A Finite Basis for 'Almost Future' Temporal Logic over the Reals

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Abstract. Kamp's theorem established the expressive completeness of the temporal modalities Until and Since for the First-Order Monadic Logic of Order (FOMLO) over Real and Natural time flows. Over Natural time, a single future modality (Until) is sufficient to express all future FOMLO formulas. These are formulas whose truth value at any moment is determined by what happens from that moment on. Yet this fails to extend to Real time domains: Here no finite basis of future modalities can express all future FOMLO formulas. In this paper we show that finiteness can be recovered if we slightly soften the requirement that future formulas must be totally past-independent: We allow formulas to depend just on the very very near-past, and maintain the requirement that they be independent of the rest - actually - of most of the past. We call them 'almost future' formulas, and show that there is a finite basis of almost future modalities which is expressively complete over the Reals for the almost future fragment of FOMLO.

1 Introduction

Temporal Logic (*TL*) introduced to Computer Science by Pnueli in [Pnu77] is a convenient framework for reasoning about "reactive" systems. This made temporal logics a popular subject in the Computer Science community, enjoying extensive research in the past 30 years. In *TL* we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are expressed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1 \ldots \varphi_k$ possibly on 'past' or 'future' points to a statement $M(\varphi_1 \ldots \varphi_k)$ on the 'present' point t_0. The rule to determine the truth of a statement $M(\varphi_1 \ldots \varphi_k)$ at t_0 is called a *Truth Table*. The choice of particular modalities with their truth tables yields different temporal logics. A temporal logic with modalities $M_1, \ldots, M_k$ is denoted by $TL(M_1, \ldots, M_k)$.

The simplest example is the one place modality $\mathsf{F}X$ saying: "X holds some time in the future". Its truth table is formalized by $\varphi_{\mathsf{F}}(t_0, X) \equiv (\exists t > t_0)X(t)$. This is a formula of the First-Order Monadic Logic of Order ($FOMLO$) - a fundamental formalism in Mathematical Logic where formulas are built using atomic propositions $P(t)$, atomic relations between elements $t_1 = t_2$, $t_1 < t_2$, Boolean

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connectives and first-order quantifiers $\exists t$ and $\forall t$. Most modalities used in the literature are defined by such *FOMLO* truth tables, and as a result every temporal formula translates directly into an equivalent *FOMLO* formula. Thus, the different temporal logics may be considered a convenient way to use fragments of *FOMLO*. *FOMLO* can also serve as a yardstick by which to check the strength of temporal logics: A temporal logic is *expressively complete* for a fragment L of *FOMLO* if every formula of L with a single free variable t_0 is equivalent to a temporal formula.

Actually, the notion of expressive completeness is with respect to the type of the underlying model since the question whether two formulas are equivalent depends on the domain over which they are evaluated. Any (partially) ordered set with monadic predicates is a model for *TL* and *FOMLO*, but the main, *canonical*, linear time intended models are the Naturals $\langle \mathbb{N}, < \rangle$ for discrete time and the Reals $\langle \mathbb{R}, < \rangle$ for continuous time.

A major result concerning *TL* is Kamp's theorem [Kam68, GHR94], which states that the pair of modalities "X *until* Y" and "X *since* Y" is expressively complete for *FOMLO* over the above two linear time canonical models.

Many temporal formalisms studied in computer science concern only future formulas - whose truth value at any moment is determined by what happens from that moment on. For example the formula X *until* Y says that X will hold from now (at least) until a point in the future when Y will hold. The truth value of this formula at a point t_0 does not depend on the question whether $X(t)$ or $Y(t)$ hold at earlier points $t < t_0$.

Over the discrete model $\langle \mathbb{N}, < \rangle$ Kamp's theorem holds also for *future formulas* of *FOMLO*: The future fragment of *FOMLO* has the same expressive power as $TL(\mathsf{Until})$ [GPSS80, GHR94]. The situation is radically different for the continuous time model $\langle \mathbb{R}, < \rangle$. In [HR03] it was shown that $TL(\mathsf{Until})$ is not expressively complete for the future fragment of *FOMLO* and there is no easy way to remedy it. In fact it was shown in [HR03] that there is no temporal logic with a finite set of future modalities which is expressively equivalent to the future fragment of *FOMLO* over the Reals.

The proof there goes (roughly) as follows: Define a sequence of future formulas $\phi_i(z)$ such that given any set B of modalities definable in the future fragment of *FOMLO* by formulas of quantifier depth at most n, the formula $\phi_{n+1}(z)$ is not expressible in $TL(B)$.

The interesting point is that these formulas are all expressible in a temporal language based on the future modality Until plus the modality K^- of [GHR94]. The formula $\mathsf{K}^-(P)$ holds at a time point t_0 if given any 'earlier' t, no matter how close, we can always come up with a t' in between ($t < t' < t_0$) where P holds. This is of course not a future modality - the formula $\mathsf{K}^-(P)$ is past-dependent. And it turns out that not only the above mentioned sequence of future formulas - but all future formulas - can be expressed (over Real time) in $TL(\mathsf{Until}, \mathsf{K}^-)$. This is a consequence of Gabbay's separation theorem [GHR94].

