Transform for Simplified Weight Computations in the Fuzzy Analytic Hierarchy Process

Manju Pandey, Nilay Khare, and S. Shrivastava

Abstract. A simplified procedure for weight computations from the pair-wise comparison matrices of triangular fuzzy numbers in the fuzzy analytic hierarchy process is proposed. A transform T:R3→R1 has been defined for mapping the triangular fuzzy numbers to equivalent crisp values. The crisp values have been used for eigenvector computations in a manner analogous to the computations of the original AHP method. The objective is to retain both the ability to capture and deal with inherent uncertainties of subjective judgments, which is the strength of fuzzy modeling and the simplicity, intuitive appeal, and power of conventional AHP which has made it a very popular decision making tool.

Keywords: Fuzzy, AHP, Triangular Fuzzy Number, Fuzzy Synthetic Extent, Weight Vector, Eigenvector, Decision Making, Optimization and Decision Making.

1 Introduction

Conventional AHP treats decision making problems as follows [1, 2, 3, 4]. All decision making problems have at least one objective or goal, set of more than one alternative or option (from which a choice of the best alternative has to be made), and a set of criteria (and possibly sub-criteria) against which these alternatives are to be compared. In conventional AHP, first, the problem objective(s) and the criteria to be considered are defined. Second, the problem is arranged into a hierarchy with the problem objective or goal at the top level, criteria and sub-criteria at the

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intermediate levels, and the alternatives or options at the final level. Fig.1 shows the hierarchy for a decision making problem with 5 criteria and 5 options or alternatives. Third, pair-wise comparisons of the criteria are made and the (normalized) eigenvector of the pair-wise comparison matrix is computed to prioritize the criteria. Fourth, for each criterion, pair-wise comparisons of the alternatives are made and the (normalized) eigenvector of the pair-wise comparison matrix is computed for ranking the alternatives with respect to a particular criterion. Fifth, weighted sum of ranks of each alternative with respect to different criteria and the corresponding criteria priorities is computed to determine overall ranks of alternatives. Last step, the alternatives are ranked in the order of rank/cost ratio.

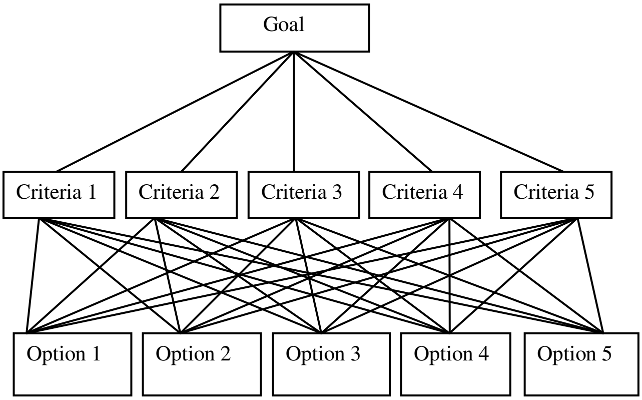

Fig. 1 Arrangement of Goal, Criteria, and Options in AHP

Conventional AHP uses a 9-point ratio scale called Saaty's scale which is used by the decision makers for assigning criteria and alternative weights during the pair-wise comparisons. The simplicity, intuitive appeal and power of AHP as a decision making tool is beautifully illustrated with a hypothetical example in [5].

Fuzzy logic was propounded by Lotfi Asker Zadeh with the objective of mathematically handling situations with inherent uncertainties and imprecision and subjective matters which are not readily amenable to mathematical modeling [6]. Fuzzy AHP is an application of the extent analysis method [7, 8, 9]. The scale for choosing preferences is based on triangular fuzzy numbers (TFNs). For prioritizing criteria and alternatives, fuzzy AHP relies on the computation of synthetic fuzzy extent values from pair-wise comparison matrices. The degree of possibility concept is used for determining the order relationship between triangular fuzzy numbers. Computations are based on fuzzy number arithmetic [10, 11, 12, 13] and fuzzy addition, subtraction, multiplication, and inverse operations are defined. The original AHP method, however, uses matrix eigenvector computations in the prioritization steps, and simple ordering and arithmetic of real numbers.

