Introduction to an Algebra of Belief Functions on Three-Element Frame of Discernment — A Quasi Bayesian Case

Milan Daniel*

Institute of Computer Science,
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, CZ – 182 07 Prague 8, Czech Republic
milan.daniel@cs.cas.cz

Abstract. The study presents an introduction to algebraic structures related to belief functions (BFs) on 3-element frame of discernment.

Method by Hájek & Valdés for BFs on 2-element frames [15,16,20] is generalized to larger frame of discernment. Due to complexity of the algebraic structure, the study is divided into 2 parts, the present one is devoted to the case of quasi Bayesian BFs.

Dempster's semigroup of BFs on 2-element frame of discernment by Hájek-Valdés is recalled. A new definition of Dempster's semigroup (an algebraic structure) of BFs on 3-element frame is introduced; and its subalgebras in general, subalgebras of Bayesian BFs and of quasi Bayesian BFs are described and analysed. Ideas and open problems for future research are presented.

Keywords: belief function, Dempster-Shafer theory, Dempster's semigroup, homomorphisms, conflict between belief functions, uncertainty.

1 Introduction

Belief functions (BFs) are one of the widely used formalisms for uncertainty representation and processing that enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence [18].

A need of algebraic analysis of belief functions (BFs) on frames of discernment with more then two elements arised in our previous study of conflicting belief functions (a decomposition of BFs into their non-conflicting and conflicting parts requires a generalization of Hájek-Valdés operation "minus") [12] motivated by series of papers on conflicting belief functions [1,6,9,17,19]. Inspired by this demand we start with algebraic analysis of BFs on 3-element frame in this study.

Here we generalize the method by Hájek & Valdés for BFs on 2-element frame [15,16,20] to larger frame of discernment. Due to complexity of the algebraic structure, the study is divided into 2 parts; the present one is devoted to the

* This research is supported by the grant P202/10/1826 of the Grant Agency of the Czech Republic. Partial support by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications" is also acknowledged.

S. Greco et al. (Eds.): IPMU 2012, Part III, CCIS 299, pp. 532–542, 2012.

special case of quasi Bayesian BFs (i.e., to the case of very simple BFs), the second part devoted to general BFs is under preparation [13].

The study starts with belief functions and algebraic preliminaries, including Hájek-Valdés method in Section 2. A Definition of Dempster's semigroup (an algebraic structure) of BFs on 3-element frame (Section 3) is followed by a study of its subalgebras in general, of Bayesian BFs and of quasi Bayesian BFs (Section 4). Ideas and open problems for future research are presented in Section 5.

2 Preliminaries

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* [18] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also [4–9]. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. There is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as of belief function.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF); in fact, it is a probability distribution on Ω. If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function* (qBBF), that is something like 'un-normalized probability distribution', but with a different interpretation. If all focal elements are nested, we speak about *consonant belief function.*

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X)$ $m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [18]. Let us recall U_n the *uniform Bayesian belief function*[1] [9], i.e., the uniform probability distribution on Ω_n, and *normalized plausibility of singletons*[2] of Bel: the BBF $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))$ $(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [2,8].

An *indecisive BF* is a BF, which does not prefer any $\omega_i \in \Omega_n$, i.e., BF which gives no decisional support for any ω_i, i.e., BF such that $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$, i.e., $(Pl\_P(Bel))(\{\omega_i\}) = \frac{1}{n}$, [10].

Let us define *Exclusive BF* as a BF such that $Pl(X) = 0$ for some $\emptyset \neq X \subset \Omega$; BF is non-exclusive otherwise, thus for non-exclusive BFs it holds true that, $Pl(\{\omega_i\}) \neq 0$ for all $\omega_i \in \Omega$. *(Simple) complementary BF* has up to two focal

[1] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [8], $0'$ comes from studies by Hájek & Valdés.

[2] Plausibility of singletons is called *contour function* by Shafer in [18], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

elements $\emptyset \neq X \subset \Omega$ and $\Omega \setminus X$. *(Simple) quasi complementary BF* has up to 3 focal elements $\emptyset \neq X \subset \Omega$, $\Omega \setminus X$ and Ω.

