Building on the Legacy of Georges Lemaître in Contemporary Cosmology

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Abstract For many years now theoretical cosmologists have been building on the rich legacy of Georges Lemaître's ideas and accomplishments in mathematically modelling the structure and dynamics of the universe. His recognition and careful demonstration of the viability and attractiveness of expanding models of the universe, along with his seminal idea of the "primeval atom," which would later be referred to as the Big Bang, are very well known. This has led directly, through other important contributions by Friedmann, Robertson and Walker, to the standard perfectly isotropic and spatially homogeneous (smooth) Friedmann-Lemaître-Robertson-Walker (FLRW) models of our universe. These are certainly a key part of Lemaître's legacy. After reviewing those models, we discuss another very important, but less well-known, contribution to fundamental cosmology, his general spherically symmetric cosmological solutions with pressure and a cosmological constant (vacuum energy, which is a leading candidate for dark energy). These models are generalizations of the FLRW models and, in general, are inhomogeneous. In recent years they have stimulated a great deal of research in the quest for further confirming the standard model, understanding the limitations of its perturbed versions, or possibly replacing it with something more adequate. Their importance for advancing our understanding and modelling of the universe, and the ways in which they are presently being studied and deployed, is described and discussed.

Lemaître's Legacy to Contemporary Cosmology

From the other contributions in this volume, we already know and appreciate very well Georges Lemaître's outstanding contributions to the foundations of contemporary physical cosmology. This recognition and appreciation has grown

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remarkably over the past 25 years. The standard perfectly isotropic and spatially homogeneous models of the universe have been often referred to as Friedmann-Robertson-Walker (FRW) models, after three of the key people who contributed to their development. Now, more commonly—especially among theoretical and mathematical cosmologists—they are called Friedmann-Lemaître-Robertson-Walker (FLRW) models, in recognition of the key role Lemaître played in their conceptualization and formulation.

Certainly, his key insight that the universe is expanding and cooling provided the essential foundations for our contemporary understanding and modelling of the universe. One very important and closely connected contribution is Lemaître's argument that it originated and expanded from an initial extremely dense quantum of mass-energy, a "primeval atom" or "cosmic egg," which predated the emergence of space and time. This, of course, is Lemaître's early version of the Big Bang (Lemaître 1931a, 1950). What strongly recommended his suggestions to Eddington, Einstein and others was his detailed demonstration that the Einstein field equations easily accommodate such expanding cosmological solutions. (Lemaître 1931b) This has led directly to the standard FLRW models, which have been so successful, and have provided the basic theoretical description of the universe. A particular FLRW model, one which is very close to spatially flat with the mass-energy density consisting of about 27 % matter (including nearly 23 % dark matter) and 73 % dark energy (with a very small amount of residual radiation energy density), has fit with great precision all the cosmologically relevant data we have so far been able to obtain—including those from the cosmic microwave background (CBR) probes, primordial-element-abundance (helium, deuterium, tritium, lithium) determinations, and redshift-distance and mass-density results. (Spergel 2003) In the next section, we shall briefly review these FLRW models and the observational results that they fit so well.

The influence of Lemaître's contributions does not stop here, however. He also solved the Einstein field equations (EFE) for much more complicated cosmologies, inhomogeneous generalizations of the FLRW models. In particular, in a remarkable paper (Lemaître 1933) he gives a detailed solution to the EFE for a spherically symmetric expanding inhomogeneous universe with both matter and vacuum energy (a cosmological constant Λ) and with pressure. This paper has been considered so important that it was reprinted in 1997 in a more accessible journal. It contains a number of cutting-edge results that anticipated much later findings.[1] The pressure-free models of this class of cosmologies are now referred to as Lemaître-Tolman-Bondi (LTB) models, and have been the subject a great deal of recent research. The full class of these general spherically symmetric models (including

[1] Among these are: the first recognition and arguments that the Schwarzschild singularity at $r = 2\,M$ (the location of the event horizon of a Schwarzschild black hole) is not an essential singularity, hut rather a coordinate singularity; a clear definition of mass for perfect fluids in general relativity; the first instance of a proven singularity theorem in cosmology, anticipating the more detailed later formulations of Penrose and Hawking; and a discussion of the gravitational processes by which "nebulae" might be formed. (Krasiński 1997a).

