8
Open and Closed Classes

8.1 Open Classes in Cantor Space

Using ordinal notation identify the ordinal 2 with the set of smaller ordinals $\{0, 1\}$. Identify set $A \subseteq \omega$ with its characteristic function $f : \omega \to \{0, 1\}$ and represent the set of these functions as 2^ω. Use the conventions of the Notation section, especially the numbering σ_y of strings $\sigma \in 2^{<\omega}$. We use the notation on trees of §3.7 and sometimes use Convention 4.1.1 of dropping the superscript 0 in defining arithmetic classes. We now deal with *classes* $\mathcal{A} \subseteq 2^\omega$, i.e., *second-order* objects, rather than just *first-order* objects like *sets* $A \subseteq \omega$.[1]

Definition 8.1.1. (i) *Cantor space* is 2^ω with the following topology (collection of open classes). For every $\sigma \in 2^{<\omega}$ define the *basic clopen class* (closed and open class)

$$(8.1) \qquad [\![\sigma]\!] = \{ f : f \in 2^\omega \ \ \& \ \ \sigma \prec f \}.$$

The *open* classes of Cantor space are unions of basic clopen classes.[2]

[1]Some material from the chapters in Part II was modified from that in the paper by Diamondstone, Dzhafarov, and Soare [2011].

[2]The classes $[\![\sigma]\!]$ are called *clopen* because they are both closed and open. Cantor space and Baire space are both *separable*. They have a countable base of open classes as above. Therefore, every open class is a union of *countably* many basic open classes. Although these are classes they are often called *open sets*, viewing the objects as reals.

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(ii) A set $A \subseteq 2^{<\omega}$ is an *open representation* of the open class

(8.2) $$[\![A]\!] \quad = \quad \bigcup_{\sigma \in A} [\![\sigma]\!].$$

(We may assume A is closed *upwards*, i.e., $\sigma \in A$ and $\sigma \prec \tau$ implies $\tau \in A$.)[3]

(iii) A class $\mathcal{A}$ is *effectively open (computably open)* if $\mathcal{A} = [\![A]\!]$ for a computable set $A \subseteq \omega$. (See Theorem 8.1.2 (i).)

(iv) A class $\mathcal{A}$ is *(lightface)* Σ^0_1 (abbreviated *(lightface)* Σ_1) if there is a computable R such that

(8.3) $$\mathcal{A} = \{\ f \ : \ (\exists x)\, R(\ f \restriction x\ )\ \}.$$

(v) A class $\mathcal{A}$ is *(boldface)* $\boldsymbol{\Sigma^0_1}$ if (8.3) holds with R replaced by R^X computable in some $X \subseteq \omega$. In this case we say $\mathcal{A}$ is $\Sigma^{0,X}_1$ or simply Σ^X_1.

Theorem 8.1.2 (Effectively Open Classes)**.** *Let* $\mathcal{A} \subseteq 2^\omega$.

(i) If $\mathcal{A} = [\![A]\!]$ *with* A *c.e., then* $\mathcal{A} = [\![B]\!]$ *for some* computable *set* $B \subseteq \omega$.

(ii) $\mathcal{A}$ *is effectively open iff* $\mathcal{A}$ *is* (lightface) Σ^0_1.

(iii) $\mathcal{A}$ *is open iff* $\mathcal{A}$ *is* (boldface) $\boldsymbol{\Sigma^0_1}$.

Proof. (i) Let $\mathcal{A} = [\![A]\!]$ with A c.e. and upward closed. Let $A = \cup_s A_s$ for a computable enumeration $\{A_s\}_{s \in \omega}$. Define a computable set B with $\mathcal{A} = [\![B]\!]$ as follows. At stage s, for every σ with $|\sigma| = s$ put σ into B if $(\exists \rho \preceq \sigma)[\rho \in A_s]$, and put σ into $\overline{B}$ otherwise. If $\sigma \in A$, then $\sigma \in A_s$ for some s, and every $\tau \succeq \sigma$ with $|\tau| \geq s$ is put into B. Hence $[\![\sigma]\!] \subseteq [\![B]\!]$. Therefore, $[\![A]\!] \subseteq [\![B]\!]$. Clearly, $B \subseteq A$ since A is upward closed. Therefore, $[\![B]\!] \subseteq [\![A]\!]$.