This future-past mixture of Until and K^- is somewhat better than the standard Until - Since basis in the following sense: Although K^- is (like Since) a past modality, it does not depend on much of the past: The formula $\mathsf{K}^-(P)$ depends just on an arbitrarily short 'near past', and is actually independent of most of the past. In this sense we may say that it is an "almost" future formula (see Section 3.1 for precise definitions).

In [HR03] it was conjectured that $TL(\mathsf{Until}, \mathsf{K}^-)$ is expressively complete for almost future formulas of *FOMLO*. Our main result here confirms this conjecture with respect to the Real time domain $(\mathbb{R}, <)$. In the full paper we extend this result to Dedekind complete time flows.

The rest of the paper is organized as follows: In Section 2 we recall the definitions of the monadic logic, the temporal logics and Kamp's theorem. In Section 3.1 we define "almost futureness", then most of the 'machinery' needed for the proof is in Sections 3.2 and 3.3, with the heart of the proof in Lemma 3.13. Section 3.4 then just puts it all together to complete the proof. Finally, Section 4 states further results and comments.

2 Preliminaries

We start with the basic definitions of First-Order Monadic Logic of Order (***FOMLO***) and Temporal Logic (***TL***), and some well known results concerning their expressive power. Fix a ***signature*** (finite or infinite) $\mathcal{S}$ of ***atoms***. We use $P, Q, R, S \dots$ to denote members of $\mathcal{S}$. Syntax and semantics of both logics are defined below with respect to such a fixed signature.

2.1 First-Order Monadic Logic of Order

Syntax: In the context of *FOMLO*, the atoms of $\mathcal{S}$ are referred to (and used) as ***unary predicate symbols***. Formulas are built using these symbols, plus two binary relation symbols, $<$ and $=$, and a finite set of ***first-order variables*** (denoted by $x, y, z, \dots$). Formulas are defined by the grammar:

$$atomic ::= \quad x < y \ \mid \ x = y \ \mid \ P(x) \qquad\qquad (\text{where } \ P \in \mathcal{S})$$

$$\varphi ::= \quad atomic \ \mid \ \neg\varphi_1 \ \mid \ \varphi_1 \vee \varphi_2 \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \exists x \varphi_1 \ \mid \ \forall x \varphi_1$$

The notation $\varphi(x_1, \dots, x_n)$ implies that φ is a formula where the x_i's are the only variables occurring free; writing $\varphi(x_1, \dots, x_n, P_1, \dots, P_k)$ additionally implies that the P_i's are the only predicate symbols that occur in φ. We will also use the standard abbreviated notation for ***bounded quantifiers***, e.g.: $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$, $(\forall x)^{\leq z}(\dots)$ denotes $\forall x((x \leq z) \rightarrow (\dots))$, $(\forall x)^{<u}_{>l}(\dots)$ denotes $\forall x((l < x < u) \rightarrow (\dots))$, etc.

Semantics: Formulas are interpreted over *structures*. A ***structure*** over $\mathcal{S}$ is a triplet $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{T}$ is a set - the ***domain*** of the structure, $<$ is an irreflexive partial order relation on $\mathcal{T}$, and $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{T})$ is the ***interpretation*** of the structure (where $\mathcal{P}$ is the powerset notation). We use the standard notation $\mathcal{M}, t_1, t_2, \dots t_n \models \varphi(x_1, x_2, \dots x_n)$. The semantics is defined in the standard way. Notice that for ***formulas with a single free first-order variable***, this reduces to:

$$\mathcal{M}, t \models \varphi(x).$$

2.2 Propositional Temporal Logics

Syntax: In the context of TL, the atoms of $\mathcal{S}$ are used as ***atomic propositions*** (also called ***propositional atoms***). Formulas are built using these atoms, and a set (finite or infinite) B of ***modality names***, where a non-negative integer ***arity*** denoted by $|\mathsf{M}|$ is associated with each $\mathsf{M} \in B$. The syntax of TL with the ***basis*** B over the signature $\mathcal{S}$, denoted by ***TL(B)***, is defined by the grammar:

$$F ::= \quad P \ | \ \neg F_1 \ | \ F_1 \vee F_2 \ | \ F_1 \wedge F_2 \ | \ \mathsf{M}(F_1, F_2, \ldots, F_n)$$

where $P \in \mathcal{S}$ and $\mathsf{M} \in B$ an n-place modality (that is, with arity $|\mathsf{M}| = n$). As usual ***True*** denotes $P \vee \neg P$ and ***False*** denotes $P \wedge \neg P$.

Semantics: Formulas are interpreted at ***time-points*** (or ***moments***) in structures (elements of the domain). The domain $\mathcal{T}$ of $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ is called the ***time domain***, and $(\mathcal{T}, <)$ - the ***time flow*** of the structure. The semantics of each n-place modality $\mathsf{M} \in B$ is defined by a 'rule' specifying how the set of moments where $\mathsf{M}(F_1, \ldots, F_n)$ holds (in a given structure) is determined by the n sets of moments where each of the formulas F_i holds. Such a 'rule' for M is formally specified by an operator $\mathcal{O}_\mathsf{M}$ on time flows, where given a time flow $\mathcal{F} = (\mathcal{T}, <)$, $\mathcal{O}_\mathsf{M}(\mathcal{F})$ is yet an operator in $(\mathcal{P}(\mathcal{T}))^n \longrightarrow \mathcal{P}(\mathcal{T})$.

