# 4
# The Arithmetical Hierarchy

## 4.1 Levels in the Arithmetical Hierarchy

In addition to the notions of computability and relative computability, the Kleene arithmetical hierarchy is one of the fundamental concepts of computability theory. In §2.1 we showed that a set $A$ is c.e. iff it has the syntactical form $\Sigma_1^0$ defined with a string of existential quantifiers. Now we define the more general notion of $\Sigma_n^0$ with $n$ alternating blocks of quantifiers. We prove that $\emptyset^{(n)} \in \Sigma_n^0 - \Sigma_{n-1}^0$ for $n > 1$. Therefore, the $\Sigma_n^0$ classes do not collapse, but rather form a hierarchy called the *arithmetical hierarchy* because these classes are definable in arithmetic. The relativized form of the hierarchy enables us to define several important special classes of sets and degrees called $high_n$ and $low_n$, some of whose properties we develop now, and more later. The arithmetical hierarchy was introduced in Kleene's paper [Kleene 1943] and was developed in Kleene's book [Kleene 1952].

**Convention 4.1.1.** We now define the $\Sigma_n^0$ and $\Pi_n^0$ sets, where the superscript 0 indicates that we are counting *number* quantifiers, not *function* quantifiers as in $\Sigma_1^1$. We rarely mention function quantifiers until Part II on open and closed classes in Cantor space. Therefore, in Part I Chapters 1–7 we usually drop the superscript 0 from $\Sigma_n^0$, $\Pi_n^0$, and $\Delta_n^0$, and abbreviate these by $\Sigma_n$, $\Pi_n$, and $\Delta_n$. Particularly in the relativized case, we write $\Sigma_n^A$ rather than $\Sigma_n^{0,A}$.

 
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**Definition 4.1.2.** (i) A set $B$ is in $\Sigma_0$ ($\Pi_0$, $\Delta_0$) iff $B$ is computable. As in Definition 2.3.1, a $\Delta_0$-*index* for $B$ is an index $e$ such that $\varphi_e = \chi_B$. (Indices for $\Sigma_n$ and $\Pi_n$ sets will be given in Definition 4.2.4.)

(ii) For $n \geq 1$, $B$ is *in* $\Sigma_n$ (written $B \in \Sigma_n$) if there is a computable relation $R(x, y_1, y_2, \ldots, y_n)$ such that

$$x \in B \iff (\exists y_1)\,(\forall y_2)\,(\exists y_3)\,\cdots\,(Q y_n)\,R(x, y_1, y_2, \ldots, y_n),$$

where $Q$ is $\exists$ for $n$ odd, and $\forall$ for $n$ even.

(iii) Likewise, $B$ is $\Pi_n$ ($B \in \Pi_n$) if

$$x \in B \iff (\forall y_1)\,(\exists y_2)\,(\forall y_3)\,\cdots\,(Q y_n)\,R(x, y_1, y_2, \ldots, y_n),$$

where $Q$ is $\exists$ or $\forall$ according to whether $n$ is even or odd.

(iv) Similarly, $B$ is $\Delta_n$ ($B \in \Delta_n$) if $B \in \Sigma_n$ and $B \in \Pi_n$.

(v) $B$ is *arithmetical* if $B \in \bigcup_n (\Sigma^0_n \cup \Pi^0_n)$.

Note that $B$ is arithmetical iff $B$ can be obtained from a computable relation by finitely many applications of projection and complementation. (See Exercise 4.1.10.)

**Definition 4.1.3.** Fix a set $A$. If we replace everywhere "computable" in Definition 4.1.2 by "$A$-computable" then we have the definition of $B$ being $\Sigma_n$ *in* $A$ (written $B \in \Sigma^A_n$), $B$ being $\Pi_n$ *in* $A$ ($B \in \Pi^A_n$), $B \in \Delta^A_n$, and $B$ being *arithmetical in* $A$.

## 4.1.1 Quantifier Manipulation

We say that a *formula* is $\Sigma_n$ ($\Pi_n$) if it is $\Sigma_n$ ($\Pi_n$) as a relation of its free variables. We assume familiarity with the usual rules of quantifier manipulation from elementary logic for converting a formula to an equivalent one in prenex normal form, consisting of a string of quantifiers (*prefix*) followed by a formula with no quantifiers (*matrix*), which will in our case be a computable relation. Using these rules we can show the following facts, which will be frequently used to prove that a particular set is in $\Sigma_n$ or $\Pi_n$. The only nontrivial fact is (vi), concerning bounded quantifiers. A *bounded quantifier* is one of the form $(Qx \leq y)F$, which abbreviates $(\forall x)\ [x \leq y \implies F]$ if $Q$ is $\forall$, and $(\exists x)\ [x \leq y \ \ \&\ \ F]$ if $Q$ is $\exists$. Part (vi) asserts that bounded quantifiers may be moved to the right past ordinary quantifiers and thus may be ignored in counting quantifier complexity.

**Theorem 4.1.4.** (i) $A \in \Sigma_n \iff \overline{A} \in \Pi_n$;

(ii) $A \in \Sigma_n(\textit{ or } \Pi_n) \implies (\forall m > n)\,[\,A \in \Sigma_m \cap \Pi_m\,]$;

(iii) $A, B \in \Sigma_n(\ \Pi_n\ ) \implies A \cup B,\ A \cap B \in \Sigma_n(\ \Pi_n\ )$;

(iv) $[R \in \Sigma_n\ \ \&\ \ n > 0\ \ \&\ \ A = \{\, x : (\exists y)\, R(x, y)\, \}] \implies A \in \Sigma_n$;

(v) $[B \leq_m A\ \ \&\ \ A \in \Sigma_n] \implies B \in \Sigma_n$;

(vi) *If* $R \in \Sigma_n(\ \Pi_n\ )$, *and* $A$ *and* $B$ *are defined by*

$$\langle x, y\rangle \in A \iff (\forall z < y)\, R(x, y, z),$$

$$\langle x, y\rangle \in B \iff (\exists z < y)\, R(x, y, z),$$

*then* $A, B \in \Sigma_n(\ \Pi_n\ )$.

*Proof.* (i) If $A = \{\, x : (\exists y_1)\,(\forall y_2)\,\cdots\, R(x, \overrightarrow{y})\,\}$, then

$$\overline{A} = \{\, x : (\forall y_1)\,(\exists y_2)\,\cdots\,\neg R(x, \overrightarrow{y})\,\}.$$

(ii) For example, if $A = \{\, x : (\exists y_1)\,(\forall y_2)\, R(x, y_1, y_2)\,\}$, then

$$A = \{\, x : (\exists y_1)\ (\forall y_2)\ (\exists y_3)\ [R(x, y_1, y_2)\ \ \&\ \ y_3 = y_3]\,\}.$$

(iii) Let $A = \{\, x : (\exists y_1)\ (\forall y_2)\ \cdots\ R(x, \overrightarrow{y})\,\}$, and

$$B = \{\, x : (\exists z_1)\,(\forall z_2)\,\cdots\, S(x, \overrightarrow{z})\,\}.$$

Then

$$x \in A \cup B \iff (\exists y_1)\ (\forall y_2)\ \cdots\ R(x, \overrightarrow{y}) \quad \vee \quad (\exists z_1)\ (\forall z_2)\ \cdots\ S(x, \overrightarrow{z}),$$

$$\iff (\exists y_1)\ (\exists z_1)\ (\forall y_2)\ (\forall z_2)\ \cdots\ [\ R(x, \overrightarrow{y}) \quad \vee \quad S(x, \overrightarrow{z})\ ],$$

$$\iff (\exists u_1)\ (\forall u_2)\ \cdots\ [\ R(x, (u_1)_0, (u_2)_0, \ldots) \quad \vee \quad S(x, (u_1)_1, (u_2)_1, \ldots)\ ],$$

where $(u)_0$ is the prime power coding as in the Notation section. Likewise, this holds for $A \cap B$.

(iv) Immediate by quantifier contraction, as in (iii).

(v) Let $A = \{\, x : (\exists y_1)\,(\forall y_2)\,\cdots\, R(x, \overrightarrow{y})\,\}$ and $B \leq_m A$ via $f$. Then

$$B = \{\, x : (\exists y_1)\,(\forall y_2)\,\cdots\, R(f(x), \overrightarrow{y})\,\}.$$

(vi) The proof is by induction on $n$. If $n = 0$, then $A$ and $B$ are clearly computable. Fix $n > 0$, suppose $R \in \Sigma_n$ and assume (vi) for all $m < n$. Then $B \in \Sigma_n$ by (iv). Now there exists $S \in \Pi_{n-1}$ such that

$$\langle x, y\rangle \in A \iff (\forall z < y)\ R(x, y, z),$$

$$\iff (\forall z < y)\ (\exists u)\ S(x, y, z, u),$$

$$\iff (\exists \sigma)\,(\forall z < y)\; S(x, y, z, \sigma(z)),$$

where $\sigma$ ranges over $\omega^{<\omega}$. Now $(\forall z < y)\; S \in \Pi_{n-1}$ by the inductive hypothesis, so $A \in \Sigma_n$. The case $R \in \Pi_n$ follows from the case $R \in \Sigma_n$ by (i). □

### 4.1.2 Placing a Set in $\Sigma_n$ or $\Pi_n$

**Proposition 4.1.5.** $\mathrm{Fin} \in \Sigma_2$.

*Proof.*

$$\begin{aligned} x \in \mathrm{Fin} &\iff W_x \text{ is finite} \\ &\iff (\exists s)\,(\forall t)\,[\, t \le s \;\vee\; W_{x,t} = W_{x,s} \,]. \end{aligned}$$

The bracketed relation of $x, s, t$ is clearly computable. □

**Proposition 4.1.6.** $\mathrm{Cof} \in \Sigma_3$.

*Proof.*

$$\begin{aligned} x \in \mathrm{Cof} &\iff \overline{W}_x \text{ is finite} \\ &\iff (\exists y)\,(\forall z)\,[\, z \le y \;\vee\; z \in W_x \,] \\ &\iff (\exists y)(\forall z)(\exists s)\,[\, z \le y \;\vee\; z \in W_{x,s} \,]. \end{aligned}$$

□

Since the final prefix depends only on the type and relative position of the quantifier symbols and sentential connectives, we frequently abbreviate these calculations by replacing previously identified predicates with strings of quantifiers indicating the classes to which they belong.

**Proposition 4.1.7.** $\{\, \langle x, y\rangle : W_x \subseteq W_y \,\} \in \Pi_2$.

*Proof.*

$$\begin{aligned} W_x \subseteq W_y &\iff (\forall z)\,[z \in W_x \implies z \in W_y] \\ &\iff (\forall z)\,[\, z \notin W_x \;\vee\; z \in W_y \,] \\ &\iff (\forall z)\,[\, \forall \;\vee\; \exists \,] \\ &\iff \forall\forall\exists\,[\, \dots \,] \\ &\iff \forall\exists\,[\, \dots \,]. \end{aligned}$$

□

**Corollary 4.1.8** (Classification of Tot)**.** (i) $\{\, \langle x, y\rangle : W_x = W_y \,\} \in \Pi_2$.

(ii) $\mathrm{Tot} = \{\, y : W_y = \omega \,\} \in \Pi_2$.