those with perfect-fluid matter and radiation with pressure and pressure gradients) are often referred to as Lemaître models. In section "Beyond the Standard FLRW Model: Lemaître and LTB Models and Their Importance" we shall briefly discuss LTB models and their critical importance for making progress in cosmology. Essentially, they provide us with the simplest inhomogeneous cosmologies. Carefully studying them and their relationships with observations will enable us to either confirm the present standard model or to modify it.

In section "Recent Research on LTB and Lemaître Cosmological Models", we shall elaborate further on LTB models, both those with and those without a cosmological constant, briefly discuss some of the outstanding recent work on them, mention their generalizations (i.e., those with nonzero pressure) and some of the applications of those generalizations, and briefly discuss perturbations of LTB models. We shall also briefly describe the types of observations that will be important for testing them. Again, it is important to emphasize that, since FLRW models are a special class within the more general class of LTB models, this work may result in confirming the standard model. One of the central aims of this effort is to determine whether or not there is significant dark energy (or a nonzero). As we have already seen, present observations require dark energy for the standard concordance FLRW model, but they do not require it for some LTB models which also fit all the present data extremely well. (Clarkson and Regis 2011; Regis and Clarkson 2012) However, there are further independent data (Hellaby 2006; Araújo and Stoeger 2009a, b; Araújo and Stoeger 2010) which may eventually be able to tell us whether the best-fit LTB model needs a cosmological constant (dark energy) or not. Favorable consideration of Λ is also part of Lemaître's legacy. In a paper in P. A. Schilpp's well-known volume, *Albert Einstein: Philosopher-Scientist,* Lemaître explains its significance, and argues compellingly for its natural suitability, both from mathematical and physical points of view. (Lemaître 1949) In section "Conclusions", we offer our concluding remarks.

The Standard FLRW Models and the Observations Supporting Them

The standard FLRW models, which have been the theoretical foundation of contemporary cosmology, are special cases of the LTB models.[2] They are as LTB models, therefore, spherically symmetric, or isotropic. But they fall into the very special class of perfectly smooth spatially homogeneous LTB models. Their mass-energy density at any given time is constant, i. e., independent of spatial position.

[2] There are some FLRW models, those for instance which describe the early radiation-dominated era of the universe, which are not—strictly speaking—special cases of LTB models, because they have non-zero pressure. Instead they are special cases of the more general Lemaître models, which allow for nonzero pressure.

Because they are spatially homogeneous, they are therefore isotropic about every point in the universe. (Kolb and Turner 1990) An FLRW universe will look exactly the same to every observer in it, no matter where he or she is located and no matter in which direction he or she looks.

Everyone knows that the actual universe is not this way. It is lumpy on all small and intermediate scales - there are stars and planets, clusters of stars, galaxies, and clusters of galaxies. FLRW obviously does not describe the universe on these scales! However, there is a great deal of evidence which suggests that on the very largest scales (averaging over large volumes greater than, say, 800 million light years in radius) the universe is almost smooth, almost-FLRW. And some of these almost- or perturbed FLRW models, like the particular one we mentioned above, fit these large- scale characteristics of the universe extremely well, and fit them on all scales at very early times (before significant structure formed). Certainly, the near smoothness of the CBR seems to be the strongest indicator of this. (Clarkson and Maartens 2010) And so this almost-FLRW universe ("almost," because it is perturbed, to account for the small deviations, on average, from constant mass-energy density on large scales due to galaxies and clusters of galaxies) has maintained its status as the standard model. How can we describe FLRW models in a little more detail? An FLRW universe is really a 3-dimensional perfectly smooth sphere of mass-energy expanding and cooling with the passage of time (it can also be contracting and heating up! See below.). The sphere of mass-energy is *not* expanding within a static, larger space. Rather 3-D space itself is expanding and dragging the mass-energy with it. The geometry of this space is very simple: it is described by a single function of time, $R(t)$, which is often referred to as the scale factor. As time elapses $R(t)$ increases or decreases - space expands or contracts. Since space contains mass-energy usually modelled as a perfect fluid, which responds adiabatically to the expansion or the contraction, the mass-energy in the universe cools as it expands, and heats up as it contracts.