(ii) Let $\mathcal{A}$ be effectively open. Then $\mathcal{A} = [\![B]\!]$ for some B computable. Define $R(\sigma)$ iff $\sigma \in B$. Now $f \in \mathcal{A}$ iff $(\exists x)\, R(f \restriction x)$. Hence, $\mathcal{A}$ is Σ^0_1. Conversely, assume $\mathcal{A}$ is Σ^0_1 via a computable R satisfying (8.3). Define $A = \{\sigma : R(\sigma)\}$. Then $\mathcal{A} = [\![A]\!]$.

(iii) Relativize the proof of (ii) to a set $X \subseteq \omega$. □

8.2 Closed Classes in Cantor Space

Recall the tree notation defined in §3.7.

[3] If $\sigma \in A$ and $\sigma \prec \tau$ but $\tau \notin A$ we may add τ to A without changing $[\![A]\!]$.

Definition 8.2.1. (i) A *tree* $T \subseteq 2^{<\omega}$ is a set closed under initial segments, i.e., $\sigma \in T$ and $\tau \prec \sigma$ imply $\tau \in T$. (By our canonical coding of strings $\sigma \in 2^{<\omega}$ we may think of T as a subset of ω.) The set of *infinite paths* through T is

$$[\,T\,] = \{\ f\ :\ (\forall n)\,[\ f \restriction n \in T\ ]\ \}. \tag{8.4}$$

(ii) A class $\mathcal{C} \subseteq 2^\omega$ is *(lightface)* Π^0_1 if there is a computable relation $R(x)$ such that

$$\mathcal{C} = \{\ f\ :\ (\forall x)\, R(\ f \restriction x\ )\ \}. \tag{8.5}$$

A class $\mathcal{C}$ is *(boldface)* $\boldsymbol{\Pi^0_1}$ if (8.5) holds for R^X computable in some $X \subseteq \omega$. This is also written $\Pi^{0,X}_1$ and is abbreviated Π^X_1.

(iii) A class $\mathcal{C} \subseteq 2^\omega$ is *effectively closed* (*computably closed*) if its complement is effectively open. A set $\mathcal{C} \subseteq 2^\omega$ is *closed* if its complement is open.

Theorem 8.2.2 (Effectively Closed Sets and Computable Trees)**.** *Fix* $\mathcal{C} \subseteq 2^\omega$*. TFAE:*

(i) $\mathcal{C} = [\,T\,]$ *for some computable tree* T*;*

(ii) $\mathcal{C}$ *is effectively closed;*

(iii) $\mathcal{C}$ *is a* Π^0_1 *class.*

Corollary 8.2.3 (Closed Sets and Trees)**.** *Fix* $\mathcal{C} \subseteq 2^\omega$*. The following are equivalent (TFAE):*

(i) $\mathcal{C} = [\,T\,]$ *for some tree* T*;*

(ii) $\mathcal{C}$ *is closed;*

(iii) $\mathcal{C}$ *is a (boldface)* $\boldsymbol{\Pi^0_1}$ *class.*

Proof. Relativize the proof of Theorem 8.2.2 to $X \subseteq \omega$. □

Remark 8.2.4. (Representing Closed Classes). The most convenient way of representing open and closed classes is with trees. If $\mathcal{C}$ is closed we choose a tree T such that $\mathcal{C} = [\,T\,]$. Define $A = \omega - T$. Then T is downward closed, A is upward closed, as sets of strings, and A defines the open set $[\![\,A\,]\!] = 2^\omega - [\,T\,] = \overline{\mathcal{C}}$. Note that the representations A and T are complementary in ω and the open and closed classes $[\![\,A\,]\!]$ and $[\,T\,]$ are complementary in 2^ω. The only difference between the effective case and general case is whether the tree T is computable or only computable in some set X.

We may imagine a path $f \in 2^\omega$ trying to climb the tree T without passing through a node $\sigma \in A$. If f succeeds, then $f \in \mathcal{C} = [\,T\,]$. However, if $f \succ \sigma$ for even *one* node $\sigma \in A$, then f falls off the tree forever and $f \notin \mathcal{C}$.

8.3 The Compactness Theorem

Particularly useful features of Cantor space are the well-known Compactness Theorem and the Effective Compactness Theorem 8.5.1, both of which lead to the study of one of our main topics, Π^0_1 classes.

