Integral Mixed Unit Interval Graphs

Van Bang Le[1] and Dieter Rautenbach[2]

[1] Institut für Informatik, Universität Rostock, Rostock, Germany
le@informatik.uni-rostock.de
[2] Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany
dieter.rautenbach@uni-ulm.de

Abstract. We characterize graphs that have intersection representations using unit intervals with open or closed ends such that all ends of the intervals are integral in terms of infinitely many minimal forbidden induced subgraphs. Furthermore, we provide a quadratic-time algorithm that decides if a given interval graph admits such an intersection representation.

Keywords: intersection graph; interval graph; proper interval graph; unit interval graph.

Classification of the topic: Graph Theory, Communication Networks, and Optimization.

1 Introduction

Interval graphs and subclasses like proper interval graphs and unit interval graphs have well studied structural [2, 10] as well as algorithmic [3–5, 12, 13] properties and occur in many applications [1, 9, 11, 14–18]. Interval graphs are the intersection graphs of closed (real) intervals and unit interval graphs are the intersection graphs of closed unit intervals.

As long as intervals of different lengths are allowed, it actually does not matter in the definition of interval graphs whether the ends of the intervals are closed or open. For unit interval graphs, this is no longer true. While Frankl and Maehara [7] proved that unit interval graphs coincide with the intersection graphs of open unit intervals, the intersection graphs of the unit intervals of different types form a strict superclass of unit interval graphs.

In two previous papers we studied the classes of intersection graphs of closed and open unit intervals [19] and of mixed unit intervals [6] where for mixed unit intervals all four combinations for the two ends, namely open-open, closed-closed, open-closed, and closed-open are allowed. Partial results in [6] naturally lead to the problem of characterizing the graphs that have intersection representations using mixed unit intervals where additionally all ends of the intervals are integers.

We refer to such graphs as *integral mixed unit interval graphs*.

Our contributions in the present paper are

– a characterization of twin-free integral mixed unit interval graphs in terms of the complete list of minimal forbidden induced subgraphs, and

J. Gudmundsson, J. Mestre, and T. Viglas (Eds.): COCOON 2012, LNCS 7434, pp. 495–506, 2012.

– a quadratic-time algorithm that decides if a given interval graph is an integral mixed unit interval graph, and if so, outputs a suitable representation.

The paper is organized as follows. In Section 2 we introduce some terminology and notation, give exact definitions, and recall some previous results. In Section 3 we study the forbidden induced subgraphs. In Section 4 we derive structural properties of the maximal cliques of integral mixed unit interval graphs. Section 5 is devoted to the representation algorithm and its analysis. Finally, in Section 6 we combine all results of the earlier sections and prove our main results.

2 Preliminaries

Let $\mathcal{M}$ be a family of sets. An *$\mathcal{M}$-representation of a graph G* is a function $M : V(G) \to \mathcal{M}$ such that for every two distinct vertices u and v of G, we have $uv \in E(G)$ if and only if $M(u) \cap M(v) \neq \emptyset$. A graph is an *$\mathcal{M}$-graph* if it has a $\mathcal{M}$-representation. Two vertices u and v in a graph G are *twins*, if they have the same closed neighborhood, that is, they are adjacent and for every vertex w in $V(G) \setminus \{u, v\}$, the vertices u and w are adjacent if and only if the vertices v and w are adjacent. Note that if u and v are twins in a graph G, then G is an $\mathcal{M}$-graph if and only if $G - u$ is an $\mathcal{M}$-graph. Thus, it suffices to consider twin-free graphs when discussing graphs admitting an $\mathcal{M}$-representation.

For two real numbers x and y, the *open interval* (x, y) is $\{z \in \mathbb{R} \mid x < z < y\}$, the *closed interval* $[x, y]$ is $\{z \in \mathbb{R} \mid x \leq z \leq y\}$, the *open-closed interval* $(x, y]$ is $\{z \in \mathbb{R} \mid x < z \leq y\}$, and the *closed-open interval* $[x, y)$ is $\{z \in \mathbb{R} \mid x \leq z < y\}$. Let

$$\begin{array}{llll}
\mathcal{I}^{--} &= \{(x,y) \mid x,y \in \mathbb{R}, x<y\}, & \mathcal{U}^{--} &= \{(x,x+1) \mid x \in \mathbb{R}\},\\
\mathcal{I}^{++} &= \{[x,y] \mid x,y \in \mathbb{R}, x\leq y\}, & \mathcal{U}^{++} &= \{[x,x+1] \mid x \in \mathbb{R}\},\\
\mathcal{I}^{-+} &= \{(x,y] \mid x,y \in \mathbb{R}, x<y\}, & \mathcal{U}^{-+} &= \{(x,x+1] \mid x \in \mathbb{R}\},\\
\mathcal{I}^{+-} &= \{[x,y) \mid x,y \in \mathbb{R}, x<y\}, & \mathcal{U}^{+-} &= \{[x,x+1) \mid x \in \mathbb{R}\},\\
\mathcal{I}^{\pm} &= \mathcal{I}^{++} \cup \mathcal{I}^{--}, & \mathcal{U}^{\pm} &= \mathcal{U}^{++} \cup \mathcal{U}^{--},\\
\mathcal{I} &= \mathcal{I}^{\pm} \cup \mathcal{I}^{-+} \cup \mathcal{I}^{+-}, & \mathcal{U} &= \mathcal{U}^{\pm} \cup \mathcal{U}^{-+} \cup \mathcal{U}^{+-}.
\end{array}$$

We allow arithmetic operations on intervals, that is, for an interval I in $\mathcal{I}$ and two real numbers x and y, we have $xI + y = \{xz + y \mid z \in I\}$.