What determines how the FLRW universe expands or contracts? The simple answer is gravity, as described by Einstein's general relativistic field equations (EFE), along with the initial conditions for the rate of expansion or contraction at one given time, the density of mass-energy at that time, and the equation of state of that mass-energy (the dependence of the pressure on the density). The mass-energy generates the gravitational field which dictates how rapidly the expansion decelerates, or accelerates, and that in turn effects a change in density, pressure and temperature. Thus, an FLRW model is uniquely determined by just two parameters, which must be determined by observations: the expansion rate at a given time, that is the Hubble parameter H, which is essentially the time-derivative of R(t), at some $t = t_0$, and the density of mass-energy at that time, together with an equation of state. Usually in late universe cosmology the equation of state is set to $p = 0$ (dust). In a much earlier radiation-dominated epoch it is just $p = (1/3)\rho$, where ρ is the mass-energy density—in this case the radiation- energy density. It turns out, as is intuitively clear, that the information given by the mass-energy density at t_0 is equivalent to that given by the deceleration parameter q_0, which is essentially the second time-derivative of *R(t)*. If the cosmological constant Λ is not zero, or there is some other form of dark clergy, then it and its effective equation of state are additional independent parameters which must be determined by observational data.

We cannot determine these cosmic parameters directly. But we can obtain the redshifts of distant galaxies, their luminosity distances, galaxy number counts, and—with some difficulty—the amount of luminous and dark matter in these galaxies and clusters of galaxies. We can also study the cosmic microwave background radiation, and its small anisotropies on different angular scales. These measurements within certain errors give us the rate of expansion now (Hubble parameter), some indication of the mass-density of the universe now, and some indirect measure of the value of the dark energy (for simplicity represented here by Λ), presuming that the universe is almost-FLRW. We can also add the data on primordial abundances of the lightest elements (hydrogen, helium, lithium) to this, to strengthen and improve the fitting, particularly with regard to the density of baryons in the universe. Although there are pieces which don't quite fit, and we don't really understand the resulting magnitude of Λ, which according to these indirect determinations amounts to 73 % of the present total mass-energy density of the universe, the overall fit of the standard FLRW model to a variety of different categories of precise cosmological data is truly remarkable. Georges Lemaître would be deeply pleased.

There are three general categories of FLRW models. Closed FLRW models expand for awhile, but eventually stop expanding and then contract under the influence of the gravitational field generated by the mass-energy it contains. For this to happen the model universe has to have enough mass-energy. Then there are the open FLRW models. These do not contain enough mass- energy to reverse the expansion into contraction, and they expand forever. If $\Lambda = 0$, the expansion constantly slows down (decelerates) but never reverses. Λ generates a *repulsive* gravitational force (because of very high negative pressure), and, unless balanced or dominated by the attractive gravitational force of normal mass-energy, will push the universe to expand more and more rapidly forever. Finally, there are flat FLRW models, which are just right in between the closed and open models. These models also expand forever, but if $\Lambda = 0$ an infinitesimal increase in mass-clergy density will lead them to collapse eventually.

And so into which category does our concordance FLRW model fall? From all indications, assuming an FLRW background model, our universe is very, very close to flat and it is very difficult to say for sure which side of flat it is (open or closed). But, if it is correct that the influence of Λ (dark energy) dominates its expansion, then either way it will certainly continue to expand forever at an accelerating rate.