*Proof.* (i) follows by Proposition 4.1.7 and Theorem 4.1.4 (iii), and (ii) follows from Proposition 4.1.7 with $W_x = \omega$. □

**Corollary 4.1.9.** Rec $\in \Sigma_3$. (Rec $:= \{e : W_e \equiv_{\mathrm{T}} \emptyset\}$ in Definition 1.6.15.)

*Proof.*

$$\begin{aligned}
x \in \text{Rec} &\iff W_x \text{ is computable} && \text{(i.e., recursive)}\\
&\iff (\exists y)\,[\, W_x = \overline{W}_y \,]\\
&\iff (\exists y)\,[\, W_x \cap W_y = \emptyset \quad \& \quad W_x \cup W_y = \omega \,]\\
&\iff \exists \ \ [\,\forall \quad \& \quad \forall\exists\,] && \text{by Corollary 4.1.8}\\
&\iff \exists\forall\exists\,[\,\dots\,].
\end{aligned}$$

□

### 4.1.3 Exercises

**Exercise 4.1.10.** Prove that $A$ is arithmetical, i.e., that $A \in \bigcup_n (\Sigma_n \cup \Pi_n)$, iff $A$ can be obtained from a computable relation by a finite number of applications of projection and complementation.

**Exercise 4.1.11.** Prove that Ext $\in \Sigma_3$ for Ext as defined in Definition 1.6.15.

**Exercise 4.1.12.** Prove that

$$\{\langle x, y\rangle : W_x \text{ and } W_y \text{ are computably separable}\} \in \Sigma_3.$$

(Recall from Remark 2.4.15 and Exercise 1.6.26 that $W_x$ and $W_y$ are *computably separable* if $W_x \subseteq R$ and $W_y \subseteq \overline{R}$ for some computable set $R$, and $W_x$ and $W_y$ are *computably inseparable* otherwise.)

**Exercise 4.1.13.** Define $A \subseteq^* B$ if $A - B$ is finite, i.e., if $A \subseteq B$ except for at most finitely many elements. Define $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$. Prove that the following are two sets are $\Sigma_3$:

$$\{\langle x, y\rangle : W_x \subseteq^* W_y\};$$

$$\{\langle x, y\rangle : W_x =^* W_y\}.$$

**Exercise 4.1.14.** Show that $\{\, x : W_x \text{ is creative} \,\} \in \Sigma_3$.

## 4.2 ⋆⋆ Post's Theorem and the Hierarchy Theorem

**Definition 4.2.1.** A set $A$ is $\Sigma_n$-*complete* ($\Pi_n$-*complete*) if $A \in \Sigma_n(\ \Pi_n\ )$ and $B \leq_1 A$ for every $B \in \Sigma_n(\ \Pi_n\ )$. (By Exercises 4.2.6 and 4.2.7 it makes no difference whether we use "$B \leq_m A$" or "$B \leq_1 A$" in the definition of $\Sigma_n$–complete and $\Pi_n$-complete.)

Note that $A$ is $\Sigma_1$-complete iff $A$ is 1-complete as defined in Definition 2.4.1. Hence, $K$ is $\Sigma_1$-complete and $\overline{K}$ is $\Pi_1$-complete. The following fundamental theorem relates the jump hierarchy of degrees from §3.4 to the arithmetical hierarchy.

### 4.2.1 Post's Theorem Relating $\Sigma_n$ to $\emptyset^{(n)}$

**Theorem 4.2.2** (Post's Theorem). *For every $n \geq 0$,*

*(i)* $B \in \Sigma_{n+1} \iff$ *$B$ is c.e. in some $\Pi_n$ set*
$\iff$ *$B$ is c.e. in some $\Sigma_n$ set*

*by Theorem 3.4.3 (vii).*

*(ii) $\emptyset^{(n)}$ is $\Sigma_n$-complete for $n > 0$;*

*(iii) $B \in \Sigma_{n+1} \iff B$ is c.e. in $\emptyset^{(n)}$;*

*(iv) $B \in \Delta_{n+1} \iff B \leq_{\mathrm{T}} \emptyset^{(n)}$.*

*Proof.* (i) ($\Longrightarrow$). Let $B \in \Sigma_{n+1}$. Then $x \in B \iff (\exists y)\, R(x,y)$ for some $R \in \Pi_n$. Hence $B$ is $\Sigma_1$ in $R$ and therefore c.e. in $R$ by Theorem 3.3.16.

(i) ($\Longleftarrow$). Suppose $B$ is c.e. in some $\Pi_n$ set $C$. Then for some $e$,

$$x \in B \iff x \in W_e^C$$

$$x \in B \iff (\exists s)\,(\exists \sigma)\,[\,\sigma \prec C \quad \& \quad x \in W_{e,s}^{\sigma}\,].$$

Clearly, $x \in W_{e,s}^{\sigma}$ is computable by Oracle Graph Theorem 3.3.8. Hence, by Theorem 4.1.4 (iv) it suffices to show that $\sigma \prec C$ is $\Sigma_{n+1}$. Now

$$\sigma \prec C \iff (\forall y < lh(\sigma))\,[\sigma(y) = C(y)]$$

$$\iff (\forall y < lh(\sigma))\,[[\sigma(y) = 1 \;\&\; y \in C] \vee [\sigma(y) = 0 \text{ and } y \notin C]]$$

$$\iff (\forall y < lh(\sigma))\,[\Pi_n \vee \Sigma_n]$$

because $C \in \Pi_n$. Hence, $\sigma \prec C$ is $\Sigma_{n+1}$ by Theorem 4.1.4 (ii), (iii), and (vi).

(ii) This is proved by induction on $n$ and is clear for $n = 1$. Fix $n \geq 1$ and assume $\emptyset^{(n)}$ is $\Sigma_n$-complete. Hence $\overline{\emptyset^{(n)}}$ is $\Pi_n$-complete. Now

$$B \in \Sigma_{n+1} \iff B \text{ is c.e. in some } \Sigma_n \text{ set by (i)}$$

$$\iff B \text{ is c.e. in } \emptyset^{(n)} \text{ by inductive hypothesis}$$

$$\iff B \leq_1 \emptyset^{(n+1)} \text{ by the Jump Theorem 3.4.3 (iii).}$$

Hence, $\emptyset^{(n+1)}$ is $\Sigma_{n+1}$-complete.

(iii) Now $\overline{\emptyset^{(n)}}$ is $\Pi_n$-complete for $n > 0$ by (ii), and (i) and (ii) imply (iii).

(iv)

$$B \in \Delta_{n+1} \iff B, \overline{B} \in \Sigma_{n+1},$$

$$\iff B, \overline{B} \text{ are c.e. in } \emptyset^{(n)}, \text{ by (iii)},$$

$$\iff B \leq_T \emptyset^{(n)}.$$

□

**Corollary 4.2.3** (Hierarchy Theorem)**.** $(\forall n > 0)[\ \Delta_n \subset \Sigma_n \quad \& \quad \Delta_n \subset \Pi_n\ ]$. *Clearly,* $\Delta_n \subseteq \Sigma_n$. *The content here is that* $\Sigma_n \not\subseteq \Delta_n$.

*Proof.* $\emptyset^{(n)} \in \Sigma_n - \Pi_n$, by Post's Theorem 4.2.2 (ii) and (iv), and the Jump Theorem 3.4.3 (ii). Likewise, $\overline{\emptyset^{(n)}} \in \Pi_n - \Sigma_n$. □

**Definition 4.2.4.** ($\Sigma_n$ and $\Pi_n$ Indices).

(i) By Definition 2.1.4, $e$ is a $\Sigma_1$-*index* for $B$ if $B = W_e$, and we also say that $e$ is a $\Pi_1$-*index* for $\overline{B}$.

(ii) For $n > 0$, by Theorem 4.2.2 (iii), $B \in \Sigma_n$ iff $B \leq_1 \emptyset^{(n)}$, say via $\varphi_e$. Then $e$ is a $\Sigma_n$-*index* for $B$ and a $\Pi_n$-*index* for $\overline{B}$.

(iii) As in Definition 2.3.1 and Definition 4.1.2 (i), a $\Delta_0$-*index* for $B$ is an index $e$ such that $\varphi_e = \chi_B$. For $n \geq 1$, a $\Delta_n$-*index* for $B$ is a pair $\langle e, i \rangle$ where $e$ is a $\Sigma_n$ index for $B$ and $i$ is a $\Pi_n$ index for $B$. (These definitions relativize to an oracle $A$.)

### 4.2.2 Exercises

**Exercise 4.2.5.** In the Limit Lemma 3.6.2, prove that we can pass effectively from an index for any one characterization (i), (ii), or (iii) to any other. An index for (i) is an $e$ such that $\varphi_e(s, x) = A_s(x)$ and $A(x) = \lim_s A_s(x)$; An index for (ii) is a $\Delta_2$-index for $A$. An index for (iii) is an $e$ such that $A = \Phi_e^K$.

**Exercise 4.2.6.** Prove that if $B \leq_m A$ and $A = \emptyset^{(n)}$ for $n \geq 1$ then $B \leq_1 A$. *Hint.* Use the Padding Lemma 1.5.2. An alternative proof is to show that $B \in \Sigma_n$ and hence $B$ is c.e. in $\emptyset^{(n-1)}$ by Post's Theorem 4.2.2. Therefore, we can apply the Jump Theorem 3.4.3.

**Exercise 4.2.7.** Prove that if $B = \emptyset^{(n)}$ for $n \geq 1$, and $B \leq_m A$, then $B \leq_1 A$. *Hint.* Use the method of Theorem 2.3.9. (By Exercise 4.2.7, in order to prove that $A$ is $\Sigma_n$-complete it suffices to prove that $\emptyset^{(n)} \leq_m A$ rather than proving $\emptyset^{(n)} \leq_1 A$.)

## 4.3 ⋆ $\Sigma_n$-Complete Sets and $\Pi_n$-Complete Sets

We have shown that $\emptyset^{(n)}$ is $\Sigma_n$-complete for all $n$. (Following Convention 4.1.1 we normally drop the superscript 0 from now on.) However, there are other $\Sigma_n$-complete sets with natural definitions which will be useful in later applications. For example, we know that $K$, $K_0$ and $K_1$ are all $\Sigma_1$-complete and we shall now show that Fin is $\Sigma_2$-complete and Cof and Rec are $\Sigma_3$-complete. Once we have classified a set $A$ as being in $\Sigma_n$ by the method of §4.1, we attempt to show that the classification is the best possible by proving that $B \leq_1 A$ for some known $\Sigma_n$-complete set $B$, thus showing that $A$ is $\Sigma_n$-complete. Recall from Definition 2.4.9 that $(A, B) \leq_m (C, D)$ via $f$ computable if $f(A) \subseteq C$, $f(B) \subseteq D$, and $f(\overline{A \cup B}) \subseteq \overline{C \cup D}$. We write "$\leq_1$" if $f$ is 1:1.

