9
Basis Theorems

9.1 Bases and Nonbases for Π^0_1-Classes

The main theme of this chapter is this: Given a nonempty Π^0_1 class $\mathcal{C}$ what are the Turing degrees of members $f \in \mathcal{C}$?

Definition 9.1.1. A nonempty Π^0_1 class $\mathcal{C}$ is *special* if it contains no computable member.

It follows that if $T \subseteq 2^{<\omega}$ is a computable tree such that $[T]$ is special, then T^{ext} must be a *perfect* tree, meaning that every $\sigma \in T^{\text{ext}}$ admits incompatible extensions in T^{ext} because any isolated path would be computable. Therefore, every special Π^0_1 class has $2^{\aleph_0}$ members.

Definition 9.1.2. (i) Let $\mathcal{D} \subseteq 2^\omega$ be a class of sets. We call $\mathcal{D}$ a *basis for* Π^0_1 *classes* if every nonempty Π^0_1 class has a member $f \in \mathcal{D}$.

(ii) Let $\mathbf{D}$ be the corresponding class of Turing degrees of sets $X \in \mathcal{D}$. Then $\mathbf{D}$ is a *basis for* Π^0_1 *classes* if $\mathcal{D}$ is. Otherwise, we call $\mathbf{D}$ a *nonbasis*.

(iii) We call $\mathbf{D}$ an *antibasis* if whenever a Π^0_1 class contains a member of every degree in $\mathbf{d} \in \mathbf{D}$, it contains a member of every degree $\mathbf{d}$.

R.I. Soare, *Turing Computability*, Theory and Applications of Computability,
DOI 10.1007/978-3-642-31933-4_9

9.2 Previous Basis Theorems for Π^0_1-Classes

In §3.7 the Low Basis Theorem and exercises included some of the following basis theorems which we now list again. By the Kreisel Basis Theorem 8.5.1 (ii) we can always find $f \le_{\mathrm{T}} \emptyset'$. In 1960 Shoenfield improved the Kreisel Basis Theorem to f *strictly* below $\emptyset'$, namely $f <_T \emptyset'$.

Theorem 9.2.1 (Kreisel-Shoenfield Basis Theorem)**.** *Every nonempty* Π^0_1 *class* $\mathcal{C}$ *has a member* $f <_T \emptyset'$*.*

Proof. Given a Π^0_1 class $\mathcal{C}$, Shoenfield considered the Π^0_1 class $\mathcal{D}$ of all $\langle f, g\rangle$ such that $f \in \mathcal{C}$ and

$$(\forall e)[\ \Phi^f_e(e)\downarrow \quad \Longrightarrow \quad \Phi^f_e(e) \ \neq \ g(e)\ ].$$

He then applied Kreisel's result to $\mathcal{D}$. □

The previous Low Basis Theorem 3.7.2 substantially generalized these results by Kreisel and Shoenfield and will itself be generalized below.

Theorem 9.2.2 (Low Basis Theorem)**.** *The low sets form a basis for* Π^0_1.

Theorem 9.2.3. *The sets of c.e. degree form a basis for* Π^0_1.

We proved this in the Effective Compactness Theorem 8.5.1 (iii). We shall see that it is false for the sets of *incomplete* c.e. degree.

9.3 Nonbasis Theorems for Π^0_1-Classes

Definition 9.3.1. If A and B are disjoint sets, then S is a *separating set* if $A \subseteq S$ and $B \cap S = \emptyset$.

Theorem 9.3.2. *(i) If* W_e *and* W_i *are disjoint c.e. sets, then the class of separating sets is a* Π^0_1*-class.*

(ii) There is a nonempty Π^0_1*-class with no computable members.*

Proof. (i) Define a computable tree T with $[\,T\,]$ the class of separating sets of W_e and W_i. For σ with $|\sigma| = s$, put σ in T if $\forall x < |\sigma|$

$$x \in W_{e,s} \implies \sigma(x) = 1 \qquad .\ \&\ . \qquad x \in W_{i,s} \implies \sigma(x) = 0.$$

Hence, $f \in [\,T\,]$ iff

$$(\forall x)[x \in W_e \implies f(x) = 1 \qquad .\ \&\ . \qquad x \in W_i \implies f(x) = 0\,].$$

(ii) Let W_e and W_i be disjoint c.e. sets which are computably inseparable as defined in Exercise 1.6.26. □

Corollary 9.3.3. *The class of computable sets is not a basis for Π^0_1 classes (i.e., $\{\mathbf{0}\}$ is a nonbasis).*

We can generalize the preceding corollary as follows.