Before concluding this brief discussion on FLRW models, we should expand on two other features. The first is that all of them contain a Big Bang, or initial singularity, at a finite time in their past (in the standard or concordance model it is 13.7 billion years ago). This is, in a manner of speaking, at the temporal "beginning" of the model, when the density, temperature and curvature go infinite. This, of course, is exactly what Lemaître expected, and predicted. It is important to note, however, that this singularity—the fact that these physical parameters go infinite—almost certainly *does not* represent the reality of that very early stage of the universe. Instead, it indicates a breakdown in the model at extremely high temperatures. We now have strong evidence that, as we go back into the past, there comes a point where we must graft onto the standard model one which is based on quantum gravity. We have not yet been able to do that adequately. When we do, then that addition will in some sense describe and explain the Planck era, the early quantum-configuration-dominated phase from which our universe began to expand and cool.

The second feature of the standard almost-FLRW model we need to explain more fully is precisely its "almost" character. This means that it possesses perturbations, or small fluctuations, to the exact concordance FLRW model the cosmologically relevant observational data support. As we have already emphasized, an exact FLRW model is perfectly homogeneous, and does not allow for any lumpiness at all, on any scale. If our universe were exactly FLRW we would not be here! Though the observations provide evidence that it is very close to the standard exact FLRW model on the largest scales, it also contains a great deal of structure on intermediate and small scales. So, using the concordance FLRW solution as the "background" model, we then add small deviations or perturbations to it, which describe the early development of lumpiness in our universe. These obey linear growth equations—as long as they are small compared with the background FLRW parameters such as matter-density. These perturbed models turn out to work very well for the very early phases of our universe, and for representing the early growth of large-scale structure of our universe. However, they cannot adequately model what happens when the density contrast of condensing structures gets to be comparable to or larger than the background density. At that point we deal with the local—not global or cosmological—evolution of those individual structures. To do so we use other methods, which are nonlinear.

In studying the early growth of the structure perturbations within the concordance FLRW background, we find that, in order to account for the kinds and patterns of structure we observe now, 13.7 billion years after the Big Bang, we need more nonbaryonic (or dark) matter than the baryonic matter we and everything we see is made of. Analyses of the CBR anisotropy data and the primordial elemental abundance data help us to set the present percentages at: 4 % of the total mass-energy density is baryonic matter, and 23 % of the total mass-energy density is non-baryonic matter. This dark matter, as the name implies, only interacts with baryonic matter and with itself very rarely, except through its gravitational influence. That's how we know it's there. Thus, we cannot see it, or observe its direct interactions with matter. We do not know yet what particles constitute this non-baryonic dark matter—only that it must be relatively cold and very weakly interacting.

We now move on to discuss Lemaître and LTB models, the broader classes of cosmologies which Lemaître pioneered. These are, as we have already mentioned, assuming much more importance and receiving much more attention in current cosmological research.

Beyond the Standard FLRW Model: Lemaître and LTB Models and Their Importance

If we keep the restriction of spherical symmetry but remove the demand for spatial homogeneity from the FLRW models, we end up with the Lemaître cosmological models, which he introduced and studied so thoroughly in his outstanding 1933 paper (Lemaître 1933). In his treatment he included both the cosmological constant, of which he was so fond, and perfect fluid matter and radiation, with pressure. A year later Tolman (1934), referring to Lemaître's paper, (Krasiński 1997a), authored his own study on the special, somewhat simpler dust ($p = 0$) cases of these Lemaître models. Much later Bondi (1947) revived interest in them with his own detailed analysis. These Lemaître models with $p = 0$ are therefore now known, as we have already mentioned above, as Lemaître-Tolman-Bondi (LTB) models.

As we have also briefly indicated, removing spatial homogeneity radically changes the model. Since LTB models are spherically symmetric, they are isotropic, but spherical symmetric and isotropic relative only to one location. There is now only a single spatial center of symmetry, which is almost always taken to be that of our position as observers. In these models the universe is *not* spherically symmetric relative to any other spatial location! This means that our position in this universe is privileged—that it is unlike any other location in our observable universe. Thus, in LTB models the cosmological, or Copernican, principle no longer holds. Confronting these models with observations, which is already being done, (Clarkson and Maartens 2010) will enable us to demonstrate whether and to what extent it does hold—that is, whether the universe is almost spatially homogeneous on the largest scales. And further, if the observable universe is close to an LTB model, how far from its center of symmetry or simply from the center of a very large under-dense region (a "void") within it could we be and still record the observational results we now have? Thus, at present the LTB models provide an apt and relatively simple foil for the standard FLRW model. At the same time they provide a spring-board for studying non-spherically symmetric perturbations to LTB and using relevant observations (e. g. of the CMB) to confront them.

