Chapter 4
Image Restoration

Assume we have access to a certain image obtained by means of a device that, in the process of *acquisition*, causes a degradation, such as *blurring*. The objective of this chapter is to show how to *restore* the image from the *degraded* image, considering specific examples[1].

In the image processing literature, the recovery of an original image of an object is called restoration or reconstruction, depending on the situation. We shall not discuss this distinction[2].

In any case, in the nomenclature of inverse problems we are using as setforth in Section 2.8, both of these cases, restoration or reconstruction, are suitably called *reconstruction* inverse problems.

We assume here that the image is in grayscale. Every shade of gray will be represented by a real number between 0 (pure black) and 1 (pure white). The original image is represented by a set of *pixels* (i, j), $i = 1, \ldots, L$ and $j = 1, \ldots, M$, which are small monochromatic squares in the plane that make up the image. Each pixel has an associated shade of grey, denoted by I_{ij} or $I(i, j)$, which constitutes an $L \times M$ matrix (or a vector of LM coordinates). A typical image can be made up of $256 \times 512 = 131\,072$ pixels. Analogously, we will denote by Y_{ij}, $i = 1, \ldots, L$, $j = 1, \ldots, M$ the grayscale of the blurring of the original image.

The inverse problems to be solved in this chapter deal with obtaining $\mathbf{I}$, the original image, from $\mathbf{Y}$, the blurred image. These *problems*[3] are of Type I, and, most of them, deal with inverse reconstruction problems, as presented in Section 2.8, which, properly translated, means that given a degraded image one wants to determine the original image, or, more realistically, to estimate it.

4.1 Degraded Images

We shall assume that the degraded image is obtained from the original image through a linear transformation. Let B be the transformation that maps the original image $\mathbf{I}$ to its degraded counterpart $\mathbf{Y}$, $\mathbf{Y} = B\,\mathbf{I}$, explicitly given by [27]

[1] The results presented in this chapter were obtained by G. A. G. Cidade [24].

[2] We just observe that reconstruction is associated with operators whose singular values are just 0 and 1 (or some other constant value), whereas when restoration is concerned the range of singular values is more extense. For understanding what are the consequences of having just 0 and 1 as singular values see Section A.4.2.

[3] See the classification in Table 2.3, page 48.

$$Y_{ij} = \sum_{i'=1}^{L} \sum_{j'=1}^{M} B_{ij}^{i'j'} I_{i'j'} \ , \quad i = 1, \ldots, L, j = 1, \ldots, M \, . \tag{4.1}$$

We call Eq. (4.1) the observation equation. We note that $\mathbf{Y}$ corresponds to the experimental data, and we call the linear operator B the *blurring matrix*[4] or, strictly speaking, blurring operator.

We assume that the blurring matrix has a specific structure

$$B_{ij}^{i+k\, j+l} = b_{kl} \, , \quad \text{for } -N \le k \le N \text{ and } -N \le l \le N \, , \tag{4.2a}$$

$$\text{and } B_{ij}^{mn} = 0, \text{ otherwise} \, . \tag{4.2b}$$

Here b_{kl}, for $-N \le k, \ l \le N$, is called the *matrix of blurring weights*, This means that blurring is equal for every position of the pixel (homogeneous blurring).

At each pixel (i, j), the blurring takes into consideration the shades of the neighbouring pixels so that the *domain of dependence* is the square neighbourhood centered in pixel (i,j) covering N pixels to the left, right, up and down (see Fig. 4.1) comprising $(2N+1)\times(2N+1)$ pixels whose values determine the value of the pixel (i,j). In particular, (b_{kl}) is a $(2N+1)\times(2N+1)$ matrix. Usually, one takes $N \ll L, M$. Moreover, the coefficients are chosen preferably in such a way that

$$\sum_{k=-N}^{N} \sum_{l=-N}^{N} b_{kl} = 1 \, , \tag{4.3}$$

with $b_{kl} \ge 0$ for $-N \le k, l \le N$.

Figure 4.1 illustrates the action of a blurring matrix with $N = 1$, on the pixel $(i, j) = (7, 3)$. This pixel belongs to the discrete grid where the image is defined. The square around it demarcates the *domain of dependence* of the shade of gray Y_{73}, of pixel (7,3) of the blurred image, in tones of gray of the pixels of the original image pixels,

$$I_{62}, I_{72}, I_{82}, I_{63}, I_{73}, I_{83}, I_{64}, I_{74}, \text{ and } I_{84} \, .$$

We can assume that b_{kl} admits *separation of variables*, in the sense that $b_{kl} = f_k f_l$, for some f_k, $-N \le k \le N$. In this case, $f_k \ge 0$ for all k, and Eq. (4.3) implies that

$$\sum_{k=-N}^{N} f_k = 1 \, . \tag{4.4}$$

Here, f_k can be one of the profiles shown in Fig. 4.2: (a) truncated gaussian, (b) truncated parabolic, or (c) preferred direction. It could also be a combination of the previous profiles or some more general weighting function.

[4] Strictly speaking $B = (B_{ij}^{i'j'})$ is not a matrix. However, it defines a linear operator.

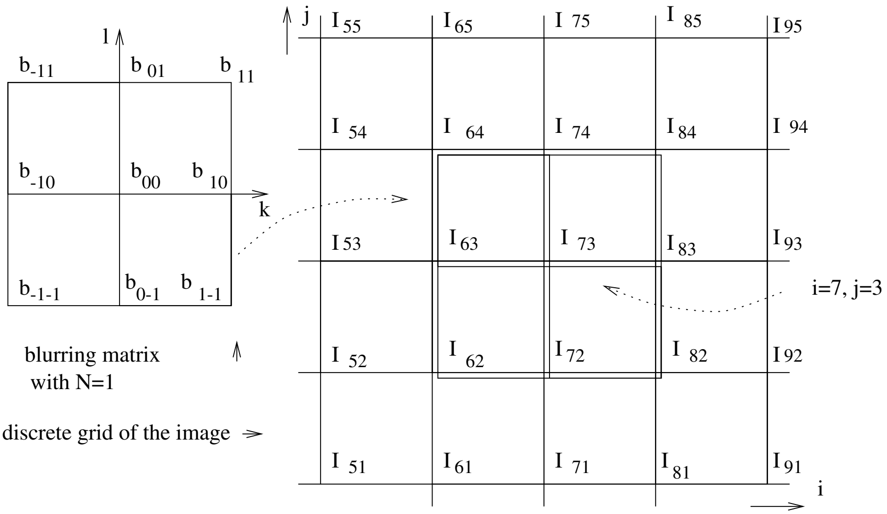

Fig. 4.1 The blurring matrix with $N = 1$ acts on the point $(i, j) = (7, 3)$ of the discrete grid of the image (*pixels*)

With respect to the situation depicted in Fig. 4.1, one possibility is to choose $f_{-1} = 1/4$, $f_0 = 1/2$ and $f_1 = 1/4$ which leads to the following matrix of blurring weights

$$\begin{pmatrix} b_{-1\,-1} & b_{-1\,0} & b_{-1\,1} \\ b_{0\,-1} & b_{0\,0} & b_{0\,1} \\ b_{1\,-1} & b_{1\,0} & b_{1\,1} \end{pmatrix} = \begin{pmatrix} f_{-1} \\ f_0 \\ f_1 \end{pmatrix} (f_{-1}\; f_0\; f_1)$$
$$= \begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} . \tag{4.5}$$

Therefore, in this case, for instance,

$$Y_{73} = \frac{1}{16} (I_{62} + 2I_{72} + I_{82} \\ +2I_{63} + 4I_{73} + 2I_{83} + I_{64} + 2I_{74} + I_{84}) \; . \tag{4.6}$$

This scheme cannot be taken all the way to the boundaries of the image. At a boundary point we cannot find enough neighbour pixels to take the weighted average. we describe two possible approaches for such situations. One simple approach here is to consider that the required pixels lying outside the image have a constant value, for example zero or one (or some other intermediate constant value). Another approach is to add the weights of the pixels that lie outside of the image to the weights of the pixels that are in the image, in a symmetric fashion. This is equivalent to attributing the pixel outside of the image the shade of gray of its symmetric pixel across the boundary (border) of the image.

