Modeling and Simulation of Forest Fire Spreading

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1 Mathematical Modeling

We consider a similar model of forest fire spreading as Asensio and Ferragut [1].

$$\frac{\partial T}{\partial t} = -\mathbf{v}\nabla T + D\Delta T + A\left(Y \exp\left(-\frac{B}{T - T_\infty}\right) - h(T - T_\infty)\right), \tag{1}$$

$$\frac{\partial Y}{\partial t} = -bY \exp\left(-\frac{B}{T - T_\infty}\right), \tag{2}$$

with temperature of fuel T, time t, wind velocity $\mathbf{v}$, diffusion coefficient D, pre-exponential factor of reaction A, mass fraction of fuel Y, coefficient due to modified Arrhenius law B, natural convection coefficient h, disappearance rate of fuel b, and ambient temperature T_∞.

2 Numerical Solution

2.1 Space and Time Discretization

We use for the space discretization a collocation method based on the sums:

$$u = \sum_{i=1}^{I} \phi(x, z_i)u_i, \quad y = \sum_{i=1}^{I} \phi(x, z_i)y_i, \quad x \in X,\ z \in Z,\ I \text{ number of points}, \tag{3}$$

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where ϕ is the trial function, X a grid representing the collocation points and Z a grid consisting of the centers of the trial functions (for more details the reader is referred to Eberle et al. [2]). Then, we apply the ansatz (3) for the temperature T and mass fraction Y and plug it in Eqs. (1) and (2):

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{I} (-\mathbf{v}\,\nabla\phi_i + D\Delta\phi_i - Ah\phi_i)\,u_i + AhT_\infty + A\sum_{i=1}^{I}\phi_i\,y_i\,\exp\left(-\frac{B}{T-T_\infty}\right), \tag{4}$$

$$\frac{\partial y}{\partial t} = -b\sum_{i=1}^{I}\phi_i\,y_i\,\exp\left(-\frac{B}{T-T_\infty}\right). \tag{5}$$

The time discretization is done by a Crank-Nicolson-scheme.

2.2 Stabilization

The above introduced solution scheme yields strongly oscillating results in the convection dominated case (Gibbs phenomenon). Thus, the method needs to be stabilized. Here, we follow the procedure of flux corrected transport of Kuzmin, Löhner, Turek [3]. In doing so, we apply the stabilization exemplary for the temperature T.
Step (1) We start with the approximation of the initial conditions and determine the according coefficients u_0 by solving the system

$$Mu_0 = T_0, \tag{6}$$

where $M = m_{ij}$ is the mass matrix given by $m_{ij} = \phi(x_i, z_j)$.

The coefficients are needed for the space discretization within the time-stepping scheme.
Step (2) Next, we consider the so-called "low-order" problem and define the lumped mass matrix M_L by

$$m_{ii} = \sum_j m_{ij} \quad \text{for} \quad i = j. \tag{7}$$

Step (3) After that we have a look at the "high-order" problem, which means we construct the operator K^H given by

$$k_{ij}^H(\phi) = -\mathbf{v}\nabla\phi(x_i, z_j) + D\Delta\phi(x_i, z_j), \tag{8}$$

which describes the convection and diffusion.
Step (4) Artificial diffusion is added now and we define the diffusion operator in the same way as by Möller [4]

$$d_{ii} = -\sum_{j \neq i} d_{ij}, \quad d_{ij} = d_{ji} = \max\{0, -k_{ij}^H, -k_{ji}^H\} \quad \text{for} \quad i < j \tag{9}$$

and the low-order operator $K^L = K^H + D$.
Step (5) The right-hand side of our convection-diffusion-reaction-problem (1) is represented by the reaction term q and we call its coefficients q_{n-1}.

$$q = A\left(Y \exp\left(-\frac{B}{T - T_\infty}\right) - h(T - T_\infty)\right). \tag{10}$$

Step (6) Following the procedure in [4] we make an approximation of the coefficients of the collocation method by

$$\overline{u} = u_{n-1} - \frac{\Delta t_n}{2} M_L^{-1}(K^L u_{n-1} - q_{n-1}). \tag{11}$$