**Definition 4.3.1.** For $n \geq 1$ define $(\Sigma_n, \Pi_n) \leq_m (C, D)$ if $(A, \overline{A}) \leq_m (C, D)$ for some $\Sigma_n$-complete set $A$, and similarly for $\leq_1$ in place of $\leq_m$. In this case we also write $\Sigma_n \leq_m C$ and $\Pi_n \leq_m D$. (By the same remark as that in Definition 4.2.1, it makes no difference whether we write "$\leq_m$" or "$\leq_1$" here.)

(This notation seems strange because $\Sigma_n$ and $\Pi_n$ are *classes* not sets. It is justified because if $(\Sigma_n, \Pi_n) \leq_m (C, D)$ then $(A, \overline{A}) \leq_m (C, D)$ and $(\overline{B}, B) \leq_m (C, D)$ for *any* $\Sigma_n$ set $A$ and $\Pi_n$ set $B$.)

### 4.3.1 Classifying $\Sigma_2$ and $\Pi_2$ Sets: Fin, Inf, and Tot

**Theorem 4.3.2.** $(\Sigma_2, \Pi_2) \leq_1$ (Fin, Tot). *Therefore,* Fin *is* $\Sigma_2$*-complete,* Inf *and* Tot *are* $\Pi_2$*-complete, and* Inf $\equiv_1$ Tot. *Hence,* Inf *and* Tot *are computably isomorphic, written* Inf $\equiv$ Tot.

*Proof.* By Proposition 4.1.5 and Corollary 4.1.8, Fin $\in \Sigma_2$ (so Inf $\in \Pi_2$) and Tot $\in \Pi_2$. Fix $A \in \Sigma_2$. Therefore, $\overline{A} \in \Pi_2$, and there is a computable relation $R$ such that

$$x \in \overline{A} \iff (\forall y)(\exists z)\, R(x, y, z).$$

Using the *s-m-n* Theorem 1.5.5, define a 1:1 computable function $f$ by

$$\varphi_{f(x)}(u) = \begin{cases} 0 & \text{if } (\forall y \leq u)\,(\exists z)\, R(x, y, z); \\ \uparrow & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} x \in \overline{A} &\implies W_{f(x)} = \omega &&\implies f(x) \in \text{ Tot}, \quad \text{but} \\ x \in A &\implies W_{f(x)} \text{ is finite} &&\implies f(x) \in \text{ Fin}. \end{aligned}$$

□

### 4.3.2 Constructions with Movable Markers

Most of the definitions of c.e. sets so far have been *static* in the sense of §2.6.2, but from now on we often give *dynamic* definitions. For example, we may define a c.e. set $B$ by a construction using a computable sequence of stages $s$ where $B_s$ represents the set of elements enumerated in $B$ by the end of stage $s$ and $B = \cup_s B_s$. To construct $B$ we concentrate on the stage $s$ approximation to the *complement* $\overline{B}$ because these are the only elements over which we still have control. Those *already* in $B$ are irretrievable. Given $B_s$, define the element $b_y^s$ for $y \in \omega$ as follows:

$$(4.1) \qquad \overline{B} = b_0 < b_1 < b_2 < \dots \qquad \& \qquad \overline{B}_s = b_0^s < b_1^s < b_2^s < \dots$$

To define $B_s$ it is often useful to imagine a sequence of *markers* $\{\Gamma_y\}_{y\in\omega}$ such that the marker $\Gamma_y$ is associated with element $b_y^s$ at the end of stage $s$. Now $b_y^s \leq b_y^{s+1}$. Therefore, we may imagine marker $\Gamma_y$ as moving upwards and being associated with a nondecreasing (possibly finite) sequence of elements $\{b_y^s\}_{s\in\omega}$ among the integers. Hence, the name *movable markers* is used in the literature.[1]

The advantage of concentrating on the marker $\Gamma_y$ rather than the element $z = b_y^s$ it is currently resting on is that for applications we may have an additional c.e. *kicking set* $V_y$ which is coordinated with the marker $\Gamma_y$. Whenever $V_y$ receives a new element, the current position of $\Gamma_y$ is enumerated in $B$. Hence, $\Gamma_y$ comes to a limit and $b_y$ exists iff $V_y$ is finite. Therefore, $\overline{B}$ is infinite iff every $V_y$ is finite. We have already implicitly used this method for $y = 0$ in Exercise 2.5.3 to prove that Inf $\leq_1$ Cof. We now illustrate the movable marker method in the following Theorem 4.3.3 but with a movable marker for every $y$ not only for $y = 0$.

### 4.3.3 Classifying Cof as $\Sigma_3$-Complete

**Theorem 4.3.3.** *Cof is* $\Sigma_3$*-complete.*

*Proof.* Fix $A \in \Sigma_3$. Now for some relation $R \in \Pi_2$, $x \in A$ iff $(\exists y)R(x,y)$. Since $R \in \Pi_2$ there is a computable function $g$ by Theorem 4.3.2 such that $R(x,y)$ iff $W_{g(x,y)}$ is infinite. Therefore,

$$(4.2) \qquad x \in A \quad \Longleftrightarrow \quad (\exists y)\,[\, W_{g(x,y)} \text{ is infinite } ].$$

[1] For more sophisticated applications, it is better to think of the markers as fixed *boxes* or *windows* sometimes arranged in some geometrical pattern, such as a matrix, a tree, or simply a line as here, through which the integers move downwards. From this point of view the boxes are fixed and the integers are moving among them, but we still have box $\Gamma_y$ associated with a nondecreasing sequence of elements $\{b_y^s\}_{s\in\omega}$.

We shall define a c.e. set $B^x$ uniformly in $x$ such that $x \in A$ iff $B^x$ is cofinite. Fix $x$. For notational convenience we drop the superscript $x$. We enumerate $B = \cup_{s\in\omega} B_s$ by stages $s$ in the following computable construction. Use the notation of §4.3.2 and (4.1). We think of $W_{g(x,y)}$ as a *kicking set* so that each new element entering $W_{g(x,y)}$ "kicks" the marker $\Gamma_y$ and forces it to move once more.

*Stage* $s = 0$. Set $B_0 = \emptyset$.

*Stage* $s+1$. Let $\overline{B}_s = \{b_0^s < b_1^s < \cdots < b_y^s < \cdots\}$. For each $y \leq s$ such that $W_{g(x,y),\,s} \neq W_{g(x,y),\,s+1}$, enumerate $b_y^s$ in $B_{s+1}$. If no such $y$ exists, define $B_{s+1} = B_s$. This ends the construction.

*Case 1.* $x \in A$. By (4.2), choose the least $y$ such that $W_{g(x,y)}$ is infinite. Now marker $\Gamma_y$ is moved infinitely often. Therefore, $\lim_s b_y^s = \infty$, and $|\overline{B}| \leq y$.

*Case 2.* $x \notin A$. By induction, fix $y$, and choose $s$ such that $W_{g(x,y),\,s} = W_{g(x,y)}$ and, for all $z < y$ such that $b_z^s = b_z$. Now $\Gamma_y$ never moves again after $s$. Hence, every marker comes to rest on $\overline{B}$, which is therefore infinite. □

### 4.3.4 Classifying Rec as $\Sigma_3$-Complete

**Definition 4.3.4.** (i) Cpl $= \{\, x : W_x \equiv_{\mathrm{T}} K \,\}$, indices of *complete* c.e. sets.

(ii) Rec $= \{\, x : W_x \equiv_{\mathrm{T}} \emptyset\}$, indices of *computable (recursive)* sets.

**Theorem 4.3.5.** $(\Sigma_3, \Pi_3) \ \leq_1 \ (\text{Cof, Cpl})$, *and* $(\Sigma_3, \Pi_3) \ \leq_1 \ (\text{Rec, Cpl})$.

**Corollary 4.3.6** (Rogers)**.** Rec *is* $\Sigma_3$*-complete.*

*Proof.* By Corollary 4.1.9 and Theorem 4.3.5 because Cof $\subseteq$ Rec and because Rec $\cap$ Cpl $= \emptyset$. □

*Proof.* (*Theorem 4.3.5*). Let $A$ be $\Sigma_3$. We define a c.e. set $B^x$ uniformly in $x$ such that

$$x \in A \iff (\exists y)\,[\, W_{g(x,y)} \text{ is infinite}\,] \iff B^x \text{ is cofinite,} \tag{4.3}$$

$$x \notin A \implies B^x \equiv_{\mathrm{T}} K. \tag{4.4}$$

Fix $x$. For notational convenience we can drop the $x$. Let $\{K_s\}_{s\in\omega}$ be a computable enumeration of $K$. The construction is now exactly the same as that of Theorem 4.3.3 except that at Stage $s+1$ we replace the second sentence by the following:

> "For each $y \leq s$ such that *either* $W_{g(x,y),s} \neq W_{g(x,y),\,s+1}$
> *or* $y \in K_{s+1} - K_s$, enumerate $b_y^s$ in $B_{s+1}$."

Now if $x \in A$ then some $W_{g(x,y)}$ is infinite and it causes $\overline{B}$ to be finite as before. If $x \notin A$ then the extra clause generates at most one extra move for marker $\Gamma_y$. Therefore, all markers move finitely often and $\overline{B}$ is infinite. The extra coding ensures that $K \leq_{\mathrm{T}} B$. Choose a stage $s$ such that marker $\Gamma_y$ has settled on $b_y^s$ by the end of stage $s$. Then $y \in K$ iff $y \in K_s$ because if $y$ enters $K$ at some stage $t > s$ then marker $\Gamma_y$ must move at stage $t$, which it cannot. □

**Remark 4.3.7.** Theorem 4.3.5 also implies the previous Theorem 4.3.3 that Cof is $\Sigma_3$-complete, and it shows that $(\Pi_3, \Sigma_3) \leq_{\mathrm{m}} (\mathrm{Cpl}, \overline{\mathrm{Cpl}})$. This does not imply that Cpl is $\Pi_3$-complete. It says exactly that Cpl is $\Pi_3$-hard, namely that a $\Pi_3$-complete set is $m$-reducible to it. Indeed Cpl is $\Sigma_4$-complete.