2.2 Belief Functions on 2-Element Frame of Discernment; Dempster's Semigroup

Let us suppose, that the reader is slightly familiar with basic algebraic notions like *a semigroup* (an algebraic structure with an associative binary operation), *a group* (a structure with an associative binary operation, with a unary operation of inverse, and with a neutral element), *a neutral element* n ($n * x = x$), *an absorbing element* a ($a * x = a$), *an idempotent* i ($i * i = i$), *a homomorphism* f ($f(x * y) = f(x) * f(y)$), etc. (Otherwise, see e.g., [4,7,15,16].)

We assume $\Omega_2 = \{\omega_1, \omega_2\}$, in this subsection. There are only three possible focal elements $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ and any normalized *basic belief assignment (bba)* m is defined by a pair $(a, b) = (m(\{\omega_1\}), m(\{\omega_2\}))$ as $m(\{\omega_1, \omega_2\}) = 1 - a - b$; this is called *Dempster's pair* or simply *d-pair* in [4,7,15,16] (it is a pair of reals such that $0 \leq a, b \leq 1, a + b \leq 1$).

Extremal d-pairs are pairs corresponding to BFs for which either $m(\{\omega_1\}) = 1$ or $m(\{\omega_2\}) = 1$, i.e., $\perp = (0, 1)$ and $\top = (1, 0)$. The set of all non-extremal d-pairs is denoted as D_0; the set of all non-extremal *Bayesian d-pairs* (i.e. d-pairs corresponding to Bayesian BFs, where $a + b = 1$) is denoted as G; the set of d-pairs such that $a = b$ is denoted as S (set of indecisive[3] d-pairs), the set where $b = 0$ as S_1, and analogically, the set where $a = 0$ as S_2 (simple support BFs). Vacuous BF is denoted as $0 = (0, 0)$ and there is a special BF (d-pair) $0' = (\frac{1}{2}, \frac{1}{2})$, see Fig. 1.

The *(conjunctive) Dempster's semigroup* $\mathbf{D}_0 = (D_0, \oplus, 0, 0')$ is the set D_0 endowed with the binary operation $\oplus$ (i.e. with the Dempster's rule) and two distinguished elements 0 and $0'$. Dempster's rule can be expressed by the formula $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ for d-pairs [15]. In D_0 it is defined further: $-(a, b) = (b, a)$, $h(a, b) = (a, b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, $h_1(a, b) = \frac{1-b}{2-a-b}$, $f(a, b) = (a, b) \oplus (b, a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2}, \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$; $(a, b) \leq (c, d)$ iff $[h_1(a, b) < h_1(c, d)$ or $h_1(a, b) = h_1(c, d)$ and $a \leq c]$ [4].

The principal properties of $\mathbf{D}_0$ are summarized by the following theorem:

Theorem 1. *(i) The Dempster's semigroup* $\mathbf{D}_0$ *with the relation* $\leq$ *is an ordered commutative (Abelian) semigroup with the neutral element* 0*;* $0'$ *is the only non-zero idempotent of* $\mathbf{D}_0$*.*
(ii) $\mathbf{G} = (G, \oplus, -, 0', \leq)$ *is an ordered Abelian group, isomorphic to the additive group of reals with the usual ordering. Let us denote its negative and positive cones as* $G^{\leq 0'}$ *and* $G^{\geq 0'}$*.*
(iii) The sets S, S_1, S_2 *with the operation* $\oplus$ *and the ordering* $\leq$ *form ordered commutative semigroups with neutral element* 0*; they are all isomorphic to the positive cone of the additive group of reals.*

---

[3] BFs (a, a) from S are called *indifferent* BFs by Haenni [14].

[4] Note, that $h(a, b)$ is an abbreviation for $h((a, b))$, similarly for $h_1(a, b)$ and $f(a, b)$.