Theorem 8.3.1 (Compactness Theorem)**.** *The following easy and well-known properties hold for Cantor Space 2^ω. The term* "compactness" *refers to any of them, but particularly to (iv).*

(i) *(Weak König's Lemma, WKL). If $T \subseteq 2^{<\omega}$ is an infinite tree, then $[\,T\,] \neq \emptyset$.*

(ii) *If $T_0 \supseteq T_1 \ldots$ is a decreasing sequence of trees with $[\,T_n\,] \neq \emptyset$ for every n, and intersection $T_\omega = \cap_{n\in\omega} T_n$, then $[\,T_\omega\,] \neq \emptyset$.*

(iii) *If $\{\mathcal{C}_i\}_{i\in\omega}$ is a countable family of closed sets such that $\cap_{i\in F} \mathcal{C}_i \neq \emptyset$ for every* finite *set $F \subseteq \omega$, then $\cap_{i\in\omega} \mathcal{C}_i \neq \emptyset$ also.*

(iv) *(Finite subcover). Any open cover $[\![\,A\,]\!] = 2^\omega$ has a finite open subcover $F \subseteq A$ such that $[\![\,F\,]\!] = 2^\omega$.*

Proof. (i) Let T be infinite. We construct a sequence of nodes $\sigma_0 \prec \sigma_1 \ldots$ such that $f = \cup_{n\in\omega} \sigma_n$ and $f \in [\,T\,]$. Define a node σ to be *large* if there are infinitely many $\tau \succ \sigma$ such that $\tau \in T$. Define $\sigma_0 = \emptyset$, which is large. Given σ_n large, one of $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ must be large by the pigeon-hole principle. (This fails for Baire space ω^ω, where there may be infinitely many possible successors none of which is large.) Let $\sigma_{n+1} = \sigma_n{}^\frown 0$ if it is large and $\sigma_{n+1} = \sigma_n{}^\frown 1$ otherwise.

(ii) Build a new tree S by putting σ of length n into S if $\sigma \in \cap_{i\leq n} T_i$ (which is also a tree). Note that S is infinite because $[\,T_n\,] \neq \emptyset$ for every n. By König's Lemma (i) there exists $f \in [\,S\,]$, but $[\,S\,] = [\,T_\omega\,]$.

(iii) Define $\widehat{\mathcal{C}_i} = \cap_{j\leq i} \mathcal{C}_j$. Hence, $\widehat{\mathcal{C}_0} \supseteq \widehat{\mathcal{C}_1} \ldots$ is a decreasing sequence of nonempty closed sets. Choose a decreasing sequence of computable trees $T_0 \supseteq T_1 \ldots$ such that $[\,T_i\,] = \widehat{\mathcal{C}_i}$ and apply (ii).

(iv) Suppose $[\![\,A\,]\!]$ is an open cover of 2^ω but $[\![\,F\,]\!] \not\supseteq 2^\omega$ for any finite subset $F \subset A$. Hence, the closed set $[T_F] = 2^\omega - [\![\,F\,]\!]$ is nonempty for all $F \subseteq A$. Therefore, $\mathcal{C} = \bigcap_{F\subset A} \; [\,T_F\,] \neq \emptyset$ by (iii), but $\mathcal{C} = 2^\omega - [\![\,A\,]\!] \neq \emptyset$. Hence, $[\![\,A\,]\!] \not\supseteq 2^\omega$. □

8.4 Notation for Trees

Recall the notation in §3.7 for a tree $T \subseteq 2^{<\omega}$:

$$T_\sigma \;=\; \{\; \tau \in T \;:\; \sigma \preceq \tau \quad \text{or} \quad \tau \prec \sigma \;\};$$

$$T^{\,\text{ext}} \;=\; \{\; \sigma \in T \;:\; (\exists f \succ \sigma)[\; f \in [\,T\,]\;]\;\}.$$

A path $f \in [\,T\,]$ is *isolated* if $(\exists \sigma)[\,[\,T_\sigma\,] \;=\; \{\,f\,\}\,]$. We say that σ *isolates* f because $[\![\,\sigma\,]\!] \,\cap\, [\,T\,] = \{f\}$ and we call σ an *atom.* If f is isolated we say it has Cantor-Bendixson rank 0. If f is not isolated, then f is a *limit point* and has rank ≥ 1. (See Definition 8.7.5 and surrounding exercises for Cantor-Bendixson rank.)

8.5 Effective Compactness Theorem

For a *computable* tree $T \subseteq 2^{<\omega}$ we can establish the following effective analogues of the Compactness Theorem 8.3.1.

