Chapter 3
Bar Element

Abstract On the basis of the bar element, tension and compression as types of basic load cases will be described. First, the basic equations known from the strength of materials will be introduced. Subsequently the bar element will be introduced, according to the common definitions for load and deformation quantities, which are used in the handling of the FE method. The derivation of the stiffness matrix will be described in detail. Apart from the simple prismatic bar with constant cross-section and material properties also more general bars, where the size varies along the body axis will be analyzed in examples [1–9] and exercises.

3.1 Basic Description of the Bar Element

In the simplest case, the bar element can be defined as a prismatic body with constant cross-sectional area A and constant modulus of elasticity E, which is loaded with a concentrated force F in the direction of the body axis (see Fig. 3.1).
The unknown quantities are

- the extension ΔL and
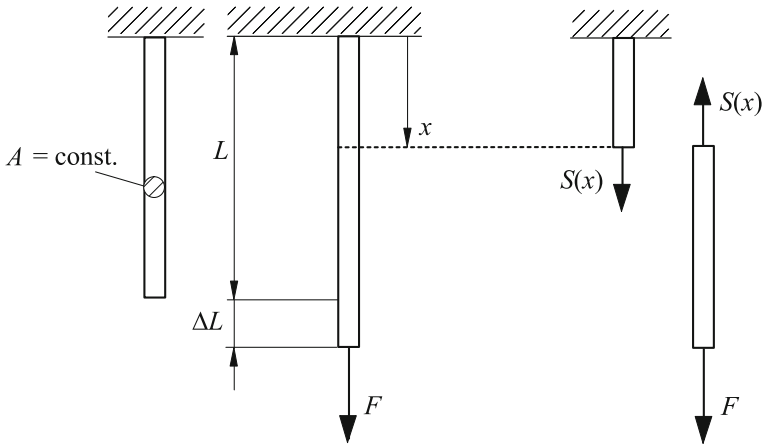- the strain ε and stress σ of the bar

dependent on the external load.
The following three basic equations are known from the strength of materials: By

$$\varepsilon(x) = \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{\Delta L}{L} \tag{3.1}$$

kinematics describes the relation between the strains $\varepsilon(x)$ and the deformations $u(x)$.
By

$$\sigma(x) = E\,\varepsilon(x) \tag{3.2}$$

Fig. 3.1 Tensile bar loaded by single force

the constitutive equation describes the relation between the stresses $\sigma(x)$ and the strains $\varepsilon(x)$ and the equilibrium condition results in

$$\sigma(x) = \frac{S(x)}{A(x)} = \frac{S(x)}{A} = \frac{F}{A}. \tag{3.3}$$

The connection between the force F and the length variation ΔL of the bar can easily be described with these three equations:

$$\frac{F}{A} = \sigma = E\varepsilon = E\frac{\Delta L}{L} \tag{3.4}$$

or with

$$F = \frac{EA}{L}\Delta L. \tag{3.5}$$

The relation between force and length variation is described as axial stiffness. Hence, the following occurs for the bar regarding the tensile loading[1]:

$$\frac{F}{\Delta L} = \frac{EA}{L}. \tag{3.6}$$

For the derivation of the differential equation the force equilibrium at an infinitesimal small bar element has to be regarded (see Fig. 3.2). A continuously distributed line load $q(x)$ acts as the load in the unit force per unit length.

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[1] The parlance tension bar includes the load case compression.
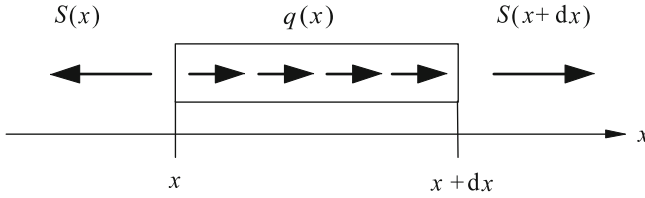
Fig. 3.2 Force equilibrium at an infinitesimal small bar element