For an interval I in $\mathcal{I}$, let $\ell(I) = \inf(I)$ and $r(I) = \sup(I)$ denote the left and right end of I, respectively.

A $\mathcal{U}$-representation I of a graph G is *integral* if $\{\ell(I(u)) \mid u \in V(G)\} \subseteq \mathbb{Z}$, that is, all ends of the intervals are integers.

Interval graphs are $\mathcal{I}^{++}$-graphs and *unit interval graphs* are $\mathcal{U}^{++}$-graphs. A graph G is a *proper interval graph* if it has a $\mathcal{I}^{++}$-representation $I : V(G) \to \mathcal{I}^{++}$ for which there are no two vertices u and v of G such that $I(u)$ is a proper subset of $I(v)$. In this case I is a *proper interval representation of* G.

A fundamental result relating these three classes of interval graphs is due to Roberts. Please refer to Figure 1 for an illustration of the Claw $K_{1,3}$.

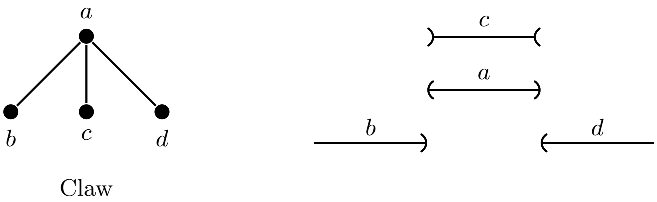

Fig. 1. The Claw $K_{1,3}$ and its four integral $\mathcal{U}$-representations

Theorem 1 (Roberts [20]). *A graph is a unit interval graph if and only if it is a proper interval graph if and only if it is a $K_{1,3}$-free interval graph.*

As mentioned in the introduction, for the definition of interval graphs, the type of the intervals does not make a difference, more precisely, if $\mathcal{M}, \mathcal{N} \in \{\mathcal{I}, \mathcal{I}^{++}, \mathcal{I}^{--}, \mathcal{I}^{+-}, \mathcal{I}^{-+}\}$, then G is a $\mathcal{M}$-graph if and only if G is a $\mathcal{N}$-graph [6, 19]. Since the Claw $K_{1,3}$ has a $\mathcal{U}^{\pm}$-representation (cf. Figure 1), the situation is different for unit interval graphs. The main two results from [6, 19] are the following. Please refer to Section 3 and Figure 2 for the definition and illustration of all graphs mentioned in these results.

Theorem 2 (Rautenbach and Szwarcfiter [19]). *A twin-free graph is a $\mathcal{U}^{\pm}$-graph if and only if it is a $\{R_0, Q_1, D_3, D_5\}$-free interval graph.*

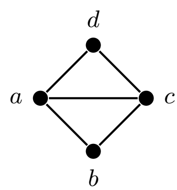

Fig. 2. The Diamond

Theorem 3 (Dourado et al. [6]). *For a diamond-free graph G, the following statements are equivalent.*

(1) *G is a $\{R_k \mid k \in \mathbb{N}_0\}$-free interval graph.*
(2) *G has an integral $\mathcal{U}$-intersection representation.*
(3) *G is a $\mathcal{U}$-graph.*

Mainly the last theorem motivated the characterization problem of the graphs that have an integral $\mathcal{U}$-representation.

3 Forbidden Induced Subgraphs

Let $\mathcal{G}$ be the class of twin-free integral mixed unit interval graphs, that is, of those twin-free graphs that have an integral $\mathcal{U}$-representation.

Let $G \in \mathcal{G}$ and let $I : V(G) \to \mathcal{U}$ be an integral $\mathcal{U}$-representation of G. For a vertex u of G, let $c(u)$ denote the number of distinct maximal cliques of G that contain u.

Since G is twin-free, the function I is necessarily injective. Hence, if H is an induced subgraph of G, then the restriction of I to $V(H)$ is an injective integral $\mathcal{U}$-representation of H, that is, even if H is not twin-free, there is an integral $\mathcal{U}$-representation of H that assigns different intervals to the vertices of H. Therefore, the minimal forbidden induced subgraphs for $\mathcal{G}$ are exactly those graphs that do not have an injective integral $\mathcal{U}$-representation while every proper induced subgraph has.