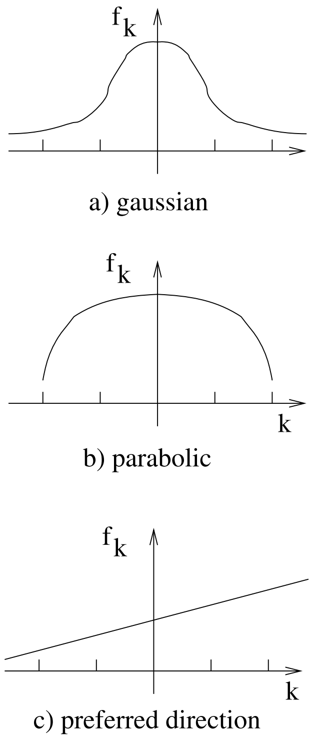

Fig. 4.2 The blurring matrix can represent several kinds of *tensorial* two-dimensional convolutions: gaussian, parabolic, with preferred direction, or a combination of them

For illustration purposes, say now that (7,3) is a pixel location on the right boundary of the image. The positions (8,4), (8,3), (8,2) are outside the image and do not correspond to any pixel. We let the weights 1/16, 2/16 and 1/16 of these positions to be added to the symmetric pixels, with respect to the boundary, respectively, (6,4), (6,3), and (6,2). The result is

$$Y_{73} = \frac{1}{16}\left(2I_{62} + 2I_{72} + 4I_{63} + 4I_{73} + 2I_{64} + 2I_{74}\right) . \tag{4.7}$$

4.2 Restoring Images

The inverse reconstruction problem to be considered here is to obtain the vector **I** when the matrix B and the experimental data **Y** are known, by solving Eq. (4.1) for **I**.

We can look at this problem as a finite dimension optimization problem. First, the norm of an image $\mathbf{I}$ is taken as the Euclidean norm of $\mathbf{I}$ thought of as a vector in $\mathbb{R}^{LM}$,

$$|\mathbf{I}| = \left(\sum_{i=1}^{L}\sum_{j=1}^{M} I_{ij}^2\right)^{\frac{1}{2}} ,$$

not the norm in the set of $L \times M$ matrices, as defined in Eq. (A5). Next, consider the *discrepancy* (or residual) vector $\mathbf{Y} - B\,\mathbf{I}$, and the functional obtained from its norm

$$R(\mathbf{I}) = \frac{1}{2}|\mathbf{Y} - B\,\mathbf{I}|^2 , \tag{4.8}$$

to which we add a Tikhonov's regularization term, [81, 26, 27, 70, 86] getting

$$\begin{aligned} Q(\mathbf{I}) &= \frac{1}{2}|\mathbf{Y} - B\,\mathbf{I}|^2 + \alpha S(\mathbf{I}) \\ &= \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{M}\left(Y_{ij} - \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl} I_{i+k,j+l}\right)^2 + \alpha S(\mathbf{I}) , \end{aligned} \tag{4.9}$$

where S is the regularization term[5] and α is the *regularization parameter*, with $\alpha > 0$. Here, B and $\mathbf{Y}$ are given by the problem. Finally, the inverse problem is formulated as finding the minimum point of Q.

Several regularization terms can be used, and common terms are[6] [25, 26, 16]:

Norm $$S(\mathbf{I}) = \frac{1}{2}|\mathbf{I} - \bar{\mathbf{I}}|^2 = \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{M}\left(I_{ij} - \bar{I}_{ij}\right)^2 \tag{4.10a}$$

Entropy $$S(\mathbf{I}) = -\sum_{i=1}^{L}\sum_{j=1}^{M}\left(I_{ij} - \bar{I}_{ij} - I_{ij}\ln\frac{I_{ij}}{\bar{I}_{ij}}\right) \tag{4.10b}$$

In both cases, $\bar{\mathbf{I}}$ is a *reference value* (an image as close as possible to the image that is to be restored), known *a priori*.

The notion of regularization and its properties are discussed in Chapter 3. The concept of reference value is introduced in Section 3.4, page 60 and the advantage of its use is explained in Section 3.6, page 63.

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[5] To improve readability we insert a comma (,) between the subscripts of I whenever adequate: $I_{i+k,\,j+l}$.

[6] Some regularization terms can be interpreted as Bregman's divergences or distances [14, 81, 25, 42]. The use of Bregman's divergences as regularization terms in Tikhonov's functional was proposed by N. C. Roberty [27], from the Universidade Federal do Rio de Janeiro. Bregman's distance was introduced in [14] and it is not a metric in the usual sense. Exercise A.14 recalls the notion of metric spaces, while Exercise A.35 presents the definition of Bregman's divergences. Some other exercises in Appendix A elucidate the concept of Bregman's divergence.

In the case now under consideration, image restoration, the reference value can be the given blurred image $\bar{\mathbf{I}} = \mathbf{Y}$, or a gray image (i.e., with everywhere constant intensity),

$$\bar{I}_{ij} = c \ , \ \text{for } i = 1, \ldots, L \ j = 1, \ldots, M \ ,$$

where c is a constant that can be chosen equal to the mean value of the intensities of the blurred image,

$$c = \frac{1}{LM}\left(\sum_{i=1}^{L}\sum_{j=1}^{M} Y_{ij}\right) \ .$$

4.3 Restoration Algorithm

Image restoration can be carried out by Tikhonov's method, with the regularization term given by the entropy functional, Eq. (4.10b). The regularized problem corresponds to the minimum point equation of the functional Q, and is a variant of the one analyzed in Chapter 3. The entropy functional chosen here renders the regularization as non-linear, differing from the one treated in the referred chapter.

Substituting the expression of $S(I)$ given in the right hand side of Eq. (4.10b) in Eq. (4.9), we obtain

$$\begin{aligned} Q(\mathbf{I}) &= \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{M}\left(Y_{ij} - \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl} I_{i+k,j+l}\right)^2 \\ &\quad -\alpha\sum_{i=1}^{L}\sum_{j=1}^{M}\left(I_{ij} - \bar{I}_{ij} - I_{ij}\ln\frac{I_{ij}}{\bar{I}_{ij}}\right) . \end{aligned} \tag{4.11}$$

To minimize this functional, the *critical point equation* is used

$$\frac{\partial Q}{\partial I_{rs}} = 0 \qquad \text{for all } r = 1, \ldots, L, \ s = 1, \ldots, M \, . \tag{4.12}$$

This is a non-linear system of LM equations and LM unknowns, I_{ij}, $i=1,\ldots,L$, $j=1,\ldots,M$. For notational convenience, let $F_{rs} = \partial Q/\partial I_{rs}$, and $\mathbf{F}$ the function

$$\mathbb{R}^{LM} \ni \mathbf{I} \mapsto \mathbf{F}(\mathbf{I}) \in \mathbb{R}^{LM} \ , \tag{4.13}$$

where, from Eq. (4.11), its (r, s)-th function[7] is given by F_{rs},

$$F_{rs} = -\sum_{i=1}^{L}\sum_{j=1}^{M}\left(Y_{ij} - \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl} I_{i+k,j+l}\right) b_{r-i,s-j} + \alpha\ln\frac{I_{rs}}{\bar{I}_{rs}} \ . \tag{4.14}$$

[7] When computing F_{rs}, we use that

$$\partial I_{ij}/\partial I_{rs} = \delta_{ir}\delta_{js} \qquad \text{and} \qquad \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl}\,\partial I_{i+k,j+l}/\partial I_{rs} = b_{r-i,s-j} \ .$$

Using this notation, the system of non-linear critical point equations of Q, Eq. (4.12), becomes

$$\mathbf{F}(\mathbf{I}) = 0\,. \tag{4.15}$$

Therefore, the inverse problem is reduced to solving Eq. (4.15). We will show how this non-linear system can be solved by the Newton's method.