In this paper, a transform $\mathbf{T}:\mathbf{R}^3 \rightarrow \mathbf{R}^1$ has been defined for mapping TFNs to equivalent crisp numbers. The transform is an attempt to represent the value of the

TFN closely. The crisp equivalents of the triangular fuzzy numbers are then used for matrix eigenvector computations and for ordering in a manner analogous to the original AHP method. It is empirically shown through some numerical examples that the priority vectors match closely those obtained from Chang's fuzzy AHP.

2 Triangular Fuzzy Numbers

A triangular fuzzy number depicted in Fig. 2 is a triplet *(l,m,u)* where $l,m,u \in \mathbf{R}$, i.e., $(l,m,u) \in \mathbf{R}^3$. The triangular fuzzy number [10][11][12][13] is defined as follows:

$$t = \begin{cases} 0, x < l \\ \dfrac{x-l}{m-l}, l \le x \le m \\ \dfrac{u-x}{u-m}, m \le x \le u \\ 0, x > u \end{cases}$$

The "value" of the TFN is "close to" or "around" *m* [13]. Membership is linear on both sides of m and decreases to zero at *l* for values less than *m* and at *u* for values greater than *m*.

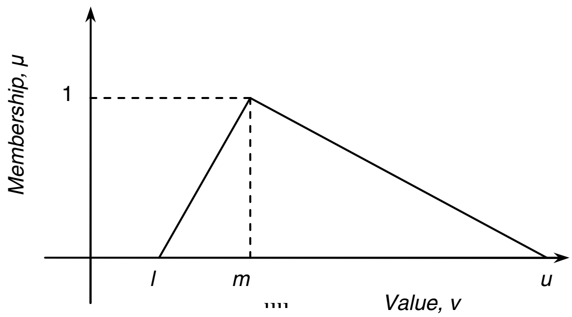

Fig. 2 Triangular Fuzzy Number

3 Transform

The origin of the axis is translated to (*m,I*) and rotated clockwise by 90º as shown in Fig.3. The transformed positive X-axis is now along the negative Y-axis in the original system and the transformed positive Y-axis is along positive X-axis of the original system.

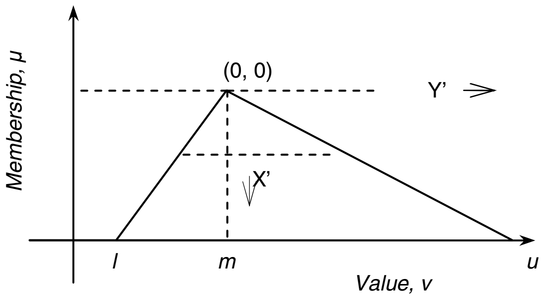

Fig. 3 Axis translated to (m,1) and rotated clockwise by 90º

Coordinates of the key points in the transformed system are in Table 1 below

Table 1 Coordinates with reference to the new coordinate system

Coordinate in Old System	**Transformed Coordinate**
(0,0)	*(1,-m)*
(l,0)	*(1, l-m)*
(m,0)	*(1,0)*
(u,0)	*(1,u-m)*
(m,1)	*(0,0)*

Following computations have been performed with reference to the new coordinate system.

- Equation of the membership line to the right of the core m is $y_R=(u-m)x$
- Equation of the membership line to the left of the core m is $y_L=(l-m)x$
- Difference is $y_R - y_L = (u-m)x - (l-m)x=(u-l)x$
- Corresponding membership is x
- Product of membership and difference functions is $x(u-l)x$.
- Averaging the product

$$\int_0^1 x(u-l)xdx = \frac{(u-l)}{3}$$

The equivalent crisp value is: (core value + average value), or,

$$v_c = m + \frac{(u-l)}{3}.$$

For example, the crisp value corresponding to the triangular fuzzy number (1,4,10) is $4+\dfrac{(10-1)}{3}=7.$

4 Method

The proposed method involves computing the normalized eigenvectors of the crisp matrices corresponding to a given TFN pair-wise comparison matrix. The elements of the crisp matrix are the crisp transforms of the corresponding elements of the TFN matrix.