The force equilibrium in the direction of the body axis delivers:

$$- S(x) + q(x)\mathrm{d}x + S(x + \mathrm{d}x) = 0. \tag{3.7}$$

After a series expansion of $S(x + \mathrm{d}x) = S(x) + \mathrm{d}S(x)$ the following occurs

$$- S(x) + q(x)\mathrm{d}x + S(x) + \mathrm{d}S(x) = 0 \tag{3.8}$$

or in short:

$$\frac{\mathrm{d}S(x)}{\mathrm{d}x} = -q(x). \tag{3.9}$$

Equations (3.1), (3.2) and (3.3) for kinematics, the constitutive equation and the equilibrium continue to apply. If the Eqs. (3.1) and (3.3) are inserted in (3.2), one obtains

$$EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x} = S(x). \tag{3.10}$$

After the differentiation and insertion of Eq. (3.9) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + q(x) = 0 \tag{3.11}$$

as the differential equation for a bar with continuously distributed load. This is a differential equation of 2nd order within the displacements. Under constant cross-section A and constant modulus of elasticity E the term simplifies to

$$EA\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + q(x) = 0. \tag{3.12}$$

3.2 The Finite Element Tension Bar

The tension bar is defined as a prismatic body with a single body axis. Nodes are introduced at both ends of the tension bar, where forces and displacements, as sketched in Fig. 3.3 are positively defined. The main objective is to achieve a stiffness relation for this element in the form
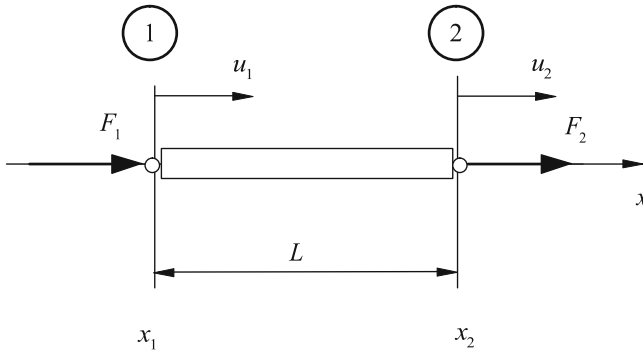
**Fig. 3.3** Definition of the finite element tension bar

$$F^{\mathrm{e}} = k^{\mathrm{e}} \cdot u_{\mathrm{p}}$$

or

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^{\mathrm{e}} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.13}$$

With this stiffness relation the bar element can be integrated in a structure. Further-more the displacements, the strains and the stresses *inside* the element are unknown. At first, an easy approach is introduced, in which the bar is modeled as a linear spring (Fig. 3.4).

Fig. 3.4 Tension bar modeled
as a linear spring
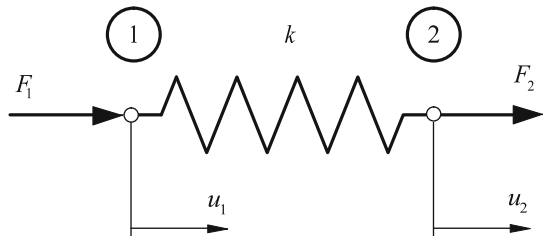
This is possible when

- the cross-sectional area A and
- the modulus of elasticity E

are constant along the body axis. The previously derived axial rigidity of the tension bar can then be interpreted as a spring constant or spring stiffness of a linear spring through

$$\frac{F}{\Delta L} = \frac{EA}{L} = k. \tag{3.14}$$

For the derivation of the stiffness relation, which is requested for the finite element method, a thought experiment is conducted. If, within the spring model at first only

the spring force F_2 is in effect and the spring force F_1 is being faded out, the equation

$$F_2 = k\Delta u = k(u_2 - u_1) \tag{3.15}$$

then describes the relation between the spring force and the length variation of the spring. If subsequently only the spring force F_1 is in effect and the spring force F_2 is being faded out, the equation

$$F_1 = k\Delta u = k(u_1 - u_2) \tag{3.16}$$

then describes the relation between the spring force and the length variation of the spring. Both situations can be superimposed and summarized compactly in matrix-form as

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.17}$$

With that the desired stiffness relation between the forces and deformations on the nodal points is derived.

The efficiency of this simple model however is limited. Thus no statements regarding the displacement, strain and stress distribution on the inside can be made. Therefore, a more elaborated model is necessary. This will be introduced in the following.

