6

Oracle Constructions and Forcing

6.1 ⋆ Kleene-Post Finite Extensions

We have seen in Chapter 5 how Post tried to solve Post's Problem 5.1.1 by defining c.e. sets A with ever thinner complements. Post himself did not live to see the refutation of this approach by [Friedberg 1958], who constructed a maximal set with the thinnest complement of all, and the construction of a *complete* maximal set by Yates, which refuted Post's approach. Post moved on to understand full Turing reducibility in [Post 1944]. He gave an excellent intuitive description of one set being Turing reducible to another.

From 1944 until his death in 1954 Post worked to understand Turing reducibility and decision problems for c.e. sets. Post in [Post 1944] page 289 introduced and later defined in [Post 1948] degrees of unsolvability (Turing degrees,) as we have presented in Definition 3.4.1. Post thought carefully about the properties of Turing reducibility and wrote extensive notes. Just before his death in 1954 he gave his notes to Kleene, who revised and expanded them and published them as [Kleene and Post 1954]. This fundamental paper clarified the properties of a Turing reduction, including the Use Principle 3.3.9, and used it to construct sets of incomparable degree below $\emptyset'$, as in Theorem 6.1.1. The paper did not directly address Post's Problem, because it was for Δ_2 sets, not Σ_1 sets, but it laid the indispensable foundation for the later solution by Friedberg and Muchnik, presented in Chapter 7, who added a computable approximation to the Kleene-Post method to obtain Turing incomparable Σ_1 sets.

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The first major contribution in [Kleene and Post 1954] is the *finite extension oracle construction*. Here we fix some oracle X, such as $X = \emptyset'$ or $X = \emptyset''$, and build a set $A \leq_{\mathrm{T}} X$ by an X-computable construction of finite initial segments $\{\sigma_s\}_{s\in\omega}$ of A with $\sigma_s \prec \sigma_{s+1}$. For example, given σ_s, index e, and argument x, we can ask the $\emptyset'$-question,

$$(\exists \rho \succ \sigma_s)(\exists y)(\exists t)\,[\ \Phi^{\rho}_{e,t}(x){\downarrow}\ =\ y\ ]?$$

If so, we can define $\sigma_{s+1} = \rho$, which guarantees that $\Phi^A_e(x) = y$ for every $A \succ \sigma_{s+1}$ by the Use Principle 3.3.9. If not, then $\Phi^A_e(x)$ diverges for every $A \succ \sigma_s$ and we can define $\sigma_{s+1} \succ \sigma_s$ in any fashion.

By the finite extension of $\sigma_{s+1} \succ \sigma_s$ we have *decided (forced)* Turing computability properties of an infinite set $A \succ \sigma_{s+1}$ not yet fully constructed. In §6.3 on generic sets we study the general case of forcing conditions which are finite initial segments. In §6.5 on least upper bounds we consider infinite matrices as forcing conditions.

Theorem 6.1.1 (Kleene-Post, 1954)**.** *There exist sets A, $B \leq_{\mathrm{T}} \emptyset'$ such that $A \mid_T B$ (i.e., $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$.) Therefore, $\emptyset <_{\mathrm{T}} A,\ B <_{\mathrm{T}} \emptyset'$.*

Proof. We shall construct functions χ_A and χ_B in stages s so $\chi_A = \cup_s \sigma_s$ and $\chi_B = \cup_s \tau_s$, where σ_s and τ_s are the finite strings constructed by the end of stage s. Since the construction of σ_s and τ_s at stage s is computable in $\emptyset'$, the sequences $\{\sigma_s\}_{s\in\omega}$ and $\{\tau_s\}_{s\in\omega}$ are $\emptyset'$-computable sequences. Therefore, $A, B \leq_{\mathrm{T}} \emptyset'$. It suffices to meet, for each e, the *requirements*:

$$R_e:\ \ A\ \neq\ \Phi^B_e \qquad \& \qquad S_e:\ \ B\ \neq\ \Phi^A_e \tag{6.1}$$

to ensure that $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$. Hence, $A \mid_T B$.

Stage $s = 0$. Define $\sigma_0 = \tau_0 = \emptyset$.

Stage $s + 1 = 2e + 1$. (*We satisfy* R_e.) Given $\sigma_s, \tau_s \in 2^{<\omega}$ of length $\geq s$. Let $n = |\sigma_s| = (\mu x)\,[\,x \notin \mathrm{dom}(\sigma_s)\,]$. Using a $\emptyset'$-oracle we test whether

$$(\exists t)\,(\exists \rho)\,[\ \rho \succ \tau_s \quad \& \quad \Phi^{\rho}_{e,t}(n){\downarrow}\ ]. \tag{6.2}$$

Note that $\rho \succ \tau_s$ is computable as a relation of strings ρ and τ_s. The second clause of (6.2) is computable by the Oracle Graph Theorem 3.3.8 (i). Therefore, (6.2) is a Σ_1 statement and can be decided computably in $\emptyset'$.

Case 1. Suppose (6.2) is satisfied. The matrix of (6.2) is computable. Find the least pair $\langle \rho, t\rangle$ satisfying that matrix. Define $\tau_{s+1} = \rho$ and $\sigma_{s+1}(n) = 1 \dot{-} \Phi^{\rho}_{e,t}(n)$ so that $\sigma_{s+1}(n) \neq \Phi^{\rho}_{e,t}(n)$.

Case 2. Suppose (6.2) fails. Then define $\sigma_{s+1} = \sigma_s{}^\frown 0$ and $\tau_{s+1} = \tau_s{}^\frown 0$.

In either case, $|\sigma_{s+1}|, |\tau_{s+1}| \geq s+1$. Therefore, $\chi_A = \cup_s \sigma_s$ and $\chi_B = \cup_s \tau_s$ are defined on all arguments. In either case, if $f \succeq \sigma_{s+1}$ and $g \succeq \tau_{s+1}$ then $f(n) \neq \Phi^g_e(n)$ by the Use Principle Theorem 3.3.9.

Stage $s+1 = 2e+2$. (*We satisfy* S_e.) Proceed exactly as above but with the roles of σ_s and τ_s replaced by τ_s and σ_s, mutatis mutandis. □

Theorem 6.1.2 (Relativized Version)**.** *For any degree* $\mathbf{c}$*, there are degrees* $\mathbf{a}$*,* $\mathbf{b}$ *such that* $\mathbf{c} \le \mathbf{a}, \mathbf{b}$ *and* $\mathbf{a}, \mathbf{b} \le \mathbf{c}'$ *and* $\mathbf{a} \mid \mathbf{b}$*.*

Proof. Fix a set $C \in \mathbf{c}$. Relativize the above proof to C, using a C'-oracle to build sets A and B such that $(A \oplus C) \mid_{\mathrm{T}} (B \oplus C)$ and $A, B \le_{\mathrm{T}} C'$. In place of (6.2) we use a C'-oracle to test whether

(6.3) $$(\exists t)\,(\exists \tau_1)\,(\exists \tau_2)\,[\ \tau_1 \succ \tau_s \quad \& \quad \tau_2 \prec C \quad \& \quad \Phi_{e,t}^{\tau_1 \oplus \tau_2}(n)\!\downarrow\ ],$$

where $\tau_1 \oplus \tau_2$ is defined to be the shortest string $\rho \in 2^{<\omega}$ such that $\rho(2x) = \tau_1(x)$ and $\rho(2x+1) = \tau_2(x)$. The obvious modification of Cases 1 and 2 ensures that $A \ne \Phi_e^{B \oplus C}$. At stage $2e+2$ we ensure that $B \ne \Phi_e^{A \oplus C}$. Finally, let $\mathbf{a} = \deg(A \oplus C)$ and $\mathbf{b} = \deg(B \oplus C)$. □

Definition 6.1.3. A countable sequence of sets $\{A_i\}_{i \in \omega}$ is *computably independent* if for each i, $A_i \not\le_{\mathrm{T}} \oplus\{A_j : j \ne i\}$, where the infinite join is defined as in Exercise 3.4.7.