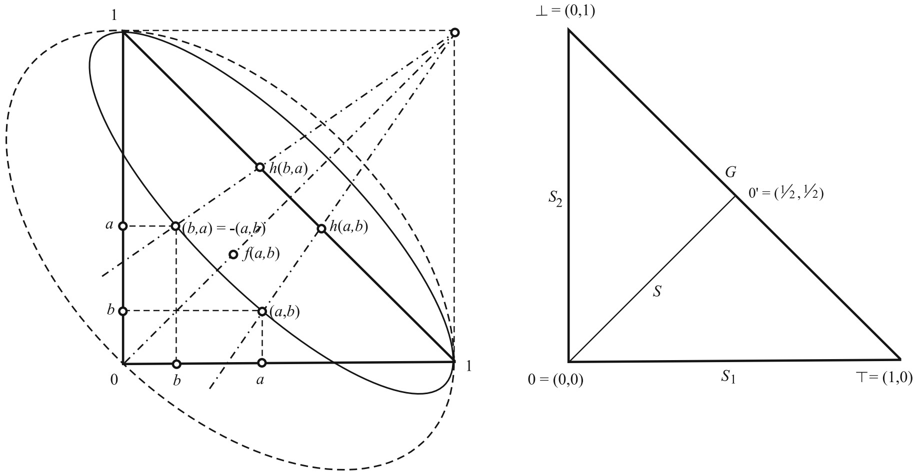

Fig. 1. Dempster's semigroup D_0. Homomorphism h is in this representation a projection to group G along the straight lines running through the point $(1,1)$. All the Dempster's pairs lying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.

(iv) h is an ordered homomorphism: $(D_0, \oplus, -, 0, 0', \leq) \longrightarrow (G, \oplus, -, 0', \leq)$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$*, i.e., the normalized plausibility of singletons probabilistic transformation.*
(v) f is a homomorphism: $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$*; (but, not an ordered one).*

For proofs see [15,16,20].

2.3 BFs on n-Element Frames of Discernment

Analogically to the case of Ω_2, we can represent a BF on any n-element frame of discernment Ω_n by an enumeration of its m values (bbms), i.e., by a $(2^n{-}2)$-tuple $(a_1, a_2, ..., a_{2^n-2})$, or as a $(2^n{-}1)$-tuple $(a_1, a_2, ..., a_{2^n-2}; a_{2^n-1})$ when we want to explicitly mention also the redundant value $m(\Omega) = a_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} a_i$. For BFs on Ω_3 we use $(a_1, a_2, ..., a_6; a_7) = (m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_3\}), m(\{\omega_1, \omega_2\})$ $m(\{\omega_1, \omega_3\}),\ m(\{\omega_2, \omega_3\}); m(\{\Omega_3\}))$.

3 Dempster's Semigroup of Belief Functions on 3-Element Frame of Discernment Ω_3

3.1 Basics

Let us sketch the basics of Dempster's semigroup of BFs on 3-element frame of discernment Ω_3 in this subsection. Following the subsection 2.3 and Hájek & Valdés' idea of the classic (conjunctive) Dempster's semigroup [15,16,20], we have a unique representation of any BF on 3-element frame by *Dempster's 6-tuple* or *d-6-tuple*[5] $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23})$, such that $0 \leq d_i, d_{ij} \leq 1$, $\sum_{i=1}^{3} d_i + \sum_{ij=12}^{23} d_{ij} \leq 1$. These can be presented them in 6-dimensional 'triangle', Fig. 2.