At first the displacement distribution $u^e(x)$ inside a bar will be described through shape functions $N(x)$ and the displacements u_p at the nodes:

$$u^e(x) = N(x)\, u_p. \tag{3.18}$$

In the simplest case, the displacement distribution is approximated linearly for the tension bar (see Fig. 3.5). With the following approach
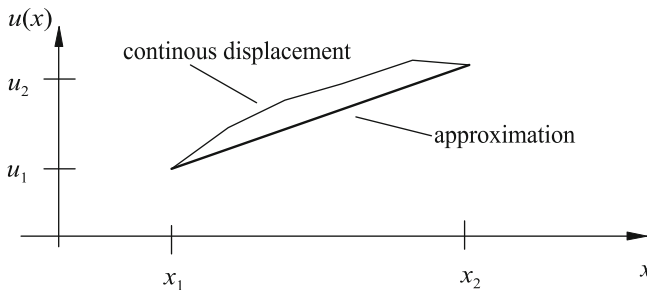
Fig. 3.5 Linear approximation of the displacement distribution in the tension bar

$$u^e(x) = \alpha_1 + \alpha_2 x \tag{3.19}$$

the displacements at the nodes

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{3.20}$$

can be described. After the elimination of α_i the following results for the displacement distribution:

$$u^e(x) = \frac{x_2 - x}{x_2 - x_1} u_1 + \frac{x - x_1}{x_2 - x_1} u_2 \tag{3.21}$$

or summarized

$$u^e(x) = \frac{1}{L}(x_2 - x)u_1 + \frac{1}{L}(x - x_1)u_2. \tag{3.22}$$

By this the shape functions $N_1(x)$ and $N_2(x)$ can be described with

$$N_1(x) = \frac{1}{L}(x_2 - x) \quad \text{and} \quad N_2(x) = \frac{1}{L}(x - x_1). \tag{3.23}$$

The displacement distribution results in a compact form in:

$$u^e(x) = N_1(x)u_1 + N_2(x)u_2 = [N_1 \ N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = N(x) \, u_p. \tag{3.24}$$

Through the kinematics relation the strain distribution results

$$\varepsilon^e(x) = \frac{d}{dx} u^e(x) = \frac{d}{dx} N(x) \, u_p = B \, u_p \tag{3.25}$$

and because of the constitutive equation the stress distribution results in

$$\sigma^e(x) = E\varepsilon^e(x) = E B u_p, \tag{3.26}$$

where the matrix B for the derivation of the shape functions is introduced. For the linear approximation of the displacement distribution the derivatives of the shape functions result in:

$$\frac{d}{dx} N_1(x) = -\frac{1}{L}, \quad \frac{d}{dx} N_2(x) = \frac{1}{L} \tag{3.27}$$

and therefore the matrix B results in

$$B = \frac{1}{L}[-1 \ \ 1]. \tag{3.28}$$

For the derivation of the element stiffness matrix the following integral has to be evaluated

$$k^e = \int_\Omega \boldsymbol{B}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{B} \, \mathrm{d}\Omega. \tag{3.29}$$

The elasticity matrix $\boldsymbol{D}$ is represented only through the modulus of elasticity E. For the tension bar, the stiffness matrix therefore results in:

$$k^e = AE \int_L \frac{1}{L} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{L} [-1 \;\; 1] \, \mathrm{d}x = \frac{EA}{L^2} L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3.30}$$

In a compact form, the element stiffness matrix is called:

$$k^e = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3.31}$$

There are also other ways to derive the stiffness matrix, which are introduced in the following sections.