6.1.1 Exercises

Exercise 6.1.4. Modify the proof of Theorem 6.1.1 to build an independent sequence $\{A_j\}_{j\in\omega}$ of sets each computable in $\emptyset'$ (indeed, $\oplus_j A_j \le_{\mathrm{T}} \emptyset'$). *Hint.* Use a finite extension $\emptyset'$-computable construction to build at stage s strings $\{\rho_j^s\}_{s\in\omega}$ such that if $A_j = \cup_s \rho_j^s$ then we meet for each e and i the requirement

$$R_{\langle e,i\rangle} \ :\ A_i \ne \Phi_e^{\oplus\{A_j \,:\, j\ne i\}}.$$

At stage $s = 0$, set $\rho_j^0 = \emptyset$ for all j. At stage $s+1 = \langle e,i\rangle + 1$ we meet requirement $R_{\langle e,i\rangle}$ as follows. Given $\{\,\rho_j^s\,\}_{j\in\omega}$, only finitely many of which are nonempty, let $n = |\rho_i^s|$, and use a $\emptyset'$-oracle to test whether there exist m and (a code number for) a finite sequence of strings $\sigma_0, \sigma_1, \ldots, \sigma_k$ such that

(6.4) $$\Phi_e^{\oplus\{\,\sigma_j \,:\, j\ne i\,\}}(n)\!\downarrow\ =\ m \quad \& \quad (\forall j \le k)\,[\ j \ne i \quad \Longrightarrow \quad \rho_j^s \prec \sigma_j\ ].$$

Now according to whether or not (6.4) holds proceed as in Theorem 6.1.1 Case 1 (letting $\rho_i^{s+1}(n) = 1 \dot{-} m$, and $\rho_j^{s+1} = \sigma_j$ for $j \ne i$), or as in Case 2 otherwise. (Be sure to make each A_i total.)

Exercise 6.1.5. A partially ordered set $\mathcal{P} = (P, \le_{\mathrm{P}})$ is *countably universal* if every countable partially ordered set is order isomorphic to a subordering of $\mathcal{P}$. Prove that there is a computable partial ordering $\le_{\mathrm{R}}$ of ω which is countably universal. *Hint.* This can be done either by considering a computably presented atomless Boolean algebra, or by a direct construction where at stage $s+1$, given a finite set P_s of elements in $\le_{\mathrm{R}}$, one obtains P_{s+1} by adding a new point for each possible order type over P_s. A Boolean algebra $\mathcal{B} = (\{ b_i \}_{i\in\omega}; \leq, \vee, \wedge, ')$ is *computably presented* if there exist a computable relation $P(i,j)$ and computable functions f, g and h such that $P(i,j)$ holds iff $b_i \leq b_j$, and such that $b_{f(i,j)} = b_i \vee b_j$, $b_{g(i,j)} = b_i \wedge b_j$, and $b_{h(i)} = b_i'$.

Exercise 6.1.6. Show that for a countable partially ordered set $\mathcal{P} = \langle P, \leq_{\mathrm{P}} \rangle$ there is a 1:1 order-preserving map from P into $\mathbf{D}(\leq \mathbf{0}')$, the degrees $\leq \mathbf{0}'$. *Hint.* By Exercise 6.1.5 we may assume $P = \omega$ and $\leq_{\mathrm{P}}$ is a computable relation. Let $\{ A_i \}_{i\in\omega}$ be as in Exercise 6.1.4. Define $f : \omega \to \mathbf{D}(\leq \mathbf{0}')$ by $f(i) = \mathbf{a}_i = \deg(\oplus A_j : j \leq_{\mathrm{P}} i)$. Show that if $i \leq_{\mathrm{P}} j$ then $\mathbf{a}_i \leq \mathbf{a}_j$ (by definition and the fact that $\leq_{\mathrm{P}}$ is computable), and if $i \not\leq_{\mathrm{P}} j$ then $\mathbf{a}_i \not\leq \mathbf{a}_j$ (by the computable independence of $\{ A_i \}_{i\in\omega}$).

Exercise 6.1.7. Prove that there are $2^{\aleph_0}$ mutually incomparable degrees. *Hint.* Recall Definition 8.2.1 of a *tree* $T \subseteq 2^{<\omega}$ and its associated trees in Definition 3.7.1. Construct a tree $T \subseteq 2^{<\omega}$ such that $f \mid_{\mathrm{T}} g$ for every pair $f, g \in [T]$ with $f \neq g$. Let $T = \cup_e T_e$ where tree $T_{e+1} \supset T_e$ and T_{e+1} is defined by induction as follows. Let $T_0 = \{ \emptyset \}$, the tree with the empty node (root) as its only member. Given T_e define L_e to be the *leaves* of tree T_e, namely

$$L_e = \{ \sigma : \sigma \in T_e \quad \& \quad (\forall \tau \succ \sigma) [\tau \notin T_e] \}.$$

Next define the *successors* to leaves,

$$S_e = \{ \sigma\hat{\ }0 : \sigma \in L_e \} \quad \cup \quad \{ \sigma\hat{\ }1 : \sigma \in L_e \}.$$

Suppose $S_e = \{ \rho_i : i \leq 2^{e+1} \}$. Fix i , $j \leq 2^{e+1}$, $i \neq j$. Use the method of Theorem 6.1.1 to replace ρ_i and ρ_j by strings $\sigma \succ \rho_i$, $\tau \succ \rho_j$ satisfying

$$(\forall f \succ \sigma)(\forall g \succ \tau)[\ \Phi_e^f \neq g \quad \& \quad \Phi_e^g \neq f\ ]. \tag{6.5}$$

Repeat this procedure for all i, $j \leq 2^{e+1}$ with $i \neq j$.