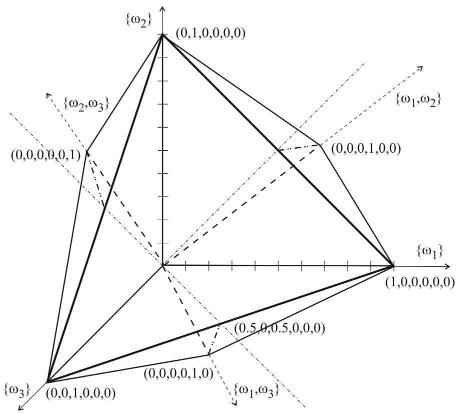

Fig. 2. General BFs on 3-element frame Ω_3

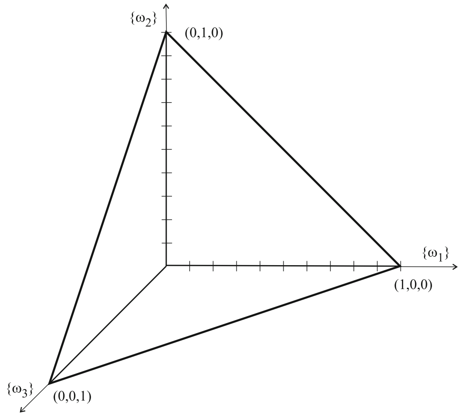

Fig. 3. Quasi Bayesian BFs on 3-element frame Ω_3

Generalizing the Hájek – Valdés terminology we obtain two special Dempster's 6-tuples $0 = (0, 0, ..., 0)$ representing the vacuous belief function (VBF) and $0' = U_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$ corresponding to the uniform distribution of bbms to all singletons. Generalization of extremal d-pairs are *categorical d-6-tuples* $(1, 0, ..., 0)$, $(0, 1, 0, ..., 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, ..., 0, 1, 0)$, $(0, ..., 0, 1)$ which represent categorical BFs on Ω_3. Further generalization of extremal (i.e. categorical) d-pairs are *exclusive d-6-tuples* $(a, b, 0, 1{-}a{-}b, 0, 0)$, $(a, 0, b, 0, 1{-}a{-}b, 0)$, $(0, a, b, 0, 0, 1{-}a{-}b)$, we can see, that the categorical 6-tuples are the special cases of exclusive 6-tuples, the most special case are categorical singletons.

There are *simple d-6-tuples* $(a, 0, ..., 0)$, $(0, a, 0, ..., 0)$, $(0, 0, a, 0, 0, 0)$, $(0, 0, 0, a, 0, 0)$, $(0, ..., 0, a, 0)$, $(0, ..., 0, a)$ corresponding to simple (support) BFs and 6 *consonant d-6-tuples* $(a, 0, 0, b, 0, 0)$, $(a, 0, 0, 0, b, 0)$, etc. corresponding to consonant BFs. We can note, that simple 6-tuples are special cases of consonant ones.

It is possible to prove that Dempster's combination $\oplus$ is defined for any pair of non-exclusive BFs (d-6-tuples) and that the set of all non-exclusive BFs is closed under $\oplus$, thus we can introduce the following version of the definition:

Definition 1. *The* (conjunctive) Dempster's semigroup $\mathbf{D}_3 = (D_3, \oplus, 0, 0')$ *is the set* D_3 *of all non-exclusive Dempster's* 6*-tuples, endowed with the binary operation* $\oplus$ *(i.e. with the Dempster's rule) and two distinguished elements* 0 *and* $0' = U_3$*, where* $0 = 0_3 = (0, 0, ..., 0)$ *and* $0' = 0'_3 = U_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$.

There is a homomorphism $h : \mathbf{D}_3 \longrightarrow \mathcal{BBF}_3 = \{Bel \in D_3 \mid Bel \text{ is BBF}\}$ defined by $h(Bel) = Bel \oplus U_3$; it holds true that $h(Bel) = Pl\_P(Bel)$ [10].

[5] For simplicity of expressions, we speak often simply on 6-tuples only.

3.2 The Extended Dempster's Semigroup

There are only single 2 extremal (categorical, exclusive) d-pairs on Ω_2, thus the extension of $\mathbf{D}_0$ to $\mathbf{D}_0^+$, (where $D_0^+ = D_0 \cup \{\bot, \top\}$ and $\bot \oplus \top$ is undefined) is important for applications, but it is not interesting from the theoretical point of view.