The integrality of the representation I immediately implies that every vertex u of G belongs to at most three maximal cliques of G, that is,

$(\mathcal{C}_1)$ *$c(u) \leq 3$ for every vertex u of G.*

where $c(u) = 3$ implies that $I(u)$ is necessarily a closed interval.

Lemma 1. *The graphs R_0, D_1, D_2, and D_3 are minimal forbidden induced subgraphs for $\mathcal{G}$.*

Proof. It follows from $(\mathcal{C}_1)$ that R_0, D_1, and D_2 are forbidden induced subgraphs for $\mathcal{G}$, because $c(a) > 3$ for each of these graphs. Suppose D_3 is an induced subgraph of some $G \in \mathcal{G}$, labelled as in Figure 4. Since G has no twins, we may assume, by symmetry, that there exists a vertex b' adjacent to a and non-adjacent to b. Now $c(a) > 3$, contradicting $(\mathcal{C}_1)$. Thus, D_3 is also a forbidden induced subgraph for $\mathcal{G}$. Finally, it is easy to verify that for each $H \in \{R_0, D_1, D_2, D_3\}$ and every $v \in V(H)$, the graph $H - v$ has an injective integral $\mathcal{U}$-representation.

Lemma 2. *The graphs S_0, T_0, K_5, and D_4 are minimal forbidden induced subgraphs for $\mathcal{G}$.*

Lemma 3. *The graph D_5 is a minimal forbidden induced subgraph for $\mathcal{G}$.*

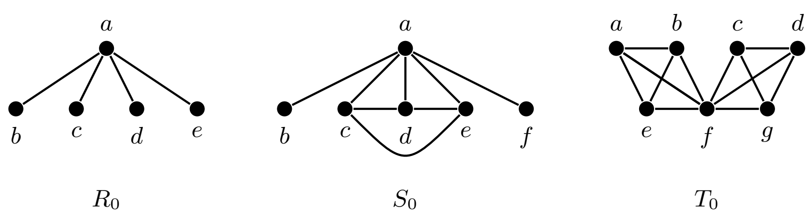

Fig. 3. The graphs R_0, S_0, and T_0

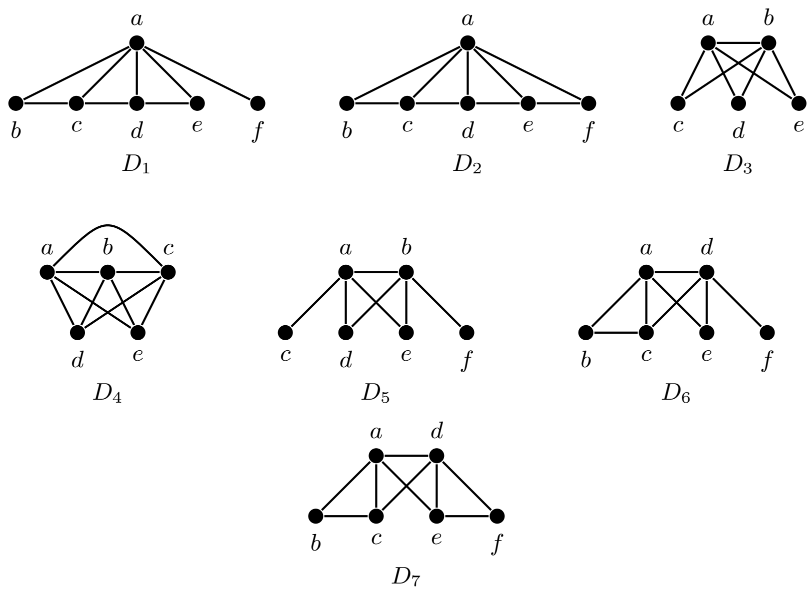

Fig. 4. The graphs D_1 to D_7

Lemma 4. *The graphs D_6 and D_7 are minimal forbidden induced subgraphs for $\mathcal{G}$.*

We now describe three infinite sequences of forbidden induced subgraphs for $\mathcal{G}$.

For $k \in \mathbb{N}$, let the graph Q_k arise from a path $a_0a_1a_2 \ldots a_{k+1}$ by adding the vertices $b_1, b_2, \ldots, b_{k+1}$, the edges a_ib_i for $1 \leq i \leq k+1$, and the edges $a_{i-1}b_i$ for $2 \leq i \leq k+1$. See Figure 5 for an illustration.
For $k \in \mathbb{N}$,

- let $\tilde{Q}_k$ arise from Q_k by adding a vertex b_0 and four edges a_0b_0, a_0b_1, a_1b_0, and b_0b_1,
- let R_k arise from Q_k by adding two vertices a_{k+2} and b_{k+2} and two edges $a_{k+1}a_{k+2}$ and $a_{k+1}b_{k+2}$,

