since compactness theorem holds over $\mathcal{C}_B$ (as $\mathcal{C}_B$ is defined by the same theory $\mathcal{T}$ as $\mathcal{C}$), it also holds over $\mathcal{C}'_B$.

Step 2: Show that ϕ is equivalent to an $\exists^B \forall^*$ sentence over $\mathcal{C}'$, hence showing the same over $\mathcal{C}$ as well.

Note: The conditions in **Step 1** imply that for every $A \in \mathcal{C}$, there exists a $D \in \mathcal{C}'$ such that $A \equiv D$. Then since compactness theorem holds over $\mathcal{C}$, it also holds over $\mathcal{C}'$.

Suppose the rank of ϕ is m. We define $\mathcal{C}'$ to be the class of graphs $G \in \mathcal{C}$ such that either (a) there exists a bound n_G (dependent on G) such that all paths in G have length less than n_G (this does not mean that G is finite – there could be infinite paths of the same length in G) or (b) there are at least $(B + m + 2)$ paths in G that are infinite in both directions. It can be shown that $\mathcal{C}'$ satisfies Property I (see [12]).

Now, to show **Step 2**, we use the following approach.

Let $P \in \mathcal{C}'$ be s.t. $P \models \phi$. Choose a core Z of P (recall that $\phi \in PSC(B)$). Let $M_P \in \mathcal{C}'_B$ be a τ_B-structure whose $\tau-$reduct is P, and in which each element of Z is assigned to some constant. Let Γ^{M_P} be the set of all $\forall^*$ sentences true in M_P.

We can show that (see [12]) if $M' \in \mathcal{C}'_B$ is such that $M' \models \Gamma^{M_P}$, then $M' \models \phi$. That is, if every finite substructure of M' is embeddable in M_P, then $M' \models \phi$. Then over $\mathcal{C}'_B$, $\Gamma^{M_P} \rightarrow \phi$. Now, since $\mathcal{C}'_B$ satisfies the compactness theorem, there exists a finite subset $\Gamma_0^{M_P}$ of Γ^{M_P} such that $\Gamma_0^{M_P} \rightarrow \phi$ over $\mathcal{C}'_B$. Note that, since $\Gamma_0^{M_P}$ is a conjunction of $\forall^*$ sentences, we can assume that $\Gamma_0^{M_P}$ is a single $\forall^*$ sentence.

Let ϕ_P be the $\tau-$sentence of the form $\exists^B\forall^*$ obtained by replacing the B constants in $\Gamma_0^{M_P}$ with B fresh variables, and by existentially quantifying these variables. We can then show that $\phi_P \rightarrow \phi$. It is also easy to see that $\phi \rightarrow \bigvee_{P \in \mathcal{C}', P \models \phi} \phi_P$, since if $P \models \phi$, then the witnesses of the $\exists$ quantifiers in ϕ_P can be chosen to be the core Z mentioned above. By the compactness theorem over $\mathcal{C}'$, there exists a finite set of structures, say $\{P_1, \cdots, P_m\}$, such that $P_i \in \mathcal{C}'$, $P_i \models \phi$ and $\phi \rightarrow \bigvee_{i=1}^{i=m} \phi_{P_i}$. Then, we have $\phi \leftrightarrow \bigvee_{i=0}^{i=m} \phi_{P_i}$ over $\mathcal{C}'$. Since each ϕ_{P_i} is of the form $\exists^B\forall^*$, the sentence $\bigvee_{i=0}^{i=m} \phi_{P_i}$ is also of the same form. This completes **Step 2** of the proof. ■

5 Conjecture Fails over Special Classes of Structures

We first look at the class $\mathcal{F}$ of all finite structures. Łoś-Tarski theorem fails over this class and hence so does Conjecture 1 (for $B = 0$). However, we have the following stronger result. We prove it for relational vocabularies (constants permitted).

Lemma 11. *For relational vocabularies, Conjecture 1 fails, over $\mathcal{F}$, for each $B \geq 0$.*

