11
Randomness and Π^0_1-Classes

11.1 Martin-Löf Randomness

In this chapter, we explore some of the relationships between Π^0_1 classes, algorithmic randomness, and computably dominated degrees.

Let μ be the Lebesgue measure on Cantor space, with which we assume the reader is familiar. For completeness, we define the measure of an open class $\mathcal{A} \subseteq 2^\omega$. Let $A \subset 2^{<\omega}$ be any set with $\mathcal{A} = [\![A]\!]$ which is prefix-free (i.e., if $\sigma \in A$ and $\tau \prec \sigma$ then $\tau \notin A$). Alternatively, let A could be the class of strings σ such that $[\![\,\sigma\,]\!] \subseteq \mathcal{A}$ and σ is minimal with respect to this property. Such an A can be seen to exist for example as follows. Since $\mathcal{A}$ is open, its complement is closed and hence is equal to $[T]$ for some tree $T \subseteq 2^{<\omega}$ (which is not necessarily computable). Then A can be taken to consist of all elements of $\overline{T}$ whose predecessors all belong to T. Now the measure of $\mathcal{A}$ is defined as

$$\mu(\mathcal{A}) = \sum_{\sigma \in A} 2^{-|\sigma|}.$$

the Lebesgue measure on Cantor space has all the same properties we are familiar with from the Lebesgue measure on the real line. Recall that a sequence of c.e. sets $A_0, A_1, \ldots$ is *uniformly c.e.* (abbreviated u.c.e.) if there exists a computable function f such that $A_n = W_{f(n)}$ for all n.

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Definition 11.1.1.

1. A sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ of subclasses of 2^ω is *uniformly (lightface)* Σ^0_1 if there exists a u.c.e. sequence $A_0, A_1, \ldots$ of subsets of $2^{<\omega}$ such that $\mathcal{A}_n = [\![A_n]\!]$ for all n.

2. A *Martin-Löf (ML) test* is a uniformly Σ^0_1 sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ of subclasses of 2^ω such that $\mu(\mathcal{A}_n) \leq 2^{-n}$ for all n.

3. A set $X \in 2^\omega$ *fails* a Martin-Löf test $\mathcal{A}_0, \mathcal{A}_1, \ldots$ if $X \in \bigcap_{n\in\omega} \mathcal{A}_n$. Otherwise, X *passes* the test.

4. A set $X \in 2^\omega$ is *Martin-Löf random (ML-random)* if it passes every Martin-Löf test.

The key point here is that the ML test must be effective in two ways. The sequence $\{\mathcal{A}_n\}_{n\in\omega}$ must be uniformly c.e., and it must converge computably fast in measure to 0. The intuition is that a non-ML-random set X is "caught" by an infinite sequence $\{\mathcal{A}_n\}_{n\in\omega}$ which reveals some of its information even though the measure of $\bigcap_n\{\mathcal{A}_n\}$ is effectively 0. For example, if the set X is computable then it is non-ML-random because it fails the ML test in which $\mathcal{A}_n = [\![X \upharpoonright n]\!]$. Schnorr proved that a set is ML-random iff it is 1-random, a closely related concept, so one may use the terms interchangeably.

11.2 A Π^0_1 Class of ML-Randoms

A Martin-Löf test $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is called *universal* if $\bigcap_{n\in\omega} \mathcal{A}_n \supseteq \bigcap_{n\in\omega} \mathcal{B}_n$ for every other Martin-Löf test $\mathcal{B}_0, \mathcal{B}_1, \ldots$. Thus, if X passes a universal test, it must pass every test, and hence

$$\bigcap_{n\in\omega} \mathcal{A}_n \;=\; \{\; X \in 2^\omega \;:\; X \text{ is not ML-random } \}.$$

This is a (lightface) Π^0_2 class and therefore an effective analogue of the (boldface) $\boldsymbol{\Pi}^0_2$ classes (i.e., G_δ classes) such as those we studied in Chapter 8, and which we shall study in the Banach-Mazur theorem in Chapter 14.

The following theorem is thus useful when trying to show that a given set is not ML-random.