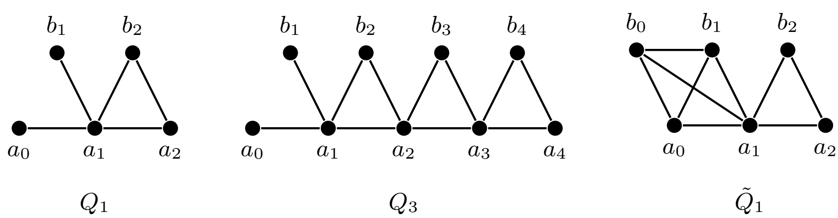

Fig. 5. The graphs Q_1, Q_3, and $\tilde{Q}_1$

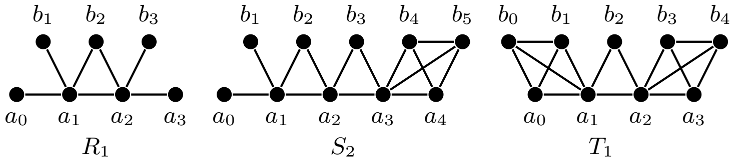

Fig. 6. The graphs R_1, S_2, and T_1

– let S_k arise from Q_k by adding three vertices a_{k+2}, b_{k+2}, and b_{k+3} and six edges $a_{k+1}a_{k+2}$, $a_{k+1}b_{k+2}$, $a_{k+1}b_{k+3}$, $a_{k+2}b_{k+2}$, $a_{k+2}b_{k+3}$, and $b_{k+2}b_{k+3}$, and
– let T_k arise from $\tilde{Q}_k$ by adding three vertices a_{k+2}, b_{k+2}, and b_{k+3} and six edges $a_{k+1}a_{k+2}$, $a_{k+1}b_{k+2}$, $a_{k+1}b_{k+3}$, $a_{k+2}b_{k+2}$, $a_{k+2}b_{k+3}$, and $b_{k+2}b_{k+3}$.

See Figures 5 and 6 for an illustration.

For $k \in \mathbb{N}$, the two vertices a_{k+1} and b_{k+1} are called the *special vertices* of Q_k and $\tilde{Q}_k$, respectively.

Lemma 5. *The graphs R_k, S_k, and T_k for $k \in \mathbb{N}$ are minimal forbidden induced subgraphs for $\mathcal{G}$.*

4 Properties of Maximal Cliques

Throughout this section, let G be a fixed connected twin-free interval graph. It is well-known [8] that there is a linear ordering of the maximal cliques of G, say $\mathcal{C} = (C_1, \ldots, C_q)$, such that every vertex of G belongs to maximal cliques that are consecutive in that ordering, that is, for every vertex u of G, there are indices $\ell(u)$ and $r(u)$ with

$$\{i \mid 1 \leq i \leq q \text{ and } u \in C_i\} = \{i \mid \ell(u) \leq i \leq r(u)\}.$$

Note that this linear ordering is unique up to reversal and that the number $c(u)$ of distinct maximal cliques of G that contain u equals $r(u) + 1 - \ell(u)$. Hence a vertex u of G is simplicial if and only if $c(u) = 1$ if and only if $\ell(u) = r(u)$.

If C and D are distinct maximal cliques of G, then $C \setminus D$ and $D \setminus C$ are both not empty, that is, for every $j \in \{1, \ldots, q\}$, there are vertices u and v such that $r(u) = \ell(v) = j$. This also implies that C_1 and C_q contain simplicial vertices.

Note that, since G is twin-free, there are no two distinct vertices u and v with $\ell(u) = \ell(v)$ and $r(u) = r(v)$.

The purpose of the present section is to derive the following structural properties of the sequence $\mathcal{C}$ that are implied by forbidding certain induced subgraphs from Section 3.

($\mathcal{C}_1$) $c(u) \leq 3$ for every vertex u of G.
($\mathcal{C}_2$) There are no two vertices u and v of G with $c(u) = c(v) = 3$ and $r(v) - r(u) = 1$.

Lemma 6. *Let G, $\mathcal{C}$, and $\ell(u)$, $r(u)$, and $c(u)$ for every vertex u of G be as above.*

(i) *If G is $\{R_0, D_1, D_2, D_3, D_4\}$-free, then ($\mathcal{C}_1$) holds.*
(ii) *If G is $\{D_5, D_6, D_7\}$-free, then ($\mathcal{C}_2$) holds.*

Proof. (i) For contradiction, we assume that $c(u_1) \geq 4$ for some vertex u_1 of G. Let $i = \ell(u_1)$. Note that $r(u_1) \geq i + 3$. Let the vertices u_2, u_3, u_4, and u_5 be such that $r(u_2) = i$, $\ell(u_3) = i + 3$, $r(u_4) = i + 1$, and $\ell(u_5) = i + 2$. Since G is R_0-free, we may assume, by symmetry, that $\ell(u_4) \leq i$. Let the vertex u_6 be such that $\ell(u_6) = i + 1$.

First, we assume that $r(u_5) = i + 2$. Since G is R_0-free, this implies $r(u_6) \geq i + 2$. If $r(u_6) = i + 2$, then $G[\{u_1, \ldots, u_6\}]$ is D_1, and, if $r(u_6) \geq i + 3$, then $G[\{u_1, u_3, \ldots, u_6\}]$ is D_3, which is a contradiction. Hence we may assume that $r(u_5) \geq i + 3$. Let the vertex u_7 be such that $r(u_7) = i + 2$.