6.2 Minimal Pairs and Avoiding Cones

Definition 6.2.1. Degrees $\mathbf{a}$ and $\mathbf{b}$ form a *minimal pair* if $\mathbf{a}$, $\mathbf{b} > \mathbf{0}$ and

$$(\forall \mathbf{c}) [\ [\ \mathbf{c} \leq \mathbf{a} \quad \& \quad \mathbf{c} \leq \mathbf{b}\ ] \quad \Longrightarrow \quad \mathbf{c} = \mathbf{0}\ ]. \tag{6.6}$$

Minimal pairs have played an important role in computability theory. Later we shall construct a minimal pair of *computably enumerable* degrees. In §6.5 we shall modify the minimal pair construction to find an exact pair of degrees for an ascending sequence of degrees as defined in Definition 6.5.2. To simplify the notation now and later we introduce a useful remark of Posner which allows us to replace pairs of indices by a single index.

Remark 6.2.2 (Posner)**.** *For all sets A and B with $A \neq B$ and all i and j, there exists e such that $\Phi_e^A = \Phi_i^A$ and $\Phi_e^B = \Phi_j^B$.*

Proof. Since $A \neq B$ they differ on some element n, say $n \in A - B$. Define the Turing reduction Φ_e^X for any X by

$$\Phi_e^X(y) = \begin{cases} \Phi_i^X(y) & \text{if } n \in X; \\ \Phi_j^X(y) & \text{otherwise.} \end{cases} \qquad \square$$

Theorem 6.2.3. *There is a minimal pair of degrees* $\mathbf{a}$, $\mathbf{b} < \mathbf{0}'$.

Corollary 6.2.4 (Theorem 6.1.1)**.** *There exist* $\mathbf{a}$, $\mathbf{b} < \mathbf{0}'$ *with* $\mathbf{a} \mid \mathbf{b}$.

Proof. If $\mathbf{a}$, $\mathbf{b}$ is a minimal pair then $\mathbf{a} \mid \mathbf{b}$. If $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} > \mathbf{0}$. $\square$

Proof of Theorem 6.2.3. It suffices to construct sets A and B unequal and computable in $\emptyset'$ satisfying for all e the following requirements.

$$(6.7)\quad N_e: \quad \Phi_e^A = \Phi_e^B \text{ total} \implies (\exists g \leq_{\mathrm{T}} \emptyset)[\, g = \Phi_e^A \,].$$

$$(6.8)\quad P_e: \quad A \neq \varphi_e \quad \text{and} \quad B \neq \varphi_e.$$

We shall use a $\emptyset'$-oracle construction to build increasing sequences of strings, $\{\sigma_s\}_{s\in\omega}$ and $\{\tau_s\}_{s\in\omega}$, and then define $A = \cup_s \sigma_s$ and $B = \cup_s \tau_s$. Define $\sigma_0 = \emptyset$ and $\tau_0 = \emptyset$.

Stage $s + 1 = 2e + 1$. (Satisfy P_e for A and B.) Given σ_s and τ_s let $x = |\sigma_s| = (\mu y)[\sigma_s(y)\uparrow]$. Ask $\emptyset'$ whether $\varphi_e(x)\downarrow$. If so, define $\sigma_{s+1}(x) = 1 \dot{-} \varphi_e(x)$ and otherwise define $\sigma_{s+1}(x) = 0$. Do likewise to ensure that τ_{s+1} and φ_e are not compatible.

Stage $s + 1 = 2e + 2$. (Satisfy N_e.) Ask $\emptyset'$ the Σ_1 question,

$$(6.9)\quad (\exists \rho \succ \sigma_s)\,(\exists \nu \succ \tau_s)\,(\exists x)\,[\, \Phi_e^\rho(x)\downarrow \; \neq \; \Phi_e^\nu(x)\downarrow \,]?$$

If so, define $\sigma_{s+1} = \rho$ and $\tau_{s+1} = \nu$. If not, define $\sigma_{s+1} = \sigma_s{}^\frown 0$ and $\tau_{s+1} = \tau_s{}^\frown 0$.

Lemma 6.2.5. $(\forall e)\,[\, \Phi_e^A = \Phi_e^B = f$ *total* $\implies$ f *is computable* $]$.

Proof. Assume $\Phi_e^A = \Phi_e^B = f$ is total. At stage $s+1 = 2e+2$, equation (6.9) could not have held, else $\Phi_e^A \neq \Phi_e^B$. Hence, for any x we can choose $\rho \succ \sigma_s$ such that $\Phi_e^\rho(x)\downarrow = y$ by the Use Principle 3.3.9 because Φ_e^A is total. Now any other $\xi \succ \sigma_s$ for which $\Phi_e^\xi(x)\downarrow = z$ must have $y = z$, else one of y and z must form a disagreement with $\Phi_e^\nu(x)$ for some $\nu \prec B$, contrary to our hypothesis that (6.9) fails. Therefore, $f(x) = y$ even though we may not have $\rho \prec f$. Since searching for the first such string ρ, which must exist, is a computable procedure, we know that f is computable. $\square$ $\square$

So far the degrees we have constructed, such as $\mathbf{0}^{(n)}$ or degrees below $\mathbf{0}'$, are comparable to $\mathbf{0}'$. We now show how to construct a degree $\mathbf{a}$ incomparable with a given degree $\mathbf{b} > \mathbf{0}$. To achieve this, $\mathbf{a}$ must avoid the *lower cone* of degrees $\{\, \mathbf{d} : \mathbf{d} \leq \mathbf{b} \,\}$ and the *upper cone* $\{\, \mathbf{d} : \mathbf{d} \geq \mathbf{b} \,\}$. The strategy for accomplishing the latter (which we play on the even stages) will be used in this chapter, and will be refined and often used in constructions of c.e. degrees, such as in the Sacks Splitting Theorem in Chapter 7.

Theorem 6.2.6 (Avoiding Cones)**.** *For every degree* $\mathbf{b} > \mathbf{0}$ *there exists a degree* $\mathbf{a} < \mathbf{b}'$ *such that* $\mathbf{a} \mid \mathbf{b}$.

Proof. Fix $B \in \mathbf{b}$. Construct f, the characteristic function of A, by a B'-computable finite extension construction, $f = \cup_s \sigma_s$, to meet the requirements R_e and S_e of Theorem 6.1.1.

Stage $s = 0$. Set $\sigma_0 = \emptyset$.

Stage $s + 1 = 2e + 1$. (Satisfy $R_e : A \neq \Phi_e^B$.) Let $n = |\sigma_s|$. With a B'-oracle determine whether $\Phi_e^B(n)$ converges, i.e., whether $\langle n, e\rangle \in K_0^B \equiv B'$. If so, define $\sigma_{s+1}(n) = 1 \dot{-} \Phi_e^B(n)$. If not, define $\sigma_{s+1}(n) = 0$.