Theorem 8.5.1 (Effective Compactness Theorem)**.** *Let $T \subseteq 2^{<\omega}$ be a computable tree.*

(i) $T^{\,\text{ext}}$ is a Π^0_1 set. Hence, $\overline{T^{\,\text{ext}}}$ is Σ^0_1, $\overline{T^{\,\text{ext}}} \leq_{\text{m}} \emptyset'$, and $T^{\,\text{ext}} \leq_{\text{T}} \emptyset'$.

(ii) (Kreisel Basis Theorem) $[\,T\,] \neq \emptyset \quad\Longrightarrow\quad (\exists f \leq_{\text{T}} \emptyset')[\; f \in [\,T\,]\;]$. *(This was generalized in the* Low Basis Theorem 3.7.2.*)*

(iii) If $f \in [\,T\,]$ is the lexicographically least member, then f has c.e. degree.

(iv) If $f \in [\,T\,]$ is isolated, then f is computable. If $[\,T\,]$ is finite, then all paths are isolated and therefore computable.

(v) Given an open cover $[\![\,A\,]\!] = 2^\omega$ with A c.e. there is finite subset $F \subseteq A$ such that $[\![\,F\,]\!] = 2^\omega$ and a canonical index for F can be found uniformly in a c.e. index for A.

Proof. (i) The formal definition of $T^{\,\text{ext}}$ in (3.22) has one function quantifier, and it is in Σ^1_1 form. Indeed is this the best we can do for Baire space ω^ω. However, for Cantor space 2^ω we can use the Compactness Theorem 8.3.1 (i) to reduce it to one arithmetical quantifier.

(8.6) $\sigma \in \overline{T^{\,\text{ext}}} \iff T_\sigma$ is finite $\iff (\exists n)(\forall \tau \succ \sigma)_{\,|\tau|=n}\;[\;\tau \notin T\;]$.

This is a Σ^0_1 condition because the second quantifier on τ is bounded by $|\tau| = n$ and acts like a finite disjunction. (See Theorem 4.1.4 (vi).)

(ii) Now use a $\emptyset'$ oracle to choose $f \in [\,T\,]$ such that $f = \cup_n \sigma_n$. Given $\sigma_n \in T^{\,\text{ext}}$, let $\sigma_{n+1} = \sigma_n\widehat{\ }0$ if $\sigma_n\widehat{\ }0 \in T^{\,\text{ext}}$, and $\sigma_n\widehat{\ }1$ otherwise.

(iii) (This gives a stronger conclusion than (ii).) Let f be the lexicographically least member of $[\,T\,]$, i.e., in the dictionary ordering $<_L$ on the alphabet $\{0, 1\}$. (Think of the tree T as growing downwards and $\sigma <_L f$ as denoting that σ is to the left of f lexicographically.) Define the following c.e. set of nodes $M \subseteq \overline{T^{\text{ext}}}$ such that $M \equiv_{\text{T}} f$:

$$M = \{\sigma : (\forall\tau)_{|\tau| = |\sigma|} [\, [\, \tau \in T \ \& \ \tau \leq_L \sigma \,] \quad \Longrightarrow \quad \tau \in \overline{T^{\text{ext}}} \,] \,\}$$

(We just wait until σ and all its predecessors of length $|\sigma|$ have appeared nonextendible. Then we put σ into M. In this way we enumerate all nodes $\tau <_L f$. Therefore, f determines a *left c.e* set, one where when σ is enumerated, all later strings τ enumerated satisfy $\sigma \leq_L \tau$.)

(iv) Choose $\sigma \in T$ with $[\, T_\sigma \,] = \{f\}$. To compute f assume we have computed $\tau = f \upharpoonright n$. Exactly one of $\tau\widehat{\ }0$ and $\tau\widehat{\ }1$ is extendible. Enumerate $\overline{T^{\text{ext}}}$ until one of these nodes appears and take the other one.