3.2.1 Derivation Through Potential

The elastic potential energy[2] of a one-dimensional problem according to Fig. 3.1 with linear-elastic material behavior results in:

$$\Pi_{\mathrm{int}} = \frac{1}{2} \int_\Omega \varepsilon_x \sigma_x \mathrm{d}\Omega. \tag{3.32}$$

If stress and strain are substituted by use of the formulations according to Eqs. (3.26) and (3.25) and if $\mathrm{d}\Omega = A\mathrm{d}x$ is taken into consideration, the following applies:

$$\Pi_{\mathrm{int}} = \frac{1}{2} \int_0^L EA \left(\boldsymbol{B}\boldsymbol{u}_{\mathrm{p}}\right)^{\mathrm{T}} \boldsymbol{B}\boldsymbol{u}_{\mathrm{p}}\mathrm{d}x. \tag{3.33}$$

If the relation for the transpose of a product of two matrices[3] is taken into account the following results

$$\Pi_{\mathrm{int}} = \frac{1}{2} \int_0^L EA\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{B}\boldsymbol{u}_{\mathrm{p}}\mathrm{d}x. \tag{3.34}$$

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[2] The form $\Pi_{\mathrm{int}} = \frac{1}{2} \int_\Omega \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} \mathrm{d}\Omega$ can be used in the general three-dimensional case, where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represents the column matrix with the stress and strain components.
[3] $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$

Since the nodal values do not represent a function of x, both column matrices can be eliminated from the integral:

$$\Pi_{\text{int}} = \frac{1}{2} \, \boldsymbol{u}_{\text{p}}^{\text{T}} \left[\int_0^L E A \boldsymbol{B}^{\text{T}} \boldsymbol{B} \mathrm{d}x \right] \boldsymbol{u}_{\text{p}}. \tag{3.35}$$

Under consideration of the $\boldsymbol{B}$ matrix definition according to Eq. (3.28) the following results for constant axial rigidity EA:

$$\Pi_{\text{int}} = \frac{1}{2} \, \boldsymbol{u}_{\text{p}}^{\text{T}} \underbrace{\left[\frac{EA}{L^2} \int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathrm{d}x \right]}_{\boldsymbol{k}^{\text{e}}} \boldsymbol{u}_{\text{p}}. \tag{3.36}$$

The last equation is equivalent to the general formulation of the potential energy of a finite element

$$\Pi_{\text{int}} = \frac{1}{2} \, \boldsymbol{u}_{\text{p}}^{\text{T}} \boldsymbol{k}^{\text{e}} \boldsymbol{u}_{\text{p}} \tag{3.37}$$

and allows the identification of the element stiffness matrix $\boldsymbol{k}^{\text{e}}$.

3.2.2 Derivation Through Castigliano's Theorem

If the stress in the formulation for the elastic potential energy according to Eq. (3.32) is substituted by use of HOOKE's law according to Eq. (3.2) and if $\mathrm{d}\Omega = A\mathrm{d}x$ is taken into consideration, the following results:

$$\Pi_{\text{int}} = \frac{1}{2} \int_L E A \varepsilon_x^2 \mathrm{d}x. \tag{3.38}$$

If now the strain is substituted using the kinematic relation according to Eq. (3.1) and introduces the approach for the displacement distribution according to Eq. (3.24), the elastic potential energy for constant axial rigidity EA finally results in:

$$\Pi_{\text{int}} = \frac{EA}{2} \int_0^L \left(\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} u_1 + \frac{\mathrm{d}N_2(x)}{\mathrm{d}x} u_2 \right)^2 \mathrm{d}x. \tag{3.39}$$

The application of CASTIGLIANO's theorem on the potential energy with reference to the nodal displacement u_1 leads to the external force F_1 on the node 1:

$$\frac{\mathrm{d}\Pi_{\mathrm{int}}}{\mathrm{d}u_1} = F_1 = EA \int_0^L \left(\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} u_1 + \frac{\mathrm{d}N_2(x)}{\mathrm{d}x} u_2 \right) \frac{\mathrm{d}N_1(x)}{\mathrm{d}x} \, \mathrm{d}x. \qquad (3.40)$$

From the differentiation regarding the other deformation parameter the following arises accordingly:

$$\frac{\mathrm{d}\Pi_{\mathrm{int}}}{\mathrm{d}u_2} = F_2 = EA \int_0^L \left(\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} u_1 + \frac{\mathrm{d}N_2(x)}{\mathrm{d}x} u_2 \right) \frac{\mathrm{d}N_2(x)}{\mathrm{d}x} \, \mathrm{d}x. \qquad (3.41)$$

Equations (3.40) and (3.41) can be summarized as the following formulation:

$$EA \int_0^L \begin{bmatrix} \frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} \\ \frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\frac{\mathrm{d}N_2(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\frac{\mathrm{d}N_2(x)}{\mathrm{d}x} \end{bmatrix} \, \mathrm{d}x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \qquad (3.42)$$

After introducing the shape functions according to Eq. (3.23) and executing the integration the element stiffness matrix, which is given in Eq. (3.31), results.