There are 6 categorical (exclusive) d-6-tuples in $\mathbf{D}_3^+$ (in the set of BFs defined over Ω_3) and many general exclusive 6-tuples (BFs) in $\mathbf{D}_3^+$, thus the issue of extension of Dempster's semigroup to all BFs is more interesting and also more important, because a complex structure of exclusive BFs is omitted in Dempster's semigroup of non-exclusive BFs, in the case of Ω_3. Nevertheless, due to the extent of this text we are concentrating only on the non-extended case in this study.

4 Subalgebras of Dempster's Semigroup

4.1 Subalgebras of $\mathbf{D_0}$ and Ideas of Subalgebras of $\mathbf{D_3}$

There are the following subalgebras of $\mathbf{D}_0$: subgroup of (non-extremal) BBFs $G = (\{BBFs\}, \oplus, -, 0')$, two trivial subgroups $0 = (\{0\}, \oplus, -, 0)$ and $0' = (\{0'\}, \oplus, -, 0')$, (other two trivial groups $\bot = (0, 1)$ and $\top = (1, 0)$ are subalgebras of $\mathbf{D}_0^+$); there are 3 important subsemigroups $S = (\{(s, s) \in D_0\}, \oplus)$, $S_1 = (\{(a, 0) \in D_0\}, \oplus)$, $S_2 = (\{(0, b) \in D_0\}, \oplus)$, further there are many subsemigroups which are endomorphic images of S_1 and S_2 by endomorphisms of $\mathbf{D}_0$, for endomorhpisms of $\mathbf{D}_0$ see [3,5] and [10]. Note that there are also other semigroups that are derived from the already mentioned subalgebras: $\mathbf{D}_0^{\geq 0}$ and $\mathbf{D}_0^{\leq 0'}$, positive and negative cones of G (i.e. $G^{\geq 0'}, G^{\leq 0'}$) with or without $0'$, versions of S, S_1, S_2 with or without absorbing elements $0'$, $(1, 0)$, $(0, 1)$, versions of S, S_1, S_2 without 0, and further $S \cup G, S_1 \cup G, S_2 \cup G$, $S \cup G_{...}, S_1 \cup G_{...}, S \cup G_{...}, S_2 \cup G_{...}, 0 \cup G, 0 \cup G_{...}, 0 \cup G_{...}, 0 \cup 0' = (\{0, 0'\}, \oplus)$, some of these subsemigroups given by union have variants without 0 and/or $0'$ with or without extremal elements $\bot$ or $\top$ (note that subalgebras with $\bot$ or $\top$ are subalgebras of extended Dempster's semigroup $\mathbf{D}_0^+$ in fact). Altogether there are many subalgebras, but there are only 4 non-trivial and really important ones: subgroup G and 3 subsemigroups S, S_1, and S_2.

From [4,15,16] we know that 0 is neutral element of $\mathbf{D}_0$, thus 0 is also neutral element of all subsemigroups of $\mathbf{D}_0$ containing 0, hence $\mathbf{D}_0$ and its subsemigroups containing 0 are monoids, i.e. we have the following observation.

Observation 1. *Dempster's semigroup* $\mathbf{D}_0$ *and its subsemigroups* S, S_1 *and* S_2 *are monoids.*

The 3-element case is much more complex. In accordance with a number of possible focal elements and a representation of BFs by d-6-tuples we cannot display general BFs on 3-element case by 3-dimensional but by 6-dimensional triangle, see Fig. 2. Also the generalization of Dempster's semigroup and its subalgebras is much more complicated, as there is a significantly greater amount of structurally more complex subalgebras. Subsequently the issue of homomorphisms of corresponding structures is more complex. Nevertheless, there is a simplified special case of quasi Bayesian BFs, which are representable by "triples" $(d_1, d_2, d_3, 0, 0, 0)$, as $d_{12} = d_{13} = d_{23} = 0$ for qBBFs, see Fig. 3. -

4.2 The Subgroups/Subalgebras of Bayesian Belief Functions

Before studying the simplified case of quasi Bayesian BFs we will utilize the results on their special case of BBFs from [10].