**Remark 4.3.8.** An alternative coding is to move the markers to prove that if $x \in \overline{A}$, then $\overline{B}$ dominates all p.c. functions and therefore $K \leq_{\mathrm{T}} B$ by Theorem 4.5.4 (ii). In Theorem 4.3.5 we have two strategies. The primary strategy $S_1$ uses $W_{g(x,y)}$ to show that if $x \in A$ then $\overline{B}$ is finite. If $x \in \overline{A}$, this primary strategy guarantees only that $\overline{B}$ is infinite. In this case we can simultaneously play the *secondary strategy* $S_2$, which ensures $B \equiv_T K$. In the $\Pi_3$ case, where $\overline{B}$ is infinite, we can code various other properties into $\overline{B}$. For example, in Chapter 5 Exercise 5.2.10 we prove that $\{e : W_e \text{ simple}\}$ is $\Pi_3$-complete.

One may imagine that the $\Pi_3$ alternative on $\overline{B}$ is an expert woodsman who goes through the forest chopping down only *certain* trees to code information. If the $\Sigma_3$ alternative holds, then the logging company comes through, cutting *all* the trees and erasing any coding done by the woodsman.

In Exercise 4.3.12 we shall prove that Ext is $\Sigma_3$-complete by defining a p.c. function $\varphi_{f(x)}$ and having a strategy $S_2$ for marker $\Gamma_y$ which guarantees that $\varphi_{f(x)}$ is not extendible to a total function $\varphi_y$ and that indeed $\Gamma_y$ bounds a counterexample $z$. In Exercise 5.2.10, the markers $\Gamma_y$, $y < e$, allow some $b_y^s \in W_e$ to enter $B$ to achieve $B \cap W_e \neq \emptyset$, so $B^x$ will be simple (see §5.2). The only restriction on the secondary strategy $S_2$ is that it must cause the marker $\Gamma_y$ to move at most finitely often so as not to *accidentally* cause $\overline{B}$ to be finite even though $x \notin A$ which is the $\Pi_3$ case.

### 4.3.5 $\Sigma_3$-Representation Theorems

The following are probably the most useful characterizations for approximating a $\Sigma_3$ set $A$, i.e., for "guessing" whether $x \in A$, and should be viewed as refinements of (4.2).

**Theorem 4.3.9** (First $\Sigma_3$-Representation Theorem). *If $A \in \Sigma_3$ then there is a computable function $g$ such that*

$$x \in A \iff (\forall^\infty y)\,[\; W_{g(x,y)} = \omega \;] \quad \textit{and} \tag{4.5}$$

$$x \in \overline{A} \iff (\forall\, y)\,[\; W_{g(x,y)} \text{ is finite } ]. \tag{4.6}$$

*Proof.* Since $A \in \Sigma_3$, let $A \leq_1 \mathrm{Cof}$ via $f$ using Theorem 4.3.3. Define $g$ by

$$z \in W_{g(x,y)} \iff (\forall u)\ [y \leq u \leq z \implies u \in W_{f(x)}]. \qquad \text{Hence,}$$

$$x \in A \implies W_{f(x)} \text{ cofinite } \implies (\exists y)(\forall z \geq y)\,[\; z \in W_{f(x)} \;]$$

$$\implies (\exists y)\ (\forall z \geq y)\ [W_{g(x,z)} = \omega]; \qquad \text{and}$$

$$x \in \overline{A} \implies W_{f(x)} \text{ coinfinite } \implies (\forall y)\ (\exists z \geq y)\ [z \notin W_{f(x)}]$$

$$\implies (\forall y)\ [W_{g(x,y)} \text{ finite}].$$

□

**Remark 4.3.10.** (*Guessing About a $\Sigma_3$ Set $A$*). To "guess" about membership in a $\Sigma_2$ set $A$, we have a computable function $f$ such that $x \in A$ iff $W_{f(x)}$ is finite. For a $\Sigma_3$ set $A$, Theorem 4.3.9 is the two-dimensional analogue where $W_{g(x,y)}$ is viewed as the $y^{\text{th}}$ row of a matrix. If $x \in A$, then almost all rows are $\omega$, and the others are finite. If $x \notin A$ then all rows are finite. The next corollary says that in the first case we may redefine the matrix so that there is a *unique* row which is infinite and that row is $\omega$.

**Theorem 4.3.11** (Second $\Sigma_3$-Representation Theorem-Uniqueness). *If $A \in \Sigma_3$ then there is a computable function $h$ such that the following lines hold:*

$$x \in A \iff (\exists\,!\, y)\,[\; W_{h(x,y)} = \omega \quad \& \quad (\forall z \neq y)[\; W_{h(x,z)} =^* \emptyset \;]\,], \tag{4.7}$$

$$x \in \overline{A} \iff (\forall y)\,[\; W_{h(x,y)} =^* \emptyset \;], \tag{4.8}$$

*where $(\exists\,!\, y)R(y)$ denotes that there exists a* unique *$y$ such that $R(y)$.*

*Proof.* $A$ is $\Sigma_3$. Choose $g(x,y)$ satisfying (4.5) and (4.6). Define

$$f(x,y,s) \;=\; y \;+\; \Sigma_{z<y}\,|\, W_{g(x,z),\,s}\,|.$$

(Think of $f(x,y,s)$ as the position at the end of stage $s$ of a movable marker $\Gamma^x_y$ which moves along the $h$ rows trying to represent row $W_{g(x,y)}$ on some $h$ row but which is bumped whenever an element appears in some $W_{g(x,z)}$ for some $z < y$.)

*Stage $s+1$.* Let $z = f(x,y,s)$. Enumerate in $W_{h(x,z)}$ all $w \in W_{g(x,y),\,s}$.

*Verification.*

*Case 1.* $x \in A$. Choose the least $y$ such that $W_{g(x,y)} = \omega$. Then $z = \lim_s f(x,y,s)$ exists, and $W_{h(x,z)} = W_{g(x,y)} = \omega$. Also, $\lim_s f(x,v,s) = \infty$ for all $v > y$ and hence $W_{h(x,u)}$ is finite for all $u > z$.

*Case 2.* $x \notin A$. For each $z$ there are at most finitely many $y$ such that $\lim_s f(x,y,s) = z$ because of the clause "$y+$" in the definition of $f(x,y,s)$. But each $g$ row $W_{g(x,y)}$ is finite. Hence, every $h$ row $W_{h(x,z)}$ is finite. □

### 4.3.6 Exercises

**Exercise 4.3.12.**$^\diamond$ Prove that $(\Sigma_3, \Pi_3) \leq_1 (\text{Cof}, \overline{\text{Ext}})$ and hence that Ext is $\Sigma_3$-complete. *Hint.* Use the notation and method of Theorem 4.3.3 to construct $\varphi_{f(x)}$ such that if $x \in A$, then $f(x) \in \text{Cof} \subset \text{Ext}$, and if $x \notin A$, then $f(x) \in \overline{\text{Ext}}$.

**Exercise 4.3.13.** Show $\{\langle x,y\rangle : W_x \text{ and } W_y \text{ are computably separable}\}$ is $\Sigma_3$-complete. *Hint.* Make $\varphi_{f(x)}$ of Exercise 4.3.12 take values $\subseteq \{\, 0, 1 \,\}$.

**Exercise 4.3.14.** Prove that $\{\langle x,y\rangle : W_x \subseteq^* W_y\}$ and $\{\langle x,y\rangle : W_x =^* W_y\}$ are each $\Sigma_3$-complete.

**Exercise 4.3.15.** Show that if $A$ is a c.e. set, then $G_m(A) \in \Sigma_3$ where

$$G_m(A) \ := \ \{ \ x \ : \ W_x \ \equiv_m \ A \ \}.$$

**Exercise 4.3.16.**$^\diamond$ (Lerman). Let $\zeta$ (zeta) denote the order type of the integers $\mathbb{Z}$ (both positive and negative in their natural order). Hence, $\zeta$ has order type $\omega^* + \omega$. A $\zeta$-representation for a set $A \subseteq \omega$ is a linear ordering

$$L_A^\zeta = \zeta + a_o + \zeta + a_1 + \ldots,$$

where $A = \{a_0, a_1, \ldots\}$ is not necessarily in increasing order and possibly with repetitions.

(i) Prove that if $L_A^\zeta$ is a computable linear ordering, i.e., the $<$ relation on it is computable, then $A \in \Sigma_3$.

(ii)$^\diamond$ Prove that if $A \in \Sigma_3$ then there is a computable ordering $L$ of order type $L_A^\zeta$.

## 4.4 Relativized Hierarchy: Low$_n$ and High$_n$ Sets

**Definition 4.4.1.** The definition of $\Sigma_n^A$ ($\Pi_n^A$) is the same as Definition 4.1.2 for $\Sigma_n$ ($\Pi_n$) except that the matrix $R$ is $A$-computable instead of computable. If $\mathbf{a} = \deg(A)$, we use the notation $\Sigma_n^{\mathbf{a}}$ in place of $\Sigma_n^A$ since the class $\Sigma_n^A$ is independent of the particular representative $A \in \mathbf{a}$.

Everything in this chapter can be relativized to an arbitrary set $A$ with virtually the same proofs, and with $\Sigma_n^A$, $\Pi_n^A$ and $A^{(n)}$ in place of $\Sigma_n$, $\Pi_n$ and $\emptyset^{(n)}$, respectively.

### 4.4.1 Relativized Post's Theorem

**Theorem 4.4.2** (Relativized Post's Theorem). *For every $n \geq 0$,*

*(i)* $A^{(n)}$ *is* $\Sigma_n^A$*-complete if* $n > 0$*;*

*(ii)* $B \in \Sigma_{n+1}^A \iff B$ *is c.e. in* $A^{(n)}$*;*

*(iii)* $B \leq_T A^{(n)} \iff B \in \Delta_{n+1}^A := \Sigma_{n+1}^A \cap \Pi_{n+1}^A$*;*

*(iv)* $B \leq_T A^{(n+1)} \iff (\exists f \leq_T A^{(n)})\,[\, B(x) = \lim_s f(x,s)\,]$.

Define $\text{Fin}^A$, $\text{Tot}^A$, and $\text{Cof}^A$ as before but with $W_e^A$ in place of $W_e$. The proofs in §4.3 relativize to $A$ and establish that $\text{Fin}^A$ is $\Sigma_2^A$-complete, $\text{Tot}^A$ is $\Pi_2^A$-complete, and $\text{Cof}^A$ and $\text{Rec}^A$ are $\Sigma_3^A$-complete, where $\text{Rec}^A$ is the set of $e$'s such that $W_e^A$ is $A$-computable ($A$-recursive). Hence, if $\mathbf{a} = \deg(A)$, then $\mathbf{a}' = \deg(A')$, $\mathbf{a}'' = \deg(\text{Fin}^A)$, and $\mathbf{a}''' = \deg(\text{Cof}^A)$.

### 4.4.2 $\text{Low}_n$ and $\text{High}_n$ Sets

In Definition 3.6.6 we introduced the low and high sets as those sets $A \leq_T \emptyset'$ whose jump $A'$ has the lowest value $\emptyset'$ and highest value $\emptyset''$. In Definition 3.4.2 (ii) we also defined the $n^{th}$ jump $A^{(n)}$ by iterating the jump $n$ times, where $A^{(0)} = A$, $A^{(1)} = A'$ and $A^{(n+1)} = (A^{(n)})'$. If $A \leq_T \emptyset'$, then by iterating the Jump Theorem 3.4.3 we know $\emptyset^{(n)} \leq_T A^{(n)} \leq_T \emptyset^{(n+1)}$.

