Stretch Factor in Wireless Sensor Networks with Directional Antennae

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Abstract. Traditional study of wireless sensor networks has relied on the assumption that sensors transmit and receive using an omnidirectional antenna. There has been some recent study using a model where sensors transmit using a directional antenna. This study has focused on the problem of finding an optimal transmission range so that there exists an orientation of the antennae at each sensor which creates a strongly connected communication network. This is known as the Antenna Orientation Problem for Strong Connectivity. In this paper we examine a similar problem: we wish to optimize not only the transmission range, but also the hop-stretch factor of the communication network (in relation to the omnidirectional model). We refer to this as the Antenna Orientation Problem with Constant Stretch Factor. We present approximations to this problem for antennae with angles $\pi/2 \leq \phi \leq 2\pi$.

Keywords: Antenna Orientation Problem, Connectivity, Directional Antenna, Stretch Factor, Wireless Sensor Networks.

1 Introduction

A wireless sensor is a computational device which transmits and receives information using a radio antenna. The traditional study of wireless sensor networks (WSNs) assumes that the sensors employ omnidirectional antennae. In this model, a sensor (the sender) can transmit messages successfully to another sensor (the receiver) if and only if the Euclidean distance between the sender and receiver is less than or equal to the transmission range of the sender. Typically it is assumed that all sensors in a WSN have the same transmission range. This leads to an undirected communication graph which is a unit disk graph (UDG), where the unit is the transmission range of the sensors. This model has been studied extensively.

There is, however, no reason a sensor cannot use directional antennae to transmit and/or receive information. Recently, there has been some study in a directional WSN model where sensors receive information omnidirectionally, but transmit in a sector of angle (or beam width) ϕ with range r.

The use of directional rather than omnidirectional antennae has many possible advantages: longer ranges are achievable with the same amount of energy; different radiation patterns might lower interference in the network and lead to increased throughput;

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there may also be an increase in the security of a network by reducing the risk of eavesdropping. These possible advantages are of particular interest in connected networks. However not all connected networks are created equal. If replacing omnidirectional antenna with directed ones leads to a network with significantly longer paths between sensors, then any possible advantages may be offset by the increased path lengths.

Study of this directional antennae model has until now been focused on the Antenna Orientation Problem for Strong Connectivity. Solutions to this problem guarantee that a WSN with directional antennae can be strongly connected, but nothing more. The resulting communication graph could have shortest paths between sensors containing a very large number of hops. This may be unavoidable depending on the distribution of the sensors. It becomes a problem if these paths are very long compared to their counterparts in the omnidirectional model, i.e., if the sensors had omnidirectional antennae instead of directional antennae, would these paths remain fairly long? We examine a problem similar to the Antenna Orientation Problem, but which places a bound on the length of paths in the directional case compared to those in the omnidirectional case.

For P, a set of points, let $U(P)$ be the UDG on P where the unit is the longest edge of the MST of P. Let $G(P)$ be a (strongly) connected (di)graph on the vertices in P.

For any two vertices u,v in P, let $d_G(u,v)$ denote the minimum number of edges of any (directed) path from u to v in G. For the rest of the paper, we refer to a path with $d_G(u,v)$ edges as a *shortest path* from u to v. We will also refer to $d_G(u,v)$ as the *hop count* from u to v.

The hop-stretch factor of $G(P)$ on $U(P)$, denoted as $\tau_G(P)$, is defined as the maximum ratio $d_G(u,v)/d_U(u,v)$ among all pairs of vertices u,v in P.

$$\tau_G(P) = \max_{\forall u,v \in P} \frac{d_G(u,v)}{d_U(u,v)}.$$

The hop-stretch factor (hereafter referred to interchangeably as simply *stretch factor*) may depend on the size of the network. This raises an interesting problem when one attempts to construct a transmission network with minimum range for a given angle which guarantees a constant stretch factor.

Antenna Orientation Problem with Constant Stretch Factor. Given a connected UDG $U(S)$ on a set of sensors S in the plane. Suppose the sensors have beam width $\phi \geq 0$. For a given hop stretch factor k, compute the minimum range, denoted by $r_k(U(S),\phi)$, so that an orientation of the antennae of the sensors of S creates a strongly connected communication digraph $G_\phi(S)$ such that $\tau_{G_\phi}(S) \leq k$.