In LTB models the inhomogeneities on each 3-D spatial slice through space-time are radial variations in matter-energy density and in other parameters and variables, e. g. the spatial curvature. As one moves outward from our central spatial location on a surface of constant time, we find that the density varies with radial coordinate distance r from our position. But there is no dependence of the density on either of the two angular coordinates, θ and φ. This is very much like the spherically symmetric concentric waves at any one moment after a rock is dropped into a pond. The key variable in the LTB models is a radial distance $R(t, r)$, often called the "areal distance," or an "area distance." It is really no longer a "scale factor," as it is in FLRW models, because it varies with the radial coordinate r on 3-D constant-time slices. The parameters which determine the model are $M(r)$, which is the gravitational mass contained within a comoving sphere of radius r (this can be related to the density), $E(r)$, which is the curvature ("closed," "open," or flat—or elliptic, hyperbolic or parabolic) on 3-D spatial slices, and $t_B(r)$, which is the "bang time," the time at which expansion starts in the model. This can be different at different values of r! (Krasiński 1997b; Pelbański and Krasiński 2006)

As we already discussed in the section "The Standard FLRW Models and the Observations Supporting Them", there is a great deal of observational support for the standard concordance FLRW model of the universe. However, there remain a number of its key features which must be studied within a broader context, and other crucial issues which must be resolved, before it is definitively confirmed as the most adequate model of our universe. It is precisely through confronting LTB models, both those with a cosmological constant and those without, with old and new observational data and through completing the related theoretical work, that many of these uncertainties and issues can be resolved. That is because the LTB models are the simplest non-perturbative inhomogeneous models that we have. Understanding the large-scale inhomogeneities they admit and their effect on cosmological observations will go a long way towards improving our confidence in our description of the universe.

The importance of LTB models can be best appreciated by summarizing some of our uncertainties and reservations about the concordance almost-FLRW model. First of all, as we have already emphasized, it must have a positive cosmological constant Λ, which represents vacuum energy, or dark energy. We really do not understand very much about vacuum energy, and even less about non-vacuum dark energy, particularly its magnitude (very small, but still apparently dominant) and its origin. We do not observe the cosmological constant or dark energy, but rather deduce it from the anomalously faint apparent luminosities of Type Ia supernovae, assuming an almost-FLRW universe. Studying these inhomogeneous LTB models and confronting them with more precise, deeper and soon-to-be-available independent data will help us determine whether or not our universe really possesses a significant dark clergy, that is a non-zero Λ, or whether, instead, the apparent gentle acceleration of the cosmic expansion is due to the influence of large-scale inhomogeneities (for instance, we may be near the center of very large underdense region, or void)(Clarkson and Maartens 2010).

Secondly, very closely related to, but broader than, this dark-energy question, is the issue of uniqueness. The concordance FLRW model is *not* the only model which provides a best-fit to the presently available data. It has been shown by a number of people that an inhomogeneous LTB model without a cosmological constant can fit all presently available data, (Mustapha et al. 1997) including CBR measurements (Clarkson and Regis 2011; Regis and Clarkson 2012). Thus, we must do further theoretical and observational work to determine which of these very different models really best describes the space-time structure and the dynamics of our universe. Obviously, to do this we need more—and new categories of—data which will distinguish among these competing models.