Following [10] we have "$-$" for any BBF $(d_1, d_2, d_3, 0, 0, 0)$, such that $d_i > 0$, and neutrality of $0' = 0'_3$, in the following sense: $-(d_1, d_2, d_3, 0, 0, 0; 0) = (x_1, x_2, x_3, 0, 0, 0; 0) = (x_1, \frac{d_1}{d_2}x_1, \frac{d_1}{d_3}x_1, 0, 0, 0; 0)$, where $x_1 = 1/(1 + \sum_{i=2}^{3} \frac{d_1}{d_i}) = 1/(1 + \frac{d_1}{d_2} + \frac{d_1}{d_3})$, such that, $(d_1, d_2, d_3, 0, 0, 0) \oplus -(d_1, d_2, d_3, 0, 0, 0) = U_3 = 0'_3$. We can prove equality of BBFs $(d_1, d_2, d_3, 0, 0, 0)$, such that $d_i > 0$ with non-exclusive BBFs, further we have definition of $\oplus$, consequently we can prove closeness of non-exclusive BBFs w.r.t. $\oplus$, hence $G_3 = (\{(d_1, d_2, d_3, 0, 0, 0) \mid d_i > 0, \sum_{i=1}^{3} d_i = 1\}, \oplus, -, 0'_3)$ is a group, i.e. subgroup of D_{3-0}. As we have 3 different non-ordered elements, without any priority, we do not have any linear ordering of G_3 in general, thus neither any isomorphism to additive group of reals in general. This is the difference of G_3 subgroup of $\mathbf{D}_{3-0}$ from G subgroup of $\mathbf{D}_0$.

There are several subalgebras of special BBFs (subalgebras both of G_3 and of $\mathbf{D}_{3-0}$). Let us start with subalgebras of BBFs $(d_1, d_2, d_2, 0, 0, 0; 0)$ where $d_2 = m(\omega_2) = m(\omega_3)$. The set of these BBFs is closed w.r.t. $\oplus$. There is $minus_{2=3}(d_1, d_2, d_2, 0, 0, 0; 0) = (\frac{d_2}{d_2+2d_1}, \frac{d_1}{d_2+2d_1}, \frac{d_1}{d_2+2d_1}, 0, 0, 0; 0) = (\frac{1-d_1}{1+3d_1}, \frac{2d_1}{1+3d_1}, \frac{2d_1}{1+3d_1}, 0, 0, 0; 0)$, for any $0 \leq d_1 \leq 1, d_2 = \frac{1}{2}(1-d_1)$, such that $(d_1, \frac{1}{2}(1-d_1), \frac{1}{2}(1-d_1), 0, 0, 0; 0) \oplus minus_{2=3}(d_1, \frac{1}{2}(1-d_1), \frac{1}{2}(1-d_1), 0, 0, 0; 0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, hence $minus_{2=3}$[6] is inverse w.r.t. $\oplus$ on the set. Thus $G_{2=3} = (\{(d_1, d_2, d_2, 0, 0, 0; 0)\}, \oplus, minus_{2=3}, 0'_3)$ is subgroup of G_3 and of $\mathbf{D}_{3-0}$. As there is a natural linear order of d_1's from 0 to 1, consequently, there is also a linear order of $G_{2=3}$, thus $G_{2=3}$ is an ordered group of BBFs. Analogically there are ordered subgroups $G_{1=3}$ and $G_{1=2}$. Based on these facts and on analogy of $G_{2=3}$, $G_{1=3}$, and $G_{1=2}$ with G, there is the following hypothesis. Unfortunately, isomorphisms of the subgroups to $(Re, +, -, 0)$ have not been observed till now.

Hypothesis 1. *$G_{2=3}$, $G_{1=3}$, and $G_{1=2}$ are subgroups of $\mathbf{D}_{3-0}$ isomorphic to the additive group of reals.*

Positive and negative cones $G_{1=2}^{\geq 0'}, G_{1=3}^{\geq 0'}, G_{2=3}^{\geq 0'}$, $G_{1=2}^{\leq 0'}, G_{1=3}^{\leq 0'}, G_{2=3}^{\leq 0'}$, $(G_{1=2}^{>0'}$, $G_{1=3}^{>0'}$, $G_{2=3}^{>0'}$, $G_{1=2}^{<0'}, G_{1=3}^{<0'}, G_{2=3}^{<0'})$ of $G_{1=2}, G_{1=3}, G_{2=3}$ with and without $0'$ are subsemigroups of G_3 and consequently also subsemigroups of $\mathbf{D}_{3-0}$.