1.1 Related Work

The Antennae Orientation Problem for Strong Connectivity was first proposed by Caragianis et al. [4]. They proved the Antenna Orientation Problem is NP-complete for angles less than $2\pi/3$. They also presented a polynomial algorithm for determining a solution for angles $\phi \geq \pi$. A similar problem was studied by Dobrev et al. [8], who studied the Antennae Orientation Problem when each sensor has more than one directional antenna. They proved the problem remains NP-Complete when each sensor has two directional antennae with sum of angles at most $9\pi/20$ even for a scaling range of 1.3. They also showed that when each sensor has $k \leq 5$ directional antennae, the upper bound on the range is bounded by $2\sin\left(\frac{\phi}{k+1}\right)$ times the optimal range, for every angle. A comprehensive survey of the antenna orientation problem is presented in [10]. In 3D, the problem was studied in [9] where the authors consider the case when each sensor has one directional antenna.

The Antenna Orientation Problem with Constant Stretch Factor was studied for the first time by Damian and Flatland [7]. They examined the particular cases when $\phi = \pi/2$ and $\phi = 2\pi/3$. They proved that $r_6(S, \pi/2) \leq 7$ and $r_5(S, 2\pi/3) \leq 5$ respectively. Recently, Bose et al.[3] studied the case when $\phi \leq \pi/3$.

A distinct model is studied in [1,2] and [5] in which sensors both transmit and receive using the same directional antenna. In [1,5] it is proven that it is always possible to create a connected graph with angle $\phi \geq \pi/3$ and unbounded range. In [2] the authors also considered how to bound the range and stretch factor. They proved that a connected graph with stretch factor 8 can be constructed with angle $\pi/2$ and range $14\sqrt{2}$ times the range necessary to create a connected UDG on the set of sensors.

1.2 Outline and Results of the Paper

The strategies used for deriving our results rely on partitioning an arbitrary set of sensors into many small subsets. We then orient these subsets independently of each other and show that these orientations lead to our desired results.

In Section 2 we describe how we can orient and connect small groups of sensors. In Section 3 we address the Antenna Orientation Problem with Constant Stretch Factor. We provide a global algorithm for beam widths $\pi \leq \phi \leq 2\pi$, and a local algorithm for beam widths $\pi/2 \leq \phi < \pi$. All approximations are in relation to the longest edge of the MST of the sensors. This is a trivial lower bound on the optimal range, as it is a lower bound for strong connectivity. The summary of our results, along with existing results is shown in Table 1.

Table 1. Results for the Antenna Orientation Problem with Constant Stretch Factor

Beam Width	Approximation Ratio of r_s	Stretch Factor	Scope	Proof
$\frac{5\pi}{3} \leq \phi < 2\pi$	1	2	Global	Theorem 1
$\pi \leq \phi < \frac{5\pi}{3}$	$2\sin(\frac{\phi}{2})$			
$\frac{\pi}{2} \leq \phi < \pi$	$4\cos(\frac{\phi}{2}) + 3$	3	Local	Theorem 2
$\phi = \frac{\pi}{3}$	$36\sqrt{2}$	10	Global	[3]
$\phi < \frac{\pi}{3}$	$4\sqrt{2}(\frac{7\pi}{\phi} - 6)$	$\lceil 8\log(\frac{2\pi}{\phi})\rceil - 1$	Global	[3]

1.3 Preliminaries and Notation

A *sensor* is an object at a point in the plane. It is able to receive transmissions omnidirectionally. A sensor with an omnidirectional antenna is able to transmit in all directions up to a distance r called the *transmission range*. A sensor with a single directional antenna is able to transmit in a sector whose angle is referred to as the *beam width* of the antenna. This antenna may be initially facing any direction, but once *oriented* the antenna is fixed in this orientation. Any point that lies within the sector defined by the antenna, regardless of its distance from the sensor, is in the sensor's *line of sight*.

For the purposes of this paper, any sensors referred to are assumed to be sensors with a single directional antenna. Furthermore, the term *sensor* may be used interchangeably to mean the location of the sensor in the plane, the sensor object itself, or the vertex representing the sensor in a graph. We may also use the terminology "orienting a sensor" to mean orienting the antenna at a sensor.