3.2.3 Derivation Through the Weighted Residual Method

In the following, the differential equation for the displacement field according to Eq. (3.13) is being considered. This formulation assumes that the axial rigidity EA is constant and it results in

$$EA \frac{\mathrm{d}^2 u^0(x)}{\mathrm{d}x^2} + q(x) = 0, \qquad (3.43)$$

whereupon $u^0(x)$ represents the exact solution of the problem. The last equation with the exact solution is exactly fulfilled at every position x on the bar and is also referred to as the *strong formulation* of the problem. If the exact solution in Eq. (3.43) is substituted through an approximate solution $u(x)$, a residual or remainder r results:

$$r = EA \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + q(x) \neq 0. \qquad (3.44)$$

Due to the introduction of the approximate solution $u(x)$ it is in general not possible anymore to fulfill the differential equation at every position x of the bar. As an alternative, it is demanded in the following that the differential equation is fulfilled over a certain length (and not at every position x anymore) and therefore ends up with the following integral demand

$$\int_0^L W(x) \left(EA \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + q(x) \right) \mathrm{d}x \overset{!}{=} 0, \qquad (3.45)$$

which is also referred to as the *inner product*. $W(x)$ as part of Eq. (3.45) represents the so-called weighting function, which distributes the error or the residual over the regarded length.

The following results through partial integration[4] of the first expression in the parentheses of Eq. (3.45)

$$\int_0^L \underbrace{W}_{f}\, EA \underbrace{\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}}_{g'}\, \mathrm{d}x = EA \left[W(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L - EA \int_0^L \frac{\mathrm{d}W(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x}\, \mathrm{d}x. \quad (3.46)$$

Under consideration of Eq. (3.45) the so-called *weak formulation* of the problem results in:

$$EA \int_0^L \frac{\mathrm{d}W(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x}\, \mathrm{d}x = EA \left[W \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L + \int_0^L W(x)q(x)\mathrm{d}x. \qquad (3.47)$$

When considering the weak form it becomes obvious that one derivative of the approximate solution was shifted to the weighting function through the partial integration and that now with reference to the derivation a symmetric form arose. This symmetry with reference to the derivation of the approximate solution and the weighting function will subsequently guarantee that a symmetric element stiffness matrix for the bar element results.

In the following, first the left-hand side of Eq. (3.47) needs to be considered to derive the element stiffness matrix for a linear bar element.

The basic idea of the finite element method now is to no longer approximate the unknown displacement distribution $u(x)$ in the total domain, but to approximately describe the displacement distribution through

$$u^{\mathrm{e}}(x) = N(x)u_{\mathrm{p}} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (3.48)$$

for a subdomain, the so-called finite element. Within the context of the finite element method the same approach as for the displacement is chosen for the weighting function:

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[4] A usual representation of the partial integration of two functions $f(x)$ and $g(x)$ is: $\int f g' \mathrm{d}x = fg - \int f' g \mathrm{d}x$.

$$W(x) = \left[N(x)\,\delta u_{\mathrm{p}}\right]^{\mathrm{T}} = \delta u_{\mathrm{p}}^{\mathrm{T}} N^{\mathrm{T}}(x) = [\delta u_1 \; \delta u_2] \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \qquad (3.49)$$

whereupon δu_i represent the so-called arbitrary or virtual displacements. The derivative of the weighting function results in

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(N\delta u_{\mathrm{p}}\right)^{\mathrm{T}} = \left(B\delta u_{\mathrm{p}}\right)^{\mathrm{T}} = \delta u_{\mathrm{p}}^{\mathrm{T}} B^{\mathrm{T}}. \qquad (3.50)$$