4.3 The Subsemigroup of Quasi-bayesian Belief Functions

Let us turn our attention to the set of all non-exclusive quasi-Bayesian belief functions $D_{3-0} = \{(a, b, c, 0, 0, 0); 0 \leq a + b + c \leq 1, 0 \leq a, b, c\}$. This includes

[6] The name $minus_{2=3}$ reflects the fact, that the operation is a generalization of Hájek-Valdés operation "minus" $(-(a, b) = (b, a))$ to $G_{2=3}$.

neutral element 0 and idempotent $0' = U_3$. Considering only non-exclusive qBBFs, $\oplus$ is always defined, closeness w.r.t. $\oplus$ is obvious, hence we have a subsemigroup (with neutral element, thus monoid) $\mathbf{D}_{3-0}$.

Subgroup G_3 of $\mathbf{D}_3$ and its subalgebras are also subalgebras of $\mathbf{D}_{3-0}$. Analogously to subsemigroups S and S_i of $\mathbf{D}_0$, there are subsemigroups $S_1 = (\{(d_1,0,0,0,0,0) \in D_{3-0}\},\oplus)$, $S_2 = (\{(0,d_2,0,0,0,0) \in D_{3-0}\},\oplus)$, $S_3 = (\{(0,0,d_3,0,0,0) \in D_{3-0}\},\oplus)$ and $S_0 = (\{(s,s,s,0,0,0) \in D_{3-0}\},\oplus)$ are subsemigroups of $\mathbf{D}_{3-0}$. and similarly also $S_{1-2} = (\{(s,s,0,0,0,0) \in D_{3-0}\},\oplus)$ (without $(\frac{1}{2},\frac{1}{2},0,0,0,0)$), S_{1-3}, and S_{2-3} of $\mathbf{D}_{3-0}$. All of them are isomorphic to the positive cone of the additive group of reals $\mathbf{Re}_{\geq 0}$. Using isomorphicity of S_1 (subsemigroup of D_0), there are simple isomorphisms $z_i : S_i \subset D_3 \longrightarrow S_1 \subset D_0$: $z_1(d_1,0,0,0,0,0) = (d_1,0)$, $z_2(0,d_2,0,0,0,0) = (d_2,0)$, $z_3(0,0,d_3,0,0,0) = (d_3,0)$. Analogously there is $z_{1-2} : S_{1-2} \subset D_3 \longrightarrow S \subset D_0$: $z_{1=2}(s,s,0,0,0,0) = (s,s)$, where S is already isomorphic to S_1 (and $\mathbf{Re}_{\geq 0}$) using Valdes' isomorphism $\varphi : S_1 \to S$ given by $\varphi(x_1,0) = (\frac{x_1}{1+x_1},\frac{x_1}{1+x_1})$, see [20].