**Definition 4.4.3.** Fix a set $A \leq_T \emptyset'$.

(i) $A$ is *low*$_n$ if $A^{(n)} \equiv_T \emptyset^{(n)}$, the lowest possible value.

(ii) $A$ is *high*$_n$ if $A^{(n)} \equiv_T \emptyset^{(n+1)}$, the highest possible value.

(iii) Let $\mathbf{D}$ denote the $\Delta_2$ degrees and $\mathbf{C}$ the c.e. degrees. A Turing degree $\mathbf{d} \in \mathbf{D}$ is *low*$_n$ or *high*$_n$ according to whether it contains a *low*$_n$ or *high*$_n$ set, since this property is degree invariant. For every $n \geq 0$, define the following subclasses of $\mathbf{D}$:

$$\mathbf{H}_n = \{\, \mathbf{d} : \mathbf{d} \in \mathbf{D} \quad \& \quad \mathbf{d}^{(n)} = \mathbf{0}^{(n+1)} \,\}$$

$$\mathbf{L}_n = \{\, \mathbf{d} : \mathbf{d} \in \mathbf{D} \quad \& \quad \mathbf{d}^{(n)} = \mathbf{0}^{(n)} \,\}.$$

(iv) A set or degree which is not *low*$_n$ or *high*$_n$ for any $n$ is *intermediate*.

Clearly, $\mathbf{L}_n \subseteq \mathbf{L}_{n+1}$ and $\mathbf{H}_n \subseteq \mathbf{H}_{n+1}$ for every $n$. Even restricted from $\mathbf{D}$ to $\mathbf{C}$ there is an intermediate c.e. degree and that the classes are strictly increasing,

$$(\forall n)\,[\ \mathbf{L}_n \subset \mathbf{L}_{n+1} \quad \& \quad \mathbf{H}_n \subset \mathbf{H}_{n+1}\ ].$$

Often we replace the $\Delta_2$ degrees $\mathbf{D}$ by the c.e. degrees $\mathbf{C}$ and use the same low/high notation, $\mathbf{L}_n/\mathbf{H}_n$, as above. Which one is intended will be clear from the context.

### 4.4.3 Common Jump Classes of Degrees

The most common jump classes of degrees are the following, with their complements (some of which are not given). In §4.7 we relate several of these classes to domination and escape properties.

| | | | |
|---|---|---|---|
| $\mathbf{H}_0$ | $=$ | $\{\mathbf{0}'\}$ | the complete degree |
| $\mathbf{L}_0$ | $=$ | $\{\mathbf{0}\}$ | the degree of $\emptyset$ |
| $\mathbf{L}_1$ | $=$ | $\{\mathbf{d} \in \mathbf{D} : \mathbf{d}' = \mathbf{0}'\}$ | low$_1$ |
| $\mathbf{L}_2$ | $=$ | $\{\mathbf{d} \in \mathbf{D} : \mathbf{d}'' = \mathbf{0}''\}$ | low$_2$ |
| $\overline{\mathbf{L}}_2$ | $=$ | $\{\mathbf{d} \in \mathbf{D} : \mathbf{d}'' > \mathbf{0}''\}$ | nonlow$_2$ |
| $\mathbf{H}_1$ | $=$ | $\{\mathbf{d} \in \mathbf{D} : \mathbf{d}' = \mathbf{0}''\}$ | high$_1$ |
| $\overline{\mathbf{H}}_1$ | $=$ | $\{\mathbf{d} \in \mathbf{D} : \mathbf{d}' < \mathbf{0}''\}$ | nonhigh$_1$. |

### 4.4.4 Syntactic Properties of High$_n$ and Low$_n$ Sets

We now develop a syntactic characterization of high and low in terms of arithmetical quantifiers. This is often useful in applying the hypothesis of high or low.

**Theorem 4.4.4** (High Theorem)**.** *For any set $A \subseteq \omega$ TFAE:*

(i) *$A$ is high* *(i.e., $\emptyset'' \leq_{\mathrm{T}} A'$, whether $A \leq_{\mathrm{T}} \emptyset'$ or not);*

(ii) $\Sigma_2 \subseteq \Delta_2^A$*;*

(iii) $\Sigma_2 \subseteq \Pi_2^A$*;*

(iv) $\overline{\emptyset^{(2)}} \leq_1 A^{(2)}$ *(i.e.,* $\mathrm{Tot} \leq_1 \mathrm{Fin}^A$*).*

*Proof.*

$$\begin{array}{llll} A \text{ is high} & \iff & \emptyset'' \leq_T A' & \\ & \iff & \emptyset'' \in \Delta_2^A & \text{by Post's Theorem 4.4.2} \\ & \iff & \Sigma_2 \subseteq \Delta_2^A & \text{because } \emptyset'' \text{ is } \Sigma_2\text{-complete} \\ & \iff & \Sigma_2 \subseteq \Pi_2^A & \text{because } \Sigma_2 \subseteq \Sigma_2^A \text{ trivially} \\ & \iff & \emptyset^{(2)} \leq_1 \overline{A^{(2)}} & \text{because } \overline{A^{(2)}} \text{ is } \Pi_2^A\text{-complete} \\ & \iff & \text{Tot} \leq_1 \text{Fin}^A & \text{because } \overline{\emptyset^{(2)}} \equiv_1 \text{Tot}. \quad \square \end{array}$$

**Theorem 4.4.5** (Low Theorem). *For any set $A \subseteq \omega$ TFAE:*

(i) *$A$ is low* *(i.e., $A' \leq_T \emptyset'$);*

(ii) $\Sigma_1^A \subseteq \Delta_2$,

(iii) $\Sigma_1^A \subseteq \Pi_2$;

(iv) $A' \leq_1 \overline{\emptyset^{(2)}}$ *(i.e., iff $K_1^A \leq_1$* Tot).

*Proof.*

$$\begin{array}{llll} A \text{ is low} & \iff & A' \leq_T \emptyset' & \\ & \iff & A' \in \Delta_2 & \text{by Post's Theorem 4.2.2} \\ & \iff & \Sigma_1^A \subseteq \Delta_2 & \text{because } A' \text{ is } \Sigma_1^A\text{-complete} \\ & \iff & \Sigma_1^A \subseteq \Pi_2 & \text{because } \Sigma_1^A \subseteq \Sigma_1^{\emptyset'} = \Sigma_2 \\ & \iff & A' \leq_1 \overline{\emptyset^{(2)}} & \text{because } \overline{\emptyset^{(2)}} \text{ is } \Pi_2\text{-complete}. \quad \square \end{array}$$

### 4.4.5 Exercises

**Exercise 4.4.6.** State and prove classifications for high$_2$ and low$_2$ similar to those in Theorems 4.4.4 and 4.4.5 for high$_1$ and low$_1$.

## 4.5 ⋆ Domination and Escaping Domination

Recall the Definition 3.5.1 of the quantifiers $(\forall^\infty x)$ and $(\exists^\infty x)$, and Definition 3.5.2 of domination and escape, which we now repeat and extend.

**Definition 4.5.1.** (i) A function $g$ *dominates* $f$, denoted by $f <^* g$, if

$$(\forall^\infty x)\,[\, f(x) < g(x)\,]. \tag{4.9}$$

A *partial* function $\theta(x)$ dominates a *partial* function $\psi(x)$ if

$$(\forall^\infty x)\,[\,\psi(x)\downarrow \quad\Longrightarrow\quad \psi(x) < \theta(x)\downarrow\,].$$

(ii) A function $f$ *escapes* (domination by) $g$ if $f \not<^* g$, i.e., if

$$(\exists^\infty x)\,[\; g(x) \leq f(x) \;]. \tag{4.10}$$

(iii) A function $g$ *majorizes* $f$, denoted by $f < g$, if

$$(\forall x)\,[\; f(x) \;<\; g(x) \;]. \tag{4.11}$$

(iv) Functions $f$ and $g$ are *almost equal,* denoted by $f =^* g$, if

$$(\forall^\infty x)\,[\; g(x) \;=\; f(x) \;].$$

(v) A class $\mathcal{C}$ of functions is *closed under finite differences* if

$$[\; g \in \mathcal{C} \quad \& \quad g =^* h \;] \qquad \Longrightarrow \qquad h \in \mathcal{C}.$$

**Proposition 4.5.2.** *Let $\mathcal{C}$ be a class of functions closed under finite differences, such as the computable functions or the $A$-computable functions for some $A$. Then for every $f$,*

$$(\exists g \in \mathcal{C})\,[\; g \;>^*\; f \;] \qquad \Longleftrightarrow \qquad (\exists h \in \mathcal{C})\,[\; h \;>\; f \;].$$

*Proof.* One direction is obvious. For the other direction, assume $g >^* f$, and find $h =^* g$ such that $h > f$. □

By Proposition 4.5.2, given such a $\mathcal{C}$ and $g \in \mathcal{C}$ with $g >^* f$, we shall assume that $g > f$. In particular, if we have a computable $g >^* f$ then we shall assume we have computable $g > f$.[2]

### 4.5.1 Domination Properties

**Definition 4.5.3.** Let $\{A_s\}_{s\in\omega}$ be a computable enumeration of c.e. set $A$.

(i) The *stage function* is the partial computable function

$$\theta_A(x) \;=\; \begin{cases} (\mu s)\,[\; x \in A_s \;] & \text{if } x \in A \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(ii) The *least modulus* as in (3.17) of Definition 3.5.4 is

$$m_A(x) = (\mu s)\,[\; A_s \upharpoonright x = \; A \upharpoonright x \;].$$

Note that $\theta_A(x)$ is *partial* but partial *computable*, while $m_A(x)$ is *total* but not computable (unless $A$ is computable).

[2] Dominate and majorize are very similar. We normally prefer *dominate* because by Proposition 4.5.2 if a computable function *g dominates f* then a computable function $h$ *majorizes f*. The negation of dominate is *escape* which gives a rich structure of nonlow$_2$ degrees in §4.5 and §4.6, but the negation of "$g$ majorizes $f$" is simply $(\exists x)\,[f(x) \geq g(x)]$, which is not useful.