We assume that any sensor, s, is able to determine its distance from other sensors, as well as the angle formed by any other two sensors with vertex s. While assuming that sensors have location awareness will satisfy these assumptions, it is not strictly necessary as sensors do not need access to a global co-ordinate system for our results. Furthermore, we assume that each sensor has the ability to communicate with all nearby sensors during the orientation process. This could be accomplished through the rotation of its directional antenna, or the use of its omnidirectional antenna to transmit as well as receive.

Let $D(a,r)$ denote the open disk centered at a with radius r and $D[a,r]$ denote the closed disk centered at a with radius r.

Definition 1 (Coverage). *Let a,b be sensors with range r. Sensor a* covers *sensor b if $b \in D[a,r]$ and b is within the line of sight of sensor a. This means that b will be a neighbour of a (although the reverse is not necessarily true). Sensor a* covers *area A if $\forall$ points $p \in A$, a sensor at p would be covered by a. A set of sensors S* covers *an area A if $\forall$ points $p \in A$, a sensor at p would be covered by at least one sensor $s \in S$*

Let a,b be two sensors. We say that sensor a can *reach* sensor b if a covers b, or a covers a sensor c which can reach b. In general terms, this means that a can reach b if there exists a directed path from a to b.

Definition 2 (k-orientation). *Let S be a set of sensors. An orientation of the antennae of S is a k-*orientation *if the directed communication graph $G(S)$ is strongly connected, and $\forall s \in S$:*

1. *$D[s,1]$ is covered by S, and*
2. *$\forall p \in D[s,1]$, the shortest path from s to a sensor covering p has length at most $k-1$ hops.*

2 Orientating Small Groups of Sensors

In this section we begin by showing how to merge k-orientations. We then relate k-orientations to stretch factor. We conclude the section by showing how we can orient various groups of small sensors to form k-orientations. These orientations will form the building blocks for our later results. Any omitted proofs can be found in the full paper.

2.1 Merging k-Orientations

Orienting small groups of sensors is the foundation for our results, however we first show how these small orientations can be put together to form an orientation for an entire graph.

Lemma 1. *Let S,T be two sets of sensors which have been oriented to form an i-orientation and a j-orientation, respectively. Suppose, without loss of generality, that $i \leq j$. If $\exists s_1 \in S, t_1 \in T$ such that s_1 covers t_1 and $\exists s_2 \in S, t_2 \in T$ such that t_2 covers s_2, then the orientations of S,T is a j-orientation of $S \cup T$.*

Proof. Let the orientations of S,T remain identical, so that the coverage and path length conditions hold for $S \cup T$. All that remains is to show that $S \cup T$ is strongly connected. This is trivial however, as both S and T are strongly connected and there is a path from S to T and vice versa.

2.2 Stretch Factor of k-Orientations

Lemma 2. *Let S be a set of sensors which have been oriented to form an i-orientation with the directed communication graph $G(S)$. The stretch factor of $G(S)$ on $UDG(S,1)$, $\tau_G(S) \leq i$.*

Proof. Assume there exists $G(S)$ such that $\tau_G(S) > i$. Therefore there must be two vertices $u,v \in S$ such that $d_G(u,v)/d_U(u,v) > i$. Since the ratio along the path in G from u to v is greater than i, this means that there must exist two vertices $a,b \in S$ such that $d_U(a,b) = 1$ and $d_G(a,b)/d_U(a,b) > i$. Therefore, $d_G(a,b) > i$. Note that since $d_U(a,b) = 1$, $b \in D[a,1]$ and vice versa. However, since $G(S)$ is the communication graph of an i-orientation of S, the shortest path between a and a sensor covering b cannot be more than $i-1$. Therefore, $d_G(a,b) \leq i$. This contradicts our assumption, and the lemma follows.

2.3 Forming k-Orientations from Small Groups of Sensors

Groups of 2 Sensors

Lemma 3. *Given two sensors u,v with beam width $\phi \geq \pi/2$. If the sensors are separated by Euclidean distance $\delta \geq 2\cos(\phi/2)$, then there exists a 2-orientation of u and v with transmission range $r = \delta + 1$.*

Proof. Suppose u,v are oriented as in Figure 1. Let c be the midpoint of the line segment uv. Each sensor beam pattern edge intersects with the intersection of the circles $D[u,1]$ and $D[v,1]$ at point i. This intersection forms a right angle triangle with c and u. Since the intersection lies on the boundary of $D[u,1]$, the hypotenuse of this triangle has length 1. Since uv bisects the sensor of beam width ϕ, the angle $\angle(ivc) = \phi/2$. Using simple trigonometry we calculate the length of the side $cv = \cos(\phi/2)$. Similarly, the side $uc = \cos(\phi/2)$, so the length of uv, $\delta = \cos(\phi/2)$.