In the following it remains to be seen that the virtual displacements can be canceled with an identical expression on the right-hand side of Eq. (3.47) and no further consideration will be necessary at this point. When considering the approaches for the displacement and the weighting function on the left-hand side of Eq. (3.47), the following results for constant axial rigidity EA:

$$EA \int_0^L \left(\delta u_{\mathrm{p}}^{\mathrm{T}} B^{\mathrm{T}}\right)\left(B u_{\mathrm{p}}\right)\mathrm{d}x \qquad (3.51)$$

or under consideration that the vector of the nodal displacement can be regarded as constant:

$$\delta u_{\mathrm{p}}^{\mathrm{T}} \, EA \underbrace{\int_0^L B^{\mathrm{T}} B \, \mathrm{d}x}_{k^{\mathrm{e}}} u_{\mathrm{p}}. \qquad (3.52)$$

The expression $\delta u_{\mathrm{p}}^{\mathrm{T}}$ can be canceled with an identical expression on the right-hand side of Eq. (3.47) and u_{p} represents the column matrix of the unknown nodal displacement. Therefore, the stiffness matrix can be calculated due to the derivative of the shape function according to Eq. (3.28) and finally the formulation according to Eq. (3.31) for the element stiffness matrix results.

In the following, the right-hand side of Eq. (3.47) is considered to derive the total load vector for a linear bar element. The first part of the right half is

$$EA \left[W \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L \qquad (3.53)$$

with the definition of the weighting function according to Eq. (3.49)

$$EA \left[\left(N\,\delta u_{\mathrm{p}}\right)^{\mathrm{T}} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L = EA \left[\delta u_{\mathrm{p}}^{\mathrm{T}} N^{\mathrm{T}} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L \qquad (3.54)$$

results, or in components

$$\delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}} E A \left[\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L . \tag{3.55}$$

The virtual displacements $\delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}$ in the last equation can be canceled with the corresponding expression in Eq. (3.52). Furthermore, the last equation represents a system of two equations, which have to be evaluated on the boundary of integration at $x = 0$ and $x = L$. The first line results in:

$$\left(N_1 E A \frac{\mathrm{d}u}{\mathrm{d}x} \right)_{x=0} - \left(N_1 E A \frac{\mathrm{d}u}{\mathrm{d}x} \right)_{x=0} . \tag{3.56}$$

Under consideration of the shape functions boundary values, meaning $N_1(L) = 0$ and $N_1(0) = 1$, the following results:

$$-E A \frac{\mathrm{d}u}{\mathrm{d}x} \Big|_{x=0} \overset{(3.10)}{=} -S(x = 0). \tag{3.57}$$

The value of the second line can be calculated accordingly:

$$E A \frac{\mathrm{d}u}{\mathrm{d}x} \Big|_{x=0} \overset{(3.10)}{=} S(x = L). \tag{3.58}$$

It must be noted that the forces S represent the internal reactions according to Fig. 3.2, hence the external loads with the positive direction according to Fig. 3.3 result from the internal reactions by reversing the positive direction on the left section and by maintaining the positive direction of the internal reaction on the right section.
The second part of the right-hand side of Eq. (3.47), meaning after canceling of $\delta \boldsymbol{u}^{\mathrm{T}}$

$$\int_0^L \boldsymbol{N}(x)^{\mathrm{T}} q(x) \mathrm{d}x \tag{3.59}$$

represents the general calculation rule for the definition of the equivalent nodal loads in the case of arbitrarily distributed loads. It should be noted at this point that the evaluation of Eq. (3.59) for a constant distributed load q results in the following load vector:

$$\boldsymbol{F}_q = \frac{qL}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{3.60}$$