Stage $s{+}1 = 2e{+}2$. (Satisfy $S_e : B \neq \Phi_e^A$.) Given σ_s, first $\emptyset'$-computably test whether the following equation holds:

$$(6.10)\qquad (\exists\sigma)(\exists\tau)(\exists x)(\exists y)(\exists z)(\exists t)$$

$$[\ \sigma_s \prec \sigma, \tau \quad \& \quad \Phi_{e,t}^{\sigma}(x)\downarrow\ = y \quad \neq \quad z\ =\ \Phi_{e,t}^{\tau}(x)\downarrow\ ].$$

If so, one of the values y or z must differ from $B(x)$. Let σ_{s+1} be the first $\sigma \succ \sigma_s$ such that $\Phi_e^\sigma(x)\downarrow\, \neq B(x)$ for some x. (This is B'-computable because $B \oplus \emptyset' \leq_{\mathrm{T}} B'$.) If (6.10) fails, we let $\sigma_{s+1} = \sigma_s\widehat{\ }0$. In this case, we claim that for any $f \succ \sigma_s$ if $\Phi_e^f = g$ is total, then g is computable (and hence $\Phi_e^f \neq B$ because $\emptyset <_{\mathrm{T}} B$). To compute $g(x)$, enumerate G_e of the Oracle Graph Theorem 3.3.8 (ii) until the first $\sigma \succ \sigma_s$ is found such that $\Phi_e^\sigma(x)$ converges. Now $g(x) = \Phi_e^f(x) = \Phi_e^\sigma(x)$, because otherwise for some τ, $\sigma_s \preceq \tau \prec f$, $\Phi_e^\tau(x)\downarrow\, \neq\ \Phi_e^\sigma(x)$, thereby satisfying (6.10). □

6.2.1 Exercises

Exercise 6.2.7. Construct an infinite sequence of degrees $\mathbf{a}_n \leq \mathbf{0}'$, $n \in \omega$, which pairwise form minimal pairs. *Hint.* Build noncomputable sets $\{A_n\}_{n\in\omega}$ meeting for all i, j, and all $m \neq n$,

$$N_{\langle m,n,i,j\rangle} : \quad \Phi_i^{A_m} = \Phi_j^{A_n} = f \ \text{ total } \quad . \implies . \quad f \ \text{ is computable.}$$

Exercise 6.2.8. (i) Fix a degree $\mathbf{c} > \mathbf{0}$. Build a degree $\mathbf{b}$ which forms a minimal pair with $\mathbf{c}$.

(ii) Given nonzero degrees $\{\mathbf{c}_n\}_{n\in\omega}$, find a sequence of degrees $\{\mathbf{a}_i\}_{i\in\omega}$ each of which is incomparable with $\mathbf{c}_n$ for every n and which pairwise form minimal pairs.

6.3 ⋆ Generic Sets

In the two preceding sections we constructed a sequence of finite functions $\sigma_s \preceq \sigma_{s+1}$ so that σ_{s+1} (and indeed all $\tau \succeq \sigma_{s+1}$) met a particular requirement. The generic construction in this section encompasses all the previous examples. Recall that in the Notation section we gave an effective index y to each string $\sigma_y \in 2^{<\omega}$ and we identify the string σ_y with its index y. Likewise, we identify a c.e. set of strings $V_e \subseteq 2^{<\omega}$ with the corresponding c.e. set of integers and use the same notation, V_e. Definition 2.6.1 defined u.c.e. and s.c.e arrays of c.e. sets. Generic sets were studied by [Jockusch 1980].

6.3.1 1-Generic Sets

Definition 6.3.1. Let $\mathbb{V} = \{V_e\}_{e\in\omega}$ be a u.c.e. sequence of c.e. sets $V_e \subseteq 2^{<\omega}$ as in Definition 2.6.1, with strings identified with their indices.

(i) We say $f \in 2^\omega$ *forces* V_e if it satisfies the *forcing* requirement,

$$(6.11) \qquad F_e : \quad (\exists \sigma \prec f)\,[\,\sigma \in V_e \quad \vee \quad (\forall \rho \succ \sigma)\,[\,\rho \notin V_e\,]\,].$$

If σ satisfies the matrix of F_e we say that σ *forces* F_e (written $\sigma \Vdash F_e$) and any $f \succ \sigma$ also *forces* F_e (written $f \Vdash F_e$).

(ii) We say f is *generic with respect to* $\mathbb{V} = \{V_e\}_{e\in\omega}$ (written $\mathbb{V}$-*generic*) if f forces V_e for every $e \in \omega$.

(iii) We say f is *1-generic* if it is generic with respect to $\{W_e\}_{e\in\omega}$. (The term "1-generic" refers to the fact that f is deciding Σ_1 statements.)

If the f satisfies the first clause $\sigma \in V_e$ of the matrix in (6.11), then we say f is *e-white* and otherwise f is *e-black*. For every e the 1-generic function f must be either e-black or e-white.

The point about a 1-generic set is that it is amorphous and difficult to describe. For example, it cannot be computable or even c.e. However, we can construct a 1-generic Δ_2 set.

6.3.2 Forcing the Jump

Occasionally, we build f as the characteristic function of a set A and we wish to control the jump A'. (At a finite stage we decide whether $e \in A'$.) We can accomplish this by meeting for all e the following requirement called *forcing the jump* $\Phi_e^A(e)$:

$$(6.12) \qquad J_e : \qquad (\exists \sigma \prec A)\,[\,\Phi_e^\sigma(e)\downarrow \qquad \vee \qquad (\forall \tau \succeq \sigma)\,[\,\Phi_e^\tau(e)\uparrow\,]\,].$$

Theorem 6.3.2 (Jockusch-Posner). *A set A is 1-generic iff A forces the jump, i.e., satisfies every jump requirement $\{J_e\}_{e\in\omega}$ of (6.12).*

Proof. ($\Longrightarrow$). Define $W_{h(e)} = \{\sigma : \Phi_e^\sigma(e)\downarrow\}$. Now A forces $W_{h(e)}$. Therefore, A forces the jump $\Phi_e^A(e)$, i.e., satisfies the requirement J_e.

($\Longleftarrow$). Define a computable function $f(e)$ by

$$\Phi^\sigma_{f(e)}(z) = \begin{cases} 1 & \text{if } (\exists s \leq |\sigma|)\,(\exists \tau \preceq \sigma)\,[\,\tau \in W_{e,s}\,], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If A meets requirement $J_{f(e)}$ of (6.12), then we can clearly see that A forces $V_e = \{\sigma : \Phi^\sigma_{f(e)}(f(e))\downarrow\}$ and A forces W_e. ☐

6.3.3 Doing Many Constructions at Once

In the preceding sections we constructed sets with several different properties: incomparable with another, half of a minimal pair, and avoiding a cone. If we now construct a 1-generic set A, then A automatically has all these properties because each property corresponds to a dense set and a 1-generic set meets every dense set of strings. Dense sets, comeager sets, and Banach-Mazur games are explained in Chapter 14. The Banach-Mazur games described there are very similar to the finite extension strategies presented in this chapter.

In Theorem 14.2.1 we shall consider finite extension arguments in the general setting of the Finite Extension Paradigm which subsumes them. This does not cover the coding argument for the Friedberg Completeness Criterion in Theorem 6.4.1 below. However, we extend our paradigm analysis to the Finite Extension Coding Paradigm in Theorem 14.2.2 which covers these examples.

6.3.4 Exercises

Exercise 6.3.3. Construct a 1-generic set $A \leq_{\mathrm{T}} \emptyset'$. *Hint.* Use a $\emptyset'$-oracle and finite extension construction as in the Kleene-Post Theorem 6.1.1 to meet all the jump requirements J_e in (6.12).