4.3.1 Solution of a System of Non-linear Equations Using Newton's Method

Consider a system of equations like the one in Eq. (4.15). Newton's method is iterative and, under certain circumstances, converges to the solution. We sketch its derivation.

An initial estimate of the solution is needed: $\mathbf{I}^o$. Then, $\mathbf{I}^{p+1}$ is defined from $\mathbf{I}^p$, with $p = 0, 1, \ldots$, using a linearization of Eq. (4.15) by means of a Taylor's series expansion of function F around $\mathbf{I}^p$.

Due to the Taylor's formula[8] of $\mathbf{F}$, we have

$$\begin{aligned}\mathbf{F}(\tilde{\mathbf{I}}) = \mathbf{F}(\mathbf{I}^p) + \sum_{m=1}^{L}\sum_{n=1}^{M} \left.\frac{\partial \mathbf{F}}{\partial I_{mn}}\right|_{I^p} \left(\tilde{I}_{mn} - I^p_{mn}\right) \\ + \mathcal{O}\left(|\tilde{\mathbf{I}} - \mathbf{I}^p|^2\right)\,,\ \text{as } \tilde{\mathbf{I}} \to \mathbf{I}^p\,.\end{aligned} \tag{4.16}$$

Here, we should be careful with the dimensions of the mathematical objects. As stated in Eq. (4.13), $\mathbf{F}$, evaluated at any point, is an element of $\mathbb{R}^{LM}$, i.e., the left side of Eq. (4.16) has LM elements. This also holds for every term $\partial\mathbf{F}/\partial I_{mn}$, for all $m = 1, \ldots, L$, $n = 1, \ldots, M$.

For Newton's method, $\mathbf{I}^{p+1}$ is defined by keeping only up to the first order term of Taylor's expansion of $\mathbf{F}$, right side of Eq. (4.16), setting the left side equal to zero (we are iteratively looking for a solution of equation $\mathbf{F} = 0$), and substituting $\tilde{\mathbf{I}}$ by $\mathbf{I}^{p+1}$. Thus, Newton's method for solution of Eq. (4.15) is

$$0 = \mathbf{F}(\mathbf{I}^p) + \sum_{m=1}^{L}\sum_{n=1}^{M} \left.\frac{\partial \mathbf{F}}{\partial I_{mn}}\right|_{I^p} \left(I^{p+1}_{mn} - I^p_{mn}\right)\,. \tag{4.17}$$

To determine $\mathbf{I}^{p+1}$, we assume that $\mathbf{I}^p$ is known. Therefore, we see that Eq. (4.17) is a system of linear equations for $\mathbf{I}^{p+1}$. This system has LM equations and LM unknowns, I^{p+1}_{mn}, where $m = 1, \ldots, L$, $n = 1, \ldots, M$.

Newton's method can be conveniently written in algorithmic form as below. Let the vector of corrections

$$\Delta\mathbf{I}^p = \mathbf{I}^{p+1} - \mathbf{I}^p\,,$$

with entries $(\Delta I^p)_{m,n}$, $m = 1, \ldots, L$, $j = 1, \ldots, M$. Choose an arbitrary tolerance (threshold) $\epsilon > 0$.

[8] Taylor's formula is recalled in Section A.5.

1. **Initialization**
   Choose an initial estimate[9] $\mathbf{I}^0$.

2. **Computation of the increment**
   For $p = 0, 1, \ldots$, determine $\Delta\mathbf{I}^p = (\Delta I^p)_{mn} \in \mathbb{R}^{M^2}$ such that[10]

$$\sum_{m=1}^{L}\sum_{n=1}^{M} \left.\frac{\partial \mathbf{F}}{\partial I_{mn}}\right|_{I^p} \Delta I^p_{mn} = -\mathbf{F}(\mathbf{I}^p)\,. \tag{4.18a}$$

3. **Computation of a new approximation**
   Compute[11]

$$\mathbf{I}^{p+1} = \mathbf{I}^p + \Delta\mathbf{I}^p\,. \tag{4.18b}$$

4. **Use of the stopping criterion**
   Compute $|\Delta\mathbf{I}^p| = |\mathbf{I}^{p+1} - \mathbf{I}^p|$. Stop if $|\Delta\mathbf{I}^p| < \epsilon$. Otherwise, let $p = p+1$ and go to step 2.

4.3.2 Modified Newton's Method with Gain Factor

Newton's method, as presented in Eq. (4.18), does not always converge. It is convenient to introduce a modification by means of a *gain factor* γ, changing Eq. (4.18b) and substituting it by

$$\mathbf{I}^{p+1} = \mathbf{I}^p + \gamma\Delta\mathbf{I}^p\,, \tag{4.19}$$

that will lead to the convergence of the method to the solution of Eq. (4.15) in a wider range of cases, if the gain factor γ is adequately chosen.

4.3.3 Stopping Criterion

The iterative computations, by means of the modified Newton's method, defined by Eqs. (4.18a) and (4.19), is interrupted when at least one of the following conditions is satisfied

$$|\Delta\mathbf{I}^p| < \epsilon_1,\ |S(\mathbf{I}^{p+1}) - S(\mathbf{I}^p)| < \epsilon_2 \ \text{ or } \ |Q(\mathbf{I}^{p+1}) - Q(\mathbf{I}^p)| < \epsilon_3\,, \tag{4.20}$$

where ϵ_1, ϵ_2 and ϵ_3 are values sufficiently small, chosen *a priori*.

[9] For the problems we are aiming at, the initial estimate, $\mathbf{I}^0$, can be, for example, $\bar{\mathbf{I}}$, i.e., the blurred image, or a totally gray image.
[10] Compare with Eq. (4.17).
[11] Given $\mathbf{I}^p$, by choosing $\Delta\mathbf{I}^p$ and $\mathbf{I}^{p+1}$ as in Eq. (4.18), it follows that $\mathbf{I}^{p+1}$ satisfies Eq. (4.17).

4.3.4 Regularization Strategy

As mentioned in Chapter 3, the regularized problem differs from the original problem. It is only in the limit (as the regularization parameter approaches zero) that the solution of the regularized problem approaches the solution of the original problem, in special circumstances determined by a specific mathematical analysis. On the other hand, as was also mentioned in the same chapter, in practice the regularization parameter should not always approach zero, since, in the inevitable presence of measurement noise, the errors in the solution of the inverse problem can be minimized by correctly choosing the value of the regularization parameter. Thus, it is necessary to find the best regularization parameter, α^*, which lets the original problem be minimally altered, and yet, that the solution remains stable. The regularization presented here is non-linear, in contrast with that defined in Section 3.3.

It is possible to develop algorithms to determine the best regularization parameter, [84, 85]. However, they are computationally costly. A natural approach is to perform numerical experiments with the restoration algorithm, to determine a good approximation for the optimal regularization parameter.