For comparison of results we demonstrate the method on three TFN matrices taken from literature [14, 15, 16]. For the TFN matrices considered, the reader is referred to the original papers which are publicly accessible on the internet. TFN matrices are mentioned in Tables 2 of [14], 1 of [15], and 2 of [16] respectively.

4.1 Numerical Example No. 1

From the TFN matrix of [14] we derive the following matrix of equivalent crisp values using the formula developed in the paper, i.e., $v_c = m + \dfrac{(u-l)}{3}$

$$\begin{bmatrix} 1 & 1.333 & 1.833 & 2.333 \\ 1.443 & 1 & 1.333 & 1.833 \\ 0.837 & 1.443 & 1 & 1.33 \\ 0.59 & 0.837 & 1.443 & 1 \end{bmatrix}$$

Normalized eigenvector corresponding to the real eigenvalue is computed as $\begin{bmatrix} 0.31 & 0.28 & 0.23 & 0.19 \end{bmatrix}^T$. Here values have been rounded to two decimal places. Difference vector between this priority vector and the priority vector computed using Chang's synthetic extent method in [14] is $\begin{bmatrix} -0.01 & 0.01 & 0.01 & 0.00 \end{bmatrix}^T$ and the root mean squared difference is 0.00866.

4.2 Numerical Example No. 2

From the TFN matrix of [15] we derive the following matrix of equivalent crisp values using the formula developed in the paper

$$\begin{bmatrix} 1 & 0.85 & 0.6 & 2.283 & 1.113 \\ 2.34 & 1 & 1.003 & 2.893 & 1.7 \\ 2.816 & 1.85 & 1 & 3.46 & 2.28 \\ 0.77 & 0.523 & 0.44 & 1 & 0.683 \\ 1.43 & 0.983 & 0.71 & 2.196 & 1 \end{bmatrix}$$

Normalized eigenvector corresponding to the real eigenvalue is $\begin{bmatrix} 0.16 & 0.25 & 0.32 & 0.09 & 0.18 \end{bmatrix}^T$. Here values have been rounded to two decimal places. Difference vector between this priority vector and the priority vector computed using Chang's synthetic extent method in [15] is $\begin{bmatrix}-0.02 & -0.02 & 0.00 & 0.04 & 0.01\end{bmatrix}^T$ and the root mean squared difference is 0.0224.

4.3 Numerical Example No. 3

From the TFN matrix of [16] we derive the following matrix of equivalent crisp values using the formula developed in the paper

$$\begin{bmatrix}1 & 2.053 & 1.48 & 2.117 & 2.167\\ 0.847 & 1 & 3.953 & 1.223 & 3.48\\ 1.263 & 0.433 & 1 & 1.437 & 1.307\\ 0.963 & 1.193 & 1.19 & 1 & 1.093\\ 1.047 & 0.633 & 1.227 & 1.253 & 1\end{bmatrix}$$

Normalized eigenvector corresponding to the real eigenvalue is $\begin{bmatrix}0.259 & 0.274 & 0.154 & 0.164 & 0.149\end{bmatrix}^T$. Here values have been rounded to three decimal places. Difference vector between this priority vector and the priority vector computed using Chang's synthetic extent method in [16] is $\begin{bmatrix}0.013 & 0.001 & -.013 & 0.000 & -0.001\end{bmatrix}^T$ and the root mean squared difference is 0.0082.

5 Advantages over Chang's Method

In Chang's method, priority vectors are computed from the pair-wise comparison matrices of triangular fuzzy numbers using the following steps. First, the fuzzy synthetic extents are computed by summing all rows which is then divided by the total sum of rows for normalization.