Exercise 6.3.4. (Jockusch-Posner) Prove that if a set A is 1-generic, then $A \oplus \emptyset' \equiv_{\mathrm{T}} A'$. Prove that there is a nonzero low degree.

Exercise 6.3.5. (Jockusch-Posner) Assume A is 1-generic.

(i) Prove that A is immune. *Hint.* Let Z be a c.e. subset of A. Define $V_e = \{\,\sigma : (\exists x \in Z)\ [\sigma(x) = 0]\,\}$ and use F_e of (6.11) to prove that Z is finite.

(ii) Prove that A is hyperimmune.

(iii) Prove that there is no noncomputable c.e. set $Z \leq_{\mathrm{T}} A$. *Hint.* Assume $Z = \Phi_i^A$ and define

$$V_e = \{\, \sigma : \ (\exists x)\,[\, \Phi_i^{\sigma}(x) = 0 \ \ \& \ \ x \in Z \,]\,\},$$

and apply requirement F_e of (6.11) to show that $\overline{Z}$ is c.e.

(iv) Prove that A_0 and A_1 are Turing incomparable where $A_0(x) = A(2x)$ and $A_1(x) = A(2x+1)$. *Hint.* To see that $A_0 \neq \Phi_e^{A_1}$ consider the c.e. set of strings

$$V_e \ = \ \{\ \sigma : (\exists x)\,[\ \Phi_e^{\sigma_1}(x)\!\downarrow\ \neq\ \sigma_0(x)\ ]\ \}$$

where $\sigma_0(x) = \sigma(2x)$ and $\sigma_1(x) = \sigma(2x+1)$.

(v) Prove that there are sets $B_i \leq_{\mathrm{T}} A$, $i \in \omega$, such that for every i, we have $B_i \not\leq_{\mathrm{T}} \oplus\{\, B_j : j \neq i\,\}$.

Exercise 6.3.6. (Shoenfield) Show there is a set $A \leq_{\mathrm{T}} \emptyset'$ which does not have c.e. degree.

Exercise 6.3.7. Given B such that $\emptyset <_T B \leq_{\mathrm{T}} \emptyset'$ find a low set A such that $A \not\geq_T B$. *Hint.* Use a $\emptyset'$-construction to build $A = \cup_s \sigma_s$. For each e designate some stage s at which you: (1) force $A \neq \varphi_e$; (2) make A satisfy the lowness requirement for Φ_e; and (3) look for e-splittings, ρ, τ extending σ_s and some x such that $\Phi_e^{\rho}(x)\!\downarrow \neq \Phi_e^{\tau}(x)\!\downarrow$. If you do not find them, then either Φ_e^A is not total or is computable.

Exercise 6.3.8. (i) Let $\{A_n\}_{n\in\omega}$ be a sequence of sets uniformly computable in $\emptyset'$, i.e., there is a $\emptyset'$-computable function g such that for all x and n, $g(n,x) \ = \ A_n(x)$. Prove that there is a set $B \leq_{\mathrm{T}} \emptyset'$ such that $(\forall n)\,[\, B \not\equiv_{\mathrm{T}} A_n\,]$. *Hint.* Ensure that B is noncomputable and for each e and n, if $\Phi_e^B = A_n$, then A_n is computable.

(ii) Give another proof of Exercise 6.3.6.

(iii) Show there is a degree $\mathbf{d} \leq \mathbf{0}'$ which is not n-c.e. and not even ω-c.e.

6.4 ⋆ Inverting the Jump

Note that for any degree $\mathbf{a}$, $\mathbf{0} \leq \mathbf{a}$ and hence $\mathbf{0}' \leq \mathbf{a}'$, i.e., any jump is above $\mathbf{0}'$. Hence, the jump, viewed as a map on degrees, has range contained in $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$. The next theorem asserts that this map is *onto* the set $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$. A degree $\mathbf{a}$ is called *complete* if $\mathbf{a} \geq \mathbf{0}'$. Hence, the result also gives a criterion for $\mathbf{a}$ being complete.

Theorem 6.4.1 (Friedberg Completeness Criterion)**.** *For every degree* $\mathbf{b} \geq \mathbf{0}'$ *there is a degree* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{b}$.

Proof. Fix $B \in \mathbf{b}$. We shall construct f, the characteristic function of A, by finite initial segments $\{\sigma_s\}_{s\in\omega}$ using a B-computable finite extension construction.

Stage $s = 0$. Set $\sigma_0 = \emptyset$.

Stage $s + 1 = 2e + 1$. (We decide whether $e \in A'$.) We meet the *forcing the jump* requirement J_e of (6.12). If A meets J_e we say that A *forces the jump* on argument e. Given σ_s, use a $\emptyset'$-oracle to test whether

$$(6.13) \qquad (\exists\sigma)\,(\exists t)\,[\ \sigma_s \prec \sigma \quad \& \quad \Phi^{\sigma}_{e,t}(e)\!\downarrow\ ].$$

(Note that the matrix is computable. Therefore, (6.13) is a Σ_1 condition and is computable in $\emptyset'$.) If (6.13) is satisfied, let σ_{s+1} be the first such σ in the standard enumeration of G_e of the Oracle Graph Theorem 3.3.8. If not, set $\sigma_{s+1} = \sigma_s$.

Stage $s+1 = 2e+2$. (We code $B(e)$ into A.) Let $n = |\sigma_s|$. Define $\sigma_{s+1}(n) = B(e)$. (This completes the construction.)

Now $f = \cup_s\sigma_s$ is total since $|\sigma_{2e}| \geq e$. Let $A = \{x : f(x) = 1\}$, and $\mathbf{a} = \deg(A)$. The construction is B-computable because at odd stages we use a $\emptyset'$-oracle, at even stages we use a B-oracle, and $\emptyset' \leq_{\mathrm{T}} B$. Since $A \oplus \emptyset' \leq_{\mathrm{T}} A'$ for any A, to prove $A' \equiv_{\mathrm{T}} B \equiv_{\mathrm{T}} A \oplus \emptyset'$ it suffices to prove the following two lemmas.