If $r(u_6) = i + 2$, then $G[\{u_1, \ldots, u_6\}]$ is D_2, which is a contradiction. If $r(u_6) \geq i + 3$, then $G[\{u_1, u_3, u_5, u_6, u_7\}]$ is D_4, which is a contradiction. Hence we may assume that $r(u_6) = i + 1$ and, by symmetry, $\ell(u_7) = i + 2$. Now $G[\{u_1, u_2, u_3, u_6, u_7\}]$ is R_0, which is a contradiction. This completes the proof of (i).

(ii) For contradiction, we assume that the vertices u_1 and u_2 are such that $c(u_1) = c(u_2) = 3$ and $r(u_2) = r(u_1) + 1$.

Let $i = \ell(u_1)$. Let the vertices u_3, u_4, u_5, and u_6 be such that $r(u_3) = i$, $\ell(u_4) = i + 3$, $r(u_5) = i + 1$, and $\ell(u_6) = i + 2$. Now $G[\{u_1, \ldots, u_6\}]$ is one of the graphs D_5, D_6, and D_7, which is a contradiction. This completes the proof of (ii).

5 The Representation Algorithm

Throughout this section, let G be a fixed connected twin-free interval graph that is not a clique. Let $\mathcal{C} = (C_1, \ldots, C_q)$ and $\ell(u)$, $r(u)$, and $c(u)$ for every vertex u of G be as in the first paragraph of Section 4. Since G is not a clique, we have $q \geq 2$.

In this section, we describe and analyze the algorithm `IntMixUniIntRep` that, given $\mathcal{C}$ as input, produces a function $I : V(G) \to \mathcal{U}$ such that $\ell(I(u)) \in \mathbb{Z}$ for every vertex u of G. We prove that I is an integral $\mathcal{U}$-representation of G provided that $\mathcal{C}$ satisfies certain structural properties and G does not contain certain induced subgraphs. The algorithm works essentially in two phases:

- In a first phase, the algorithm determines a path $P : v_0 \ldots v_{k+1}$ in G (cf. `ClosedVertices`). To the vertices of this path it assigns the intervals $I(v_i) = [i, i + 1]$ for $i \in \{0, \ldots, k + 1\}$ (cf. line 2 of `IntMixUniIntRep`).

– In a second phase, it processes the maximal cliques of G according to the ordering given by $\mathcal{C}$ (cf. line 3 of `IntMixUniIntRep`). When it processes the maximal clique C_i, then $I(u)$ is defined for all vertices u of G with $\ell(u) = i$, that is, it specifies the unit interval for those vertices that appear in C_i for the first time and do not belong to P. (cf. line 4 of `IntMixUniIntRep`).

Recall that a vertex u is simplicial if and only if $c(u) = 1$.

Procedure `ClosedVertices`

let v_0 be a simplicial vertex in C_1
$i := 0$; $j := 1$
repeat
  $i := i + 1$
  let v_i be a vertex in $C_j \setminus \{v_0, \ldots, v_{i-1}\}$ with maximum $r(v)$
  $j := r(v_i)$
until $j = q$
$k := i$
let v_{k+1} be a simplicial vertex in C_q

If $i \in \{0, \ldots, k-1\}$, then $r(v_i) < q$ and the connectivity of G implies $r(v_{i+1}) > r(v_i)$. This implies that `ClosedVertices` necessarily terminates. Clearly, by the choice of the vertices, $P : v_0v_1 \ldots v_{k+1}$ is a path in G.

Lemma 7. *If $\mathcal{C}$ satisfies $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, then the vertices $v_0, \ldots, v_{k+1}$ selected by* `ClosedVertices` *satisfy the following properties.*

(i) $r(v_i) = \ell(v_{i+1})$ *for* $i \in \{0, \ldots, k\}$.

(ii) *The vertices $v_0, \ldots, v_{k+1}$ are uniquely determined.*

(iii) *Each maximal clique C_j of G contains one or two vertices from $V(P)$. Furthermore,*
- *if C_j contains only one vertex from $V(P)$, say v_i, then $j \in \{2, \ldots, q-1\}$, $\ell(v_i) = j - 1$, and $r(v_i) = j + 1$, and*
- *if C_j contains two vertices from $V(P)$, then $C_j \cap V(P) = \{v_i, v_{i+1}\}$ for some $i \in \{0, \ldots, k\}$ and $r(v_i) = \ell(v_{i+1}) = j$.*

(iv) $c(u) \leq 2$ *for every vertex u in $V(G) \setminus V(P)$.*

(v) *For every $j \in \{1, \ldots, q\}$, there are at most two vertices u with $\ell(u) = j$ that do not belong $V(P)$, that is, $|C_j \setminus (C_{j-1} \cup V(P))| \leq 2$ for every $j \in \{1, \ldots, q\}$. Furthermore, if $C_j \setminus (C_{j-1} \cup V(P))$ contains two distinct vertices u and v for some $j \in \{1, \ldots, q\}$, then $j \in \{2, \ldots, q-1\}$ and $\{c(u), c(v)\} = \{1, 2\}$.*