The semantics of $TL(B)$ formulas is then defined inductively: Given a structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ and a moment $t \in \mathcal{M}$ (read $t \in \mathcal{M}$ as $t \in \mathcal{T}$), define when a formula F ***holds*** in $\mathcal{M}$ at t - notation: $\mathcal{M}, t \models F$ - as follows:

- $\mathcal{M}, t \models P$ iff $t \in \mathcal{I}(P)$, for any propositional atom P.
- $\mathcal{M}, t \models F \vee G$ iff $\mathcal{M}, t \models F$ or $\mathcal{M}, t \models G$; similarly ("pointwise") for $\wedge$, $\neg$.
- $\mathcal{M}, t \models \mathsf{M}(F_1, \ldots, F_n)$ iff $t \in [\mathcal{O}_\mathsf{M}(\mathcal{T}, <)](T_1, \ldots, T_n)$ where $\mathsf{M} \in B$ is an n-place modality, $F_1, \ldots, F_n$ are formulas and $T_i =_{def} \{s \in \mathcal{T} : \mathcal{M}, s \models F_i\}$.

Truth tables: Practically most standard modalities studied in the literature can be specified in $FOMLO$: A $FOMLO$ formula $\varphi(x, P_1, \ldots, P_n)$ (with a single free first-order variable x and with n predicate symbols P_i) is called an ***n-place first-order truth table***. Such a truth table φ ***defines*** an n-ary modality M (whose semantics is given by an operator $\mathcal{O}_\mathsf{M}$) iff for any time flow $(\mathcal{T}, <)$, for any $T_1, \ldots, T_n \subseteq \mathcal{T}$ and for any structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$ where $\mathcal{I}(P_i) = T_i$:

$$[\mathcal{O}_M(\mathcal{T}, <)](T_1, \ldots, T_n) = \{t \in \mathcal{T} : \mathcal{M}, t \models \varphi(x, P_1, \ldots, P_n)\}$$

Example 2.1. *Below are truth-table definitions for the well known "**Eventually**", the (binary)* ***strict*-Until** *and* ***strict*-Since** *of [Kam68] and for* K^- *of [GHR94]:*

- $\Diamond$ *("**Eventually**") defined by:* $\varphi_\Diamond(x, P) =_{def} (\exists x')_{>x} P(x')$
- **Until** *defined by :* $\varphi_{\mathsf{Until}}(x, P, Q) =_{def} (\exists x')_{>x}(Q(x') \wedge (\forall y)^{<x'}_{>x} P(y))$
- **Since** *defined by:* $\varphi_{\mathsf{Since}}(x, P, Q) =_{def} (\exists x')^{<x}(Q(x') \wedge (\forall y)^{<x}_{>x'} P(y))$
- K^- *defined by:* $\varphi_{\mathsf{K}^-}(x, P) =_{def} (\forall x')^{<x}(\exists y)^{<x}_{>x'} P(y)$

We use infix notation for the binary modalities Until and Since: P Until Q denotes $\mathsf{Until}(P, Q)$, meaning "there is some future moment where Q holds, and P holds all along till then". The ***non-strict*** version Until^{ns} requires that P should hold at the "present moment" as well. The formula $\mathsf{K}^-(P)$ holds at the "present moment" t_0 iff given any earlier $t < t_0$ - no matter how close - there is a moment t' in between ($t < t' < t_0$) where the formula P holds.

2.3 Kamp's Theorem

We are interested in the relative expressive power of *TL* (compared to *FOMLO*) over the class of ***linear structures***. Major results in this area are with respect to the subclass of ***Dedekind complete structures*** - where the order is Dedekind complete, that is, where every non empty subset (of the domain) which has an upper bound has a least upper bound.

Equivalence between temporal and monadic formulas is naturally defined: $F \equiv \varphi(x)$ iff for any $\mathcal{M}$ and $t \in \mathcal{M}$: $\mathcal{M}, t \models F \Leftrightarrow \mathcal{M}, t \models \varphi(x)$. We will occasionally use $\equiv_{\mathcal{L}}$ / $\equiv_{\mathcal{DC}}$ / $\equiv_{\mathcal{C}}$ to distinguish equivalence over linear / Dedekind complete / any class $\mathcal{C}$ of structures.

Definability: A temporal modality is definable in *FOMLO* iff it has a *FOMLO* truth table; a temporal formula F is definable in *FOMLO* over a class $\mathcal{C}$ of structures iff there is a monadic formula $\varphi(z)$ such that $F \equiv_{\mathcal{C}} \varphi(z)$. In this case we say that φ ***defines*** F over $\mathcal{C}$. Similarly, a monadic formula $\varphi(z)$ may be definable in $TL(B)$ over $\mathcal{C}$.

Expressive completeness/ equivalence: A temporal language $TL(B)$ (as well as the basis B) is expressively complete for (a fragment of) *FOMLO* over a class $\mathcal{C}$ of structures iff all monadic formulas (of that fragment) $\varphi(z)$ are definable over $\mathcal{C}$ in $TL(B)$. Similarly, one may speak of expressive completeness of *FOMLO* for some temporal language. If we have expressive completeness in both directions between two languages - they are ***expressively equivalent***.