**Theorem 4.5.4** (Domination Properties). *Let $\{A_s\}_{s\in\omega}$ be an enumeration of a c.e. set $A$ and $f$ a total function.*

*(i)* *If $f$ dominates $\theta_A(x)$ then $A \leq_{\mathrm{T}} f$.*

*(ii)* *For any $D \leq_{\mathrm{T}} \emptyset'$,*

$$D \equiv_{\mathrm{T}} \emptyset' \iff (\exists f \leq_{\mathrm{T}} D)[\, f \text{ dominates every partial computable function}\,].$$

*(iii)* *If $f$ dominates $m_A(x)$ then $A \leq_{\mathrm{T}} f$.*

*(iv)* *If $\{B_s\}_{s\in\omega}$ is an enumeration of a c.e. set $B$ and $m_A(x)$ dominates the least modulus function $m_B(x)$, then $B \leq_{\mathrm{T}} A$.*

*Proof.* (i) $(\forall^\infty x)\,[\, x \in A \iff x \in A_{f(x)}\,]$.

(ii) $(\Longleftarrow)$ By (i) because $f$ dominates $\theta_K(x)$.

(ii) $(\Longrightarrow)$ Build $f \leq_{\mathrm{T}} \emptyset'$ by using $\emptyset'$ to determine for a given input $x$ which $\varphi_e(x)$ converge for $e \leq x$. Then define $f(x)$ to exceed all these values.

(iii) $(\forall^\infty x)\,[\, x \in A \iff x \in A_{f(x)}\,]$.

(iv) $(\forall^\infty x)\,[\, x \in B \iff x \in B_{m_A(x)}\,]$. ☐

These are only the simplest facts about domination. In §4.5.2 and throughout the book we develop many more domination properties, and extend (ii) to an elegant characterization by Martin in Theorem 4.5.6 of functions which dominate all *total* computable functions. Escape properties are more subtle, but in Theorem 4.6.2 we characterize functions which escape $\emptyset'$-computable functions and we use these in computable model theory.

## 4.5.2 Martin's High Domination Theorem

The first few levels of the high/low degree hierarchy, especially the high$_1$, low$_1$, and low$_2$ degrees and their complements, have many important applications. In addition to the syntactic characterization of high degrees in Theorem 4.4.4, we now give the very useful characterization (Theorem 4.5.6) by Martin in terms of dominating functions. The following characterization of high degrees gives useful characterizations in §4.7 and §4.8 for uniform enumerations of the computable functions and properties of those $\Delta_2$ sets which are low$_2$ or nonlow$_2$. Later we consider low$_2$ and nonlow$_2$ sets. We now extend the domination notions from Definition 3.5.2.

**Definition 4.5.5.** $f$ is *dominant* if $f$ dominates every (total) computable function; an *infinite set* $A = \{a_0 < a_1 < \cdots\}$ is *dominant* if its principal function $p_A$ dominates every (total) computable function, where $p_A(n) = a_n$.

**Theorem 4.5.6** (High Domination Theorem, Martin, 1966b)**.** *A set $A$ is high ($\emptyset'' \leq_T A'$) iff there is a dominant function $f \leq_T A$.*

*Proof.* By Theorem 4.3.2 we know that Tot $\equiv_T$ $\emptyset''$. Hence, by the Limit Lemma 3.6.8 relativized to $A$, we have $\emptyset'' \leq_T A'$ iff there is an $A$-computable $\{0,1\}$-valued function $g(e,s)$ such that $\lim_s g(e,s) = \text{Tot}(e) := \chi_{Tot}(e)$.

($\Longrightarrow$). Assume $\emptyset'' \leq_T A'$. Given $g(e,s)$ as above we define a dominant function $f \leq_T A$ as follows:

*Stage s.* (To define $f(s)$). For all $e \leq s$ define $t(e)$ and $f(s)$ as follows:

$$t(e) \;=\; (\mu t > s)\,[\; g(e,t) = 0 \quad \vee \quad (\forall x \leq s)\,[\; \varphi_{e,t}(x)\!\downarrow\; ]\;],$$

$$f(s) \;=\; \max\{\; t(e) : e \leq s \;\}.$$

Note that $t(e)$ exists because if $\varphi_e$ is not total, then $\lim_t g(e,t) = 0$. If $\varphi_e$ is total, then $\lim_t g(e,t) = 1$, and therefore $f(s) > \varphi_e(s)$ for a.e. $s$. (Recall by Definition 1.6.17 that if $\varphi_{e,t}(x) = y$ then $e, x, y < t$.)

($\Longleftarrow$). Assume $f \leq_T A$ is dominant. Define an $A$-computable function $g(e,s)$ such that $\lim_s g(e,s) = \text{Tot(e)}$ as follows:

$$g(e,s) \;=\; \begin{cases} 1 & \text{if } (\forall z \leq s)\,[\; \varphi_{e,f(s)}(z)\!\downarrow\; ]; \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Note that if $\varphi_e$ is total, then so is $\theta_e(y) = (\mu s)\,(\forall z \leq y)\,[\; \varphi_{e,s}(z)\!\downarrow\; ]$. Thus, $f(y)$ dominates $\theta_e(y)$. Therefore, $g(e,s) = 1$ for a.e. $s$. If $\varphi_e$ is not total, then $\varphi_e(y)$ and $\theta_e(y)$ diverge for some $y$, and $g(e,s) = 0$ for all $s \geq y$. ☐

## 4.5.3 Exercises

**Exercise 4.5.7.** Give another proof of Martin's Theorem 4.5.6. *Hint.* Assume $A' \geq_T \emptyset''$. Using Theorem 4.4.4 (iv) fix a computable function $g$ such that $\varphi_e$ is total iff $W^A_{g(e)}$ is finite. Use an $A$-computable construction to define $f \leq_T A$. To define $f(s)$ first wait for all $e \leq s$ until either $\varphi_e(s)\!\downarrow$ or $W^A_{g(e)}$ receives a new element.

**Exercise 4.5.8.** Let $A$ be coinfinite, nonhigh, and c.e. Prove that $A$ has a computable enumeration $\{A_s\}_{s\in\omega}$ that is *diagonally correct*, that is, $(\exists^\infty s)\,[\; a^s_s = a_s\;]$, where $\overline{A}_s = \{a^s_0 < a^s_1 < \cdots\}$ and $\overline{A} = \{a_0 < a_1 < \cdots\}$.

## 4.6 Characterizing Nonlow$_2$ Sets $A \leq_T \emptyset'$

Fix $A \leq_T \emptyset'$ and relativize the previous proof to the cone $\{B : A \leq_T B\}$ with base $A$ in place of $\emptyset$ and with $\emptyset' \geq_T A$ as a set in this cone. We obtain the following useful escape property characterizing nonlow$_2$ sets $A \leq_T \emptyset'$.

**Theorem 4.6.1** (Relativized Domination Theorem, Martin, 1966b). *Fix $A \leq_T \emptyset'$. Then $A'' \leq_T \emptyset''$ (i.e., $A$ is low$_2$) if and only if there is a function $g \leq_T \emptyset'$ which dominates every total function $f \leq_T A$.*

*Proof.* Fix $A \leq_T \emptyset'$. Relativize Martin's Theorem 4.5.6 to the cone of sets $\{X : X \geq_T A\}$. Now $A$ is low$_2$ ($A'' \equiv_T \emptyset''$) iff $\emptyset'$, viewed as a member of this cone, is high in the cone, namely iff one jump of $\emptyset'$, that is, $\emptyset''$, reaches $A''$ in Turing degree, because $A''$ is the double jump of the base $A$ of the cone. By Martin's Theorem 4.5.6 this occurs iff there is a function $g \leq_T \emptyset'$ which is dominant relative to $A$-computable functions, so that $g$ dominates every total function $f \leq_T A$. □

**Corollary 4.6.2** (Nonlow$_2$ Escape Theorem). *Fix $A \leq_T \emptyset'$. Then $A$ is nonlow$_2$ ($A'' > \emptyset''$) iff for every function $g \leq_T \emptyset'$ there is a function $f \leq_T A$ which escapes $g$ in the sense of* (4.10), *i.e.,*

(4.13)
*(Nonlow$_2$ Escape)* $$(\forall g \leq_T \emptyset')\,(\exists f \leq_T A)\,(\exists^\infty x)\,[\, g(x) \leq f(x)\,].$$

*Proof.* This is the contrapositive of Theorem 4.6.1. □

We have stated (4.13) separately for the sake of the list of properties in Theorem 4.7.1.

### 4.6.1 Exercises

**Exercise 4.6.3.**$^{\diamond\diamond}$ (Csima, Hirschfeldt, Knight, Soare, 2004). Identify a string $\sigma_y$ with its code number $y$. A set $A$ satisfies the *isolated path property* if for every computable tree $T \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_T A)\,(\forall \sigma \in T)\,[\, g_\sigma \in [\,T_\sigma\,] \quad \& \quad g_\sigma \text{ is isolated }],$$

i.e., for every $x \in T$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending $\sigma$, which is an isolated path of the closed set $[\,T\,]$. Prove that every nonlow$_2$ set $A \leq_T \mathbf{0}'$ satisfies the isolated path property.

**Exercise 4.6.4.**$^{\diamond\diamond}$ (Csima, Hirschfeldt, Knight, Soare, 2004). A set $A$ satisfies the *tree property* if for every computable tree $T \subseteq 2^{<\omega}$ with no terminal nodes, and every uniformly $\Delta_2$ sequence of subsets $\{S_i\}_{i\in\omega}$ all dense in $[\,T\,]$,

$$(\exists g \leq_T A)\,(\forall \sigma \in T)\,(\forall i)\,(\exists \tau \in S_i)\,[\; \sigma \prec g_\sigma \quad \& \quad \tau \prec g_\sigma \quad \& \quad g_\sigma \in [\,T\,]\,].$$

Prove that every nonlow$_2$ set $A \leq \mathbf{0}'$ satisfies the tree property.

## 4.7 Domination, Escape, and Classes of Degrees

Martin's Theorem 4.5.6 gave a remarkable connection between high degrees $\mathbf{H}_1$ and dominant functions, and the Nonlow$_2$ Escape Theorem 4.6.2 produced an escape characterization for $\overline{\mathbf{L}}_2$ degrees. Now we summarize the previous properties in the following Theorem 4.7.1.

Recall the Definition 4.5.1 of dominate and escape and the common jump classes in §4.4.3. The contrapositive of Martin's High Domination Theorem 4.5.6 is

$$(4.14) \quad A' \not\geq_{\mathrm{T}} \emptyset'' \quad \iff \quad (\forall g \leq_{\mathrm{T}} A)(\exists f \leq \emptyset)\,(\exists^{\infty} x)\,[\, g(x) \;\leq\; f(x)\,].$$

In this case we say "$f$ escapes $g$." We say that a set $A$ satisfying the right-hand side has the *escape property*. Martin's equation (4.14) says that the degrees satisfying the escape property are exactly the nonhigh$_1$ degrees.

However, this definition does not require that we be able to *uniformly* find an index $i$ with $\varphi_i = f$ given an index $e$ with $g = \Phi_e^A$. Roughly, if we can uniformly find $i$ from $e$, then the $A$ satisfies the *Uniform Escape Property (UEP)*. We now summarize the domination and escape characterizations so far. (The redundancy of these properties is intensional, *e.g.*, our stating a property on one line and its negation on the next, so that we can later refer to a specific property by its line number here, because we intend to further develop both domination and escape.)