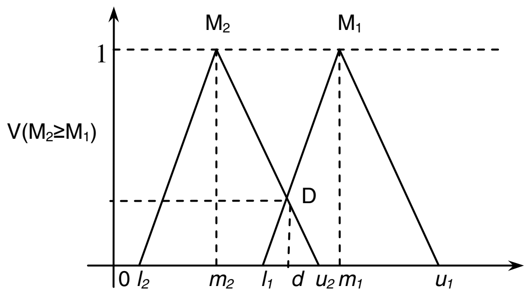

Fig. 4 Ordering of two triangular fuzzy numbers in Chang's method

In the second step, these fuzzy synthetic extent values are ordered by computing the degrees of possibility, V, of each fuzzy number being greater than the other in the fuzzy synthetic extent vector.

The fuzzy synthetic extents S_i of the first step are computed for a matrix $\tilde{P}_{mn}$ of triangular fuzzy numbers using the equation

$$S_i = \frac{\sum_{j=1}^{n} \tilde{P}_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{P}_{ij}}$$

In the second step, if M_1 and M_2 are two triangular fuzzy numbers as shown in Fig. 4 above, then the degree of possibility, V, of $M_2 \geq M_1$ is given by

$$V(M_2 \geq M_1) = \begin{cases} 1, & \text{if } m_2 = m_1 \\ 0, & \text{if } l_1 = u_2 \\ \dfrac{(l_1 - u_1)}{(m_2 - u_2) - (m_1 - l_1)}, & \text{otherwise} \end{cases}$$

For a m x n matrix of triangular fuzzy numbers, Chang's method involves m x $(n-1)$ fuzzy addition operations and m fuzzy division operations for computing the fuzzy synthetic extents. Also in the ordering step, it involves $m(m-1)/2$ degree of possibility, i.e., V calculations.

Chang's method has the following limitations:

- The method requires knowledge of fuzzy sets and fuzzy arithmetic involving triangular fuzzy numbers.
- The method involves ordering fuzzy numbers, that is, the ranking of fuzzy numbers based on the computation of degree of possibility. This is not intuitive.
- It does not extend classical AHP method of Saaty for priority vector computations.

Considering the above, the advantages of the method of this paper over Chang's method can be summarized as follows:

- The method does not require any knowledge of fuzzy arithmetic involving triangular fuzzy numbers.
- The method involves ordering numbers on the real line, which is intuitive, has a definite geometric interpretation, and is well understood.

- After the transform is applied to individual elements of the triangular fuzzy number matrix, computation proceeds exactly as in Saaty's classical analytic hierarchy process method.

Therefore, the method is easy to follow and adopt for a decision maker making the transition from classical AHP to fuzzy AHP. The results of the method closely match those of Chang's method as has been demonstrated in three numerical examples.

6 Conclusions

The transform and method discussed in this paper concern a simplified technique for computing the priority vectors from the pair-wise comparison matrices of triangular fuzzy numbers in the context of the fuzzy analytic hierarchy method. The resulting priority vectors closely match those obtained from Chang's fuzzy synthetic extent analysis which is the backbone of the fuzzy AHP method. It can be seen that with the sole exception of ranks 3 and 4 of numerical example 3 which get interchanged between the method developed in this paper and Chang's method, the order is preserved, and no other discrepancy in the order relation of the priorities is observed. In the exception mentioned also, it needs to be noted that the priorities assigned to 3 and 4 are the same if truncated to two decimal places by Chang's method, and therefore these ranks should be expectedly closer. The similar results are obtained by a rather simple transformation of the triangular fuzzy numbers in the pair-wise comparison matrices to equivalent crisp numbers and then proceeding in a manner similar to conventional AHP. Unlike Chang's method, the method does not require knowledge of fuzzy arithmetic, involves ordering of crisp real numbers in place of ordering fuzzy numbers, and involves priority vector computations using the familiar method of conventional AHP. Present method therefore has the simplicity, intuitive appeal and power of conventional AHP while retaining the ability to capture and deal with subjective information which is characteristic of fuzzy modeling.

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