Theorem 9.3.4 (Jockusch and Soare, 1972a, Theorem 4)**.** *The class of sets of incomplete c.e. degree is not a basis for Π^0_1 classes (i.e., the class of c.e. degrees $\mathbf{d} < \mathbf{0}'$ is a nonbasis).*

Proof. Let A be the Post simple set of Theorem 5.2.3. Then $\overline{A}$ and every infinite subset $S \subseteq \overline{A}$ is effectively immune via $f(x) = 2x + 1$, and therefore is not of incomplete c.e. degree by Exercise 5.4.6. Furthermore, $\overline{A}$ is computably bounded by $f(x) = 2x$ and therefore $\overline{A}$ is not hyperimmune by Theorem 5.3.3. Let $\{F_x\}_{x\in\omega}$ be a disjoint strong array witnessing that $\overline{A}$ is not hyperimmune. Define the Π^0_1 class

$$\mathcal{C} = \{\ S : S \cap A \ = \ \emptyset \quad \& \quad (\forall x)[\ F_x \cap S \neq \emptyset\ ]\}.$$

This produces a nonempty Π^0_1 class $\mathcal{C}$ containing only infinite subsets of $\overline{A}$ and therefore having no members of incomplete c.e. degree. □

Note that $\mathcal{C}$ has no c.e. members and no members of incomplete c.e. degree.

9.4 The Super Low Basis Theorem (SLBT)

The proof of the Low Basis Theorem 3.7.2 gives even more information about the jump f' than was explicitly claimed, but explaining it requires some definitions.

Definition 9.4.1. A set $A \leq_{\mathrm{T}} \emptyset'$ is *super low* if $A' \leq_{\mathrm{tt}} \emptyset'$ or equivalently if A' is ω-c.e. by Theorem 3.8.8.

Theorem 9.4.2 (Super Low Basis Theorem (SLBT))**.** *Every nonempty Π^0_1 class $\mathcal{C} \subseteq 2^\omega$ has a member A which is super low and indeed A' is 2^{e+1}-c.e.*

We now give what was historically the first proof of the SLBT from c. 1969, by Jockusch and Soare. This unpublished result was subsequently obtained independently by others.

Proof. We construct a computable a sequence of strings $\{\sigma_s\}_{s\in\omega}$ such that $A := \lim_s \sigma_s$ is super low. Fix a computable tree T with $[T] = \mathcal{C}$. Define the computable tree,

$$U_{e,s} \quad = \quad \{\ \sigma :\ \Phi^{\sigma}_{e,\,s}\ (e){\uparrow}\ \} \tag{9.1}$$

Let $T_{0,\,s} = T$ for all s. For every s given $T_{e,s}$: (1) define $T_{e+1,\,s} = T_{e,\,s} \cap U_{e,s}$, the e-black strings, if the latter contains a string σ of length s; and (2) define $T_{e+1,\,s} = T_{e,\,s}$, the e-white strings, otherwise.

To visualize this e-strategy, fix e and the previous tree $T_{e,s}$. Begin by playing the e-black strategy of choosing σ_s to be e-black if possible until for some n all nodes of length n are e-white. In other words, try to outrun letting $\Phi_e^\sigma(e)\downarrow$ as long as possible. This may involve many changes in σ_s but no change in the e-black strategy. During this phase nest the i-strategies within the e-strategy for all $i > e$.

If ever there is a stage when there is an n such that all strings of length n are e-white, then make *one* change of e-strategy from e-black to e-white. Thereafter, the e-strategy exerts no influence on the i-strategies for $i > e$. To prove that this construction succeeds define the following computable function.

$$\widehat{g}(e,s) := \begin{cases} 1 & \text{if} \quad \Phi_{e,s}^{\sigma_s}(e)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\widehat{g}(e,s)$ is computable. Fix e and assume by induction that $g(j) = \lim_s \widehat{g}(j,s)$ for all $j < e$ and that $g(j) = A'(j)$. Now the e-strategy begins in the e-black case and $\sigma_s \neq \sigma_{s+1}$ only if σ_s becomes e-white. If this happens finitely often, then the final σ_s is e-black and $\lim_s \widehat{g}(e,s) = 0 = A'(e)$. If it happens infinitely often, then the e-white nodes cover T_e. By compactness there is a finite subcover and therefore an n when all strings of length n are e-white. At this point we change once from the e-black to the e-white strategy. Thereafter, σ_s never changes, $\widehat{g}(e,s) = g(e) = A'(e)$.