Thirdly, the application of almost-FLRW models is generally assumed from the outset without complete justification. There is fairly good evidence that the universe is isotropic, and *some but much less* observational evidence that the universe *may be* spatially homogeneous on the largest scales. We cannot directly observe spatial homogeneity, as we have no direct observational access to the cosmological extent of any one constant time slice. This partial evidence for almost spatial homogeneity has given us the "green light" to assume it, and use FLRW models as the basis for our description of the universe. The fact that they fit the data encourages us to continue using them. However, as we have just seen, we do not yet have adequate justification for doing so! What we really need to do—to confirm our use of FLRW models - is assume that the universe is represented by a less-special, more general model, like an LTB model, with or without a Λ, and then demonstrate from enough mutually independent types of data that our universe is almost-FLRW on the largest scales. We cannot accomplish this simply by considering perturbations to FLRW. This is why use of LTB and Lemaître models is so important, and has generated so much recent interest and investigation. As is now obvious, the main issue here is the cosmological principle. Is the universe almost spatially homogeneous on the largest scales, or not?

Fourthly, it is clear that the universe is *not* almost-FLRW on intermediate and small scales. And we really do not know, even now, what the smallest scale is on which the universe can be fit by the concordance (standard) almost-FLRW model. Using the more general LTB models will help us do that. And fifthly, any careful study of the growth of structure in our universe really needs to be supplemented and checked by studying non-perturbative deviations from FLRW. These would all be important questions for Lemaître, were he still working with us today.

Now we briefly turn to consider a sampling of the outstanding work that is being done on LTB and Lemaître models and on their perturbations.

Recent Research on LTB and Lemaître Cosmological Models

The amount of significant research centering on or employing LTB and Lemaître models has accelerated very noticeably over the past 25 years. This is certainly due to the need to explore descriptions and models of our universe and its structures that take us beyond considerations of FLRW and perturbed or almost-FLRW models. A critical part of that research has been directed towards linking the free parameters of those models to observations, so that it becomes clearer what types of independent observational data is needed to constrain them, and what data would indicate that an LTB model is an almost-FLRW model. At the end of the last section, we summarized the principal reasons for this need to pursue our cosmological research beyond almost-FLRW models. It is impossible in this chapter to summarize adequately all the important work that has and is being done in this area. Here I shall simply provide some examples of, and reference to, some of the more notable contributions.

For discussion we can conveniently divide results and contributions on these models into several categories: First, careful expositions and treatments of LTB and Lemaître models and their characteristics and properties; second, their connection with cosmologically relevant observables; thirdly, their direct confrontation with various types of cosmologically relevant observational data and comparison with FLRW models, including more focused contributions exploring their use in validating whether and to what extent the cosmological principle holds; fourthly, explorations of perturbations to LTB models; and finally application of the Lemaître models to the formation of structures within the universe. We shall briefly discuss and reference work in each of these areas.

Apart from the foundational papers of Lemaître (1933) and Bondi (1947) the best recent treatments of LTB and the more general Lemaître models (those perfect fluid models with isotropic or even anisotropic pressure and pressure gradients—Lemaître's paper considers both) are found in Krasiński's *Inhomogeneous Cosmological Models*, chapters "Georges Lemaître: The Priest Who Invented the Big Bang and 'The Wildest Speculation of All': Lemaître and the Primeval-Atom Universe" (Krasiński 1997b), and especially in Plebański and Krasiński's *An Introduction to General Relativity and Cosmology,* Chapter. "18" (Pelbański and Krasiński 2006) and in Bolejko et al.'s *Structures in the Universe by Exact Methods* (Bolejko et al. 2010). These treatments particularly the latter two, discuss all the principal characteristics of these models, including their free parameters, horizons, big-bang structure, shell-crossings, etc., and provide relatively up-to-date references to important results. A very helpful paper by Wainwright and Andrews (2009) gives a brilliant treatment of the dynamics of LTB models, including their asymptotic behavior and of the spherically symmetric perturbations (both growing and decaying modes) to FLRW and their relationship to LTB parameters.