Consider any point $p \in D[u,1]$. The farthest p can be from v is $\delta + 1$. Similarly, the farthest any point $q \in D[v,1]$ can be from u is $\delta + 1$. Therefore $r = \delta + 1$ is a sufficient transmission range to ensure that both $D[u,1]$ and $D[v,1]$ are covered.

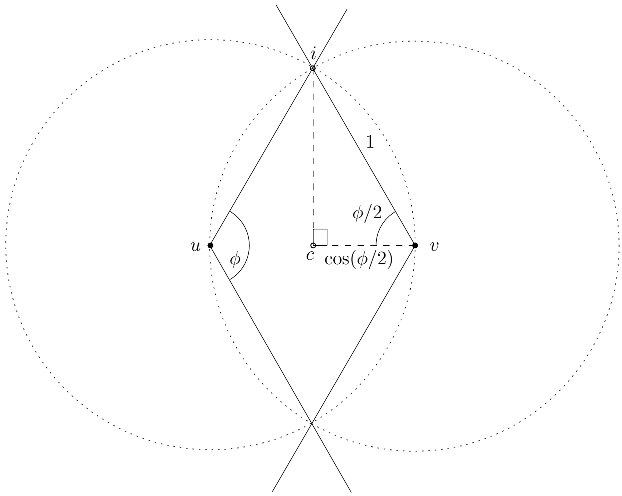


Fig. 1. Antenna orientation of two sensors with beam width $\frac{2\pi}{3} \leq \phi < \pi$

In the proposed antennae orientation, u covers v and vice versa, so each are connected by a path of length 1. Therefore this orientation of u and v is a 2-orientation.

A more specific version of the following result was first given in [6]. We derive a more general version and include the proof for completeness.

Lemma 4. *Given two sensors u, v with beam width $\phi \geq \pi$. Suppose the Euclidean distance between them is δ. There exists a 2-orientation of u and v with transmission range* $\max(1, \delta, \sqrt{1 + \delta^2 - 2\delta\cos(\phi)})$.

Groups of 3 Sensors

Lemma 5. *Given three sensors u, v and w with beam width $\phi \geq \pi/2$. If two of the sensors are separated by Euclidean distance $\delta \geq 2\cos(\phi/2)$, there is a 3-orientation of $\{u, v, w\}$ with transmission range $r = \delta + 1$.*

Lemma 6. *Given a set S of $n \geq 3$ sensors with beam width $\phi \geq \pi$. Suppose $\exists c \in S$ such that the maximum distance between c, and any other sensor $s \in S - \{c\}$ is δ. If all the sensors $s \in S - \{c\}$ are contained within a sector centered at c with angle ϕ, there is a 2-orientation of S with transmission range $r = \max(1, \delta, \sqrt{1 + \delta^2 - 2\delta\cos(\phi)})$.*

Groups of 4 Sensors

Lemma 7. *Given a set S of four sensors with beam width $\phi \geq \pi/2$. Suppose the maximum Euclidean distance between any two sensors in S is δ. There is a 3-orientation of S with transmission range $r = \delta + 1$.*

Proof. Let us first consider the case where the sensors have infinite transmission range. Label the sensors $t,u,v,w \in S$ such that the greatest pairwise Euclidean distance between sensors occurs between t and w. We then obtain one of two cases (Figure 2).

Case (a): Both sensors u and v are on the same side of the line tw. We orient the antennae of t and w to cover the half plane above (and including) the line tw. We then orient the antennae of u and v to cover the half plane below tw. Since t and w cover the half plane above tw, and u and v cover the half plane below tw, the entire plane is covered. Since t and w are the two sensors farthest apart, both u and v must lie between t and w. Therefore, t covers u, v, and w. Similarly w covers t, u and v. The sensors u and v each cover one of t and w. Therefore t and w cover each other, and u and v each cover one of t or w and are covered by both. This means the induced graph is strongly connected.