(v) Assume $[\![\, A \,]\!] = 2^\omega$ with A c.e. Enumerate A until a finite set $F \subseteq A$ is found with $[\![\, F \,]\!] = 2^\omega$ by the Compactness Theorem (iv). We can search until we find it. □

Remark 8.5.2. Note that the *conclusions* in the Effective Compactness Theorem 8.5.1 have various levels of effectiveness even though the *hypotheses* are all effective. In (v) if $[\![\, A \,]\!]$ covers 2^ω then the passage from A to F is *computable* because we simply enumerate A until F appears (as with any Σ_1 process). However, if $[\![\, A \,]\!]$ *fails* to cover 2^ω then the complementary closed class $[\, T \,] = 2^\omega - [\![\, A \,]\!]$ is nonempty. Then (ii) gives a path $f \in [\, T \,]$ with $f \leq_{\text{T}} \emptyset'$ and (iii) even produces a path of c.e. degree, but neither produces a *computable* path f because, given an extendible string $\sigma \prec f$, the process for the proof of König's Lemma in Theorem 8.3.1 (i) does not computably determine whether to extend to $\sigma\widehat{\ }0$ or $\sigma\widehat{\ }1$. In Theorem 9.3.2 we shall construct a computable tree with paths but no computable paths.

8.6 Dense Open Subsets of Cantor Space

The following important notion of *dense sets* will be developed more later.

Definition 8.6.1. Let $\mathcal{S}$ be Cantor space 2^ω.

(i) A set $\mathcal{A} \subseteq \mathcal{S}$ is *dense* if $(\forall\sigma)\,(\exists f \succ \sigma)\,[\, f \in \mathcal{A} \,]$.

(ii) An open set $\mathcal{A} \subseteq \mathcal{S}$ is *dense open* if

$$(\forall\tau)(\exists\sigma \succeq \tau)(\forall f \succ \sigma)\,[\, f \in \mathcal{A} \,]. \tag{8.7}$$

(iii) A class $\mathcal{B} \subseteq \mathcal{S}$ is G_δ, i.e., boldface $\mathbf{\Pi^0_2}$, if $\mathcal{B} = \cap_i \mathcal{A}_i$, a countable intersection of open sets $\mathcal{A}_i$.

To be *dense* $\mathcal{A}$ must contain a point f in every basic open set $[\![\,\sigma\,]\!]$. To be *dense open* $\mathcal{A}$ must contain an *entire basic open set* $[\![\,\tau\,]\!] \subseteq [\![\,\sigma\,]\!]$ for every basic open set $[\![\,\sigma\,]\!]$. Notice that a set is dense open iff it is both dense and open.

After open and closed sets, much attention has been paid in point set topology to G_δ sets. If the sets $\mathcal{A}_i$ are *dense open* sets, then they have special significance. In §14.1.3 we shall explore Banach-Mazur games for finding a point $f \in \cap_i \mathcal{A}_i$ where the $\mathcal{A}_i$ are *dense open* sets. This is the paradigm for the *finite extension* constructions in Chapter 6, where we used the method to construct sets and degrees meeting an infinite sequence of "requirements." Meeting a requirement R_i amounts to meeting the corresponding dense open set $\mathcal{A}_i$.

8.7 Exercises

Exercise 8.7.1. We use the notation and definitions of §8.1, including the open representation A of $[\![\,A\,]\!]$ and the closed representation $T = \overline{A}$ of the closed set $[\,T\,] = 2^\omega - [\![\,A\,]\!]$, and we use the tree notation of §3.7 on the Low Basis Theorem.

(i) Define the open representation A to be the set of strings σ containing at least *two* 0's, and let $T = \overline{A}$. Describe the paths $f \in [\,T\,]$. Which are the limit points and which are the isolated ones?

(ii) Next define the open representation A to be the set of strings σ containing at least *three* 0's, and let $T = \overline{A}$. Describe the paths $f \in [\,T\,]$. (See Exercise 8.7.9 for the Cantor-Bendixson rank of these points, which gives much deeper insight into the structure of $[\,T\,]$ when three 0's are replaced by n 0's.)

Exercise 8.7.2. Prove that if T is computable and $[\,T\,]$ has exactly *one* limit point f, then $f \leq_{\mathrm{T}} \emptyset''$.

Exercise 8.7.3. Prove that there is a computable tree $T \subset 2^{<\omega}$ such that $[T]$ contains a unique limit point $f \equiv_{\mathrm{T}} \emptyset''$.

Exercise 8.7.4. (i) Define

$$\Gamma(T) \;=\; \{\, \sigma \;:\; \text{card}\,([T_\sigma]) < \infty \,\}, \tag{8.8}$$

i.e., the nodes σ with only finitely many paths $f \in [\,T\,]$ with $f \succ \sigma$.

