Equilibrium Analysis of the Markovian Queues with Repairs and Vacations

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Abstract A single server Markovian queue with repairs and vacations is studied in this paper. Under a linear reward-cost structure, we investigate the behavior of customers with various levels of information regarding the system state. Equilibrium strategies for the customers under different levels of information are derived and the stationary behaviors of the system under these strategies are investigated.

Keywords Equilibrium strategies • Breakdowns • Vacations

1 Introduction

Studies about the economic analysis of queueing systems can go back at least to the pioneering work of Naor [5] who analyzed customers optimal strategies in an observable M/M/1 queue with a simple reward-cost structure. After the work of Naor, Edelson and Hildebrand [3] investigated this model. Then surveys on this subject were showed in [1, 2, 4] respectively. The paper is organized as follows. In Sect. 2, we describe the dynamics of the model and the reward-cost structure. In Sect. 3, we derive the equilibrium strategies for the customers under different levels of information. Finally, in Sect. 4, some conclusions are given.

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2   Model Description

We consider a single-server queue in which customers arrive according to a Poisson process with rate λ. We assume that the service times are exponentially distributed with rate u. The server alternates between on and off periods that are also exponentially distributed at rates ξ and θ respectively.

And the server is deactivated and begins a vacation as soon as the queue becomes empty. When a new customer arrives at an empty system, a setup process starts for the server to be activated. The time required to setup is also exponentially distributed with rate β. We assume that inter-arrival times, service times and setup times are mutually independent.

We represent the state at time t by the pair $(N(t); I(t))$, where $N(t)$ denotes the number of customers in the system and $I(t)$ denotes the state of the server. Hereinafter, we define a breakdown period at state 0 and a work period at state 1 as well as a vacation period at state 2. We assume that every customer receives reward of R units for completing service. Moreover, there exists a waiting cost of C units per time unit.

In the next sections we obtain customer optimal strategies for joining/balking. We distinguish two cases with respect to the level of information available to customers at their arrival instants:

(1) Fully observable case: Customers are informed about the queue length as well as the sever state $(N(t), I(t))$;
(2) Fully unobservable case: Customers are not informed about the queue length $N(t)$ nor the sever state $I(t)$.

3   Equilibrium Strategies

In this section we show that there exist equilibrium strategies in the two information cases that we defined above.

3.1   Fully Observable Case

We begin with the fully observable case in which there exist equilibrium strategies of threshold type.

Theorem 1. *In the fully observable M/M/1 queue with repairs and vacations, there exist a triple of thresholds*

$$(n_e(0), n_e(1), n_e(2)) = \left(\left\lfloor \left(\frac{R}{C} - \frac{1}{\theta} \right) \frac{\theta\mu}{\theta + \xi} \right\rfloor - 1, \left\lfloor \frac{R}{C} \frac{\theta\mu}{\theta + \varepsilon} \right\rfloor - 1, \right.$$
$$\left. \left\lfloor \left(\frac{R}{C} - \frac{1}{\beta} \right) \frac{\theta\mu}{\theta + \varepsilon} \right\rfloor - |1 \right). \tag{1}$$

Such that the strategy 'While arriving at time t, observe $(N(t), I(t))$; enter if $N(t) \leq n_e(I(t))$ and balk otherwise' is a weakly dominant strategy.

Proof. Consider an arriving customer, his expected net reward if he enter is $S(n; i) = R - CT(n; i)$, where $T(n; i)$ denotes his expected mean sojourn time given that he finds the system at state $(N(t); I(t))$ upon his arrival. We have

$$T(n, 0) = \frac{1}{\theta} + T(n, 1), \quad T(n, 2) = \frac{1}{\beta} + T(n, 1), \qquad n = 1, 2, \ldots, \tag{2}$$

$$T(n, 1) = \frac{1}{\mu + \varepsilon} + \frac{\varepsilon}{\mu + \varepsilon} T(n, 0) + \frac{\mu}{\mu + \varepsilon} T(n - 1, 1), \qquad n = 2, 3, \ldots. \tag{3}$$

We denote T_0 by the expected mean service time of a customer, then we get

$$T_0 = \frac{\mu}{\mu + \varepsilon} \cdot \frac{1}{\mu} + \frac{\varepsilon}{\mu + \varepsilon} \left(\frac{1}{\theta} + T_0 \right), T(1, 1)$$
$$= \frac{\mu}{\mu + \varepsilon} \left(\frac{1}{\mu} + T_0 \right) + \frac{\varepsilon}{\mu + \varepsilon} T(1, 0), T(0, 2) = \frac{1}{\beta} + T_0.$$

Then the following equations are obtained easily

$$T(n, 0) = \frac{1}{\theta} + (n + 1) \frac{\theta + \xi}{\theta\mu}, T(n, 1) = (n + 1) \frac{\theta + \xi}{\theta\mu}, \qquad n = 1, 2, \ldots, \tag{4}$$

$$T(n, 2) = \frac{1}{\beta} + (n + 1) \frac{\theta + \xi}{\theta\mu}, \qquad n = 0, 1, \ldots. \tag{5}$$

By solving $S(n, i) \geq 0$ for n, we obtain that customers decide to enter if and only if $n \leq n_e(I(t))$ where $n_e(0)$, $n_e(1)$ and $n_e(2)$ are given by Theorem 1.

3.2 Fully Unobservable Case

In this section we proceed the fully unobservable case. With a mixed strategy, an arriving customer joins with a certain probability q, and the effective arrival rate, or joining rate is λq.