Case (b): u and v are on opposing sides of the line tw. We label u as the point whose projection on the line tw is closest to t. We can then orient the antennae of t and w so that they cover the entire plane except the shaded areas in Fig 2(b). The antenna of u (similarly v) can then be oriented to cover the shaded area adjacent to w (t) as well as sensors v,w (t,u). Collectively, the sensors t,u,v and w cover the entire plane. In this orientation, t covers u, u covers w, w covers v, and v covers t. Therefore the sensors are strongly connected.

In both cases, the antennae are oriented so that they cover the plane, and so that the sensors are strongly connected. Now consider any point $p \in D[t,1] \cup D[u,1] \cup D[v,1] \cup D[w,1]$. The farthest p can be from any of t, u, v, or w is $\delta + 1$. Therefore a transmission range of $r = \delta + 1$ is sufficient to obtain a k-orientation in both cases.

Furthermore, an examination of the orientations reveals that the shortest path between any two of the sensors is at most 2. Therefore these are both 3-orientations. Since we always obtain one of the two cases, the proof follows.

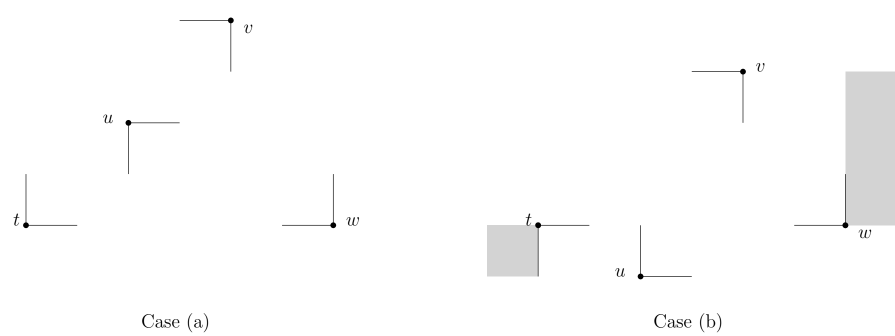


Fig. 2. Antenna orientation of four sensors with beam width $\phi \geq \frac{\pi}{2}$

Corollary 1. *Given a set S of $n \geq 4$ sensors with beam width $\phi \geq \pi/2$. Suppose the maximum Euclidean distance between any two sensors in S is δ. There is a 3-orientation of S with transmission range $r = \delta + 1$.*

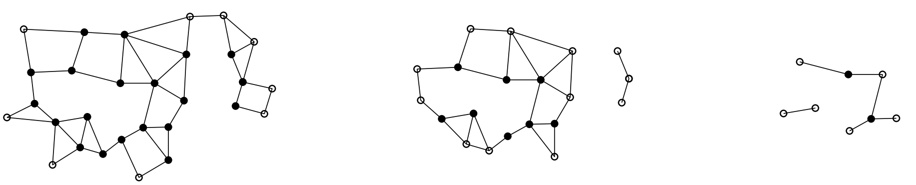
3 Orienting Antennae with Constant Stretch Factor

In this section we will present some approximations for the Antenna Orientation Problem with Constant Stretch Factor. These approximations will have transmission ranges dependant on the beam width of the sensors, the desired stretch factor, as well as whether the orientation is to be found with local or global information.

3.1 Orienting Antennae of Beam Width $\phi \geq \pi$ with Constant Stretch Factor

Theorem 1. *Given a connected UDG $U(S)$ on a set of sensors S each with one directional antenna of beam width $\phi \geq \pi$. There exists an antennae orientation of S with range* $\max(1, 2\sin(\phi/2))$ *which creates a connected transmission network $G_\phi(S)$ such that $\tau_{G_\phi}(S) \leq 2$.*

Proof. For any graph H, denote $C(H)$ as the vertices of the convex hull of each of the connected components of H. We then define the following hierarchical structure, Q: $Q_0(V_0, E_0) = U(S)$, $Q_{k+1}(V_{k+1}, E_{k+1}) = Q_k[V_k - C(Q_k)]$ (the subset of Q_k induced by $V_n - C(Q_n)$). Intuitively, every iteration of the structure is the previous graph with the convex hulls of its connected components peeled away. We note that every iteration is a proper subset of the previous: $Q_{k+1} \subset Q_k$, unless $Q_{k+1} = Q_k = \emptyset$. The construction of such a hierarchical structure is illustrated in Figure 3.