Step (7) Next, we modify the right-hand side of problem (1) by applying Zalesak's algorithm [5] for which we need to calculate the residuum r and the weights α to get q_{n-1}^*. The algorithm considers only the next neighbors i of every collocation point

$$P_j^+ = \sum_{i \neq j} \max_{i=1,\dots,N}\{0, r_{ij}\},\ P_j^- = \sum_{i \neq j} \min_{i=1,\dots,N}\{0, r_{ij}\}, \tag{12}$$

$$Q_j^+ = \max\{0, \max_{i=1,\dots,N}(\overline{u}_j - \overline{u}_i)\},\ Q_j^- = \min\{0, \min_{i=1,\dots,N}(\overline{u}_j - \overline{u}_i)\}, \tag{13}$$

$$R_j^+ = \min_{i=1,\dots,N}\left\{1, \frac{m_i Q_i^+}{P_i^+}\right\},\ R_j^- = \min_{i=1,\dots,N}\left\{1, \frac{m_i Q_i^-}{P_i^-}\right\}, \tag{14}$$

$$\alpha_{ij} = \min_{i=1,\dots,N}\left\{1, \frac{R_i^+}{R_j^-}\right\} \text{ for } r_{ij} > 0 \text{ and } \alpha_{ij} = \min_{i=1,\dots,N}\left\{1, \frac{R_i^-}{R_j^+}\right\} \text{ else.} \tag{15}$$

Step (8) Now we are able to determine the coefficients

$$u_n = u_{n-1} - \frac{\Delta t_n}{2} M_L^{-1}(K^L u_{n-1} - q_{n-1} - q_{n-1}^*) \tag{16}$$

Step (9) Finally, we use these coefficients to get solutions for the temperature T and the mass fraction of the fuel Y with the stabilized method.

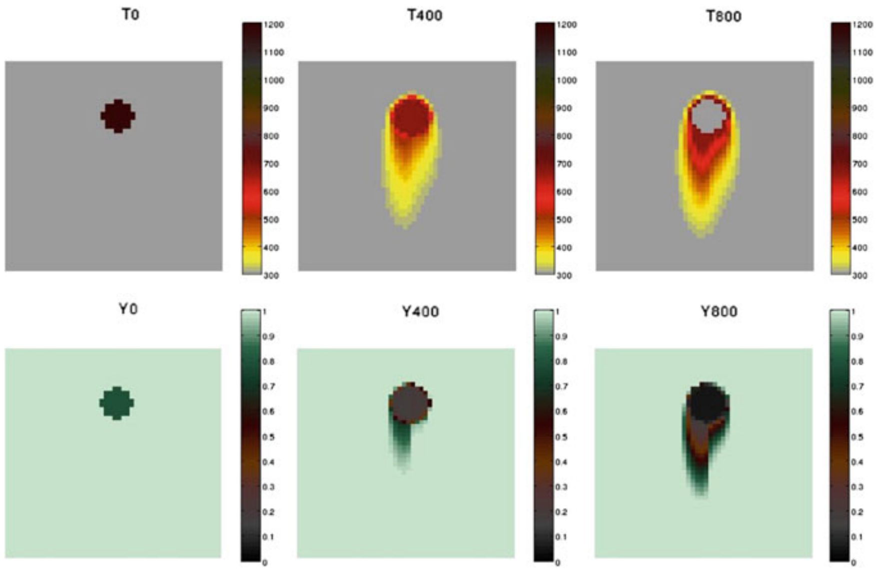

Fig. 1 In the first row the temperature of fuel T in $[K]$ is plotted for the initial temperature, after 400 and 800 time steps and accordingly in the second row we see the mass fraction of fuel Y

3 Numerical Simulation

Figure 1 shows first simulations for two different fuel types (type 1 on the left-hand side and type 2 on the right-hand side) and wind directed to the south. We can see the fire spreads faster for the fuel type 1 and due to the wind its shape is elliptic.

References

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