Theorem 11.2.1 (Martin-Löf, 1966)**.** *There exists a universal Martin-Löf test.*

Proof. Let $\{V^0_n\}_{n\in\omega}$, $\{V^1_n\}_{n\in\omega}$, ...be an effective listing of all uniformly c.e. subsets of $2^{<\omega}$. Let $\mathcal{B}^e_n = [\![V^e_n]\!]$ where we stop enumerating if the measure exceeds 2^{-n}. Then $\{\mathcal{B}^e_n\}_{n\in\omega}$ for $e \in \omega$ lists all ML tests. Define $\mathcal{A}_n = \mathcal{B}^e_{e+n+1}$. Then the $\{\mathcal{A}_n\}$ are uniformly c.e. and $\mu(\mathcal{A}_n) \;\leq\; 2^{-n}$.

$$\mu(\mathcal{A}_n) \;=\; \Sigma_e\; \mu(\mathcal{B}^e_{n+e+1}) \leq \Sigma_e\; 2^{-n+e+1} \;=\; 2^{-n}.$$

Therefore, $\{\mathcal{A}_n\}_{n\in\omega}$ is a universal ML test. $\square$

Notice that this implies that the class of ML-randoms has measure 1. Indeed, each member of a universal Martin-Löf test $U_0, U_1, \ldots$ is an open set covering $\{X \in 2^\omega \ : \ X$ is not ML-random$\}$, implying that

$$\mu(\{X \in 2^\omega : X \text{ is not ML-random}\}) \ \leq \ \mu(U_n) \ \leq \ 2^{-n}$$

for all n. Essentially the same argument, in reverse, yields the following:

Corollary 11.2.2. *(F. Stephan) There is a nonempty* Π^0_1 *class all of whose elements are ML-random.*

Proof. Let $U_0, U_1, \ldots$ be a universal Martin-Löf test. For every $n > 0$, U_n is a proper Σ^0_1 subclass of 2^ω, implying that $\overline{U_n}$ is a nonempty Π^0_1 class. By the definition of a universal Martin-Löf test,

$$\overline{U}_n \subseteq \bigcup_{n\in\omega} \overline{U_n} = \overline{\bigcap_{n\in\omega} U_n} = \{X \in 2^\omega : X \text{ is ML-random}\},$$

as desired. $\square$

From this and the various basis theorems in Chapter 9, we can conclude that there are ML-random sets which are of c.e. degree, hyperimmune-free (computably dominated), low, even superlow, and of PA degree. However, any set which is ML-random and of PA degree must be of degree $\geq \mathbf{0}'$.

11.3 Π^0_1 Classes and Measure

Given the measure-theoretic definition of ML-randomness, it is natural to ask about the measure of Π^0_1 classes containing ML-randoms. The following theorem gives a full answer to this question.

Theorem 11.3.1. *Let $\mathcal{C}$ be a Π^0_1 class. If $\mu(\mathcal{C}) = 0$, then $\mathcal{C}$ contains no ML-random sets.*

Proof. Suppose $\mathcal{C}$ has measure 0. Let $T \subseteq 2^{<\omega}$ be a tree such that $\mathcal{C} = [T]$, and for each $n \in \omega$, let $\mathcal{A}_n = [\![\{\sigma \in T : |\sigma| = n\}]\!]$. Then $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is a nested sequence of open classes whose intersection is the measure 0 class $\mathcal{C}$, so it must be that $lim_n\ \mu(\mathcal{A}_n) = 0$. As the sequence $\{\mathcal{A}_n\}_{n\in\omega}$ is given by a strong array of finite sets of strings, the map $n \mapsto \mu(\mathcal{A}_n) \in \mathbb{Q}$, the rationals, is computable. Therefore, we can find a computable function p such that $\mu(\mathcal{A}_{p(n)}) \leq 2^{-n}$ for all n. Now since $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is uniformly Σ^0_1, $\mathcal{A}_{p(0)}, \mathcal{A}_{p(1)}, \ldots$ is a Martin-Löf test. But for all $f \in \mathcal{C}$, $f \in \bigcap_{n\in\omega} \mathcal{A}_{p(n)}$, so f is not ML-random. $\square$

Note that we can view this as a generalization of the remark earlier that any computable set is not ML-random beginning with a similar sequence defined by strings of length n.