**Theorem 4.7.1.** *Fix a degree* $\mathbf{d} \leq \mathbf{0}'$.

*(i)* $\mathbf{d} = \mathbf{0}' \iff (\exists g \leq \mathbf{d})[\ g$ *dominates all p.c. functions* $]$.

*(ii)* $\mathbf{d} < \mathbf{0}' \iff (\forall g \leq \mathbf{d})(\exists\, \theta$ *p.c.* $)[\ \theta$ *escapes* $g\ ]$.

*(iii)* $\mathbf{d} \in \mathbf{H}_1 \iff (\exists g \leq \mathbf{d})(\forall f \leq \mathbf{0})[\ g$ *dominates* $f\ ]$.

*(iv)* $\mathbf{d} \in \overline{\mathbf{H}}_1 \iff (\forall g \leq \mathbf{d})(\exists f \leq \mathbf{0})[\ f$ *escapes* $g\ ]$.

*(v)* $\mathbf{d} \in \mathbf{L}_2 \iff (\exists g \leq \mathbf{0}')(\forall f \leq \mathbf{d})[\ g$ *dominates* $f\ ]$.

*(vi)* $\mathbf{d} \in \overline{\mathbf{L}}_2 \iff (\forall g \leq \mathbf{0}')(\exists f \leq \mathbf{d})[\ f$ *escapes* $g\ ]$.

*Proof.* Theorem 4.5.4 (ii) establishes (i) and (ii), Martin's Domination Theorem 4.5.6 establishes (iii) and (iv), the NonLow$_2$ Escape Theorem 4.6.2 proves (v) and (vi). □

## 4.8 Uniform Enumerations of Functions and Sets

Theorem 4.8.2 will relate nicely to the previous Martin Theorem 4.5.6 on dominant functions and high degrees. Also, the notions we now introduce in Definition 4.8.1 have proved useful in other areas of computability theory, computable model theory, and models of arithmetic.

**Definition 4.8.1.** (i) If $f(x,y)$ is a binary function then

$$f_y \text{ denotes } \lambda x\,[\,f(x,y)\,]. \tag{4.15}$$

As in analytic geometry, we imagine a two-dimensional plane with horizontal coordinate $x$ and vertical coordinate $y$. We view $\lambda x, y\,[\,f(x,y)\,]$ as specifying a *matrix* with entry $f(x,y)$ at the location $(x,y)$. For vertical coordinate $y \in \omega$ we view $f_y$ as the $y^{th}$ *row* according to our notation (4.15).

(ii) Let $\mathcal{C}$ be a class of (unary) functions and $\mathbf{a}$ be a degree. Then $\mathcal{C}$ is called $\mathbf{a}$-*uniform* ($\mathbf{a}$-*subuniform*) if there is a binary function $f(x,y)$ of degree $\leq \mathbf{a}$ such that

$$\mathcal{C} = \{\, f_y \,\}_{y\in\omega} \qquad (\text{respectively, } \mathcal{C} \subseteq \{\, f_y \,\}_{y\in\omega}\,).$$

Therefore, $f$ *uniformly* lists the rows $\{\, f_y \,\}_{y\in\omega}$. In the uniform case these are exactly the rows of $\mathcal{C}$. In the subuniform case $\mathcal{C}$ may be a proper subclass of these rows.

## 4.8.1 Limits of Functions

Given $f(x,y)$ as in (4.15), we may need to take limits in both the $x$ and $y$ directions. For example, if $\{A_y\}_{y\in\omega}$ is a uniformly computable sequence of computable sets then the vertical limit $B(x) = \lim_y A_y(x)$ is a $\Delta_2$ set as in the Limit Lemma 3.6.2. Now suppose that $A_y = W_{f(y)}$ where $f(x)$ is the computable function in the proof of Theorem 4.3.2. Hence, $W_{f(y)}$ is finite if $W_y$ is finite and $W_{f(y)} = \omega$ otherwise. Define $C(y) = \lim_x A_y(x)$. Now

$$C(y) \;= \lim_x A_y(x) \;=\; \mathrm{Tot}(y).$$

Hence, $C' \geq_{\mathrm{T}} 0''$, a useful fact in many infinite injury constructions such as the Thickness Lemma, because any set $D$ thick in $A$ also satisfies $D' \geq_{\mathrm{T}} 0''$.

## 4.8.2 $A$-uniform Enumeration of the Computable Functions

The next useful characterization follows from Martin's Theorem 4.5.6.

**Theorem 4.8.2** (Jockusch, 1972a). *If* $\mathbf{d}$ *is any degree, then statements* (i)–(iv) *are equivalent:*

(i) $\mathbf{d}' \geq \mathbf{0}''$

(ii) *the computable functions are* $\mathbf{d}$-*uniform;*

(iii) *the computable functions are* $\mathbf{d}$-*subuniform;*

(iv) *the computable sets are* $\mathbf{d}$-*uniform.*

*If* $\mathbf{d}$ *is c.e., then* (i)–(iv) *are each equivalent to*

(v) *the computable sets are* $\mathbf{d}$*-subuniform.*

*Proof.* The implications (ii) $\Longrightarrow$ (iii), (ii) $\Longrightarrow$ (iv), and (iv) $\Longrightarrow$ (v) are immediate.

(i) $\Longrightarrow$ (ii). By Martin's Theorem 4.5.6 choose a dominant function $g$ of degree $\leq \mathbf{d}$. Define $f(\langle e,i\rangle,x) = \varphi_{e,i+g(x)}(x)$ if $\varphi_{e,i+g(y)}(y)\downarrow$ for all $y \leq x$ and $f(\langle e,i\rangle,x) = 0$ otherwise. Now either $f_{\langle e,i\rangle} = \varphi_e$ is a total function, or $f_{\langle e,i\rangle}$ is finitely nonzero. In either case $f_{\langle e,i\rangle}$ is computable. If $\varphi_e$ is total then $g(x)$ dominates $\theta(x) = (\mu s)[\, \varphi_{e,s}(x)\downarrow\,]$, so $\varphi_e = f_{\langle e,i\rangle}$ for some $i$.

(iii) $\Longrightarrow$ (i). Let $f(e,x)$ be a function of degree $\leq \mathbf{d}$ such that every computable function is an $f_e$. Define $g(x) = \max\{f_e(x) : e \leq x\}$. Then $g$ is dominant, so $\mathbf{d}' \geq \mathbf{0}''$ by Martin's Theorem 4.5.6.

(iv) $\Longrightarrow$ (i). By Theorem 4.3.2 and Exercise 4.3.12 we have

$$(\mathrm{Tot}, \overline{\mathrm{Tot}}) \leq_m (\mathrm{Tot}, \overline{\mathrm{Ext}})$$

via some computable function $g$. Assume $f$ has degree $\leq \mathbf{d}$ and that the $f_e$'s are exactly the computable characteristic functions. Then for all $e$,

$$\begin{aligned} e \in \mathrm{Tot} \quad &\Longleftrightarrow (\exists i)\,[\, f_i \text{ extends } \varphi_{g(e)}\,] \\ &\Longleftrightarrow (\exists i)\,(\forall x)\,(\forall y)\,(\forall s)\,[\, \varphi_{g(e),\,s}(x) = y \quad \Longrightarrow \quad f_i(x) = y\,]. \end{aligned}$$

Thus, $\mathrm{Tot} \in \Sigma_2^A$. But $\mathrm{Tot} \in \Pi_2 \subseteq \Pi_2^A$. Therefore, $\mathrm{Tot} \in \Delta_2^A$. Hence, $\mathbf{0}'' \leq \mathbf{d}'$ by the Relativized Post's Theorem 4.4.2.

(v) $\Longrightarrow$ (i). (The following resembles the proof that the computable functions are not uniformly computable.) Assume that $\mathbf{d}$ is c.e. but (i) is false and $f(e,x)$ is any function of degree $\leq \mathbf{d}$. We must construct a $\{0,1\}$-valued computable function $h \neq f_e$ for all $e$. Since $\deg(f) \leq \mathbf{0}'$ there is a computable function $\hat{f}(e,x,s)$ such that $f(e,x) = \lim_s \hat{f}(e,x,s)$ and a modulus function $m(e,x)$ for $\hat{f}$ which has degree $\leq \mathbf{d}$ by the Modulus Lemma 3.6.3. Let $p(x) = \max\{m(e,\langle e,x\rangle) : e \leq x\}$. Since $\deg(p) \leq \mathbf{d}$ and (i) fails, there is a computable function $q(x)$ which $p(x)$ fails to dominate. Define $h(\langle e,x\rangle) = 1 \dot{-} \hat{f}(e,\langle e,x\rangle,q(x))$. Then $h$ is a computable function and $h(\langle e,x\rangle) \neq f_e(\langle e,x\rangle)$ whenever $x \geq e$ and $q(x) \geq p(x)$. (Exercise 4.9.6 on $\Pi_1^0$-classes shows that the hypothesis $\mathbf{d}$ c.e. is necessary for this part.) □

**Corollary 4.8.3** (Jockusch)**.** *If* $\mathbf{d} < \mathbf{0}'$ *is c.e. then the class of c.e. sets of degree* $\leq \mathbf{d}$ *is not* $\mathbf{d}$*-uniform.*

*Proof.* If $\mathbf{d}$ is a counterexample, then the computable sets are $\mathbf{d}$-subuniform. Therefore, $\mathbf{d}' = \mathbf{0}''$ by (v) $\Longrightarrow$ (i) of Theorem 4.8.2. However, since the c.e. sets of degree $\leq \mathbf{d}$ are $\mathbf{d}$-uniform, they are $\mathbf{0}'$-uniform and so $\mathbf{d}'' = \mathbf{0}''$ by a later result. $\square$

## 4.9 $^{\oslash}$ Characterizing Low$_2$ Sets $A \leq_{\mathrm{T}} \emptyset'$

**Definition 4.9.1.** *The* $0'$*-uniform property of* $A$ asserts:

$$U(A): \quad (\exists f \leq_{\mathrm{T}} \emptyset')\,[\,\{\, Y : Y \leq_{\mathrm{T}} A \,\} = \{\, f_e \,\}_{e \in \omega}\,], \tag{4.16}$$

where $f_e = \lambda x\,[\,f(x,e)\,]$ as in (4.15) and is viewed as the $e^{th}$ row of the matrix with characteristic function $f(x,e)$. (We identify a set $Y$ with its characteristic function $\chi_Y$.)

The uniformity property $U(A)$ asserts that there is a $\emptyset'$-computable matrix $f \leq_{\mathrm{T}} \emptyset'$ whose rows $\{\, f_e \,\}_{e\in\omega}$ are exactly the sets $Y \leq_{\mathrm{T}} A$.