As Until and Since are definable in *FOMLO*, it follows that *FOMLO* is expressively complete for $TL(\mathsf{Until}, \mathsf{Since})$. The fundamental theorem of Kamp shows that for Dedekind complete structures the opposite direction holds as well:

Theorem 2.2 ([Kam68]). *$TL(\mathsf{Until}, \mathsf{Since})$ is expressively equivalent to FOMLO over Dedekind complete structures.*

This was further generalized by Stavi who introduced two new modalities Until' and Since' and proved that $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}', \mathsf{Since}')$ and *FOMLO* have the same expressive power over all linear time flows [GPSS80, GHR94].

2.4 In Search of a Finite Basis for Future Formulas

We use standard ***interval*** notations and terminology for subsets of the domain of a structure $\mathcal{M} = (\mathcal{T}, <, \mathcal{I})$, e.g.: $(t, \infty) =_{def} \{t' \in \mathcal{T} | t' > t\}$; similarly we define $(t, t'), [t, t'), (t, \infty), [t, \infty)$, etc., where $t < t'$ are the ***endpoints*** of the interval. The ***sub-structure*** of $\mathcal{M}$ restricted to an interval is defined naturally. In particular: $\mathcal{M}|_{>t_0}$ denotes the sub-structure of $\mathcal{M}$ restricted to (t_0, ∞): Its domain is

(t_0, ∞) and its order relation and interpretation are those of $\mathcal{M}$, restricted to this interval. $\mathcal{M}|_{\geq t_0}$ is defined similarly with respect to $[t_0, \infty)$. If structures $\mathcal{M}, \mathcal{M}'$ have domains $\mathcal{T}, \mathcal{T}'$, and if I is an interval of $\mathcal{M}$, with endpoints $t_1 < t_2$ in $\mathcal{M}$, such that $I \cup \{t_1, t_2\} \subseteq \mathcal{T} \cap \mathcal{T}'$ and the order relations of both structures coincide on $I \cup \{t_1, t_2\}$ - we will say that I is a ***common interval*** of both structures. This is defined similarly for intervals with ∞ or $-\infty$ as either endpoint. Two structures ***coincide*** on a common interval iff the interpretations coincide there. Two structures ***agree*** on a formula at a given time-point (or along a common interval) iff the formula has the same truth value at that point (or along that interval) in both structures.

Definition 2.3 (Future / past formulas and modalities). *A formula (temporal, or monadic with a single free first-order variable) F is (**semantically**):*

*A **future** formula iff whenever two linear structures coincide on a common interval $[t_0, \infty)$ they agree on F at t_0.*
*A **pure future** formula iff whenever two linear structures coincide on a common interval (t_0, ∞) they agree on F at t_0.*
***Past** and **pure past** formulas are defined similarly.*

*A temporal modality is a first-order **future (past) modality** iff it is definable in FOMLO by a future (past) truth table.*

Note that 'future' can be characterized also syntactically: A formula $\varphi(x_0)$ is a future formula iff it is equivalent to a formula with all quantifiers relativized to $[x_0, \infty)$, that is, all quantifiers are of the form $(\forall x)_{\geq x_0}(\dots)$ or $(\exists x)_{\geq x_0}(\dots)$.

Looking at their truth tables, it is easy to verify that Until is a future modality and Since is a past modality. This pair $\{\mathsf{Until}, \mathsf{Since}\}$ forms an expressively complete (finite) basis in the sense of Kamp's theorem. Do we have a finite basis of future modalities which is expressively complete for all future formulas? Here are some answers:

Theorem 2.4 ([GPSS80]). *$TL(\mathsf{Until})$ is expressively equivalent to the future fragment of FOMLO over discrete time flows (Naturals, Integers, finite).*

Theorem 2.5 ([HR03]). *There is no temporal logic with a finite basis of **future** modalities which is expressively equivalent to the future fragment of FOMLO over Real time flows.*

Theorem 2.6 ([GHR94]). *$TL(\mathsf{Until}, \mathsf{K}^-)$ is expressively complete for the future fragment of FOMLO over Dedekind complete time flows.*[1]

Here we don't have expressive equivalence, as not all $TL(\mathsf{Until}, \mathsf{K}^-)$ formulas are future formulas. Theorem 2.6 offers a finite basis $\{\mathsf{Until}, \mathsf{K}^-\}$, but just like Kamp's $\{\mathsf{Until}, \mathsf{Since}\}$ - this is a 'mixed' future-past basis. [HR03] points out that in spite of its 'past' nature, K^- is "almost" a future modality because it depends just on an arbitrarily small portion of the near past, and is independent of most of the past. It is conjectured there that this "almost future basis" 'generates' only such "almost future formulas", and that it generates ***all*** of them. In this paper we show that this conjecture holds over the Real time domain $(\mathbb{R}, <)$.

[1] This follows [GHR94]'s work along the proof of their separation theorem (10.3.20).

3 A Finite Basis for Almost Future Formulas over $\mathbb{R}$

In Section 3.1 below we define almost future formulas. In Section 3.2 we refine a result of [Hod99], then the most technical part of the proof is in Section 3.3, with the heart of the proof in Lemma 3.13. Section 3.4 finally puts it all together to complete the proof.