(a) $Q_0(V, E)$ and $C(Q_0)$ (b) $Q_1 = Q_0[V - C(Q_0)]$ and $C(Q_1)$ (c) Q_2 and $C(Q_2)$

Fig. 3. Construction of Q for a given graph $G(V, E)$. $C(Q_i)$ denoted by hollow points

We want to use induction to show that we can find a suitable orientation for some Q_i and that given an orientation for any Q_k we can find an orientation for Q_{k-1}. We do this by ensuring that each sensor is either: *oriented* so that it part of a k-orientation, or the sensor is *convex*. We say that a sensor s is convex in a graph H if s is on the convex hull of $N_H(s) \cup \{s\}$, where $N_H(s)$ is the set of neighbours of s in H. Intuitively, s is convex in H if a line can be drawn through s such that all neighbours of s in H are on one side of the line. Since the sensors have beam width $\phi \geq \pi$, the antennae of convex sensors can be positioned so that they cover all their neighbours. Therefore if all sensors in S are either oriented or convex, there exists an orientation of S which is strongly connected.

Let i be the smallest value such that $Q_i = Q_{i+1} = \emptyset$. Consider Q_{i-1}. It must contain at least one sensor, and all sensors are on the convex hull of their connected components, so they are all convex. Therefore we have a valid orientation for Q_{i-1}.

Consider now Q_{i-2}. We know that all the sensors in $C(Q_{i-2})$ are convex. What about the remaining sensors? None have yet been oriented, some may remain convex, but some may not. It is the sensors that are no longer convex which prevent an orientation from being achieved. For each sensor s which is no longer convex, there must exist at least one sensor in $C(Q_{i-1})$ which is a neighbour of s in $U(S)$ (otherwise s would still be convex). Suppose each such unoriented, non-convex sensor selects one of its neighbours in $C(Q_{i-1})$ and requests to orient with it. Let us now consider each sensor in $C(Q_{i-1})$. A sensor may receive no requests to orient, in which case it remains convex. It may receive a single request, in which case the two sensors orient themselves according to Lemma 4 to form a 2-orientation. It may also receive multiple requests, in which case the sensors can orient themselves according to Lemma 6 to form a 2-orientation. After each sensor of $C(Q_{i-1})$ has taken the appropriate action, we now have the case where all sensors are either oriented or convex, so we have a valid orientation for Q_{i-2}.

Consider now Q_{k-1}. Assume that we have a valid orientation for Q_k. We know that all the sensors in $C(Q_{k-1})$ are convex. What about the remaining sensors? Some sensors may have already been oriented, some may remain convex, but some may not. Using the same method as above, we can orient all the unoriented, non-convex sensors with sensors in $C(Q_{k-1})$ to achieve a valid orientation of Q_{k-1}.

Since we have proven we can find an orientation for Q_{i-2}, and since we can find a valid orientation for Q_{k-1} given a valid orientation for Q_k, we know that we can find a valid orientation $G_\phi(S)$ for $Q_0 = U(S)$.

All sensors in the orientation $G_\phi(S)$ are either part of a 2-orientation or convex. Sensors in a 2-orientation can reach all their neighbours in $U(S)$ is at most two hops. Convex sensors can directly reach all their neighbours in $U(S)$. This means that $\tau_{G_\phi}(S) \le 2$.

3.2 Orienting Antennae of Beam Width $\pi/2 \le \phi < \pi$ with Constant Stretch Factor

Definition 3. *We define the annulus graph $A(P,r,R)$ on a set of points P as the straight line graph where two points a,b at distance d are connected if and only if $r \le d \le R$.*

Theorem 2. *Given a connected UDG, $U(S,1)$, on a set of sensors S each with beam width $\pi/2 \le \phi < \pi$. There exists an antenna orientation of S with range at most $4\cos(\phi/2) + 3$ which creates a strongly connected communication graph $G_\phi(S)$ such that $\tau_{G_\phi}(S) \le 3$. This communication graph can be constructed in constant time.*

Proof. Let $\mathcal{A} = A(S, 2\cos(\phi/2), 2\cos(\phi/2)+1)$ be an annulus graph on S.

Claim. Let u be any point in S and v the farthest point in S from u. If $d(u,v) \ge 2\cos(\phi/2)$, then the degree of u in A, $d_{\mathcal{A}}(u)$ is at least 1.