4.3.5 Solving Sparse Linear Systems Using the Gauss-Seidel Method

We now present an iterative method suitable for solving the system of equations (4.18a).

From Eq. (4.14), we obtain

$$C_{mn}^{rs} = \frac{\partial F_{rs}}{\partial I_{mn}} = \sum_{k=-N}^{N} \sum_{l=-N}^{N} b_{kl} b_{r-m+k,s-n+l} + \frac{\alpha}{I_{rs}} \delta_{rm} \delta_{sn} , \tag{4.21}$$

for $r, m = 1, 2, \ldots, L$ and $s, n = 1, 2, \ldots, M$. Here, we also use the equations that can be found in the footnote on page 90.

The linear system of equations given by Eqs. (4.18a) and (4.21) is *banded*,[12] the length of it (distance from the non-zero elements to the diagonal) varying with the order of the blurring matrix represented in Eqs. (4.1) and (4.2). The *diagonal* of fourth order tensor C is given by the elements C_{mn}^{rs}, of C, such that $r = m$ and $s = n$, i.e., by elements C_{mn}^{mn}. For some types of blurring operator, it is guaranteed that the diagonal is dominant for the matrix C of the linear system of Eq. (4.18a).

Due to the large number of unknowns to be computed (for example, if $LM = 256 \times 512$), and the features of matrix C, that we just described, an iterative method

[12] Every point of the image is related only to its neighbours, within the reach of the blurring matrix.

is better suited to solve the system of Eq. (4.18a), and we choose the Gauss-Seidel method.[13]

Putting aside the term of the diagonal in Eq. (4.18a), we obtain the correction term of a Gauss-Seidel iteration,

$$\Delta I_{rs}^{p,q+1} = -\frac{1}{(\partial F_{rs}/\partial I_{rs})|_{I^{p,q}}} \left(F_{rs}|_{I^{p,q}} + \sum_{\substack{m=1\\ m\neq r}}^{L} \sum_{\substack{n=1\\ n\neq s}}^{M} \left.\frac{\partial F_{rs}}{\partial I_{mn}}\right|_{I^{p,q}} \Delta I_{mn}^{p,\tilde{q}} \right). \tag{4.23}$$

Here q is the iteration counter of the Gauss-Seidel method and $\tilde{q}$ can be q or $q+1$. This is so because, depending on the form the elements of $\Delta\mathbf{I}^{p,q}$ are stored in the vector of unknowns $\Delta\mathbf{I}$, for every unknown of the system, characterized by specific r, s, it will use the previous value of the unknowns (i.e., the value computed in the previous iteration, $\Delta\mathbf{I}^{p,q}$) in some (m,n) positions, or the present values $\Delta\mathbf{I}^{p,q+1}$, in other (m,n) positions, computed in the current iteration, $q+1$.

For this problem, we can set the initial estimate to zero, $\Delta\mathbf{I}^{p,0} = 0$.

4.3.6 Restoration Methodology

We summarize here the methodology adopted throughout this chapter:

1. *Original problem.* Determine vector $\mathbf{I}$ that solves the equation
   $B\mathbf{I} = \mathbf{Y}$;

[13] We recall here the Gauss-Seidel method, [35]. Consider the system

$$A\mathbf{x} = \mathbf{b} \ . \tag{4.22}$$

Let D be the diagonal matrix whose elements in the diagonal coincide with the diagonal entries of A. Let L and U denote, respectively, the lower and upper-triangular matrices, formed by the elements of A. Then,

$$A = L + D + U \ ,$$

and the system can be rewritten as

$$(L+D)\mathbf{x} = -U\mathbf{x} + \mathbf{b} \ .$$

Denoting by $\mathbf{x}^q$ the q-th iteration (approximation of the solution), the *Gauss-Seidel method* is: given $\mathbf{x}^0 = \mathbf{x}_0$, arbitrarily chosen, let

$$(L+D)\mathbf{x}^{q+1} = -U\mathbf{x}^q + \mathbf{b} \ ,$$

for $q = 0,1,2\ldots$, until convergence is reached. Using index notation, we have

$$x_i^{q+1} = a_{ii}^{-1}\left\{ b_i - \left(\sum_{j<i} a_{ij}x_j^{q+1} + \sum_{j>i} a_{ij}x_j^q \right) \right\} \ .$$

It is expected that $\lim_{q\to+\infty} \mathbf{x}^q = \mathbf{x}$, where $\mathbf{x}$ denotes the solution of Eq. (4.22). This can be guaranteed under special circumstances.

2. *Alternative formulation.* Minimize the function
   $R(\mathbf{I}) = \frac{1}{2}|\mathbf{Y} - B\mathbf{I}|^2$;

3. *Regularized problem.* Minimize the function
   $Q(\mathbf{I}) = R(\mathbf{I}) + \alpha S(\mathbf{I})$;

4. *Critical point.* Determine the critical point equation
   $\nabla Q(\mathbf{I}) = 0$;

5. *Critical point equation solution.* use modified Newton's method, to solve the non-linear critical point system of equations;

6. *Linear system solution.* use Gauss-Seidel method to solve the linear system of equations which appears in Newton's method
   $C(\mathbf{I}^{p+1} - \mathbf{I}^p) = -F(\mathbf{I}^p)$, where C is from Eq. (4.21).

In the following sections, we will present three examples of the application of this methodology to the restoration of a photograph, a text and a biological image.

4.4 Photo Restoration

In Fig. 4.3b, it is shown the blurring of the original 256 × 256 pixels image, presented in Fig. 4.3a, due to a blurring matrix consisting on a Gaussian weight, with a dependence domain of 3 × 3 points, that is, b_{kl} satisfies Eq. (4.3) and

$$b_{kl} \quad \propto \quad \exp\left(-\frac{r_k^2 + r_l^2}{2\sigma^2}\right) \ , \tag{4.24}$$

where σ is related[14] to the bandwidth and $r_k = |k|$.

The space of shades of gray, [0,1], is discretized and coded with 256 integer values, between 0 and 255, where 0 corresponds to black and 255 to white[15]. The histograms in Fig. 4.3, present the frequency distribution of occurrence of every (discrete) shade in the image. For example, if in 143 (horizontal axis) the frequency is 1003 (vertical axis), it means that there are 1003 pixels in the image with shade 143.

Figure 4.3c exhibits the photograph's restoration, done without regularization, stopped in the 200-th iteration of the Newton's method, and in Fig. 4.3d the regularized restoration ($\alpha = 0.06$).

The histogram of the image restored with regularization is, qualitatively, the one closer to the histogram of the original image. This corroborates the evident improvement in the image restored with regularization.

The behaviour of functionals Q, R and S defined in Eqs. (4.8)–(4.10) is recorded in Fig. 4.4. The minimization of functional Q is related to the maximization of the entropy functional, S.

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[14] The symbol $\propto$ means that the quantities are proportional, that is, if $a \propto b$, then there is a constant c such that $a = cb$.

[15] The discretization of the shade space is known as *quantization*.