3.1 Almost Future Formulas

Definition 3.1 (Almost future formulas, modalities, bases). *A formula (monadic, temporal) F is an almost future formula iff whenever two linear structures coincide on a common interval (t, ∞) they agree on F all along (t, ∞). A temporal modality is almost future iff it has an almost future truth table in FOMLO. A basis is almost future iff all its modalities are.*

Clearly, all pure future formulas are in particular future formulas and all future formulas are almost future. Note that we can give an alternative (equivalent) definition for future and pure future formulas in the style of Definition 3.1 as follows (compare with Definition 2.3): A formula F is

- ***Future*** iff whenever two linear structures coincide on a common interval $[t, \infty)$ they agree on F all along $[t, \infty)$.
- ***Pure future*** iff whenever two linear structures coincide on a common interval (t, ∞) they agree on F all along $[t, \infty)$.

In the sequel we will be interested in "Real structures" - these are structures over time domains isomorphic to the Real time flow $(\mathbb{R}, <)$. We denote this class of structures by $\mathcal{R}$. Note that if $\mathcal{M} \in \mathcal{R}$, then for every $t \in \mathcal{M}$, the structure $\mathcal{M}|_{>t}$ is also in $\mathcal{R}$.

Remark 3.2. *The next two facts and the lemma below follow immediately:*
1. *If an almost future formula holds at t_0 in a substructure $\mathcal{M}|_{>t}$ of some $\mathcal{M} \in \mathcal{R}$ where $t < t_0$ - then it holds there in $\mathcal{M}$ as well.*
2. *If an almost future formula holds at t_0 in a structure $\mathcal{M} \in \mathcal{R}$ then it holds at t_0 in all substructures $\mathcal{M}|_{>t}$ where $t < t_0$.*

Lemma 3.3. *If a basis B is almost future then so are all of $TL(B)$ formulas. In particular:* Until, K^- *and all the formulas of* $TL(\mathsf{Until}, \mathsf{K}^-)$ *are almost future.*

Example: Consider the following property: "Any open interval (t, t_0) contains a proper subinterval (t_2, t_1) such that P (an atomic property) holds at the ends t_1 and t_2, but doesn't hold anywhere inside (t_2, t_1)". This is an almost future property expressible in *FOMLO*. In $TL(\mathsf{Until}, \mathsf{K}^-)$ it is expressed by:

$$\mathsf{K}^-(P \wedge (\neg P \ \mathsf{Until}\ P))$$

Our main result states that, with respect to the class $\mathcal{R}$, ***any*** almost future property expressible in *FOMLO* can be translated to $TL(\mathsf{Until}, \mathsf{K}^-)$:

Main Theorem 3.4. *$TL(\mathsf{Until}, \mathsf{K}^-)$ is expressively equivalent to the almost future fragment of FOMLO over the class of Real structures.*

As Until and K^- are definable in *FOMLO*, the expressive completeness of almost future *FOMLO* for $TL(\mathsf{Until}, \mathsf{K}^-)$ over all linear structures (and in particular over Real ones) follows immediately by Lemma 3.3. For the opposite direction we have to show how almost future monadic formulas translate into $TL(\mathsf{Until}, \mathsf{K}^-)$. Most of our effort will now be in finding such a translation.

In the rest of the paper we highlight the core of the proof, omitting less significant technical details. A detailed proof can be found in the full paper.

3.2 Decomposition Formulas

Both expressive completeness proofs of [GPSS80] (for Theorem 2.4 above) and of [Hod99] (for Kamp's Theorem 2.2) go through manipulating monadic formulas to reach an equivalent formula in some standard form that can then be translated to the target temporal language. We follow the same track:

Definition 3.5 ([Hod99] Decomposition formulas). [2] *A FOMLO formula is **basic** over $TL(B)$ (where B is any temporal basis) iff it is a boolean combination of: (1) Atomic FOMLO formulas and (2) FOMLO formulas definable over Dedekind complete structures in $TL(B)$. A formula of the form: $\exists\bar{x}\forall y\chi(\bar{x}, y, z)$ where $\bar{x}$ is a tuple of first-order variables and χ is basic over $TL(B)$ is called a **decomposition formula** over $TL(B)$.*

Theorem 3.6 ([Hod99]). *Every FOMLO formula $\varphi(z)$ is equivalent over Dedekind complete structures to a positive boolean combination of decomposition formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$:*

$$\varphi(z) \equiv_{\mathcal{DC}} \bigvee_i \bigwedge_j \exists\bar{x}\forall y\chi_{ij}(\bar{x}, y, z), \text{ where } \chi_{ij} \text{ are basic over } TL(\mathsf{Until}, \mathsf{K}^-).$$

Targeting at Kamp's theorem, [Hod99] formulates this theorem and the preceding definition with respect to $TL(\mathsf{Until}, \mathsf{Since})$; yet, the proof there actually uses Since in a restricted form: 'X Since $True$', which is equivalent to $\neg\mathsf{K}^-(\neg X)$. Thus, the proof actually holds for $TL(\mathsf{Until}, \mathsf{K}^-)$ as well.