One of the largest bodies of work on LTB models is that of exploring how LTB models, and their anisotropic perturbations, can be determined by observational data, taking their inspiration from the key paper by Kristian and Sachs in 1966 (Kristian and Sachs 1966). Some of this work has been pursued employing the usual orthogonal 3 + 1 coordinate formulation (O(3+1)), in which space-time is foliated into space-like hypersurfaces along the normal time-coordinate [this is better than "O-3 + 1"]. But significant other theoretical work has used "observational coordinates" (OC), through which space-time is foliated into past light-cones (PLC) along the observer's world line, such that each instant of time labels a PLC. The motivation for this formulation is the ease with which key observations, e.g. redshifts and angular-diameter (observer-area) distances can be related to the metric variables on our PLC, and the functional relation between red-shift and the null radial coordinate obtained [eliminate space between "red-" and "shift"]. All the data coming to us via electromagnetic radiation is arrayed on our PLC.

In both formalisms, the work that has been done demonstrates in detail how to solve the LTB EFE field equations uniquely given idealized data, consisting of galaxy redshifts, angular-diameter distances (to which luminosity distances can be easily converted), and galaxy number counts (which in principle, but with difficulty, will give the mass-energy density as a function of redshift). Some later papers using one or the other framework add the maximum of the angular-diameter distance and time-drift of galaxy redshifts as eventually feasible measurements which can in principle provide independent information to constrain the value of the cosmological constant (Hellaby 2006; Araújo and Stoeger 2009b) and the cosmic mass-energy density, (Araújo and Stoeger 2010) respectively.

Two key papers which have carried out this work in O(3+1) and give the flavor of the approach are those of Mustapha, Hellaby and Ellis (Mustapha et al. 1997) and Hellaby (2001). Hellaby in his important 2006 paper (Hellaby 2006) and later in his 2010 paper with Alfedeel (Alfedeel and Hellaby 2010) shows how in general LTB and Lemaître models the cosmic mass is defined, even when the pressure is non-zero, what the apparent horizon is and why it is important, and derives the simple algebraic relationship connecting the value of the cosmological constant Λ with the maximum of the angular-diameter distance and the gravitational mass a sphere of that radius contains. It is this relationship which, in principle, enables Λ to be determined from observational data. It is worth noting, in this context, that the definition of mass in these models was one of the important results Lemaître presented in his 1933 paper. (Lemaître 1933) In two other papers Lu and Hellaby (2007) and, a year later, McClure and Hellaby (2008) demonstrate how to obtain a LTB metric with $\Lambda = 0$ from simulated data by numerical integration, and study the stability of some of the key equations.

The foundational paper for the OC approach is the 1985 paper by Ellis et al. (1985). There the motivation and relationships between the metric variables and the observables are laid out carefully, and theorems proving the uniqueness of solutions to the general EFE given an idealized data set consisting of galaxy redshifts, angular-diameter (observer-area) distances, galaxy number counts, cosmological proper motions, and integrated null-shear measurements are explained. The latter two, of course, are zero in an exactly isotropic space-time. A non-zero Λ is not included, nor the discussion of the supplementary independent observations needed to determine its value. The present status of this approach and detailed accounts of the results so far for LTB cosmologies are given in the two recent papers by Araújo and Stoeger already cited (Araújo and Stoeger 2009b, 2010). Two other recent papers, by Hellaby and Alfedeel (2009) and by Araújo and Stoeger (2011) compare and contrast teh O(3+1) and the OC approaches to determining the LTB metric of the universe from observations.

There has been a great deal of research directed towards confirming the cosmological principle to determine whether our universe is almost spatially homogeneous on cosmic scales. Typically, such work attempts to fit a universe with a large local under-dense region (or void) to the cosmological data, including various kinds of measurements of the CBR. In most cases, a full LBT and/or Lemaître model analysis is not used, but instead either a low-density FLRW or LBT dust model (a "void") surrounding the observer matched to a higher density FLRW model at high redshifts. All of this work has been referenced and critically reviewed in great detail in Clarkson and Regis's paper (Clarkson and Regis 2011). They have also provided a Lemaître model framework and analysis for further investigation of this important problem, along with guidelines for meeting some of the important challenges in resolving it. Some of the other papers which have considered real observational data within a LTB and Lemaître framework to see if such models fit the observations are those of Tomita and Inoue (2009) Tomita (2010) Dunsby et al.