Proof: We refer to [1] for the counterexample χ for Łoś-Tarski in the finite. Let τ be the vocabulary of χ (i.e. $\{\leq, S, a, b\}$) along with a unary predicate U. Let us call an element x as having colour 0 in a structure if $U(x)$ is true in the structure and having colour 1 otherwise. Let φ be a sentence asserting that there are exactly B elements having colour 0 and these are different from a and b. Then consider $\phi = \neg\chi \wedge \varphi$. Check that since $\neg\chi$ is preserved under substructures in the finite, in any model of ϕ, the B elements of colour 0 form a core of the model w.r.t. ϕ. Then $\phi \in PSC(B)$. Suppose ϕ is equivalent to ψ given by $\exists^B\bar{x}\forall^n\bar{y}\,\beta$ where β is quantifier-free. Observe that in any model of ϕ and ψ, any witness for ψ must include all the B elements of colour 0 (else the substructure formed by the witness would not model φ and hence ϕ, though it would model ψ). Consider the structure $M = (\{0, 1, \ldots, B + 2n + 3\}, \leq, S, a, b, U)$ where $\leq$ is the usual linear order on numbers, S is the (full) successor relation of $\leq$, $a = 0, b = B + 2n + 3$ and $U = \{1, \ldots, B\}$. Now $M \not\models \phi$ since $M \not\models \neg\chi$. Consider M_1 which is identical to M except that $S(B + n + 1, y)$ is false in M_1 for all y. Then $M_1 \models \phi$ so that $M_1 \models \psi$. Any witness $\bar{a}$ for ψ must include all the B colour 0 elements of M_1. Then choose exactly the same value, namely $\bar{a}$, from M to assign to $\bar{x}$. Choose any $\bar{b}$ as $\bar{y}$ from M. Check that it is possible to choose $\bar{d}$ as $\bar{y}$ from M_1 s.t. $M(\bar{a}\bar{b})$ is isomorphic to $M_1(\bar{a}\bar{d})$ under the isomorphism f given by $f(0) = 0, f(B+2n+3) = B+2n+3, f(a_i) = a_i$ and $f(b_i) = d_i$ where $\bar{a} = (a_1, \ldots, a_B)$, $\bar{b} = (b_1, \ldots, b_n)$ and $\bar{d} = (d_1, \ldots, d_n)$. Then since $M_1 \models \beta(\bar{a}, \bar{d})$, $M \models \beta(\bar{a}, \bar{b})$. Then M models ψ, and hence ϕ. But that is a contradiction. ■

The example expressed by χ can also be written as a sentence in a purely relational vocabulary. Then one can do a similar proof as above to show that for purely relational vocabularies too, for each $B \geq 0$, Conjecture 1 fails over $\mathcal{F}$ (see [12]).

So far, in all the cases we have seen, it has always been the case that Conjecture 1 and Łoś-Tarski theorem either are both true or are both false. We then finally have the following result which is our first instance of a class of structures over which *Łoś-Tarski theorem holds but the conjecture fails*.

Theorem 9. *Over the class $\mathcal{C}$ of graphs in which each graph is a finite collection of finite* undirected *paths, for each $B \geq 2$, there is a sentence $\phi \in PSC(B)$ which is not equivalent to any $\exists^B \forall^*$ sentence. However, Łoś-Tarski theorem holds over $\mathcal{C}$.*

Proof: Łoś-Tarski theorem holds from the results of Dawar et al. over bounded degree structures [2]. As a counterexample to the conjecture for $B \geq 2$, consider the property D which asserts that there are at least B paths in the graph (0 length included). It can be shown (see [12]) that D is equivalent to the following condition D' parametrized by B: (The number of nodes of degree 0) + $\frac{1}{2}\times$ (the number of nodes of degree 1) $\geq B$. Then given B, take ϕ to be the sentence expressing D' for B. We reason out for the case of $B = 2$ since for the other cases an analogous reasoning can be done (see [12]).

Every model N of ϕ has at least 2 paths of length ≥ 0. Consider set A formed by an end point of one path and an end point of the other path. Check that A is a core of N w.r.t. ϕ so that $\phi \in PSC(2)$. Suppose ϕ is equivalent over $\mathcal{C}$ to $\psi = \exists^2 \bar{x}\ \forall^n \bar{y}\ \beta$ where β is quantifier-free. Consider a model N of ϕ having exactly 2 paths each of length $\geq 5n$. Then since $N \models \psi$, consider the witnesses a_1, a_2 for ψ. It cannot be that a_1, a_2 are both from the same path else the path by itself would be a model for ψ and hence ϕ. Now consider a structure M containing a single path that is of length $\geq 5n$ with end points p_1, p_2. If a_1 (resp. a_2) is at a distance of $\leq n$ from any end point in N, choose a point b_1 (resp. b_2) at the same distance from p_1 (resp. p_2) in M. Else choose b_1 (resp. b_2) at a distance of $n+1$ from p_1 (resp. p_2). Choose any $\bar{d}$ as $\bar{y}$ from M. Check that it is possible to choose $\bar{e}$ as $\bar{y}$ from N s.t. $M(b_1 b_2 \bar{d})$ is isomorphic to $N(a_1 a_2 \bar{e})$ under the isomorphism f given by $f(b_i) = a_i$, $f(d_j) = e_j$ where $\bar{d} = (d_1, \ldots, d_n)$ and $\bar{e} = (e_1, \ldots, e_n)$. Since $N \models \beta(a_1, a_2, \bar{e})$, $M \models \beta(b_1, b_2, \bar{d})$. Then M models ψ, and hence ϕ. Contradiction. ■

Interestingly however, the conjecture holds over $\mathcal{C}$ for $B = 1$. We also give a simpler proof for the case of $B = 0$ i.e. Łoś-Tarski over $\mathcal{C}$ (see [12]).

6 Conclusion and Future Work

For future work, we would like to investigate cases for which combinatorial proofs of Conjecture 1 can be obtained. This would potentially improve our understanding of the conditions under which combinatorial proofs can be obtained for the Łoś-Tarski theorem as well. An important direction of future work is to investigate whether the conjecture holds for important classes of finite structures for which the Łoś-Tarski theorem holds. Examples of such classes include those considered by Atserias et al in [2]. We have also partially investigated how preservation theorems can be used to show FO inexpressibility for many typical examples (see [13]). We would like to pursue this line of work as well in future.

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