[GPSS80] introduces a specific form of decomposition formulas where the basic $\chi(\bar{x}, y, z)$ is 'split' into $TL(B)$-definable formulas that 'talk' about a sequence of moments (represented by the tuple $\bar{x}$) and about the sequence of intervals 'marked' by these points:

Definition 3.7 ([GPSS80] $\overleftarrow{\exists\forall}$-formulas). *A FOMLO formula is a **$\overleftarrow{\exists\forall}$-formula** over $TL(B)$ iff it is of the form:*

[2] [Hod99]'s definitions are more general; this simplified version is sufficient for us.

$$
\begin{array}{lll}
\psi(z) := \exists x_n \ldots \exists x_1 \exists x_0 & & \\
\quad [(x_n < x_{n-1} < \cdots < x_1 < x_0 = z) & & \text{“Ordering”} \\
\quad \wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) & & \text{“All } \alpha_j \text{'s hold at the points } x_j\text{”} \\
\quad \wedge \bigwedge_{j=0}^{n-1} [(\forall y)_{>x_{j+1}}^{<x_j} \beta_j(y)] & & \text{“Each } \beta_j \text{ holds along } (x_{j+1}, x_j)\text{”} \\
\quad \wedge \ (\forall y)^{<x_n} \beta_n(y)] & & \text{“} \beta_n \text{ holds everywhere 'before' } x_n\text{”}
\end{array}
$$

where α_j, β_j are FOMLO formulas definable over Dedekind complete structures in $TL(B)$.

Notation 3.8. *Having a particular interest in $\overleftarrow{\exists\forall}$-formulas over $TL(\mathsf{Until}, \mathsf{K}^-)$, we shortly call them $\overleftarrow{\exists\forall}$**-formulas**. We use the notation $\psi^n(z)$ to explicitly reflect the length of the quantifier prefix; and we use the abbreviated notation $\psi^n = (\langle\alpha_0, \beta_0\rangle, \ldots, \langle\alpha_n, \beta_n\rangle)$ for a $\overleftarrow{\exists\forall}$-formula as above, with α_i, β_i definable over Dedekind complete structures in $TL(\mathsf{Until}, \mathsf{K}^-)$.*

The following can be derived from Theorem 3.6 by standard logical equivalences:

Proposition 3.9. *Every FOMLO formula $\varphi(z)$ is equivalent over Dedekind complete structures to a finite disjunction of $\overleftarrow{\exists\forall}$-formulas.*

3.3 Formulas That Hold "Regardless of Most of the Past"

A formula F "holds in $\mathcal{M}$ at t_0 regardless of most of the past" if we can truncate the past as close we wish to the left of t_0, and F persistently holds at t_0 in all such truncated structures. As we will not be using here the dual notion of "holding regardless of most of the future" - we will shortly say that F "almost-holds in $\mathcal{M}$ at t_0". Formally:

Definition 3.10 ('Almost holds'). *Given $\mathcal{M} \in \mathcal{R}$ and $t_0 \in \mathcal{M}$, and given a formula (monadic, temporal) F: F **almost-holds** in $\mathcal{M}$ at t_0 iff for every $t < t_0$ in $\mathcal{M}$ there is a $t' \in (t, t_0)$ such that $\mathcal{M}|_{>t'}, t_0 \models F$.*

Remark 3.11.
1. *If a formula F is almost future, $\mathcal{M} \in \mathcal{R}$ and $t_0 \in \mathcal{M}$ then: F holds in $\mathcal{M}$ at t_0 iff it almost-holds there.*
2. *In general, it might be the case that a formula F (which is not almost future) almost-holds in some $\mathcal{M}$ at t_0, yet F does not hold in $\mathcal{M}$ at t_0. Example: "P always held in the past" ($(\forall x)^{<z} P(x)$). Similarly, "P once held in the past" ($(\exists x)^{<z} P(x)$) demonstrates the converse situation.*

Lemma 3.12. *If a finite disjunction of FOMLO formulas $\varphi(z) = \bigvee \psi_i(z)$ is almost future, then for any $\mathcal{M} \in \mathcal{R}$ and $t_0 \in \mathcal{M}$:*

$$\mathcal{M}, t_0 \models \varphi(z) \text{ iff some } \psi_i(z) \text{ almost-holds in } \mathcal{M} \text{ at } t_0 \qquad (1)$$

Proof. Given an almost future $\varphi(z) = \bigvee \psi_i(z)$ and $t_0 \in \mathcal{M} \in \mathcal{R}$ as above:

Proof of $\Leftarrow$***:*** Let $t < t_0$. Assume that some $\psi_i(z)$ almost-holds in $\mathcal{M}$ at t_0, then there is a $t' \in (t, t_0)$ such that $\mathcal{M}|_{>t'}, t_0 \models \psi_i(z)$, hence $\mathcal{M}|_{>t'}, t_0 \models \varphi(z)$, and as φ is almost future - $\mathcal{M}, t_0 \models \varphi(z)$ as well (Remark 3.2 (1)).

Proof of $\Rightarrow$***:*** Assume that $\mathcal{M}, t_0 \models \varphi(z)$, then (by Remark 3.2 (2)) for every $t < t_0$ in $\mathcal{M}$: $\mathcal{M}|_{>t}, t_0 \models \varphi(z)$, hence for every $t < t_0$:

$$\mathcal{M}|_{>t}, t_0 \models \psi_i(z) \text{ for some index } i \tag{2}$$

Now, assume to the contrary that none of the disjuncts ψ_i almost-holds in $\mathcal{M}$ at t_0. Then for each i there is a point - denote it by t_i - such that $t_i < t_0$ and for all $t' \in (t_i, t_0)$: $\psi_i(z)$ does not hold in $\mathcal{M}|_{>t'}$ at t_0. Let $\bar{t}$ denote the largest ('latest') t_i (we started off with a finite disjunction) and let $t \in (\bar{t}, t_0)$. Then for each i: $t_i \leq \bar{t} < t < t_0$, and therefore for each i: $\psi_i(z)$ does not hold in $\mathcal{M}|_{>t}$ at t_0. This contradicts (2) above. Thus, we conclude that (at least) one of the disjuncts ψ_i ***does*** almost-hold in $\mathcal{M}$ at t_0. □

The above lemma motivates us to seek a way to express in $TL(\mathsf{Until}, \mathsf{K}^-)$ the fact that "a formula almost-holds in $\mathcal{M}$ at t_0". The main technical lemma below shows that this is possible for $\overleftarrow{\exists}\forall$-formulas.