Lemma 6.4.2. *$A' \leq_{\mathrm{T}} B$.*

Lemma 6.4.3. *$B \leq_{\mathrm{T}} A \oplus \emptyset'$.*

Proof of Lemma 6.4.2. Since the construction is B-computable, the sequence $\{\sigma_s\}_{s\in\omega}$ is B-computable. To decide whether $e \in A'$, B-computably determine using $\emptyset' \leq_{\mathrm{T}} B$ whether (6.13) holds for σ_{2e}. If so, $e \in A'$, and otherwise $e \notin A'$ because no $\sigma \succeq \sigma_s$ has $\Phi^{\sigma}_e(e)$ defined. □

Proof of Lemma 6.4.3. We show $\{\sigma_s\}_{s\in\omega}$ is an $(A \oplus \emptyset')$-computable sequence. This suffices because $B(e)$ is the last value of σ_{2e+2}. The proof is by induction on s. Given $\{\sigma_s : s \leq 2e\}$, use a $\emptyset'$-oracle to compute σ_{2e+1}. If $n = |\sigma_{2e+1}|$ then $\sigma_{2e+2} = \sigma_{2e+1}{}^\frown A(n)$, so σ_{2e+2} is computed from σ_{2e+1} using an A-oracle. □

This completes the proof of Theorem 6.4.1. □

Theorem 6.4.4 (Relativized Friedberg Completeness Criterion)**.** *For every degree* $\mathbf{c}$,

$$F_1(\mathbf{c}): \quad (\forall\mathbf{b})\,[\,\mathbf{b} \geq \mathbf{c}' \implies (\exists\mathbf{a})\,[\,\mathbf{a} \geq \mathbf{c}\ \&\ \mathbf{a}' = \mathbf{a} \cup \mathbf{c}' = \mathbf{b}\,]\,].$$

Proof. Do the proof of Theorem 6.4.1 with $\mathbf{c}$ and $\mathbf{c}'$ in place of $\mathbf{0}$ and $\mathbf{0}'$. □

Corollary 6.4.5. *For every $n \geq 1$, and every degree* $\mathbf{c}$,

$$F_n(\mathbf{c}): \quad (\forall \mathbf{b})\,[\,\mathbf{b} \geq \mathbf{c}^{(n)} \implies (\exists \mathbf{a})\,[\,\mathbf{a} \geq \mathbf{c} \;\&\; \mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{c}^{(n)} = \mathbf{b}\,]\,].$$

Proof. To prove $(\forall \mathbf{c})F_n(\mathbf{c})$ holds for all $n \geq 1$, use induction on n and the fact that $F_{n+1}(\mathbf{c})$ follows from $F_n(\mathbf{c})$ and $F_1(\mathbf{c}^{(n)})$. ☐

Although Theorem 6.4.1 demonstrates a pleasant property of the jump operator, it also demonstrates an unpleasant property, namely that the jump as a map on degrees is not 1:1. To see this, apply Theorem 6.4.1 with $\mathbf{b} = \mathbf{0}''$ to obtain $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}''$. Clearly, $\mathbf{a} \mid \mathbf{0}'$, yet they have the same jump. It is also possible to have $\mathbf{a} < \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$. (It is easy to see that the jump is 1:1 on *sets*.)

6.4.1 Exercises

Exercise 6.4.6. [Jockusch-Shore, 1983] Prove that for any $i \in \omega$ and any B such that $\emptyset' \leq_{\mathrm{T}} B$ there exists A such that

$$A \oplus W_i^A \equiv_{\mathrm{T}} A \oplus \emptyset' \equiv_{\mathrm{T}} B.$$

Note that Theorem 6.4.1 is a special case of this setting where i is defined by $W_i^X = X'$. *Hint.* Do the proof of Theorem 6.4.1 but in (6.12) replace $\Phi_e^\rho(e)\!\downarrow$ by $e \in W_i^\rho$ for $\rho = \sigma$ or τ. (Note that this construction is uniform in B and in any j such that $\Phi_j^B = \emptyset'$.)

Exercise 6.4.7. Prove that

$$(\forall \mathbf{b} \geq \mathbf{0}')\;(\exists \mathbf{a}_0)\;(\exists \mathbf{a}_1)\;[\;\mathbf{a}_0 \mid \mathbf{a}_1 \;\&\; \mathbf{a}_0' = \mathbf{a}_0 \cup \mathbf{0}' = \mathbf{b} = \mathbf{a}_1 \cup \mathbf{0}' = \mathbf{a}_1'\;].$$

Hint. Combine the constructions of Theorems 6.1.1 and 6.4.1 to handle four types of requirements, the two types from Theorem 6.1.1 and the two from Theorem 6.4.1. As in that theorem at stage $2e+2$, code $B(e)$ into *both* of A_0 and A_1.

6.5 Upper and Lower Bounds for Degrees

Every nonempty finite set of degrees has a least upper bound (lub). In this section we show that this is false for greatest lower bounds (glb's). Hence, the degrees do not form a lattice, but merely an upper semi-lattice.

Definition 6.5.1. (i) For any set A define the *ω-jump* of A,

$$A^{(\omega)} \;=\; \{\; \langle x, n\rangle \;:\; x \in A^{(n)} \;\}.$$

In Exercise 6.5.9 we show that this is well-defined on degrees. Therefore, we can define the induced ω-jump on *degrees* $\mathbf{a}^{(\omega)} = \deg(A^{(\omega)})$ for $A \in \mathbf{a}$.

(ii) An infinite sequence of degrees $\{\mathbf{a}_n\}_{n\in\omega}$ is *ascending* if $\mathbf{a}_n \leq \mathbf{a}_{n+1}$ for all n and *strictly ascending* if $\mathbf{a}_n < \mathbf{a}_{n+1}$ for all n. For example, $\mathbf{0}, \mathbf{0}^{(1)}, \mathbf{0}^{(2)}, \ldots$ is strictly ascending, and $\mathbf{0}^{(\omega)}$ is a natural upper bound for the sequence, although by the next theorem the sequence has no lub.

Definition 6.5.2. If $\{\mathbf{a}_n\}_{n\in\omega}$ is an ascending sequence of degrees then upper bounds $\mathbf{b}$ and $\mathbf{c}$ are an *exact pair* for the sequence if for every degree $\mathbf{d}$,

$$[\,\mathbf{d} \leq \mathbf{b} \quad \& \quad \mathbf{d} \leq \mathbf{c}\,] \quad \Longrightarrow \quad (\exists n)\,[\,\mathbf{d} \leq \mathbf{a}_n\,].$$

Theorem 6.5.3 (Kleene-Post-Spector)**.** *For every ascending sequence of degrees,* $\{\mathbf{a}_n\}_{n\in\omega}$*, namely* $\mathbf{a}_n \leq \mathbf{a}_{n+1}$*, there exist upper bounds* $\mathbf{b}$*and* $\mathbf{c}$ *which form an exact pair for the sequence.*

Corollary 6.5.4. *No infinite strictly ascending sequence of degrees, i.e.,* $\mathbf{a}_n < \mathbf{a}_{n+1}$*, has a least upper bound.* □

Corollary 6.5.5. *There are degrees* $\mathbf{b}$ *and* $\mathbf{c}$ *with no greatest lower bound.* □

Before proving Theorem 6.5.3 we make some definitions and introduce some new notation.

Definition 6.5.6. For any set $A \subseteq \omega$ define the *y-section* of A,

$$A^{[y]} = \{\langle x, z\rangle : \langle x, z\rangle \in A \quad \& \quad z = y\} \quad \text{and} \tag{6.14}$$

$$A^{[<y]} = \bigcup\{\, A^{[z]} : z < y\,\}. \tag{6.15}$$

(Using the pairing function we can identify A with a subset of $\omega \times \omega$ and view $A^{[y]}$ as the y^{th} *row* of A. We use the square bracket notation $A^{[y]}$ to distinguish from the y^{th} jump $A^{(y)}$.)