Proof. Assume there exist two points u and v such that $d(u,v) \ge 2\cos(\phi/2)$ and $d_{\mathcal{A}}(u) = 0$. Since U is connected we can always find a path $P = u = u_0, u_1, ...u_k = v$. Observe that $d(u_i,u_{i+1}) \le 1$ for all i. Therefore, at least one vertex in the path $u_1, ...u_k = v$ is at distance between $2\cos(\phi/2)$ and $2\cos(\phi/2)+1$ since P crosses the annulus of u. This contradicts the assumption, therefore $d_{\mathcal{A}}(u) > 0$.

Given two sensors u,v in S, we say that they form a *2-group* if $d(u,v) \geq 2\cos(\phi/2)$. By Lemma 3 there exists a 2-orientation of the antennae at u and v with angle ϕ and range $d(u,v)+1$.

Given three sensors u,v,w in S where $\delta = \max(d(u,v),d(u,w),d(w,v))$. We say that u,v,w form a *3-group* if $\delta \geq 2\cos(\phi/2)$. By Lemma 5 there exists a 3-orientation of u,v,w with range $\delta+1$.

We say that two points u,v are *close* if $d(u,v) \leq 4\cos(\phi/2)+2$. Given $S' \subseteq S$, we say that S' is a *4-strong subset* if there exists four sensors $u,v,w,x \in S'$ such that $\forall a,b \in S'$, a is close to b. By Corollary 1 we can find a 3-orientation of u,v,w and x with range $4\cos(\phi/2)+3$.

If no two vertices are distance at least $2\cos(\phi/2)$ apart, either U is a 4-strong set, or there are three or fewer sensors in U. If U is a 4-strong set, we can orient it according to Corollary 1. If U consists of only three sensors u,v,w, then they can be oriented so that u covers v, v covers w and w covers u. If there are two or fewer sensors, the orientation is trivial. In the rest of the proof we assume that there exist two vertices separated by distance at least $2\cos(\phi/2)$.

Let M be a maximal matching of $\mathcal{A}$. Consider the following geometric graph $G = (S,E)$ where $\{a,b\} \in E$ if and only if a is an unmatched sensor and b is the nearest matched sensor to a.

Claim. For each edge $\{a,b\}$ in G, $d(a,b) \leq 2\cos(\phi/2)+1$

Proof. From the first claim we know that each point has degree at least one in $\mathcal{A}$. Therefore, a point is only unmatched if all its neighbours in $\mathcal{A}$ are matched. Thus, $d(a,b) \leq 2\cos(\phi/2)+1$.

Let $\{u,v\}$ be any edge in M and $N_G(u)$ denote the neighbours of u in G. Since M is a matching, clearly u,v are not incident to any other edge in M. From our definition of G, $v \notin N_G(u)$. Furthermore, $\forall a,b \in M, N_G(a) \cap N_G(b) = \emptyset$. Since every sensor in U is incident to at least one edge in either M or G, the previous conditions mean that we can partition the graph based on the edges in M. For each edge $\{u,v\}$ in M, we define a subset $S'_{\{u,v\}} = \{u,v\} \cup N_G(u) \cup N_G(v)$. Each subset is non-empty since it must contain u,v. As mentioned previously, every sensor will be part of one and only one subset. This is therefore a valid partition. We will show that each subset can be oriented to form a 3-orientation.

Without loss of generality assume that $|N_G(u)| \leq |N_G(v)|$. There are now multiple cases we may encounter.

Case 1: $|N_G(u)| = |N_G(v)| = 0$.
In this case, $S'_{\{u,v\}}$ forms a 2-group.

Case 2: $|N_G(u)| = 0$ **and** $|N_G(v)| \geq 1$.
If $|N_G(v)| = 1$, $S'_{\{u,v\}}$ forms a 3-group. Otherwise, it is a 4-strong subset.