Main Lemma 3.13. *For every $\overleftarrow{\exists}\forall$-formula $\psi(z)$ there is a $TL(\mathsf{Until}, \mathsf{K}^-)$ formula F_ψ such that for every structure $\mathcal{M} \in \mathcal{R}$ and every $t_0 \in \mathcal{M}$:*

$$\mathcal{M}, t_0 \models F_\psi \textit{ iff } \psi(z) \textit{ almost-holds in } \mathcal{M} \textit{ at } t_0 \tag{3}$$

Proof. Let $\psi^n(z) = (\langle \alpha_0, \beta_0 \rangle, \ldots, \langle \alpha_n, \beta_n \rangle)$ be a $\overleftarrow{\exists}\forall$-formula (see Notation 3.8), and let A_i, B_i be $TL(\mathsf{Until}, \mathsf{K}^-)$ formulas defining α_i, β_i ($\alpha_i \equiv_{\mathcal{DC}} A_i$; $\beta_i \equiv_{\mathcal{DC}} B_i$). Define $TL(\mathsf{Until}, \mathsf{K}^-)$ formulas $G_0^{\psi^n}, G_1^{\psi^n}, \ldots, G_n^{\psi^n}, G_{n+1}^{\psi^n}$ and F_{ψ^n} as follows:

$$\begin{aligned}
&G_0^{\psi^n} := A_0 \\
&G_{j+1}^{\psi^n} := A_{j+1} \wedge (B_j \ \mathsf{Until}\ G_j^{\psi^n}) \ \text{ - for } j = 0, 1, \ldots, n-1 \\
&G_{n+1}^{\psi^n} := B_n \ \mathsf{Until}\ G_n^{\psi^n} \\
&F_{\psi^n} := A_0 \wedge \neg \mathsf{K}^-(\neg B_0) \wedge \bigwedge_{j=1}^{n+1} \mathsf{K}^-(G_j^{\psi^n})
\end{aligned}$$

Now let $t_0 \in \mathcal{M}$, and show that F_{ψ^n} satisfies the required property (3). The $\Leftarrow$ direction follows directly from definitions. For the $\Rightarrow$ direction: Assume that $\mathcal{M}, t_0 \models F_{\psi^n}$. Let $t < t_0$. To show that ψ^n(z) almost-holds in $\mathcal{M}$ at t_0 we must find a $t' \in (t, t_0)$ such that $\mathcal{M}|_{>t'}, t_0 \models \psi^n(z)$.

First, as $\mathcal{M}, t_0 \models \neg \mathsf{K}^-(\neg B_0)$ we have an interval $(\tilde{t}, t_0)$ where B_0 holds and $t < \tilde{t} < t_0$. Second, as $\mathcal{M}, t_0 \models \mathsf{K}^-(G_{n+1}^{\psi^n})$, we have a $t' \in (\tilde{t}, t_0)$ where $G_{n+1}^{\psi^n}$ holds, that is: $\mathcal{M}, t' \models (B_n \ \mathsf{Until}\ G_n^{\psi^n})$. We will find points $t_1, \ldots, t_n, t_{n+1}$ in $\mathcal{M}$ such that (i) $t < \tilde{t} < t' = t_{n+1} < t_n < \cdots < t_1 < t_0$ and (ii) for each $0 \leq i \leq n$: B_i holds in $\mathcal{M}$ along (t_{i+1}, t_i) and $\mathcal{M}, t_i \models G_i^{\psi^n}$ (and thus, in particular

$\mathcal{M}, t_i \models A_i$). Then, as A_i, B_i are almost future (Lemma 3.3), the same holds in the substructure $\mathcal{M}|_{>t'}$ as well (Remark 3.2 (2)). And as $A_i \equiv_{\mathcal{DC}} \alpha_i$; $B_i \equiv_{\mathcal{DC}} \beta_i$, we conclude that $\mathcal{M}|_{>t'}, t_0 \models \psi^n(z)$.