Proposition 1. *Consider the fully unobservable model of the M/M/1 queue with repairs and vacations, the expected mean sojourn time of a customer who decides to enter is given by*

$$E[W] = \frac{(\xi\beta - \xi\theta - \theta^2)\lambda q + \theta(\xi\beta + \theta\beta + \theta\mu)}{\theta\beta[\theta\mu - \lambda q(\xi + \theta)]}. \tag{6}$$

Proof. Let $p(n, i)$ be the stationary distribution of the corresponding system and the balance equations are presented below.

$$(\lambda q + \theta)p(1, 0) = \xi p(1, 1), \tag{7}$$

$$(\lambda q + \theta)p(n, 0) = \lambda q p(n - 1, 0) + \xi p(n, 1), \quad n = 2, 3, \ldots, \tag{8}$$

$$(\mu + \xi + \lambda q)p(1, 1) = \theta p(1, 0) + \mu p(2, 1) + \beta p(1, 2), \tag{9}$$

$$(\mu + \varepsilon + \lambda q)p(n, 1) = \theta p(n, 0) + \mu p(n + 1, 1) + \beta p(n, 2) \\ + \lambda q p(n - 1, 1), n = 2, 3, \ldots, \tag{10}$$

$$\lambda q p(0, 2) = \mu p(1, 1) \tag{11}$$

$$(\beta + \lambda q)p(n, 2) = \mu p(n - 1, 2), \quad n = 1, 2, \ldots. \tag{12}$$

Define the partial stationary probability generating function of the system as $G_i(z) = \sum_{n=0}^{\infty} p(n, i)z^n, (|z| \le 1, i = 0, 1, 2)$ and we can get $G_0(z)$, $G_1(z)$ and $G_2(z)$ by solving the above balance equations. Then the mean number of the customers in the system is as follows

$$E[N] = \sum_{n=0}^{\infty} n(p(n, 0) + p(n, 1) + p(n, 2)) = G'_0(z)|_{z=1} + G'_1(z)|_{z=1} + G'_2(z)|_{z=1}$$

$$= \frac{(\xi\beta - \xi\theta - \theta^2)\lambda^2 q^2 + \theta(\xi\beta + \theta\beta + \theta\mu)\lambda q}{\theta\beta[\theta\mu - \lambda q(\xi + \theta)]}. \tag{13}$$

Hence, the expected mean sojourn time of a customer who decides to enter upon his arrival can be obtained by using Little's law $E[W] = E[N]/\lambda q$.

Theorem 2. *In the fully unobservable model of the M/M/1 queue with repairs and vacations, a unique Nash equilibrium mixed strategy 'enter with probability q_e' exists, where q_e is given by*

$$q_e = \begin{cases} q_e^*, & if \quad C\left(\frac{1}{\beta} + \frac{\theta + \xi}{\theta\mu}\right)\frac{\lambda(\xi\beta - \xi\theta - \theta^2) + \theta(\xi\beta + \theta\beta + \theta\mu)}{\theta\beta[\theta\mu - \lambda(\xi + \theta)]} \\ \\ 1, & if \qquad R \geq C\frac{\lambda(\xi\beta - \xi\theta - \theta^2) + \theta(\xi\beta + \theta\beta + \theta\mu)}{\theta\beta[\theta\mu - \lambda(\xi + \theta)]} \end{cases},$$

$$where \quad q_e^* = \frac{R\theta^2\mu\beta - C\theta(\xi\beta + \theta\beta + \theta\mu)}{\lambda\left[C(\xi\beta - \xi\theta - \theta^2) + R\theta\beta(\xi + \theta)\right]}. \tag{14}$$

Proof. We consider a tagged customer at his arrival instant. If he decide to enter the system, the expected net benefit he gets is

$$S(q) = R - C\frac{(\xi\beta - \xi\theta - \theta^2)\lambda q + \theta(\xi\beta + \theta\beta + \theta\mu)}{\theta\beta[\theta\mu - \lambda q(\xi + \theta)]}, \tag{15}$$

where $q \in [0, 1]$. We have

$$S(0) = R - C\left(\frac{1}{\beta} + \frac{\theta + \xi}{\theta\mu}\right), \tag{16}$$

$$S(1) = R - C\frac{\lambda(\xi\beta - \xi\theta - \theta^2) + \theta(\xi\beta + \theta\beta + \theta\mu)}{\theta\beta[\theta\mu - \lambda(\xi + \theta)]}. \tag{17}$$

The expected mean sojourn time is strictly increasing for $q \in (0, 1)$, so when $R \in \left(C\left(\frac{1}{\beta} + \frac{\theta+\xi}{\theta\mu}\right), C\frac{\lambda(\xi\beta-\xi\theta-\theta^2)+\theta(\xi\beta+\theta\beta+\theta\mu)}{\theta\beta[\theta\mu-\lambda(\xi+\theta)]}\right)$, (15) has a unique root in (0,1) which gives the first branch of (14). When $R \in \left[C\frac{\lambda(\xi\beta-\xi\theta-\theta^2)+\theta(\xi\beta+\theta\beta+\theta\mu)}{\theta\beta[\theta\mu-\lambda(\xi+\theta)]}, \infty\right)$, $S(q)$ is positive for every q. In other words, the unique equilibrium point is $q_e = 1$ in this case, which gives the second branch of (26).

4 Conclusion

In this paper we studied the equilibrium strategies of the M/M/1 queues with repairs and vacations. The equilibrium balking strategies were investigated for the two kinds of queues. To the author's knowledge, this is the first time that repairs and vacations are introduced into the queue system at the same time.

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