Theorem 11.3.2 (Kucera). *Let $\mathcal{C}$ be a Π^0_1 class. If $\mu(\mathcal{C}) > 0$, then every ML-random set computes a member of $\mathcal{C}$.*

Proof. Suppose $\mathcal{C}$ has positive measure and let X be a ML-random set. Let V_0 be a prefix-free c.e. subset of $2^{<\omega}$ such that $\overline{\mathcal{C}} = [\![V_0]\!]$. For each $n \in \omega$, let $V_{n+1} = [\![\{\sigma\widehat{\ }\tau : \sigma \in V_n \;\&\; \tau \in V_0\}$, and let $\mathcal{A}_n = [\![V_n]\!]$. Notice that for all n, V_n is prefix-free since V_0 is, so we have

$$\begin{aligned}
\mu(\mathcal{A}_{n+1}) &= \textstyle\sum_{\sigma \in V_{n+1}} 2^{-|\sigma|} \\
&= \textstyle\sum_{\sigma \in V_n} \sum_{\tau \in V_0} 2^{-|\sigma\tau|} \\
&= \textstyle\sum_{\sigma \in V_n} 2^{-|\sigma|} \sum_{\tau \in V_0} 2^{-|\tau|} \\
&= \mu(\mathcal{A}_n)\mu(\mathcal{A}_0).
\end{aligned}$$

It follows that $\mu(\mathcal{A}_n) = \mu(\mathcal{A}_0)^{n+1} = \mu(\overline{\mathcal{C}})^{n+1}$, and hence that $\lim_n \mu(\mathcal{A}_n) = 0$ because $\mu(\overline{\mathcal{C}}) = 1 - \mu(\mathcal{C}) < 1$. Since $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is uniformly Σ^0_1, and the measures $\mu(\mathcal{A}_n)$ converge to zero faster than the (computable) function $p(n) = q^n$, where $q > \mu(\mathcal{A}_0)$ is rational, there is some subsequence of the sequence $\{\mathcal{A}_n\}$ which is a Martin-Löf test. Since X is ML-random, it is not in the intersection of this test, so $X \notin \mathcal{A}_n$ for some least n. If $n = 0$, then $X \notin \overline{\mathcal{C}}$ and hence $X \in \mathcal{C}$. If $n > 0$, since $X \in \mathcal{A}_{n-1}$, we can choose $\sigma \in V_{n-1}$ such that $\sigma \prec X$. Since no $\tau \in V_0$ can satisfy $\sigma\widehat{\ }\tau \prec X$, it follows that $Y = \{x - |\sigma| : x \in X \;\&\; x \geq |\sigma|\} \notin \mathcal{A}_0$ as $X = \sigma\widehat{\ }Y$. Thus, $Y \in \mathcal{C}$, which, since $Y \equiv_T X$, completes the proof. □

We saw in Chapter 9 that the PA degrees are precisely those which, for every nonempty Π^0_1 class, bound the degree of a member of that class. The preceding theorem can be seen as saying that the degrees of ML-random sets are precisely the analogues of PA degrees with respect to Π^0_1 classes of positive measure. This is a surprising fact because, in most other settings, the PA degrees and degrees of ML-random sets behave very differently. It is fact that if a set X is both ML-random and of PA degree, then $X \geq_T \emptyset'$ although we do not prove it.

11.4 Randomness and Computable Domination

We conclude by looking at applications of some of the ideas from computable domination to two other notions studied in the area of algorithmic randomness. We begin with the following.

Definition 11.4.1. [Terwijn and Zambella] (i) A set X is *computably traceable* if there is a computable function p such that, for each $f \leq_T X$, there is a computable function h with $|D_{h(n)}| \leq p(n)$ and $f(n) \in D_{h(n)}$ for all n.

(ii) A set X is *c.e. traceable* if there is a computable function p such that, for each $f \leq_T X$, there is a computable function h with $|W_{h(n)}| \leq p(n)$ and $f(n) \in W_{h(n)}$ for all n.