Furthermore, assume by induction that for $e-1$ there are at most 2^e stages when $\widehat{g}(e-1,s) \neq \widehat{g}(e-1,s+1)$. The e-strategy adds one more to each so that there are at most 2^{e+1} stages when $\widehat{g}(e,s) \neq \widehat{g}(e,s+1)$. (This is the same injury pattern as for the Friedberg-Muchnik finite injury construction.) □

9.5 The Computably Dominated Basis Theorem

The key idea in the next theorem is to use a $\emptyset''$ oracle to build a member f of a given Π_1^0 class with the property that we can decide whether Φ_e^f is total or not at a definite stage of the construction. This differs from the proof of the Low Basis Theorem, where we needed only a $\emptyset'$ oracle to similarly decide whether $\Phi_e^f(e)$ converges or not. In both cases, however, we use the same technique (known as *forcing with* Π_1^0 *classes*) of continually pruning an infinite computable tree while preserving certain desired properties.

Recall that a function f is *computably dominated* (*hyperimmune-free*) if every function $h \equiv_{\mathrm{T}} f$ is dominated by some computable function g. (See also Definition 5.6.1.)

Theorem 9.5.1 (Computably Dominated Basis Theorem, Jockusch and Soare, 1972b)**.** *Every nonempty Π^0_1 class has a member f which is low$_2$ and computably dominated.*

Proof. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$. We build a sequence of infinite computable trees

$$T = T_0 \supseteq T_1 \supseteq \cdots$$

as follows. Given T_e, define for each $x \in \omega$ the set

$$U_{e,x} = \{\sigma \in T_e : \Phi^\sigma_{e,|\sigma|}(x)\uparrow\},$$

noting that this is a computable subtree of T_e whose index as such can be found effectively from e, x, and an index for T_e. Now $\emptyset''$ can determine whether any of these subtrees is infinite, since this amounts to answering the following Σ^0_2 question:

$$(\exists x)(\forall n)(\exists \sigma)_{|\sigma|=n}\ [\ \sigma \in U_{e,x}\ ]?$$

If so, let $T_{e+1} = U_{e,x}$ for the least x such that $U_{e,x}$ is infinite, and otherwise let $T_{e+1} = T_e$. In the former case, $\Phi^f_e(x)\uparrow$ for all $f \in [T_{e+1}]$, so Φ^f_e is not total, and in the latter, $\Phi^f_e(y)\downarrow$ for all y and all $f \in [T_{e+1}]$, so Φ^f_e is total.

As usual, take $f \in \bigcap_{e\in\omega} [T_e]$. Then $\emptyset''$ can compute the set Tot^f of all $e \in \omega$ such that Φ^f_e is total, and hence also $f'' \equiv_T \mathrm{Tot}^f$, because the above construction was $\emptyset''$-effective. Therefore, whether or not $e \in \mathrm{Tot}^f$ was decided during the construction at a finite stage. Hence, f is low$_2$. To show that f is computably dominated, let h be an f-computable function and fix e such that $h = \Phi^f_e$. In particular, Φ^f_e is total, so during the construction it must have been that $U_{e,x}$ was finite for all x. Hence, for every x, there must exist an n such that $\Phi^\sigma_{e,|\sigma|}(x)\downarrow$ for all $\sigma \in T_e$ of length n; let n_x be the least such n for a given x. Since T_e is computable, we can effectively find n_x for every x, meaning that the function

$$g(x) = \max\{\Phi^\sigma_{e,|\sigma|}(x) : |\sigma| = n_x\ \wedge\ \sigma \in T_e\}$$

is computable. Note that g bounds h. □

Note that if $\mathcal{C}$ is a *special* Π^0_1 class, i.e., one with no computable members, then the above theorem yields a low$_2$ nonlow$_1$ member $f \in \mathcal{C}$, because no noncomputable, computably dominated f can be computable in $\emptyset'$, let alone be low, as we saw in Theorem 5.6.7.