Definition 6.5.7. (i) Given sets A and B, for every y the *thickness requirement* for y states

$$T_y: \quad A^{[y]} \; =^* \; B^{[y]} \tag{6.16}$$

where $X =^* Y$ denotes that $(X - Y) \cup (Y - X)$ is finite.

(ii) A subset $A \subseteq B$ is a *thick subset* of B, written $A \subseteq_{\text{thick}} B$, if T_y is satisfied for all y.

Thick subsets will be very useful here and in later constructions of c.e. sets and degrees, such as the thickness lemma and infinite injury constructions.

Definition 6.5.8. Partial functions θ, ψ are *compatible*, which we write as *compat*(θ, ψ), if they have a common extension, i.e., if there is no x for which $\theta(x)$ and $\psi(x)$ are defined and unequal. Otherwise, they are *incompatible*.

Proof of Theorem 6.5.3. Choose $A_y \in \mathbf{a}_y$ for each y and then define $A = \{\langle x, y\rangle : x \in A_y\}$, so that $\langle x, y\rangle \in A^{[y]}$ iff $x \in A_y$. We shall construct characteristic functions f and g of sets B and C which are thick in A (so that $A_y \equiv_{\mathrm{T}} B^{[y]} \leq_{\mathrm{T}} B$, and likewise for C). This ensures that $\mathbf{b} = \deg(B)$ and $\mathbf{c} = \deg(C)$ are upper bounds for $\{\, \mathbf{a}_n \,\}_{n \in \omega}$. For all y we must meet the thickness requirements,

$$T^B_y : \quad B^{[y]} \ =^* \ A^{[y]},$$

$$T^C_y : \quad C^{[y]} \ =^* \ A^{[y]}.$$

We must also meet, for all e and i, the exact pair requirements,

$$R_{\langle e,i\rangle} : \quad \Phi^B_e = \Phi^C_i \ \text{ total } \quad \Longrightarrow \quad (\exists y)\,[\ \Phi^B_e \leq_{\mathrm{T}} A_y\ ]$$

by looking for "e-splittings" as we did in proving Theorem 6.2.3.

Let σ_s, τ_s, B_s, and C_s be the portions of f, g, B, and C defined by the end of stage s of the following construction.

Stage $s = 0$. Set $\sigma_0 = \tau_0 = \emptyset$.

Stage $s + 1$. Assume that σ_s and τ_s are defined on $\omega^{[<s]}$ and assume that:

$$(6.17) \qquad (\forall y < s)\,[\ B^{[y]}_s \ =^* \ C^{[y]}_s \ =^* \ A^{[y]}\ ]; \qquad \text{and}$$

$$(6.18) \qquad (\mathrm{dom}(\sigma_s) - \omega^{[<s]}) \ =^* \ \emptyset \ =^* \ (\mathrm{dom}(\tau_s) - \omega^{[<s]}).$$

Step 1. (Satisfy $R_{\langle e,i\rangle}$ for $s = \langle e, i\rangle$.) If

$$(6.19) \qquad (\exists\sigma)\,(\exists\tau)\,(\exists x)\,(\exists t)\,[\ \mathrm{compat}(\sigma, \sigma_s) \quad \& \quad \mathrm{compat}(\tau, \tau_s) \\ \& \quad \Phi^{\sigma}_{e,t}(x)\!\downarrow \ \neq \ \Phi^{\tau}_{i,t}(x)\!\downarrow\ ],$$

then let σ and τ be the first such strings and extend σ_s to $\widehat{f} = \sigma_s \cup \sigma$ and τ_s to $\widehat{g} = \tau_s \cup \tau$. Otherwise, let $\widehat{f} = \sigma_s$, and $\widehat{g} = \tau_s$. Note that $\sigma_s \equiv_{\mathrm{T}} A^{[<s]} \equiv_{\mathrm{T}} \tau_s$ by (6.17) and (6.18). Hence, $\mathrm{compat}(\sigma, \sigma_s)$ is an $A^{[<s]}$-computable relation on σ. (Note that for $s > 0$, Step 1 requires an $A'_{s-1} \equiv_{\mathrm{T}} (A^{[<s]})'$ oracle.)

Step 2. (Satisfy T^B_s and T^C_s.)

Let $\sigma_{s+1} = \widehat{f}$ on $\mathrm{dom}(\widehat{f})$. On all $x \in \omega^{[s]} - \mathrm{dom}(\widehat{f})$ define $\sigma_{s+1}(x) = A(x)$. Let $\tau_{s+1} = \widehat{g}$ on $\mathrm{dom}(\widehat{g})$ and $\tau_{s+1}(x) = A(x)$ for all $x \in \omega^{[s]} - \mathrm{dom}(\widehat{g})$. By (6.18), σ_s (and hence $\widehat{f}$) is already defined on at most finitely many elements of $\omega^{[s]}$, and similarly for τ_s, so $B^{[s]}_{s+1} =^* A^{[s]} =^* C^{[s]}_{s+1}$, and f and g are now defined on $\omega^{[\leq s]}$. This ends the construction.

If (6.19) holds, then $\Phi^B_e \neq \Phi^C_i$. If (6.19) fails and $\Phi^B_e = \Phi^C_i = h$ is total, then for $s = \langle e, i\rangle$ we shall show that $h \leq_{\mathrm{T}} A^{[<s]}$. Notice that $A^{[<s]} \leq_{\mathrm{T}} A_s$

because

$$A^{[<s]} \equiv_{\mathrm{T}} A^{[0]} \oplus \cdots \oplus A^{[s-1]} \equiv_{\mathrm{T}} A_0 \oplus \cdots \oplus A_{s-1} \leq_{\mathrm{T}} A_s.$$

To $A^{[<s]}$-computably determine $h(x)$, find the first string σ in some enumeration of $\{\sigma : \Phi_e^\sigma(x)\downarrow\}$ such that $\mathrm{compat}(\sigma, \sigma_s)$ and set $h(x) = \Phi_e^\sigma(x)$. Now $h(x) = \Phi_e^f(x)$, or else for some $\sigma' \prec f$, $\mathrm{compat}(\sigma', \sigma_s)$ holds and $\Phi_e^{\sigma'}(x)\downarrow = y \neq \Phi_e^\sigma(x)$, so (6.19) holds for either σ or σ' and for any $\tau \prec C$ such that $\Phi_i^\tau(x)$ converges. □

6.5.1 Exercises

Exercise 6.5.9. Let the ω-jump $A^{(\omega)}$ be defined as in Definition 6.5.1. Prove that if $A \equiv_{\mathrm{T}} B$, then $A^{(\omega)} \equiv_{\mathrm{T}} B^{(\omega)}$. *Hint.* To show that $B^{(\omega)} \leq_{\mathrm{T}} A^{(\omega)}$ we must prove that $B^n \leq_{\mathrm{T}} A^{(\omega)}$ uniformly in n. Apply the Jump Theorem 3.4.3 (vi) to show that $B^{(n)} \equiv_{\mathrm{T}} A^{(n)}$ uniformly in n.

Exercise 6.5.10. Show that the proof of Theorem 6.5.3 automatically produces sets B and C computable in $\oplus\{\, A'_y \,\}_{y\in\omega}$.