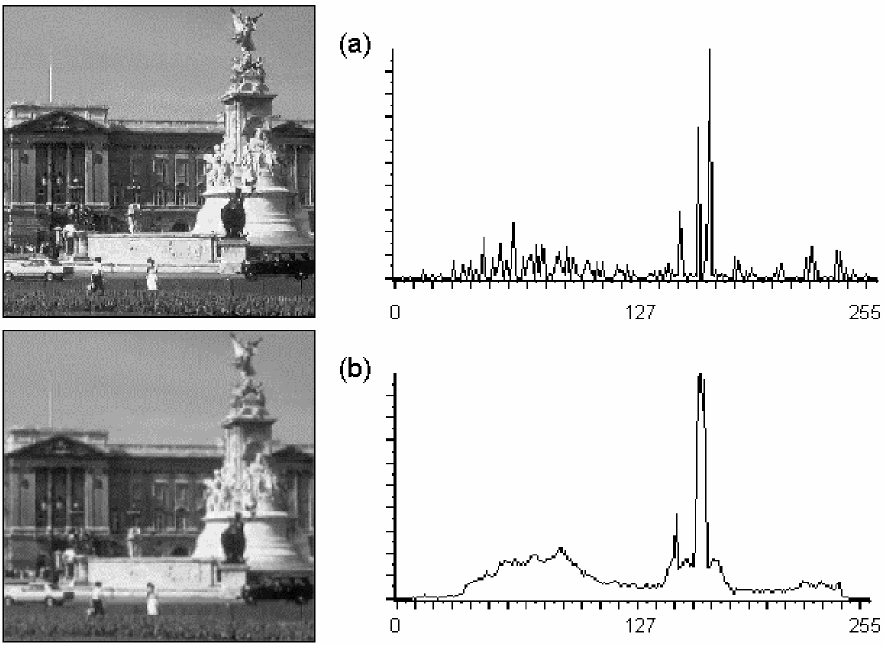

Fig. 4.3 Restoration of an artificially modified image, from a Gaussian with 3×3 points and $\sigma^2 = 1$. Images are on the left and their shade of gray histograms are on the right. (a) original image; (b) blurred image. (Author: G. A. G. Cidade from the Universidade Federal do Rio de Janeiro).

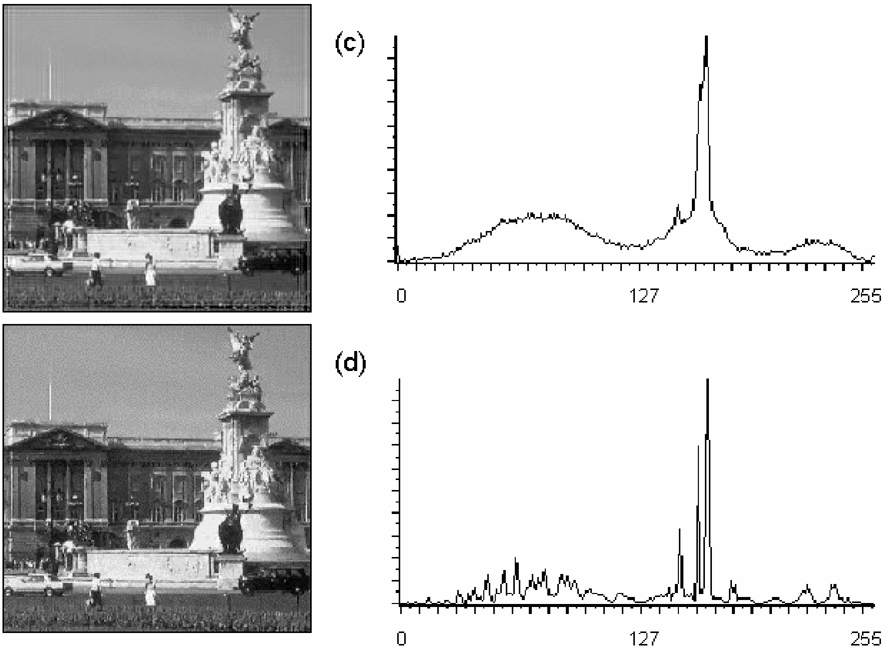

Fig. 4.3 (Cont.) Restoration of an artificially modified image, from a Gaussian with 3×3 points and $\sigma^2 = 1$. (c) restored image without regularization ($\alpha = 0,\ \gamma = 0.1$); (d) restored image with regularization ($\alpha = 0.06,\ \gamma = 0.1$). (Author: G. A. G. Cidade).

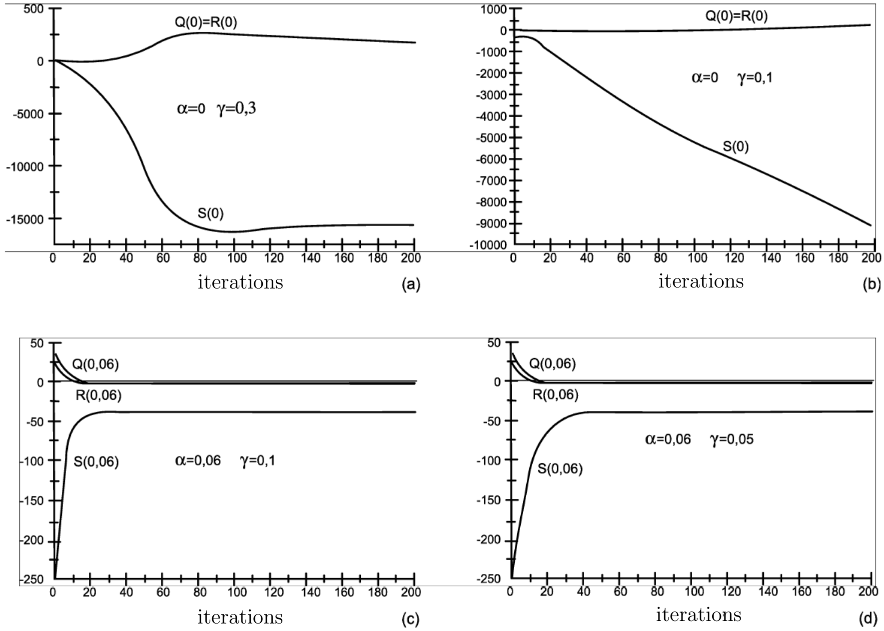

Fig. 4.4 Behaviour of the functionals Q, S and R during the iterative process. (a) $\alpha = 0$ and $\gamma = 0.3$; (b) $\alpha = 0$ and $\gamma = 0.1$; (c) $\alpha = 0.06$ and $\gamma = 0.1$; (d) $\alpha = 0.06$ and $\gamma = 0.05$. (Author: G. A. G. Cidade).

When the regularization parameter is not present, $\alpha = 0$, it is observed that the proposed algorithm diverges, even when one uses a gain factor in the corrections of the intensity in Newton's iterative procedure, Eq. (4.19), as can be seen in Figs. 4.4a,b.

Figures 4.4c,d show the convergence of the algorithm, that naturally is achieved faster for the largest gain factor, $\gamma = 0.1$. In this case, approximately 20 iterations are needed. On the other hand, 40 iterations will be necessary if $\gamma = 0.05$. Functional R, shown in these figures, corresponds to half the square of the norm of the *residuals*, defined as the difference between original blurred image, $\mathbf{Y}$, and restored blurred image, $B\mathbf{I}$, given by

$$R(\mathbf{I}) = \frac{1}{2}|\mathbf{Y} - B\mathbf{I}|^2 .$$

4.5 Text Restoration

Figures 4.5a and b present an original text (256×256 pixels) and the result of its blurring by means of a Gaussian blurring matrix with 5×5 points and $\sigma^2 = 10$.

Text restoration is shown in Fig. 4.5c. There are *border effects*, that is, structures along the border that are not present neither in the original text nor in the blurred image. These occur due to inadequate treatment of pixels near the border (boundary) of the image. This effect can be minimized by considering reflexive conditions at the borders, or simply by considering null the intensity of elements outside the image. See Exercises 4.2, 4.4.