9.6 Low Antibasis Theorem

For the purposes of the following theorem, we will say that a set $S \subseteq 2^{<\omega}$ is *isomorphic to* $2^{<\omega}$ provided there is a bijection $g : 2^{<\omega} \to S$ such that for all $\sigma, \tau \in 2^{<\omega}$, $\sigma \preceq \tau$ if and only if $g(\sigma) \preceq g(\tau)$. Notice that if a tree T has a subset isomorphic to $2^{<\omega}$ via a computable such bijection, then $[T]$ has a member of every degree. Indeed, for every real X, we have $Y = \cup_n g(X \upharpoonright n) \in [T]$. Clearly, $Y \leq_T X$, while to compute $X(n)$ from Y for a given n we search for a $\sigma \in 2^{<\omega}$ until we find one of length greater than n with $g(\sigma) \subset Y$, and then $\sigma(n) = X(n)$.

Theorem 9.6.1 (Low Antibasis Theorem, Kent and Lewis, 2009)**.** *Every* Π^0_1 *class that has a member of every nonzero low degree has one of every degree.*

Proof. [1] Fix a nonempty Π^0_1 class $\mathcal{C}$ not containing a member of every degree and let $T \subseteq 2^{<\omega}$ be a computable tree such that $\mathcal{C} = [T]$. We define a noncomputable low set A such that for all $e \in \omega$,

$$(9.2) \qquad \Phi^A_e = h \in 2^\omega \quad \Longrightarrow \quad [\ h \leq_{\mathrm{T}} \emptyset \quad \vee \quad h \notin [T]\ ].$$

In particular, $[T]$ has no member $h \equiv_T A$. We obtain A as $\cup_s \sigma_s$ where $\sigma_0 \preceq \sigma_1 \preceq \cdots$ are built in a $\emptyset'$-construction. Write $\Phi^\rho_e = \tau$ if

$$(\forall x < |\tau|)[\ \Phi^\rho_e(x)\downarrow = \tau(x)\ ].$$

Let $\sigma_0 = \emptyset$. At stage s+1 we are given σ_s.

Stage s+1 = 3e. Let $n = |\sigma|$. Using $\emptyset'$, define $\sigma_{s+1} \succ \sigma_s$ such that $\sigma_{s+1}(n) \neq \varphi_e(n)$.

Stage s+1 = 3e+1. Ask $\emptyset'$ whether there exists $\rho \succ \sigma_s$ such that $\Phi^\rho_e(e)$ converges. If so, define σ_{s+1} to be the least such ρ, and define $\sigma_{s+1} = \sigma_s$ otherwise.

Stage s+1 = 3e+2. There are two cases.

Case 1. There exist strings $\alpha \succ \sigma_s$ and τ such that $\Phi^\alpha_e = \tau$ and $\tau \notin T$. In this case let σ_{s+1} be the least such α.

Case 2. Otherwise. In this case it follows that if $\Phi^A_e = h$ total, then $h \in [T]$. We proceed as follows. For a given σ define the c.e. set

$$\begin{aligned} V_\sigma \quad = \quad & \{\langle \alpha, \beta \rangle : [\sigma \prec \alpha, \beta] \\ & \&\ \ (\exists \rho)(\exists \tau)[\ \Phi^\alpha_e = \rho \quad \& \quad \Phi^\beta_e = \tau\ ] \\ & \&\ \ (\exists x < \min\{|\rho|, |\tau|\})[\ \rho(x)\downarrow \neq\ \tau(x)\downarrow\ ]\}. \end{aligned}$$

(We say that $\langle \alpha, \beta \rangle$ form an *e-splitting of* σ.) Using $\emptyset'$ we search for a $\sigma \succ \sigma_s$ such that $V_\sigma = \emptyset$. We claim that this search must succeed, and we define $\sigma_{s+1} = \sigma$ for the least such σ found.

[1] This proof is due to Dzhafarov and Soare with comments by Jockusch.