3.3 Sample Problems and Supplementary Problems

3.3.1 Sample Problems

3.1. Tension Bar with Variable Cross-Section

So far the cross-section $A(x)$ was assumed to be constant along the body axis. As an enhancement to that the cross-section needs to be variable. The cross-section $A(x)$ should change linearly along the body axis. The modulus of elasticity is still regarded to be constant. Unknown is the stiffness matrix (Fig. 3.6).
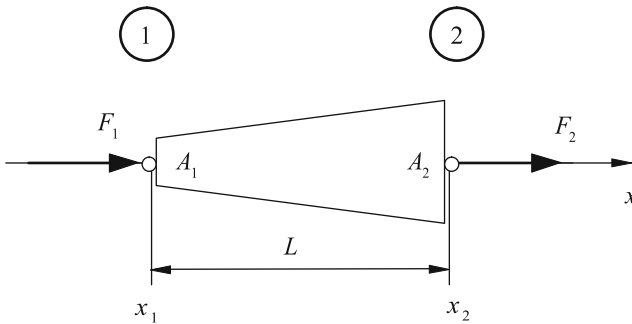

Fig. 3.6 Tension bar with variable cross-section

3.1 Solution

The integral

$$k^{\mathrm{e}} = \int_{\Omega} B^{\mathrm{T}} D B \, \mathrm{d}\Omega \tag{3.61}$$

has to be evaluated to derive the element stiffness matrix. The displacement distribution should be approximated linearly, as in the derivation above. Nothing changes for the shape functions or their derivatives. The following results for matrix B

$$B = \frac{1}{L}[-1 \quad 1]. \tag{3.62}$$

In contrast to the prismatic bar with constant cross-section, the area $A(x)$ remains under the integral. The constant modulus of elasticity E in

$$k^{\mathrm{e}} = \int_{L} \frac{1}{L}\begin{bmatrix} 1 \\ -1 \end{bmatrix} E \frac{1}{L}[-1 \quad 1] A(x) \, \mathrm{d}x \tag{3.63}$$

can be drawn in front of the integral. It remains:

$$k^e = \frac{E}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_L A(x)\, dx. \tag{3.64}$$

The linear course of the cross-section can be described through the following:

$$A(x) = A_1 + \frac{A_2 - A_1}{L} x. \tag{3.65}$$

After the execution of the integration

$$\int_L A(x)\, dx = \int_L \left[A_1 + \frac{A_2 - A_1}{L} x \right] dx = \frac{1}{2}(A_1 + A_2)\, L \tag{3.66}$$

the stiffness matrix

$$k^e = \frac{E}{L} \frac{1}{2}(A_1 + A_2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.67}$$

for a tension bar with linear changeable cross-section results.

3.2. Tension Bar Under Dead Weight

Given is a bar with length L with constant cross-section A, constant modulus of elasticity E and constant density ρ along the bar axis. The bar is now loaded through its dead weight (see Fig. 3.7).

Fig. 3.7 Tension bar under dead weight
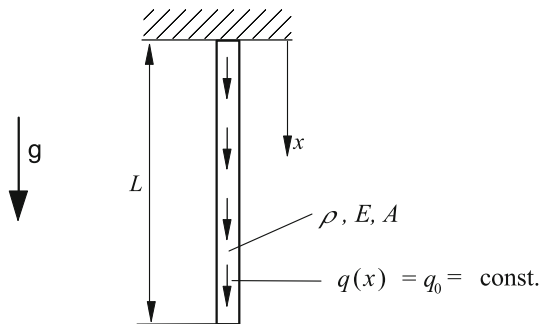
Unknown are:

1. The analytical solution and
2. the finite element solution for a single bar element with linear approximation of the displacement distribution.

3.2 Solution

Analytical Solution for the Tension Bar Under Dead Weight

Equation (3.13) is the basis for the solution. The dead weight force needs to be interpreted as a continuously distributed load $q(x)$, which is constant throughout the length of the bar:

$$q(x) = q_0 = \rho g A. \tag{3.68}$$

Starting from the differential equation of 2nd order

$$EA\frac{d^2 u(x)}{dx^2} = EAu''(x) = -q_0, \tag{3.69}$$

one receives the first derivative of the displacement through a one-time integration

$$EA\frac{du(x)}{dx} = EAu'(x) = -q_0 x + c_1, \tag{3.70}$$

and due to a further integration one receives the function of the displacement:

$$EAu(x) = -\frac{1}{2} q_0 x^2 + c_1 x + c_2. \tag{3.71}$$

The constants of integration c_1 and c_2 are adjusted through the boundary conditions. The displacement is zero at the fixed support and the following applies:

$$u(x = 0) = 0 \Rightarrow c_2 = 0. \tag{3.72}$$

The end of the bar is without force and the following results from Eq. (3.70):

$$EAu'(x = L) = 0 \Rightarrow -q_0 L + c_1 = 0 \Rightarrow c_1 = q_0 L. \tag{3.73}$$