Proof. (i) For $i = 0$ or $i = k$, the desired statement follows easily because v_0 and v_{k+1} are simplicial vertices with $\ell(v_0) = r(v_0) = 1$ and $\ell(v_{k+1}) = r(v_{k+1}) = q$. Now let $i \in \{1, \dots, k-1\}$. By line 5 of `ClosedVertices`, we have $r(v_i) \geq \ell(v_{i+1})$. Therefore, for contradiction, we assume that $r(v_i) > \ell(v_{i+1})$. As noted above, we have $r(v_{i+1}) > r(v_i)$. By the choice of v_i, this implies $\ell(v_i) < \ell(v_{i+1})$. By $(\mathcal{C}_1)$ and since G is twin-free, this implies that $c(v_i) = c(v_{i+1}) = 3$ and $r(v_i) = \ell(v_{i+1}) + 1$, which yields a contradiction to $(\mathcal{C}_2)$.

(ii) Since G is twin-free, each of the cliques C_1 and C_q contains exactly one simplicial vertex, which implies that v_0 and v_{k+1} are uniquely determined. If v_i has already been determined and $r(v_i) < q$, then $i \leq k$. Now part (i) and the twin-freeness of G imply that v_{i+1} is uniquely determined.

(iii) This follows immediately from part (i) and the observation $r(v_{i+1}) > r(v_i)$ for $i \in \{0, \dots, k-1\}$.

(iv) In view of $(\mathcal{C}_1)$, we may assume, for contradiction, that $c(u) = 3$ for some $u \in V(G) \setminus V(P)$. Let $j = \ell(u)$.

If there is exactly one vertex v_i from $V(P)$ with $v_i \in C_j$, then part (iii) implies $\ell(v_i) < j < r(v_i)$. This implies $c(v_i) = 3$ and $r(u) = r(v_i) + 1$, which yields a contradiction to $(\mathcal{C}_2)$.

If there are two vertices from $V(P)$ in C_j, then part (iii) implies that $C_j \cap V(P) = \{v_i, v_{i+1}\}$ for some $i \in \{0, \dots, k\}$ such that $j = \ell(v_{i+1})$. Since G is twin-free, this implies $c(v_{i+1}) \leq 2$, which yields a contradiction to the choice of v_{i+1}.

(v) This follows from part (iv) and the twin-freeness of G.

Lemma 8. *Let $\mathcal{C}$ satisfy $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ and let I be the function defined by* `IntMixUniIntRep`*.*

(i) *For every two distinct vertices u and v of G, if $\{u, v\} \cap V(P) \neq \emptyset$, then $uv \in E(G)$ if and only if $I(u) \cap I(v) \neq \emptyset$.*
(ii) *For every two distinct vertices u and v of G, if $uv \in E(G)$, then $I(u) \cap I(v) \neq \emptyset$.*

Lemma 9. *Let $\mathcal{C}$ satisfy $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. Just after an execution of line 16 of* `IntMixUniIntRep` *that defines $I(u)$, there is an induced subgraph H of G such that*

- *$\ell(v) < \ell(u)$ for every $v \in V(H) \setminus \{u, v_{i+1}\}$, that is, $I(v)$ is already defined for every vertex v of H,*
- *$u, v_i, v_{i+1} \in V(H)$,*
- *H is*
  - *either K_4,*
  - *or Q_k for some $k \in \mathbb{N}$ such that u and v_{i+1} are the special vertices of H,*
  - *or $\tilde{Q}_k$ for some $k \in \mathbb{N}$ such that u and v_{i+1} are the special vertices of H.*

Proof. We prove the statement by induction on j where j and i are as in the considered execution of line 16 of `IntMixUniIntRep`.

Algorithm `IntMixUniIntRep`

1. run `ClosedVertices` to compute $P : v_0 \dots v_{k+1}$
2. **for** $i := 0$ **to** $k+1$ **do** $I(v_i) := [i, i+1]$
3. **for** $j := 1$ **to** q **do**
4.   **for** each $u \in V(G) \setminus V(P)$ with $\ell(u) = j$ **do**
5.   **if** $|C_j \cap V(P)| = 1$ **then**
6.   let $C_j \cap V(P) = \{v_i\}$ for some $i \in \{0, \dots, k+1\}$
7.   **if** u is simplicial
8.   **then** $I(u) := (i, i+1)$
9.   **else** $I(u) := (i, i+1]$
10.   **endif**
11.   **if** $|C_j \cap V(P)| = 2$ **then**
12.   let $C_j \cap V(P) = \{v_i, v_{i+1}\}$ for some $i \in \{0, \dots, k\}$
13.   **if** u is simplicial **then**
14.   **if** there is no $v \in V(G) \setminus V(P)$ with $\ell(I(v)) = i$
15.   **then** $I(u) := (i, i+1]$
16.   **else** $I(u) := [i+1, i+2)$
17.   **else** $I(u) := [i+1, i+2)$
18.   **endif**
19.   **endfor**
20. **endfor**