(ii) If $T = T^{\text{ext}}$, define the set of *splitting nodes*,

$$\mathcal{S}(T) \;=\; \{\, \sigma : (\exists \rho \in T)(\exists \tau \in T)\,[\,\sigma \prec \rho \;\;\&\;\; \sigma \prec \tau \;\;\&\;\; \rho \,|\, \tau\,]\,\}, \tag{8.9}$$

the nodes σ which *split* in T in the sense that some ρ and τ split σ, where $\rho \mid \tau$ denotes that $(\exists x)\,[\,\rho(x)\!\downarrow \neq \tau(x)\!\downarrow\,]$.

(i) Prove that if $\sigma \in \Gamma(T)$ and $f \in [\,T_\sigma\,]$ then f is isolated.

(ii) Prove that if $f \in [\,T\,]$ is not isolated then every $\sigma \prec f$ lies in $S(T)$.

(iii) Prove that $\mathcal{S}(T)$ is Σ_1 in T and hence $\mathcal{S}(T) \leq_{\mathrm{T}} T\,'$.

Definition 8.7.5. (Cantor-Bendixson Derivative for tree T). Fix a tree T. For $\sigma \in T$ define the *Cantor-Bendixson rank* $r(\sigma)$ of σ relative to T.

$$D^0(T) = T.$$

$$D^{\alpha+1}(T) = D^{\alpha}(T) - \Gamma(D^{\alpha}(T)) \text{ for } \Gamma \text{ defined in (8.8).}$$

$$D^{\lambda}(T) = \bigcap\{D^{\alpha}(T) : \alpha < \lambda\} \text{ for } \lambda \text{ a limit ordinal.}$$

$$r(\sigma) = (\mu\alpha)[\,\sigma \in D^{\alpha}(T) - D^{\alpha+1}(T)\,].$$

If there is no such α, define $r(\sigma) = \infty$.

Definition 8.7.6. (Cantor-Bendixson derivative for closed set $\mathcal{A}$). If $\mathcal{A}$ is a closed set, choose a tree T such that $[T] = \mathcal{C}$ and let $r(\sigma)$ be the rank above for tree T. If $r(\sigma) = \alpha$ and σ isolates f in $D^{\alpha}(T)$, then define $r(f) = \alpha$. If there is no such α then define $r(f) = \infty$.

The derivative of a closed set $\mathcal{C}$ is the set of all points which are not isolated points of $\mathcal{C}$, and we are iterating this derivative. Note that derivative of a closed set is closed.

Exercise 8.7.7. Prove that Definition 8.7.6 for the Cantor-Bendixson derivative of a closed set does not depend on the choice of the tree such that $[T] = \mathcal{C}$. Take any two trees T_1 and T_2 such that $[T_1] = [T_2] = \mathcal{C}$ and prove that the tree derivative of Definition 8.7.5 gives the same rank in both trees for any $f \in \mathcal{C}$. *Hint.* Keep applying the fact that $T_1^{\mathrm{ext}} = T_2^{\mathrm{ext}}$.

Exercise 8.7.8. $^{\diamond}$ (i) Prove that $D^{\alpha}(T)$ is a tree and hence $[D^{\alpha}(T)]$ is closed subset of $\mathcal{A} = [T]$.

(ii) Show that there is an ordinal β such that $D^{\beta}(T) = D^{\alpha}(T)$ for all $\alpha > \beta$. Define $D^{\infty}(T) = D^{\beta}(T)$. Prove that there is an $\alpha < \omega_1$ such that $D^{\alpha}(T) = D^{\infty}(T)$. We call $D^{\infty}(T)$ and $[D^{\infty}(T)]$ the *perfect kernel*.

(iii) Prove that either $D^{\infty}(T) = \emptyset$ or else $D^{\infty}(T)$ is a perfect tree, namely every $\sigma \in D^{\infty}(T)$ splits as defined above. In this case $D^{\infty}(T)$ has $2^{\aleph_0}$ many infinite paths.

(iv) Let β be as in (ii). Prove that $[D^{\alpha}(T)] - [D^{\alpha+1}(T)]$ is countable for every $\alpha < \beta$. Therefore, $\cup_{\alpha<\beta}[D^{\alpha}(T)]$ is countable, namely $[T] - [D^{\infty}(T)]$ is countable.

Exercise 8.7.9. Define the open representation A as in Exercise 8.7.1 and define $T = \overline{A}$.

(i) Analyze the Cantor-Bendixson rank of all points $f \in [\,T\,]$.

(ii) How does the rank change if we define A to be all strings having at least n 0's?

(iii) Define a computable tree T such that $[T]$ has a point of rank ω.