Suppose the claim is false. We shall contradict the assumption that $[T]$ does not have a member of every degree. Define a map $h : 2^{<\omega} \mapsto 2^{<\omega}$ as follows. Let $h(\emptyset) = \sigma_{s+1}$. Having defined $h(\sigma)$ for some σ, search computably for the least member $\langle \alpha, \beta \rangle$ of the nonempty c.e. set V_σ. Then define $h(\sigma^\frown 0) = \alpha$ and $h(\sigma^\frown 1) = \beta$. Now define $g : 2^{<\omega} \mapsto T$ by letting $g(\sigma) = \Phi_e^{h(\sigma)}$ for all σ. Since Case 1 does not hold, it is clear that $g(\sigma) \in T$. Therefore, g defines an isomorphic copy of $2^{<\omega}$ in T, contrary to hypothesis.

The first two types of stages guarantee that $A = \cup_s \sigma_s$ is a low noncomputable set. It remains to prove the following lemma.

Lemma 9.6.2. *If $\Phi_e^A = h$ is total, then h is computable or $h \notin [T]$.*

Proof. If Case 1 held at Stage $s+1 = 3e+2$, then h would not be in $[T]$. So suppose Case 2 held. By construction, $\sigma_{s+1} \preceq A$ was such that $V_{\sigma_{s+1}} = \emptyset$. In other words, there are no e-splittings above σ_{s+1}. Thus, to compute $h(n)$ find the first $\alpha \succeq \sigma_{s+1}$ such that $\Phi_e^\alpha(n)\downarrow$. Now $\Phi_e^\alpha(n) = \Phi_e^A(n) = h(n)$, else there would have been an e-splitting above σ_s. □

□

Corollary 9.6.3. *If $\mathcal{C}$ is a nonempty Π_1^0 class which does not have a member of every degree, then there are infinitely many low degrees with no members in $\mathcal{C}$.*

Proof. Combine the proof of this theorem with Exercise 6.3.7, where we avoided the cone above a nonzero low degree and repeat for infinitely many low degrees uniformly below $\mathbf{0}'$. □

There are two notable features of the proof of the Low Antibasis Theorem 9.6.1. As in Exercise 6.3.7 we do not try to force the functional to be undefined. We merely look for e-splittings, which is a Σ_1 process, and then apply Lemma 9.6.2 if we cannot find them. Second, we do not actually build the computable bijection g but we *threaten* to. This is analogous to constructing a simple set A below a noncomputable c.e. set C where we threatened to build a computable characteristic function $g = C$. We did not build all of g but enough of g to force C to permit elements to enter A.

9.7 Proper Low$_n$ Basis Theorem

The following generalization of the Low Basis Theorem says that, up to degree, the restriction of the jump operator to any special Π_1^0 class is surjective. The trick used for pushing the jump of the member up to the desired set is like the one used in the standard proof of the Friedberg Completeness Criterion.

The following theorem was stated with proof by Jockush and Soare in 1972 after Theorem 2.1 and later by Cenzer in 1999.

Theorem 9.7.1. *For every set $A \geq_T \emptyset'$, every special Π^0_1 class has a member f satisfying $f \oplus \emptyset' \equiv_T f' \equiv_T A$.*

Proof. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$. We build a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \cdots$ as follows. Let T_e be given. If e is even, define T_{e+1} from T_e as in the proof of the Low Basis Theorem. If e is odd, say $e = 2i + 1$, note that T_e^{ext} must be perfect since $\mathcal{C}$ is special, so $\emptyset'$ can find the smallest extendible nodes $\sigma, \tau \in T_e$ such that $\sigma(x) = 0$ and $\tau(x) = 1$ for some x. Let T_{e+1} consist of all the nodes in T_e comparable with σ or τ, depending on whether $A(i) = 0$ or $A(i) = 1$, respectively.

Take $f \in \bigcap_{e \in \omega} [T_e]$. If e is even, T_{e+1} can be obtained from T_e computably in $\emptyset'$, and hence both $f \oplus \emptyset'$-effectively and A-effectively because $A \geq_T \emptyset'$. If e is odd, say $e = 2i + 1$, then to obtain T_{e+1} from T_e we need an oracle for $\emptyset'$ to find the extendible nodes σ and τ and the position x on which they disagree, and then an oracle for A since we need to know $A(i)$. But in this case, $i \in A$ iff $f(x) = 1$, so an oracle for f suffices to determine whether to let T_{e+1} consist of the nodes comparable with σ or the nodes comparable with τ. Since f' is decided during the construction, we consequently have that $f \oplus \emptyset' \leq_T f' \leq_T A \leq_T f \oplus \emptyset'$, as desired. $\square$