A simple text has, essentially, but two shades: black and white. This is reflected by the histograms of the original and restored texts (Fig. 4.5). However, the blurred text exhibits gray pixels, as shown by its histogram Fig. 4.5b. Notice that the original and restored texts can be easily read, unlike the blurred text.

4.6 Biological Image Restoration

In this section we consider a biological image restoration, which consists of an example of an inverse problem involving a combination of identification and reconstruction problems (problems P_2 and P_3, Section 2.8).

Results of applying the methodology described in this chapter to a real biological image of 600 nm × 600 nm are presented in Fig. 4.6. The image represents an erythroblast being formed, under a leukemic pathology. This image has been acquired by means of an atomic force microscope at the Institute of Biophysics Carlos Chagas Filho, of the Universidade Federal do Rio de Janeiro [25].

In the inverse problem presented here, the original image, $\mathbf{I}$, is being restored together with the choice of the blurring operator given by matrix B. This is an ill-posed problem to determine $\mathbf{I}$, since neither the blurring matrix B is known, in contrast with the problems treated in the previous sections, nor the original image is known.

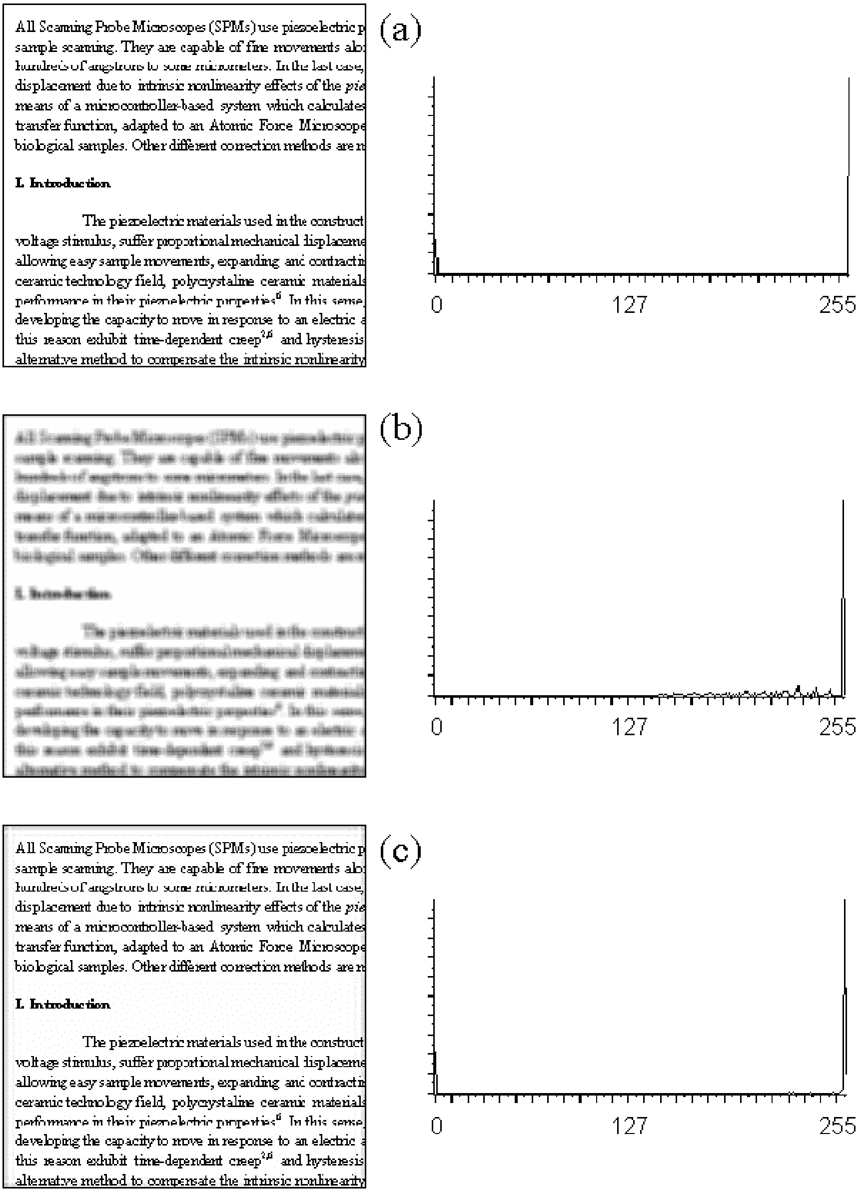

Fig. 4.5 Restoration of a text blurred by means of a Gaussian. (a) original text; (b) blurred text; (c) restored text with regularization parameter $\alpha = 0.2$, and gain factor $\gamma = 0.1$. (Author: G. A. G. Cidade).

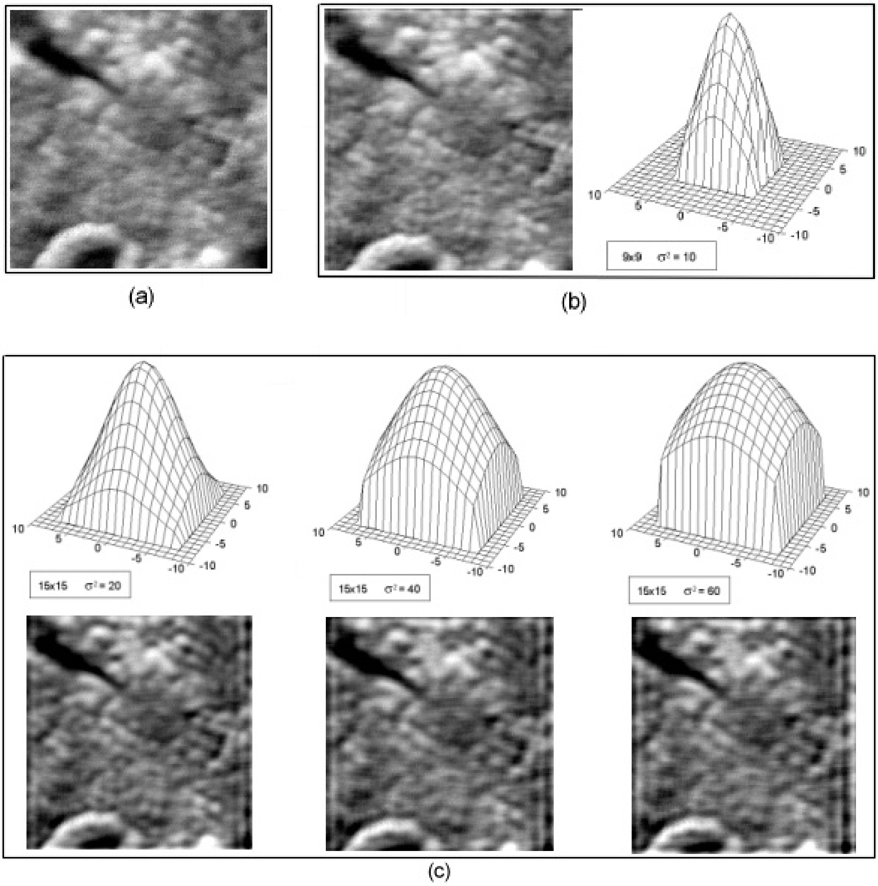

Fig. 4.6 Restorarion of an image obtained by means of an atomic force microscope, together with the blurring matrix identification . (a) original image; (b) restored image with $\alpha = 0.03$, $\gamma = 0.2$, Gaussian with 9×9 points and $\sigma^2 = 10$; (c) restored image with $\alpha = 0.03$, $\gamma = 0.2$, and Gaussians with 15×15 points and $\sigma^2 = 20, 40$ and 60. (Author: G. A. G. Cidade. Images acquired with an Atomic Force Microscope at the Instituto de Biofísica Carlos Chagas Filho of the Universidade Federal do Rio de Janeiro.)