In view of line 14 of `IntMixUniIntRep`, just before the considered execution of line 16 of `IntMixUniIntRep`, there is a vertex $v \in V(G) \setminus V(P)$ with $\ell(I(v)) = i$, which implies that $\ell(v) < \ell(u)$ and $j, i \geq 1$. By Lemma 7, we have $\ell(v_i) \in \{j-1, j-2\}$. By Lemma 8, the vertex v is adjacent to v_i, that is, v and v_i both lie in a maximal clique of G. If $\ell(v) < \ell(v_i)$, then, by Lemma 7, $\ell(v_{i-1}) = \ell(v) - 1$ and $\ell(v_i) = \ell(v) + 1$ and in view of `IntMixUniIntRep` we obtain $I(v) = (i-1, i]$, which is a contradiction. Hence $\ell(v) \geq \ell(v_i)$. If $v \in C_j$, then $H = G[\{v_i, v_{i+1}, u, v\}]$ is K_4. Hence, we may assume that $r(v) < j = \ell(u)$.

If $\ell(v) > \ell(v_i)$, then, by Lemma 7, $\ell(v_i) = j - 2$, $\ell(v) = r(v) = j - 1$, and $|C_{j-1} \cap V(P)| = 1$. Now $H = G[\{v_{i-1}, v_i, v_{i+1}, v, u\}]$ is Q_1 such that u and v_{i+1} are the special vertices of H. Hence we may assume that $\ell(v) = \ell(v_i)$.

If $c(v) = 2$, then, by Lemma 7, $\ell(v_i) = j - 2$, $\ell(v) = j - 2$, $r(v) = j - 1$, and $|C_{j-2} \cap V(P)| = 2$. Let the vertex w be such that $\ell(w) = j - 1$. If $r(w) = j - 1$, then $H = G[\{v_{i-1}, v_i, v_{i+1}, w, u\}]$ is Q_1 such that u and v_{i+1} are the special vertices of H. If $r(w) > j - 1$, then $H = G[\{v_i, v_{i+1}, w, u\}]$ is K_4. Hence, we may assume that $c(v) = 1$.

Since $\ell(I(v)) = i$, we obtain that $I(v)$ was defined by an earlier execution of line 16 of `IntMixUniIntRep`. By induction, this implies that, just after $I(v)$ was defined, there was an induced subgraph H' of G with the desired properties. Now $H = G[\{V(H') \cup \{u, v_{i+1}\}\}]$ has the desired properties.

Lemma 10. *Let $\mathcal{C}$ satisfy $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ and let I be the function defined by `IntMixUniIntRep`.*

If G is $\{K_5\} \cup \{R_i \mid i \in \mathbb{N}\} \cup \{S_i \mid i \in \mathbb{N}_0\} \cup \{T_i \mid i \in \mathbb{N}_0\}$-free, then I is an integral $\mathcal{U}$-representation of G.

6 Harvest

In this section we prove our two main results.

Theorem 4. *If G is a twin-free connected interval graph that is not a clique, and $\mathcal{C}$ is as in the first paragraph of Section 4, then the following statements are equivalent.*

(1) *G is $\{D_1, \dots, D_7\} \cup \{K_5\} \cup \{R_i \mid i \in \mathbb{N}_0\} \cup \{S_i \mid i \in \mathbb{N}_0\} \cup \{T_i \mid i \in \mathbb{N}_0\}$-free.*
(2) *$\mathcal{C}$ satisfies $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ and G is $\{K_5\} \cup \{R_i \mid i \in \mathbb{N}\} \cup \{S_i \mid i \in \mathbb{N}_0\} \cup \{T_i \mid i \in \mathbb{N}_0\}$-free.*
(3) *G has an integral $\mathcal{U}$-representation.*

Proof. By Lemma 6, the first statement implies the second. By Lemma 10, the second statement implies the third. Finally, by Lemmas 1, 2, 3, 4, and 5, the third statement implies the first.

It is straighforward yet tedious to derive from Theorem 4 the complete list of all minimal forbidden induced subgraphs of integral mixed unit interval graphs by considering the forbidden induced subgraphs of interval graphs and all minimal twin-free supergraphs of the graphs mentioned in (1) of Theorem 4. We leave the details to the reader.

Furthermore, Theorem 4 directly implies Theorem 3: Let G be a diamond-free graph satisfying (1) of Theorem 3. Note that we may assume that G is twin-free. This easily implies that G is K_4-free. Hence G satisfies (1) in Theorem 4, and therefore G satisfies (2) in Theorem 3. In Theorem 3, the implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (1) follows by noting that all R_k are forbidden induced subgraphs even for the class of $\mathcal{U}$-graphs.