If the constants of integration c_1 and c_2 are inserted with the term for the distributed load, the following results for the displacement field along the bar axis

$$u(x) = \frac{1}{EA}\left[-\frac{1}{2}q_0 x^2 + q_0 L x\right] = \frac{\rho g L^2}{E}\left[-\frac{1}{2}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)\right] \tag{3.74}$$

and the strain field

$$\varepsilon(x) = \frac{du(x)}{dx} = \frac{\rho g L}{E}\left[1 - \frac{x}{L}\right] \tag{3.75}$$

and the stress field

$$\sigma(x) = E\varepsilon(x) = \rho g L\left[1 - \frac{x}{L}\right]. \tag{3.76}$$

FE Solution for the Tension Bar Under Dead Weight

The basis for the finite element solution is the stiffness relation

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2} q_0 L \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.77}$$

with a linear approximation of the displacement distribution. If the formulations

$$k = \frac{EA}{L}, \quad q_0 = \rho g A \tag{3.78}$$

are inserted for the stiffness k and the distributed load q_0, the following compact form results

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2} \frac{\rho g L^2}{E} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{3.79}$$

from which, the displacement at the lower end of the bar

$$u_2 = \frac{1}{2} \frac{\rho g L^2}{E} \tag{3.80}$$

can be read off, after introducing the boundary condition ($u_1 = 0$). The displacement at the lower end of the bar matches with the analytical solution. The displacement is linearly distributed on the inside of the bar. The error towards the analytical solution with a quadratic distribution can be minimized or eliminated through the use of more elements or elements with quadratic shape functions.

3.3. Tension Bar Under Dead Weight, Two Elements

Given is the tension bar with length L under dead weight, as in Exercise 3.2. For the determination of the solution on the basis of the FE method, *two* elements with linear shape functions should be used.

3.3 Solution

The basis for the solution is the single stiffness relation for the bar under consideration of a distributed load. One receives the total stiffness relation with two elements through the development of two single stiffness relations.[5] With the formulations for the stiffness k and the distributed load q_0

$$k = \frac{EA}{L}, \quad q_0 = \rho g A \tag{3.81}$$

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[5] Here the FE solution is shown in brief. A detailed derivation for the development of a total stiffness matrix, for the introduction of boundary conditions and for the identification of the unknown is introduced in Chap. 7.

a compact form results

$$\frac{EA}{\frac{1}{2}L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2} \rho g A \frac{1}{2} L \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix}. \tag{3.82}$$

The first line and column can be eliminated due to the boundary conditions ($u_1 = 0$). It remains a system of equations with two unknowns

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{8} \frac{\rho g L^2}{E} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{3.83}$$

after a short transformation

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{8} \frac{\rho g L^2}{E} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \tag{3.84}$$

the displacement at the end node

$$u_3 = \frac{1}{2} \frac{\rho g L^2}{E} \tag{3.85}$$

and through insertion into Eq. (3.84) the displacement at the mid-node

$$u_2 = \frac{1}{2} \left[\frac{1}{2} + \frac{1}{8} \right] \frac{\rho g L^2}{E} = \frac{3}{8} \frac{\rho g L^2}{E} \tag{3.86}$$

can be identified.

3.3.2 Supplementary Problems

3.4. Tension Bar with Quadratic Approximation

Given is a prismatic tension bar with length L, with constant cross-section AL, and modulus of elasticity E. In contrast to the derivation above, the displacement distribution on the inside of the bar element needs to be approximated through a quadratic shape function. Unknown is the stiffness matrix.

3.5. Tension Bar with variable Cross-Section and Quadratic Approximation

The cross-section $A(x)$ changes linearly along the body axis. The modulus of elasticity is constant further on. The displacement distribution on the inside of the bar element needs to be approximated through quadratic shape functions.

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