If we knew the original image, or if we knew several images and their blurred counterparts, we could adapt the methodology employed to solve the model problem considered in Chapter 1 to *identify* the blurring matrix.

In the absence of a more quantitative criterion, the blurring matrix identification is performed qualitatively, taking into consideration the perception of medical *specialists*, in relation to the best informative content, due to different restorations, obtained from assumed blurring matrices.

To find the blurring matrix in a less conjectural way, one can use *Gaussian topographies* (see Fig. 4.6) to represent, as much as possible, the geometric aspect of the tip of the microscope, which is the main cause for the degradation that the image undergoes. In other words, we assume that the class of blurring operators is known, i.e., *we characterize the model*, being the specific model identified simultaneously with the restoration of the image.

We used blurring matrices of 9×9, 15×15 and 21×21 points, with different values for σ^2: 10, 20, 40 and 60.

From Fig. 4.6, it can be concluded that we gain more information with Gaussian topographies of 15×15 than with those of 9×9. For the tests with 21×21 points the results were not substantially better.

Figure 4.6c, resulting from the procedure, is considerably better than the blurred image. As in the previous section, the border effects here present are also due to inadequately considering the outside neighbour elements of the border of the image.

Exercises

4.1. Show that Eq. (4.4) is valid.

4.2. Assume that pixel (7,3) is located at the upper right corner of the image. Following the deduction of Eq. (4.7), and assuming symmetry, show that we should take

$$Y_{73} = \frac{1}{16}\left(4I_{62} + 4I_{72} + 4I_{63} + 4I_{73}\right) = \frac{1}{4}\left(I_{62} + I_{72} + I_{63} + I_{73}\right) .$$

4.3. Consider a 3×3 blurring weight matrix[16]

$$b = \begin{pmatrix} b_{-1\,-1} & b_{-1\,0} & b_{-1\,1} \\ b_{0\,-1} & b_{0\,0} & b_{0\,1} \\ b_{1\,-1} & b_{1\,0} & b_{1\,1} \end{pmatrix} .$$

Let **I** be an image and **Y** its blurred counterpart. Assume we use symmetric conditions on the boundary. Work out explicit formulae for the blurred image **Y** tone of grays, Y_{ij}, if

(a) pixel (i,j) is in the interior of the image;

[16] Working with indices is sometimes very cumbersome. Several of the following exercises proposes practicing a little 'indices mechanics'...

(b) pixel (i,j) is in the image's lower boundary;

(c) pixel (i,j) is in the image's lower left corner.

4.4. Do a similar problem as Exercise (4.3), however, instead of using a symmetric condition at boundaries and corners, assume that outside the image, pixels have a uniform value, denoted by I_{ext}.

4.5. Given two pixels (i,j) and (i',j'), we define their *distance* by

$$d((i,j),(i',j')) = \max\{|i - i'|, |j - j'|\},$$

where max of a finite set of real numbers denotes the largest one in the set.

(a) Give explicitly the pixels that comprise the *circle* centered at (i,j) and radius 1,

$$C_{(i,j)}(1) = \{(i',j') \mid d((i,j),(i',j')) = 1\} .$$

(b) Determine the *disk* $B_{(i,j)}(2)$, centered at (i,j) and radius 2,

$$B_{(i,j)}(2) = \{(i',j') \mid d((i,j),(i',j')) \leq 2\} .$$

(c) Sketch the sets $C_{(i,j)}(1)$ and $B_{(i,j)}(2)$.

(d) Give, in the same way, the circle with center (i,j) and radius N.

4.6. (a) Let

$$\mathcal{P} = \{(i',j'),\ i',j' = 1,\ldots,M\} = \{i,\ i = 1,\ldots,M\}^2 ,$$

be the set of pixels of a square image. Given a set of pixels $\mathcal{S} \subset \mathcal{P}$, define the distance of pixel (i,j) to set $\mathcal{S}$ by

$$d((i,j),\mathcal{S}) = \min\{d((i,j),(i',j')),\ (i',j') \in \mathcal{S}\} ,$$

where min of a finite set of real numbers denotes the smallest one in the set. The *right boundary* of the image is

$$R = \{(i',j') \in \mathcal{P},\ \mid i' = M\} = \{(M,j'),\ j' = 1,\ldots,M\} .$$

Determine $d((i,j),R)$.

(b) Likewise, define respectively, L, the left, U, the top, and D, the bottom image boundaries, and compute $d((i,j),L)$, $d((i,j),U)$, and $d((i,j),D)$.

(c) The boundary of the image is defined by $\mathcal{B} = L \cup T \cup R \cup D$. Compute $d((i,j),\mathcal{B})$.
Hint. Use functions max and/or min to express your answer.

4.7. Assume that B is a blurring operator with form given by Eq. (4.2).

(a) Let (i,j) be a fixed *interior* pixel (not on the boundary of the image), and at a distance at least N from the boundary of the image. In particular, $B_{(i,j)}(N) \subset \mathcal{P}$. Show that

$$Y_{ij} = \sum_{k=-N}^{N} \sum_{l=-N}^{N} b_{kl} I_{i+k,j+l} . \tag{4.25}$$

Hint. Change, for instance, the index of summation i', by k, with $i' = i + k$, in Eq. (4.1), that is 'center' the summation around i.

(b) The structure of the blurring operator cannot be taken all the way to the boundary of the image. This is why in (a), the pixel (i,j) is restricted to the pixels of distance N or more from the boundary. Verify this. If (i,j) has distance less than N, we cannot compute Y_{ij} using Eq. (4.25).

4.8. Given a blurring operator B, as in Eq. (4.1), define the *domain of dependence* of pixel (i,j) as the set of pixels of the image $\mathbf{I}$ that contribute to the value of the pixel in the blurred image, Y_{ij}, that is,

$$\mathcal{D}_{(i,j)} = \left\{(i',j') \text{ such that } B_{ij}^{i'j'} \neq 0\right\} .$$

Assume that the blurring operator B has the structure specified in Eq. (4.2). Determine for which pixels $(i,j) \in \mathcal{P}$, one has

$$\mathcal{D}_{(i,j)} = B_{(i,j)}(N) .$$

4.9. The blurring operator B has the structure presented in Eq. (4.2). Let β be the $(2N + 1) \times (2N + 1)$ matrix given by

$$\beta = \begin{pmatrix} b_{-N,-N} & \cdots & \cdots & \cdots & b_{-N,N} \\ \vdots & \ddots & \vdots & & \vdots \\ b_{0,-N} & & b_{0,0} & & b_{0,N} \\ \vdots & & \vdots & \ddots & \vdots \\ b_{N,-N} & \cdots & \cdots & \cdots & b_{N,N} \end{pmatrix} .$$

(a) Give an expression for the entries of β, $\beta_{i'j'}$, in terms of b_{kl}. That is, determine $k(i')$ and $l(j')$ such that

$$\beta_{i'j'} = b_{k(i')}b_{l(j')}, \text{ for } i',j' = 1,\ldots,2N + 1 .$$

(b) For pixel (i,j), define the $(2N + 1) \times (2N + 1)$ matrix $\mathcal{I} = \mathcal{I}^{(i,j)}$, given by

$$\mathcal{I} = \mathcal{I}^{(i,j)} = \begin{pmatrix} I_{i-N,j-N} & \cdots & I_{i-N,j} & \cdots & I_{i-N,j+N} \\ \vdots & \ddots & \vdots & & \vdots \\ I_{i,j-N} & & I_{i,j} & & I_{i,j+N} \\ \vdots & & \vdots & \ddots & \vdots \\ I_{i+N,j-N} & \cdots & I_{i+N,j} & \cdots & I_{i+N,j+N} \end{pmatrix} . \tag{4.26}$$

Give an expression for the entries of $\mathcal{I} = \mathcal{I}^{(i,j)}$, $\mathcal{I}^{(i,j)}_{i',j'}$, in terms of the entries I_{kl} of the image $\mathbf{I}$, similar to what was done in (a).