(2010), and Clarkson and Maartens (2010). Despite the many technical difficulties in carrying out this work and the controversy surrounding it, the preliminary results, as we have already mentioned above, clearly show that there are inhomogeneous models that can fit the presently available cosmological data, without a non-zero Λ (Regis and Clarkson 2012; Clarkson and Regis 2011). Once other independent types of cosmologically relevant data are available, e. g. the maximum of the angular-diameter distance and its redshift, and better measurements of the cosmic mass-energy density (possibly through redshift time-drift data), we will be able to determine whether we do need a non-zero Λ and whether the concordance FLRW model or some inhomogeneous model end up winning this competition. It is worth mentioning in this context that Uzan et al. (2008) have used an LTB background to suggest how the time-drift of galaxy redshifts can be used to test whether there is almost spatial homogeneity on the largest scales.

Another important area of research on focuses on the behavior of LTB and Lemaître perturbations. This work is crucial for understanding and modelling formation and development of galaxies and clusters of galaxies in inhomogeneous cosmologies. It turns out that this subject is extremely difficult. However, a good start has been made by Zibin (2008) Clarkson, Clifton and February, (Clarkson et al. 2009) Clarkson and Maartens (2010) and others. An excellent overview of this subject, its progress and challenges is given in the Clarkson and Maartens paper.

Finally, a number of people, notably Krasiński, Hellaby and Bolejko, have pioneered the exact use of LTB and Lemaître models to study the local formation of structure voids, clusters of galaxies, etc. without resorting to linear perturbation treatments. Though such applications suffer from certain obvious limitations (e.g. their inability to deal with rotation), they also enjoy many advantages. A very recent and important reference for this research is the book by Bolejko et al. (2010). It is also worth noting that Bolejko and Stoeger (2010) have recently shown that certain LTB and Lemaître models - those with spatial curvature very much less than the Ricci curvature induced by the mass-energy density—can undergo spontaneous temporary homogenization. This process may be important in achieving the necessary homogeneity conditions for initiating inflation in the very early universe.

Conclusions

In our discussion we have reviewed the central ground-breaking contributions of Georges Lemaître to cosmology, emphasizing his outstanding but perhaps less well-known theoretical work in solving and interpreting the solutions to the general spherically symmetric Einstein field equations. These results have provided the foundations for the study of inhomogeneous cosmological models—the more restricted Lemaître-Tolman-Bondi (LTB) models, where the pressure is assumed to be zero, the Lemaître models for more general perfect fluids, and non-spherically symmetric perturbations to these exact solutions. It is apparent, from what we have seen here, that, beyond the well-appreciated and fundamental developments of Lemaître's work in Big Bang cosmology and particularly in the standard spatially homogeneous FLRW models, there has been an explosion of important work expanding on his contributions to the study of inhomogeneous models—both for elucidating and confirming the large-scale structure of our universe, and for modeling more carefully the formation of galaxies and clusters of galaxies.

In particular, we have discussed and summarized the recent and ongoing research connecting cosmologically relevant observations to LTB models, in order to increase and confirm our understanding of the principal features of our universe. Since FLRW models are special cases of the more general LTB and Lemaître models, this makes perfect sense. Such research is even more urgent, given our ignorance and uncertainty about dark energy and the cosmological constant, which seems to be absolutely necessary for an adequate almost-FLRW description of space-time, our inability to directly confirm large-scale spatial inhomogeneity, our need for further, more precise and deeper observational data, and the strong indications we have that the standard concordance FLRW model is not a unique best-fit to the data. As Clarkson and Maartens comment at the end of their outstanding paper (Clarkson and Maartens 2010), "Only by developing a family of inhomogeneous spacetimes to the same level of sophistication as the standard concordance model, and directly comparing them side by side, will we really be able to understand whether Λ is real, or actually a consequence of our homogeneity assumption." This basic open approach motivates all the work in this area—to confirm, increase, and, if need be, modify our understanding and our description of the universe, and to learn what their limitations are, in particular the length scales below which they are inapplicable. Lemaître would be in thorough accord with this philosophy. That, too, is part of his legacy to contemporary cosmology.

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