Exercise 6.5.11. Show that in the proof of Theorem 6.5.3 if B is any upper bound for the A_y sets then we can modify Steps 1 and 2 to construct C such that B and C satisfy the same requirements as before.

Exercise 6.5.12. Let $\mathbf{I}$ be a countable ideal contained in the Turing degrees $\mathbf{D}$. Prove that there exist degrees $\mathbf{b}$, $\mathbf{c}$ such that for all $\mathbf{a} \in \mathbf{D}$,

$$\mathbf{a} \in \mathbf{I} \quad \Longleftrightarrow \quad [\, \mathbf{a} \leq \mathbf{b} \quad \& \quad \mathbf{a} \leq \mathbf{c} \,].$$

We call $\mathbf{b}$ and $\mathbf{c}$ an *exact pair* for the ideal $\mathbf{I}$ as in Definition 6.5.2, and "ideal" is defined in the Notation section.

Exercise 6.5.13. $^\diamond$ (K. Lange). Fix an infinite computable tree $T \subseteq 2^{<\omega}$. Fix a set $\mathcal{A} = \{A_n\}_{n\in\omega} \subseteq [\,T\,]$ of computable paths through T (not necessarily closed) but *dense* in T in the sense that

$$(\forall \sigma \in T)(\exists A_n \succ \sigma)[\, A_n \in \mathcal{A} \,].$$

For some degree $\mathbf{d}$, a $\mathbf{d}$-*basis* for $\mathcal{A}$ is a sequence of paths $X = \{B_n\}_{n\in\omega} \subseteq [\,T\,]$ and a function $f \leq_{\mathrm{T}} \mathbf{d}$ such that $\varphi_{f(n)} = B_n$, i.e., $\mathbf{d}$ can uniformly compute a Δ_0-index for every path in $\mathcal{A}$, viewed as a row in the $\mathbf{d}$-computable matrix $B = \oplus_n B_n$.

(i) If the set $\mathcal{A}$ of isolated paths of T is dense in T, prove that $\mathcal{A}$ has a $\mathbf{0}'$-basis.

(ii) Prove that if $\mathcal{A}$ has a $\mathbf{0}'$-basis $X = \{A_n\}_{n\in\omega}$, then there is a sequence $\{B_n\}_{n\in\omega}$ such that $B = \oplus_n B_n$ is low and the collection of paths $\{B_n\}_{n\in\omega}$

equals the collection of paths $\{A_n\}_{n\in\omega}$, although the sequences may not be the same.

Hint for (ii). Given a $\mathbf{0}'$-basis $X = \{A_n\}_{n\in\omega}$, use a $\mathbf{0}'$-construction to build another basis $Y = \{B_n\}_{n\in\omega}$ having the same rows $\{A_n\}$ as X but perhaps in a different order. Simultaneously, force the jump of the matrix $B = \oplus_n B_n$ so that B is low. Search only through strings σ such that $(\forall j \le |\sigma|)[\sigma^{[j]} \in T]$, where $\sigma^{[j]}(x) = \sigma(\langle x, j\rangle)$. Now extend these $\sigma^{[j]}$ on the B side by $\mathbf{0}$-effectively choosing some row on the A side extending $\sigma^{[j]}$ and filling this row in on the B side.

Remark 6.5.14. Note that if $A \le_{\mathrm{T}} \emptyset'$ and $S = A'$ then $S \ge_{\mathrm{T}} \emptyset'$ and S is c.e. in $\emptyset'$. Therefore, the jump map takes the degrees $\mathbf{a} \le \mathbf{0}'$ *into* the degrees c.e. in and above $\mathbf{0}'$. The next theorem proves that this map is *onto.*

Exercise 6.5.15. $^\diamond$ (Shoenfield Jump Inversion Theorem). Fix S such that $\emptyset' \le_{\mathrm{T}} S$ and S is c.e. in $\emptyset'$, namely such that S is c.e. in and above (c.e.a.) in $\emptyset'$. Construct $A \le_{\mathrm{T}} \emptyset'$ such that $A' \equiv_{\mathrm{T}} S$. *Hint.* Define a $\emptyset'$-sequence $\{\sigma_s\}_{s\in\omega}$ of $\{0,1\}$-valued partial functions such that $\sigma_s \preceq \sigma_{s+1}$ and $\lim_s \ \sigma_s = \chi_A$. We ensure that $S \le_{\mathrm{T}} A'$ by arranging that for all y, $\lim_x \ A(\langle x, y\rangle) = \chi_S(y)$. We ensure $A' \le_{\mathrm{T}} S$ by forcing the jump $\Phi_e^A(e)$. Fix a $\emptyset'$-computable enumeration $\{\, S_s \,\}_{s\in\omega}$ of S such that $|S_{s+1} - S_s| = 1$. Let $\sigma_0 = \emptyset$. The following is a $\emptyset'$-construction.

Stage $s+1$. Assume that if $y \in S_s$ then $\sigma_s(\langle x, y\rangle) = 1$ for almost every x, and otherwise $\sigma_s(\langle x, y\rangle)\downarrow$ for at most finitely many x and $\sigma_s(\langle x, y\rangle)\uparrow$ for all other x.

Step 1. Now σ_{s+1} has a computable domain and is computable on its domain. Hence, we can $\emptyset'$-computably test for each $e \le s$ which has not yet been forced in A' whether

$$(\exists t)\,(\exists\sigma)\,[\mathrm{compat}(\sigma, \sigma_s) \quad \& \quad \Phi^{\sigma}_{e,t}(e)\downarrow \\ \& \quad (\forall y < e)\ (\forall x)\ [\ \langle x, y\rangle \notin \mathrm{dom}(\sigma_s) \quad \Longrightarrow \quad \sigma(\langle x, y\rangle) = 0\ ]\,]. \tag{6.20}$$

If so, choose the least e and the least corresponding string σ. Define $\tau_{s+1} = \sigma_s \cup \sigma$ and say that e is *forced into* A'. Otherwise, define $\tau_{s+1} = \sigma_s$.

Step 2. Enumerate the next element $z \in S_{s+1} - S_s$. Define

$$\sigma_{s+1}(\langle x, y\rangle) = \\ \begin{cases} \tau_{s+1}(\langle x, y\rangle) & \text{if } \langle x, y\rangle \in \ \mathrm{dom}(\tau_{s+1}); \\ 1 & \text{if } y = z \text{ and } \langle x, y\rangle \notin \ \mathrm{dom}(\tau_{s+1}); \\ 0 & \text{if } y \notin S_{s+1}, \langle x, y\rangle \le s, \langle x, y\rangle \notin \ \mathrm{dom}(\tau_{s+1}). \end{cases}$$

The last clause is to ensure that if $y \notin S$ then $\lim_x \ A(\langle x, y\rangle) = 0$. To see that $A' \le_{\mathrm{T}} S$, fix e, assume that membership of $i \in A'$ has been decided for all $i < e$, and find s such that $S_s \restriction e = S \restriction e$. Show that if e has not been forced into A' by stage s, then $e \notin A'$, i.e., it has been forced out of A'.