15
Gale-Stewart Games

15.1 Gale-Stewart Games and Open Games

Gale-Stewart games illustrate the applications of Π^0_1-classes. In a Gale-Stewart game there are two players who alternately choose elements $a_i \in \{0, 1\}$. Player I chooses a_0, then player II chooses a_1, and so on. The infinite sequence f chosen, namely $f(n) = a_n$, is the particular *play* of the game. We fix ahead of time a set $\mathcal{A} \subseteq 2^\omega$. In game $\mathcal{G}(\mathcal{A})$ player I wins if the play $f \in \mathcal{A}$ and II wins otherwise. A *winning strategy* for player I is a function g on finite positions in the game, namely strings $\sigma \in 2^{<\omega}$ of even length (nodes at which I is to play), such that $g(\sigma) \in \{0, 1\}$ and if I follows strategy g then he wins the game. Likewise, a winning strategy for player II is defined on strings σ of odd length (where Player II is to play), and guarantees a win for player II. The game $\mathcal{G}(\mathcal{A})$ is *determined* if one player or the other has a winning strategy. The first easy theorem about these games is that $\mathcal{G}(\mathcal{A})$ is determined if $\mathcal{A}$ is open, namely boldface $\mathbf{\Sigma_1}$. This means that $\mathcal{A} = [\![\, A \,]\!]$ for some set $A \subseteq \omega$ as defined in (8.2). We can play as if this were an *effectively* open set by fixing the parameter A as an oracle. We analyze the computable content of this game and the winning strategies.

Theorem 15.1.1 (Gale-Stewart, 1953). *If $\mathcal{A} \subseteq 2^\omega$ is open, then the game $\mathcal{G}(\mathcal{A})$ is determined.*

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Proof. Let $\mathcal{A} = [\![A]\!]$ as in (8.2). Define an open set $[\![B]\!] \supseteq [\![A]\!]$, namely a certain A-c.e. set $B \subseteq 2^{<\omega}$, $B \supseteq A$, by induction as follows,

$$\sigma \in A \implies \sigma \in B \tag{15.1}$$

$$|\sigma| \text{ even} \quad \& \quad (\exists i)\,[\sigma\widehat{\ }i \in B\,] \quad . \implies \sigma \in B \tag{15.2}$$

$$|\sigma| \text{ odd} \quad \& \quad (\forall i)\,[\sigma\widehat{\ }i \in B\,] \quad . \implies \sigma \in B. \tag{15.3}$$

The set B represents the nodes σ from which Player I has a winning strategy to eventually get into the open set $\mathcal{A}$. In (15.1) if $\sigma \in A$ then $\sigma \in B$ because Player I has already ensured that $f \in [\![A]\!]$. Now $|\sigma|$ even means that Player I is to play next. Therefore, if (15.2) holds, then there is an immediate extension $\tau = \sigma\widehat{\ }i \in B$ which Player I can play. Hence, by moving from σ to $\tau \in B$ Player I can ensure inductively that he has a winning strategy from position σ. The case $|\sigma|$ odd, namely Player II to play, is similar but *every* extension $\tau = \sigma\widehat{\ }i$ must be in B or else Player II can move to avoid nodes in B.

Note that B is an A-computably enumerable set. (This uses the compactness of 2^ω because in ω^ω for (15.3) we would have to examine infinitely many i before putting σ into B.) Choose an A-computable tree T such that $[T] = 2^\omega - [\![B]\!]$. If T is infinite then Player II has a winning strategy g which consists of always choosing nodes $\sigma \notin B$. This strategy g amounts to choosing a path on $[T]$. This strategy is not necessarily A-computable because although the tree T is A-computable, the tree of extendible nodes T^{ext} is only computable in A' by the Effective Compactness Theorem 8.5.1. If the tree T is finite, then player I has a winning strategy $h \leq_{\text{T}} A$. $\square$

15.1.1 Exercises

Exercise 15.1.2. Now assume that A is computable with $\chi_A = \varphi_k$.

(i) Prove that if Player I has a winning strategy h, then Player I has a computable winning strategy.

(ii) Prove that if Player II has a winning strategy, then he has a winning strategy $g \leq_{\text{T}} \emptyset'$.

(iii) Prove that if Player II has a winning strategy g then he has a low winning strategy h, namely such that $h' \equiv_{\text{T}} \emptyset'$. (You must consider the Π^0_1 class of *strategies* for player II, not just the class of plays.)

(iv) (Slaman) Prove that (i) is not uniform. Use the Recursion Theorem to prove there is no total computable function ψ such that for all k, if $\chi_A = \varphi_k$, then $\psi(k)$ converges, and if Player I has a winning strategy then $\varphi_{\psi(k)}$ is an effective winning strategy for Player I.

15.1.2 Remarks on the Axiom of Determinacy

D.A. Martin [1975] proved determinacy for all Borel sets. The Axiom of Determinacy (AD) asserts that all games are determined. The assumption that definable sets are determined plays an important role in set theory. Full AD contradicts the Axiom of Choice (AC) but nevertheless is an important tool. A *cone* of degrees is a set of degrees of the form $\{\mathbf{d} : \mathbf{d} \geq \mathbf{a}\}$ for some degree $\mathbf{a}$.

Theorem 15.1.3 (Martin, 1968)**.** *If AD holds, then every set of degrees either contains a cone or is disjoint from a cone.*

Proof. Given a set $\mathcal{A}$ of degrees, let $\mathcal{A}^*$ be the class of sets whose degrees are in $\mathcal{A}$. Suppose Player I has a winning strategy for $G(\mathcal{A}^*)$. That strategy has a degree $\mathbf{a}$ and every degree $\mathbf{b} \geq \mathbf{a}$ must be in $\mathcal{A}$. Choose any function $f \in \mathbf{b}$, let Player II play according to f and Player I according to the winning strategy. The final outcome will be a function of degree $\mathbf{b}$. The outcome must be in $\mathcal{A}^*$ because Player I was following a winning strategy. Hence, $\mathbf{b} \in \mathcal{A}$. Similarly, if Player II has a winning strategy of degree $\mathbf{d}$ then all degrees $\mathbf{c} \geq \mathbf{d}$ lie in $\overline{\mathcal{A}}$. $\square$