Case 3: $|N_G(u)| = 1$ **and** $|N_G(v)| \geq 1$.
Let $x \in N_G(u)$. If $d(x,v) \leq 2\cos(\phi/2)+1$, $S'_{\{u,v\}}$ is a 4-strong subset. If not, we consider three possible cases:

- $|N_G(v)| = 1$. Let $y \in N_G(v)$. If $d(y,u) \leq \cos(\phi/2) + 1$, $S'_{\{u,v\}}$ is a 4-strong subset. Otherwise, $\{x,v\}$ forms a 2-group and $\{y,u\}$ forms a 2-group. If two groups are formed, we remove $S'_{\{u,v\}}$ from the partition and add $\{x,v\}$ and $\{y,u\}$.
- $|N_G(v)| = 2$. Let y and z be the elements in $N_G(v)$. If $\max(d(y,u), d(z,v)) \leq 2\cos(\phi/2) + 1$, $S'_{\{u,v\}}$ is a 4-strong subset. Otherwise, $\{x,v\}$ forms a 2-group and $\{u,y,z\}$ forms a 3-group. If two groups are formed, we remove $S'_{\{u,v\}}$ from the partition and add $\{x,v\}$ and $\{u,y,z\}$.
- $|N_G(v)| \geq 3$. Let z,w be the nearest sensor of v in $N_G(v) \setminus \{y\}$. $N_G(v) \cup \{u\}$ is a 4-strong subset and $\{x,v\}$ forms a 2-group. We remove $S'_{\{u,v\}}$ from the partition and add $N_G(v) \cup \{u\}$ and $\{x,v\}$.

Case 4: $|N_G(u)| = 2$ **and** $|N_G(v)| \geq 2$**.**
If there are two sensors x,y in $N_G(u) \cup N_G(v)$ that are at distance at most $2\cos(\phi/2) + 1$ of u and v, $S'_{\{u,v\}}$ is a 4-strong subset. Otherwise, $N_G(u) \cup \{v\}$ forms a 3-group. If $d_G(v) = 2$, $N_G(v) \cup \{u\}$ forms a 3-group, otherwise it is a 4-strong subset. If two groups are formed remove $S'_{\{u,v\}}$ from the partition and add $N_G(v) \cup \{v\}$ and $N_G(v) \cup \{u\}$.

Case 5: $|N_G(u)| \geq 3$ **and** $|N_G(v)| \geq 3$**.**
$N_G(u) \cup \{u\}$ is a 4-strong subset and $N_G(v) \cup \{v\}$ is a 4-strong subset. We remove $S'_{\{u,v\}}$ from the partition and add $N_G(v) \cup \{u\}$ and $N_G(v) \cup \{v\}$.

Once we have oriented every subset in the partition, we observe that they now all consist of 2-groups, 3-groups and 4-strong subsets. As mentioned previously, we know that we can form 3-orientations for each of these. Therefore we now have a partition of the sensors of U such that each sensor is a part of a 3-orientation. Note that the transmission range required to orient any of the sensor groups was always less than or equal to $2\cos(\phi/2) + 3$. What is left to show is that U is a 3-orientation. Consider any edge $\{a,b\} \in U$. Suppose a,b are not in the same 3-orientation. Since $b \in D[a,1]$, there is a sensor in a's 3-orientation which covers b. Similarly, there is a sensor in b's 3-orientation which covers a. Therefore, by Lemma 1 these 3-orientations can be combined to form a larger 3-orientation of which both a and b are members. Since U is connected, this process can be repeated until all sensors are a part of the same 3-orientation. Therefore there must exist some orientation $G_\phi(S)$ of the sensors of S which is a 3-orientation. Therefore by Lemma 2, there must exist some orientation $G_\phi(S)$ of the sensors of S such that $\tau_{G_\phi}(S) \leq 3$.

Regarding the complexity, a maximal matching can be constructed in constant time [11] and each other step is local.

One may ask whether we can improve our result by considering the annulus graph $G = A(P, 2\cos(\phi/2), 2\cos(\phi/2) + 1 - \varepsilon)$. However, we cannot guarantee minimum degree greater than zero on G and consequently the properties of the unmatched vertices do not hold.

4 Conclusion

In this paper we have examined issues relating to connectivity in the directional antenna model. There remain unanswered questions relating to this problem. Can approximations can be found for angles between $\pi/2$ and $\pi/3$? Can tighter bounds on range and/or stretch factor be found? How does the Antenna Orientation Problem with Constant Stretch Factor relate to the Antenna Orientation Problem for Strong Connectivity? How would multiple antennae per sensor affect connectivity? How does the problem change if Euclidean stretch factor is considered instead of hop-stretch factor?

The properties of this model may be of particular interest for questions such as: How would routing work? What level of sender and receiver interference would be expected? These are interesting questions and are worthy of study.

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