The idea of computably traceable is that there is for any function $f \leq_{\mathrm{T}} X$ a strong array of "boxes" $D_{h(n)}$ such that the value $f(n)$ lies in box $D_{h(n)}$. In addition, there is a single computable function $p(n)$ which uniformly bounds the size of the boxes over all such f. The idea of c.e. traceable is the same except with a weak array $W_{h(n)}$ in place of a strong array. This is the analogous change in weakening h-simple to hh-simple by replacing a strong array by a weak one.

Clearly, every computably traceable set is c.e. traceable, and it can be shown that this implication is strict (see Downey and Hirschfeldt [2010]). On the other hand, the following theorem shows that the reverse implication is true if we restrict ourselves to sets of computably dominated degree.

Theorem 11.4.2 (Kjos-Hanssen, Nies, and Stephan, 2005)**.** *If X is a set of computably dominated degree, then X is c.e. traceable if and only if it is computably traceable.*

Proof. Let X be a c.e. traceable set of computably dominated degree, and let p be a bound as in Definition 11.4.1 (ii). Given $f \leq_T X$, let h_0 be a computable function with $|W_{h_0(n)}| \leq p(n)$ and $f(n) \in W_{h_0(n)}$ for all n. Define a function g by

$$g(n) = (\mu s)[\ f(n) \in W_{h_0(n),s}\ ],$$

so that g is total and X-computable. By Theorem 5.6.2 (ii), there exists a computable function h_1 with $h_1(n) \geq g(n)$ for all n. If we define h by letting $h(n)$ be the canonical index of the finite set $W_{h_0(n),h_1(n)}$, we have

$$|D_{h(n)}| = |W_{h_0(n),h_1(n)}| \leq |W_{h_0(n)}| \leq p(n)$$

and $f(n) \in W_{h_0(n),h_1(n)} = D_{h(n)}$. Hence, X is computably traceable. $\square$

We obtain a similar result by looking at the following notion of randomness due to Kurtz. In view of Theorem 11.3.1 (i), it is implied by ML-randomness, and, as above, it can be shown that this implication is strict.

Definition 11.4.3. A *Kurtz test* is an effective sequence of clopen classes $\{\mathcal{A}_n\}_{n\in\omega}$ such that

$$(\forall n)[\ \mu(\mathcal{A}_n)\ <\ 2^{-n}\ ].$$

A set X is *Kurtz random* or *weakly 1-random* if it passes every Kurtz test.

Kurtz tests are equivalent to Π^0_1 classes of measure 0 in a uniform way. Therefore, a set X is weakly 1-random iff X avoids all Π^0_1 classes of measure 0 iff X is contained in every Σ^0_1 class of measure 1.

Theorem 11.4.4 (Nies, Stephan, and Terwijn, 2005)**.** *If X is a set of computably dominated degree, then X is ML-random if and only if it is weakly 1-random.*

Proof. Let X be a set of computably dominated degree which is not 1-random. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be a Martin-Löf test which X does not pass, and let f be a computable function such that $\mathcal{A}_n = [\![W_{f(n)}]\!]$ for all n. Define a function g by

$$g(n) = (\mu s)(\exists \sigma \prec X)[\, \sigma \in W_{f(e),s} \,],$$

noting that since $X \in [\![W_{f(n)}]\!]$ for all n, g is total and X-computable. By Theorem 5.6.2 (ii), there exists a computable function h with $h(n) \geq g(n)$ for all n. Define

$$\mathcal{C} = \bigcap_{n \in \omega} W_{f(n),\, h(n)}.$$

Therefore, $\mathcal{C}$ is a Π^0_1 class with $X \in \mathcal{C}$ and

$$\mu(\mathcal{C}) \leq \mu([\![W_{f(n),h(n)}]\!]) \leq \mu(S_n) = 2^{-n}$$

for all n. Hence, $\overline{\mathcal{C}}$ is a Σ^0_1 class of measure 1 not containing X, so X is not weakly 1-random. □

It follows by a result of Kurtz (see [Downey and Hirschfeldt 2010]), that every hyperimmune degree contains a set which is weakly 1-random but not 1-random. Thus, the degrees separating these two randomness notions are *precisely* the hyperimmune degrees.