It remains to show there are points t_i as above. For t_{n+1} we simply pick t'. Next, we construct t_n: We have $\mathcal{M}, t_{n+1} \models (B_n \ \mathsf{Until}\ G_n^{\psi^n})$, hence, $G_n^{\psi^n}$ holds at some $t'' > t_{n+1}$ and B_n holds along (t_{n+1}, t''). Now, if $t'' < t_0$ denote: $t_n = t''$. Otherwise, as $\mathcal{M}, t_0 \models \mathsf{K}^-(G_n^{\psi^n})$, there is a $t^* \in (t_{n+1}, t_0)$ where $G_n^{\psi^n}$ holds - and in this case denote: $t_n = t^*$. In any case, we have $t < \tilde{t} < t' = t_{n+1} < t_n < t_0$, B_n holds along (t_{n+1}, t_n) and $\mathcal{M}, t_n \models G_n^{\psi^n}$. Repeat the above arguments (induction, down-counting from t_n to t_1) to construct the rest of the t_i's. Finally, B_0 clearly holds along (t_1, t_0) and $\mathcal{M}, t_0 \models A_0$, so the points t_i indeed satisfy (i) and (ii) as required. □

3.4 Putting It All Together

Lemma 3.13 renders the desired semantics-preserving translation over Real structures for almost future *FOMLO* formulas. Now we are ready to complete the proof of our main result (Theorem 3.4):

Given an almost future $\varphi(z)$ in *FOMLO*, we will construct a $TL(\mathsf{Until}, \mathsf{K}^-)$ formula F_φ such that:

$$\varphi(z) \equiv_{\mathcal{R}} F_\varphi \tag{4}$$

1. Given an almost future $\varphi(z)$, by Proposition 3.9 we have: $\varphi(z) \equiv_{\mathcal{DC}} \bigvee \psi_i(z)$ where ψ_i are $\overleftarrow{\exists\forall}$-formulas.
2. By Lemma 3.13 each disjunct ψ_i has a 'representative' F_{ψ_i} in $TL(\mathsf{Until}, \mathsf{K}^-)$ that satisfies property (3) of the lemma, or - in other words - that asserts that "$\psi_i(z)$ almost-holds in a Real structure $\mathcal{M}$ at t_0". Define:
$$F_\varphi =_{def} \bigvee F_{\psi_i}$$
3. Notice that so far we haven't used the fact that φ is almost future: Steps 1 and 2 above hold for any monadic $\varphi(z)$. Now verify that (4) above indeed holds: Let $t_0 \in \mathcal{M} \in \mathcal{R}$. By Lemma 3.12 (and this is the point where the "almost futureness" of φ is crucial), $\mathcal{M}, t_0 \models \varphi(z)$ iff there is an index i such that $\psi_i(z)$ almost-holds in $\mathcal{M}$ at t_0, in other words - by Lemma 3.13 - iff there is an i such that $\mathcal{M}, t_0 \models F_{\psi_i}$, that is, iff $\mathcal{M}, t_0 \models F_\varphi$.

4 Further Results and Comments

We have shown expressive equivalence of $TL(\mathsf{Until}, \mathsf{K}^-)$ and almost future *FOMLO* over time flows isomorphic to the Reals. The notion of past, future, almost future formulas is defined with respect to the class of all linear structures. One may as well consider similar notions relative to specific classes of structures. For example, a formula is a future formula over $\mathcal{R}$ (the class of Real structures) if any pair of Real structures that coincide on the future of some point t agree on the formula at t. Clearly, every future formula over the class of all linear structures is also a future formula over $\mathcal{R}$. The converse doesn't hold: "There is a first-moment and P holds there" for example, is unsatisfiable over $\mathcal{R}$, and therefore a future formula over $\mathcal{R}$, but this is not a future formula over Natural time domains. We actually proved a stronger result: Every formula which is almost future over $\mathcal{R}$ has a $TL(\mathsf{Until}, \mathsf{K}^-)$-equivalent over $\mathcal{R}$.

It is decidable whether a formula $\varphi(x)$ is almost future over $\mathcal{R}$. Indeed let $\varphi^{\text{rel}}_{>x'}(x)$ be obtained from φ by relativization of all quantifiers to (x', ∞). A formula φ is almost future over $\mathcal{R}$ iff $\forall x\big((\forall x')^{<x}(\varphi(x) \leftrightarrow \varphi^{\text{rel}}_{>x'}(x))\big)$ is valid over $\mathcal{R}$. Since the validity of a *FOMLO* formula over $\mathcal{R}$ is decidable [BG85], we conclude that it is decidable whether a formula is almost future over $\mathcal{R}$.

In the full paper we prove expressive equivalence of $TL(\mathsf{Until}, \mathsf{K}^-)$ and almost future *FOMLO* over all Dedekind complete structures. Lifting the proof from the Reals to Dedekind complete orders requires careful handling of subtleties that don't appear in the Reals. In Dedekind complete structures there are three types of points: A structure may have a least element or not - a "first moment". A non-first moment is a "successor" if it has a "latest earlier moment" and a "left-limit" otherwise. The fact that in $\mathbb{R}$ all points are left-limits simplifies the proof. The translation presented in Section 3.4 works fine for left-limit points in Dedekind complete structures as well, but fails for successors and first moments. These two types of points need different (but simpler) handling. The core of the proof - handling left-limit points - is the same as presented in this paper.

Over linear structures in general, $\{\mathsf{Until}, \mathsf{K}^-\}$ is not expressive enough: It is not a basis for almost future formulas. Stavi generalized Kamp's theorem by enhancing $\{\mathsf{Until}, \mathsf{Since}\}$ to obtain a basis expressively equivalent to *FOMLO* over linear time [GHR94]. Unfortunately, $\{\mathsf{Until}, \mathsf{K}^-\}$ cannot be extended in a similar manner: In the full paper we show that no finite basis of almost future modalities is expressively equivalent to almost future *FOMLO* over linear time.

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