**Theorem 4.9.2.** *If* $A \leq_{\mathrm{T}} \mathbf{0}'$ *is low*$_2$ *then* $U(A)$ *holds, i.e., the* $A$*-computable functions (and hence also* $A$*-computable sets) are* $\mathbf{0}'$*-uniform.*

*Proof.* Let $A$ be low$_2$ i.e., $A'' \leq_{\mathrm{T}} \emptyset''$. Hence, $\mathrm{Tot}^A \leq_{\mathrm{T}} \emptyset''$. Let $\widehat{g}(e,s)$ be a $\emptyset'$-computable function whose limit $g(e) = \lim_s \widehat{g}(e,s)$ is the characteristic function of $\mathrm{Tot}^A$. Now, using a $\emptyset'$ oracle, find for every $e$ and $x$

$$(\mu t > x)\,[\;\; \Phi^A_{e,t}(x)\!\downarrow \quad \vee \quad \widehat{g}(e,t) = 0 \;\;].$$

If the first case holds, define $h(x,e) = \Phi^A_{e,t}(x)$, and in the second case define $h(x,e) = 0$. This produces $h \leq_{\mathrm{T}} \emptyset'$. Let $\omega^{<\omega}$ be $\{\tau_i\}_{i\in\omega}$. Define $f \leq_{\mathrm{T}} \emptyset'$ by

$$f(x, \langle e, i\rangle) = \begin{cases} \tau_i(x) & \text{if } x < |\tau_i| \ ; \\ h(x,e) & \text{if } x \geq |\tau_i| \ . \end{cases}$$

For every $e$, if $\Phi^A_e$ is total, then $\Phi^A_e =^* h_e$ and $\Phi^A_e = f_{\langle e,i\rangle}$ for some $i$. $\square$

**Corollary 4.9.3.** *If* $X \leq_{\mathrm{T}} \mathbf{0}'$ *is low*$_2$*, then there is a computable function* $\widehat{f}(x,y,s)$ *such that the limit* $f(x,y) = \lim_s \widehat{f}(x,y,s)$ *exists for all* $x$ *and* $y$*, and*

$$\{Y : Y \leq_{\mathrm{T}} X\} = \{\, f_y : y \in \omega \,\}. \tag{4.17}$$

*Proof.* Apply Theorem 4.9.2 to see that $f(x,y) \leq_{\mathrm{T}} \emptyset'$ exists and apply the Limit Lemma 3.6.2 to derive $\widehat{f}(x,y,s)$. $\square$

For a fixed low$_2$ set $X$, we can think of $f(x,y)$ as a $\emptyset'$-*matrix* with rows $\{\, f_y \,\}_{y\in\omega}$, which is approximated at every stage $s$ in our computable construction by $\lambda\, x\, y\,[\,\widehat{f}(x,y,s)\,]$, and which in the limit correctly gives (4.17). We can often use the *dynamic matrix* approximation

$$\{\; \lambda\, e\, y\,[\;\widehat{f}(e,y,s)\;]\;\}_{s\in\omega}$$

to show that a low$_2$ set resembles a computable set.

**Proposition 4.9.4.** *Set $A$ satisfies $U(A)$ iff $A \leq_T \mathbf{0}'$ and $A$ is low$_2$.*

*Proof.* ($\Longleftarrow$). Apply Theorem 4.9.2.

($\Longrightarrow$). If $f$ is a computable function satisfying (4.17) then $Y = A$ *itself* is one of the rows $f_y$ for some $y$, but $f \leq_T \mathbf{0}'$, so $A \leq_T \mathbf{0}'$. Using $f \leq_T \mathbf{0}'$ we can define a $\mathbf{0}'$-function which dominates every $A$-computable function. Now $A'' \leq_T \mathbf{0}''$ by Theorem 4.6.1. □

## 4.9.1 Exercises

**Exercise 4.9.5.** Give another proof of Theorem 4.9.2 using domination. *Hint.* If $A$ is low$_2$ then $\emptyset'$ is high over $A$. Relativize Theorem 4.8.2 to $A$, replacing $\emptyset$ by $A$ and $A$ by $\emptyset'$. By Theorem 4.6.1, choose a $\emptyset'$-function $g$ which dominates every total $A$-computable function $\Phi_e^A$. Since $A \leq_T \emptyset'$ we can $\emptyset'$-computably define:

$$f(\langle e, i\rangle, x) \;=\; \begin{cases} \Phi^A_{e,\, i+g(x)}(x) & \text{if} \quad (\forall y \leq x)[\ \Phi^A_{e,\, i+g(y)}(y)\downarrow\ ] \\ 0 & \text{otherwise.} \end{cases}$$

Either $f_{\langle e,i\rangle} = \Phi_e^A$ is a total function, or $f_{\langle e,i\rangle}$ is finitely nonzero. If $\Phi_e^A$ is total then $g(x)$ dominates $c(x) = (\mu s)\,[\Phi^A_{e,s}(x)\downarrow]$.

**Exercise 4.9.6.** [Jockusch] Show that the hypothesis $\mathbf{d}$ c.e. in the proof of (v) $\Longrightarrow$ (i) of Theorem 4.8.2 was necessary by proving that there is a (non-c.e.) degree $\mathbf{d}$ such that $\mathbf{d}' = \mathbf{0}'$ and the computable sets are $\mathbf{d}$-subuniform. *Hint.* Apply the Low Basis Theorem 3.7.2 to the $\Pi^0_1$ class $\mathcal{C} \subseteq 2^\omega$ defined by

$$\begin{aligned} f \in \mathcal{C} \iff\quad & \mathrm{rng}(f) \subseteq \{0,1\} \quad \& \\ & (\forall e)\,(\forall x)\,[\,\varphi_e(x)\downarrow \implies f(\langle e,x\rangle) = \min\{1, \varphi_e(x)\}\,] \end{aligned}$$

to obtain some $f \in \mathcal{C}$ of low degree.

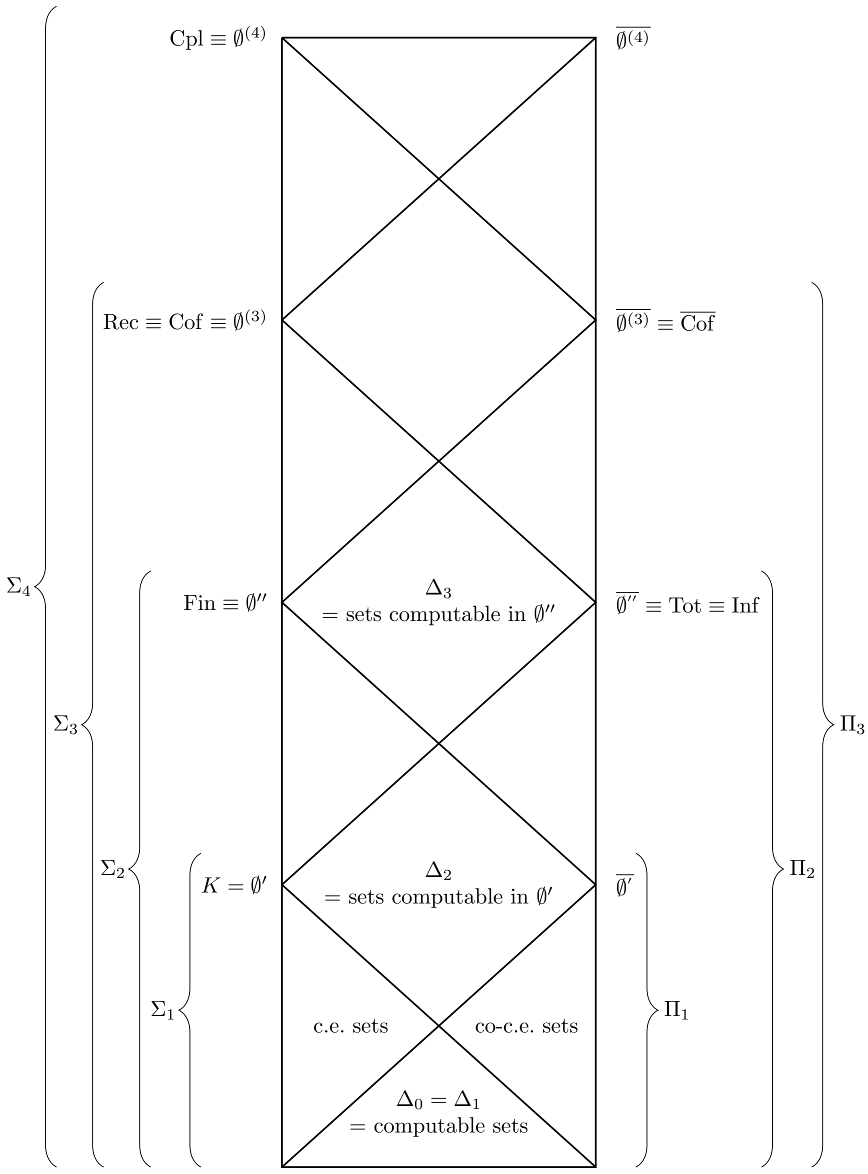


Figure 4.1. Arithmetical hierarchy of sets of integers

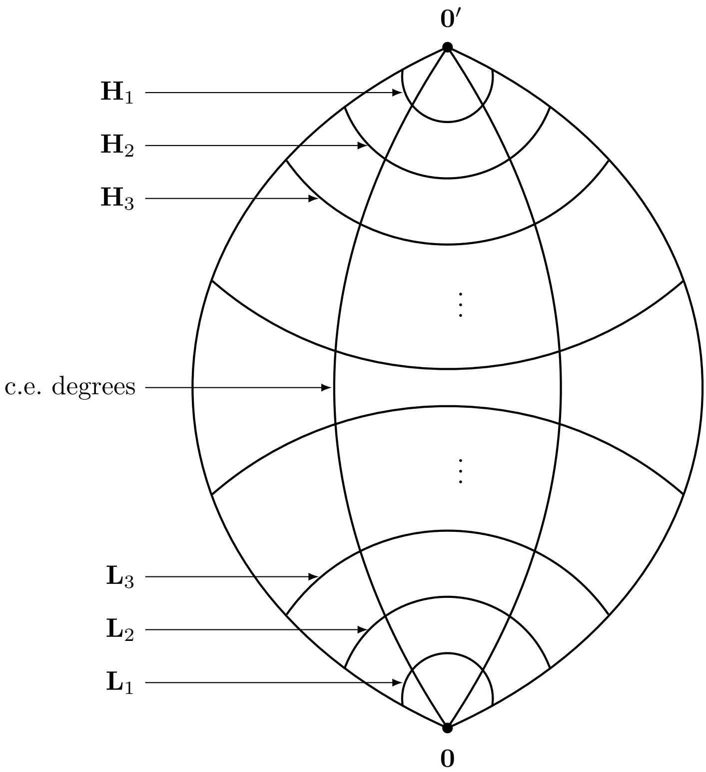


Figure 4.2. High and low degrees