Theorem 5. *There is a quadratic-time algorithm that, given an interval graph G, decides if G has an integral $\mathcal{U}$-representation, and if so, outputs such a representation for G.*

Proof. Let G be an interval graph. Since all twins of G can be detected in time $O(|V(G)|^2)$, we may assume that G is twin-free. Note that a linear ordering $\mathcal{C} = (C_1, \dots, C_q)$ of the maximal cliques of G can be computed in linear time (cf. [10]). If some C_j has more than four vertices, G does not have an integral $\mathcal{U}$-representation (cf. Lemma 2). Otherwise, we compute $c(u), \ell(u)$, and $r(u)$ for $u \in V(G)$ in linear time in an obvious way, and run `IntMixUniIntRep` to get the function I. Note that, since $|C_j| \leq 4$ for all j, `IntMixUniIntRep` has linear running time. Now we test whether I is an intersection representation of G or not by constructing the graph $H = (V(G), \{uv \mid I(u) \cap I(v) \neq \emptyset\})$ and checking if $G = H$, that is, checking $N_G(v) = N_H(v)$ for all $v \in V(G)$. This can be done in time $O(|V(G)|^2)$. If $G = H$, then I is an integral $\mathcal{U}$-representation of G. Otherwise, Lemma 10 and Theorem 4 imply that G has no such a representation.

References

1. Halldórsson, M.M., Patt-Shamir, B., Rawitz, D.: Online Scheduling with Interval Conflicts, in. In: Proceedings of the 28th Annual Conference on Theoretical Aspects of Computer Science (STACS), pp. 472–483 (2011)
2. Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph Classes: A Survey, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA (1999)
3. Corneil, D.G., Kim, H., Natarajan, S., Olariu, S., Sprague, A.P.: Simple linear time recognition of unit interval graphs. Inf. Process. Lett. 55, 99–104 (1995)
4. Corneil, D.G., Olariu, S., Stewart, L.: The ultimate interval graph recognition algorithm? (Extended abstract). In: Proceedings of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 175–180 (1998)
5. Corneil, D.G.: A simple 3-sweep LBFS algorithm for the recognition of unit interval graphs. Discrete Appl. Math. 138, 371–379 (2004)
6. Dourado, M.C., Le, V.B., Protti, F., Rautenbach, D., Szwarcfiter, J.L.: Mixed unit interval graphs. manuscript (2011)
7. Frankl, P., Maehara, H.: Open-interval graphs versus closed-interval graphs. Discrete Math. 63, 97–100 (1987)
8. Fulkerson, D.R., Gross, O.A.: Incidence matrices and interval graphs. Pacific J. Math. 15, 835–855 (1965)
9. Goldberg, P.W., Golumbic, M.C., Kaplan, H., Shamir, R.: Four strikes against physical mapping of DNA. J. Comput. Biol. 2, 139–152 (1995)
10. Golumbic, M.C.: Algorithmic Graph Theory and Perfect Graphs, Amsterdam, The Netherlands. Annals of Discrete Mathematics, vol. 57 (2004)
11. Heggernes, P., Suchan, K., Todinca, I., Villanger, Y.: Characterizing Minimal Interval Completions. In: Thomas, W., Weil, P. (eds.) STACS 2007. LNCS, vol. 4393, pp. 236–247. Springer, Heidelberg (2007)
12. Hell, P., Shamir, R., Sharan, R.: A fully dynamic algorithm for recognizing and representing proper interval graphs. SIAM J. Comput. 31, 289–305 (2001)
13. Herrera de Figueiredo, C.M., Meidanis, J., Picinin de Mello, C.: A linear-time algorithm for proper interval graph recognition. Inf. Process. Lett. 56, 179–184 (1995)
14. Kaplan, H., Shamir, R.: Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. SIAM J. Comput. 25, 540–561 (1996)
15. Kendall, D.G.: Incidence matrices, interval graphs, and seriation in archaeology. Pacific J. Math. 28, 565–570 (1969)
16. Kratsch, D., Stewart, L.: Approximating Bandwidth by Mixing Layouts of Interval Graphs. In: Meinel, C., Tison, S. (eds.) STACS 1999. LNCS, vol. 1563, pp. 248–258. Springer, Heidelberg (1999)
17. Krokhin, A.A., Jeavons, P.G., Jonsson, P.: The Complexity of Constraints on Intervals and Lengths. In: Alt, H., Ferreira, A. (eds.) STACS 2002. LNCS, vol. STACS 2002, pp. 443–454. Springer, Heidelberg (2002)
18. Papadimitriou, C.H., Yannakakis, M.: Scheduling interval-ordered tasks. SIAM J. Comput. 8, 405–409 (1979)
19. Rautenbach, D., Szwarcfiter, J.L.: Unit Interval Graphs - A Story with Open Ends. In: European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2011). Electronic Notes in Discrete Mathematics, vol. 38, pp. 737–742 (2011)
20. Roberts, F.S.: Indifference graphs. In: Harary, F. (ed.) Proof Techniques in Graph Theory, pp. 139–146. Academic Press (1969)