4.10. (a) Given $m \times n$ matrices, A and B, define the following *pointwise matrix product*, giving rise to a $m \times n$ matrix, $C = A : B$ where $C_{ij} = A_{ij} \cdot B_{ij}$. Compute $\beta : \mathbf{I}$, where β is defined in Exercise 4.9, and $\mathbf{I}$ is an image.

(b) Compute the pointwise matrix product, $\tilde{\beta} : \mathcal{I}^{(7,3)}$, between matrix given by Eq. (4.5), denoted here by $\tilde{\beta}$, and $\mathcal{I}^{(7,3)}$, defined in Eq. (4.26), when $N = 1$, and $(i,j) = (7,3)$.

(c) Let $S(A)$ be the sum of all elements of matrix A,

$$S(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} .$$

Compute $S(\tilde{\beta} : \mathcal{I}^{(7,3)})$ and compare with Eq. (4.6).

(d) Show that Eq. (4.25) can be written as

$$Y_{ij} = S\left(\tilde{\beta} : \mathcal{I}^{(i,j)}\right) .$$

4.11. The entropy regularization, Eq. (4.10b), makes use of the function $s(x) = x_0 - x + x \ln \frac{x}{x_0}$, with $x_0 = \bar{I}_{ij}$ and $x = I_{ij}$. Take, for concreteness, $x_0 = 1/2$.

(a) Compute and sketch s'.

(b) Compute and sketch s''. Show that $s''(x) > 0$ for all x. Conclude that s is strictly convex[17].

(c) Sketch s.

4.12. Let $\mathbf{f} = (f_p)$ be a vector in $\mathbb{R}^n$ with entries f_p, and m a constant in $\mathbb{R}$. Consider Csiszár measure, [42],

$$\Theta_q = \frac{1}{1+q} \sum_p f_p \frac{f_p^q - m^q}{q} , \tag{4.27}$$

and Bregman divergence, [14],

$$\mathcal{B}_{\Theta_q}\left(f, \overline{f}\right) = \Theta_q(f) - \Theta_q(\overline{f}) - \langle \nabla\Theta_q(\overline{f}), f - \overline{f} \rangle .$$

(a) Show that $\theta_q(x) = x\frac{x^q - m^q}{q}$, is strictly convex.

(b) Show that the family of Bregman divergence, parametrized by q, is

$$\mathcal{B}_{\Theta_q} = \frac{1}{q+1} \sum_p \left[f_p \frac{f_p^q - \overline{f}_p^q}{q} - \overline{f}_p^q (f_p - \overline{f}_p) \right] .$$

[17] The concept of convexity of functions is recalled in Exercise A.34, page 216.

(c) Derive the following family of regularization terms, parametrized by q

$$\begin{aligned} S(\mathbf{I}) &= \mathcal{B}_{\Theta_q}(\mathbf{I},\bar{\mathbf{I}}) \\ &= \sum_{i=1}^{L}\sum_{j=1}^{M}\left[I_{ij}^q \frac{I_{ij}^q - \bar{I}_{ij}^q}{q} - \bar{I}_{ij}^q (I_{ij} - \bar{I}_{ij}) \right] . \end{aligned} \tag{4.28}$$

Here, each value of q yields different regularization terms such as the ones defined in Eq. (4.10). These regularization terms may be used in the last term of Eq. (4.9), [27].

Hint. Define $f_p = I_{ij}$ as the estimated value of the shade of gray for the image at the pixel (i,j), and $\bar{I}_{ij}$ its corresponding reference value. Observe that the term m^q in Eq. (4.27) cancels out in the derivation steps of Eq. (4.28).

(b) Setting $q = 1$, derive Eq. (4.10a), from the family of regularization terms given by Eq. (4.28).

(c) Considering the limit $q \to 0$, in Eq. (4.28), check that

$$\lim_{q\to 0} \mathcal{B}_{\Theta_q}(\mathbf{I},\bar{\mathbf{I}}) = S(\mathbf{I}) ,$$

where $S(\mathbf{I})$ is given by Eq. (4.10b).

Hint. Recall that $\lim_{q\to 0}(x^q - 1)/q = \ln x$.

4.13. Use the same notation as in Exercise 4.12.

(a) Show that $\theta_q(x) = \frac{x^q - m^q}{q}$, when $q > 1$, and $\theta_q(x) = -\frac{x^q - m^q}{q}$, when $0 < q < 1$, are strictly convex functions.

(b) Show that the family of Bregman divergence, parametrized by q, is

$$\mathcal{B}_{\Theta_q} = \sum_p \left[\frac{f_p^q - \bar{f}_p^q}{q} - \bar{f}_p^{q-1}(f_p - \bar{f}_p) \right] ,$$

when $q > 1$, and determine the corresponding expression when $0 < q < 1$.

(c) Derive the following family of regularization terms, parametrized by q

$$\begin{aligned} S(\mathbf{I}) &= \mathcal{B}_{\Theta_q}(\mathbf{I},\bar{\mathbf{I}}) \\ &= \sum_{i=1}^{L}\sum_{j=1}^{M}\left[\frac{I_{ij}^q - \bar{I}_{ij}^q}{q} - \bar{I}_{ij}^{q-1} (I_{ij} - \bar{I}_{ij}) \right\} . \end{aligned} \tag{4.29}$$

when $q > 1$. Derive also the expression when $0 < q < 1$.

(b) Setting $q = 2$, derive Eq. (4.10a), from the family of regularization terms given by Eq. (4.29).

(c) Considering the limit $q \to 0$, in Eq. (4.29), check the relation between

$$\lim_{q\to 0} \mathcal{B}_{\Theta_q}(\mathbf{I},\bar{\mathbf{I}}) \, ,$$

and $S(\mathbf{I})$ given by Eq. (4.10b).

4.14. (a) Derive Eq. (4.14), from Eq. (4.11).

(b) Derive Eq. (4.21), from Eq. (4.14).

4.15. (a) Show that for a general value $q > 0$, Eq. (4.14) is written as

$$F_{rs} = -\sum_{i=1}^{M}\sum_{j=1}^{M}\left(Y_{ij} - \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl} I_{i+k,j+l}\right) b_{r-i,s-j} + \frac{\alpha}{q}\left(I_{rs}^{q} - \bar{I}_{rs}^{q}\right) \, . \tag{4.30}$$

(b) Considering the limit $q \to 0$, derive Eq. (4.14) from Eq. (4.30).

4.16. Show that for a general value $q > 0$ Eq. (4.21) is written as

$$C_{mn}^{rs} = \frac{\partial F_{rs}}{\partial I_{mn}} = \sum_{k=-N}^{N}\sum_{l=-N}^{N} b_{kl} b_{r-m+k,s-n+l} + \alpha I_{rs}^{q-1} \delta_{rm} \delta_{sn} \, .$$

4.17. Equation (4.24) states a proportionality that b_{kl} has to satisfy.

(a) Let c denote the constant of proportionality for a 3×3 blurring matrix. Determine it.

(b) Show that b_{kl} in Eq. (4.24) admits a separation of variables structure.

(c) Let c denote the constant of proportionality for a $N \times N$ blurring matrix. Obtain an expression for it.