For subsemigroup S_0 in $\mathbf{D}_3$ we can use isomorphicity of S_1 verified in the previous paragraph, further we have to define new isomorphism $\varphi_3 : S_1 \to S$ given by $\varphi_3(d_1,0,0,0,0,0) = (\frac{d_1}{1+2d_1},\frac{d_1}{1+2d_1},\frac{d_1}{1+2d_1},0,0,0)$ for $0 \leq d_1 \leq 1$, where $\varphi_3^{-1}(s,s,s,0,0,0) = (\frac{s}{1-2s},0,0,0,0,0)$ for $0 \leq s \leq \frac{1}{3}$. Let us verify the homomorphic properties: we have to verify $\varphi_3((a,0,0,0,0,0) \oplus (b,0,0,0,0,0)) \overset{?}{=} \varphi_3(a,0,0,0,0,0) \oplus \varphi_3(b,0,0,0,0,0)$: $\varphi_3((a,0,0,0,0,0) \oplus (b,0,0,0,0,0)) = \varphi_3(a+b-ab,0,0,0,0,0) = (c,c,c,0,0,0)$, where $c = \frac{a+b-ab}{1+2a+2b-2ab}$;
$(u,u,u,0,0,0) \oplus (v,v,v,0,0,0) = (w,w,w,0,0,0)$, where $w = \frac{u+v-5uv}{1-6uv}$, thus $\varphi_3(a,0,0,0,0,0) \oplus \varphi_3(b,0,0,0,0,0) = (\frac{a}{1+2a},\frac{a}{1+2a},\frac{a}{1+2a},0,0,0) \oplus (\frac{b}{1+2b},\frac{b}{1+2b},\frac{b}{1+2b},0,0,0) = (s,s,s,0,0,0)$, where $s = \frac{\frac{a}{1+2a}+\frac{b}{1+2b}-5\frac{a}{1+2a}\frac{b}{1+2b}}{1-6\frac{a}{1+2a}\frac{b}{1+2b}} = \frac{a+b+ab}{1+2a+2b-2ab} = c$. Hence φ_3 is really a homomorphism, i.e. we have the following lemma:

Lemma 1. *S_0 is subsemigroup of $\mathbf{D}_{3-0}$ isomorphic to the positive cone of the additive group of reals extended with ∞.*

Let us consider subsemigroup $\mathbf{D}_{1-2=3} = (\{(d_1,d_2,d_2,0,0,0\},\oplus)$ now. Analogously to $G_{2=3}$, $d_2 = d_3$, but $d_1 + 2d_2 \leq 1$ here. Thus $G_{2=3}$ is proper subalgebra of $\mathbf{D}_{1-2=3}$. There are subsemigroups S_1,$S_{2=3} = (\{(0,d_2,d_2)\},\oplus)$ and S_0, we have already seen that S_1 and S_0 are isomorphic to $\mathbf{Re}_{\geq 0}$ and $\mathbf{Re}^+_{\geq 0}$, the same holds also for $S_{2=3}$ using simple isomorphism $z : S_{2=3} \longrightarrow S \subset D_0$, such that $z(0,d_2,d_2) = (d_2,d_2)$. A structure of the subsemigroup $\mathbf{D}_{1-2=3}$ is very similar to that of $\mathbf{D}_0$, we can even extend the operation $minus_{2=3}$ from $G_{2=3}$ to the entire $\mathbf{D}_{1-2=3}$, where $minus_{2=3}(d_1,d_2,d_2,0,0,0) = (x_1,x_2,x_2)$, such that $x_1 = d_1 + 2d_2 - 2\frac{2d_1+d_2-d_1^2-2d_2^2-3d_1d_2}{3-d_1-5d_2}$, $x_2 = \frac{2d_1+d_2-d_1^2-2d_2^2-3d_1d_2}{3-d_1-5d_2}$. Assuming validity of Hypothesis 1, the subsemigroup $\mathbf{D}_{1-2=3} = (\{(d_1,d_2,d_2,0,0,0\},\oplus,minus_{2=3},0,U_3)$ is isomorphic to Dempster's semigroup $\mathbf{D}_0$. The same for $\mathbf{D}_{2-1=3}$ and $\mathbf{D}_{3-1=2}$.

We can observe that subsemigroups $\mathbf{D}_{1-2} = (\{(d_1,d_2,0,0,0,0\},\oplus)$, $\mathbf{D}_{1-3}$, $\mathbf{D}_{2-3}, S_{1-2}, S_{1-3}, S_{2-3}$ are not included $\mathbf{D}_{3-0}$ (due to exclusive BBFs, e.g. $(d_1,1-d_1,0,0,0,0)$ for $\mathbf{D}_{1-2}$), thus they are subalgebras of $\mathbf{D}^+_{3-0}$ only.

We can summarize the properties of subsemigroup $\mathbf{D}_{3-0}$ of qBBFs as:

Theorem 2. *(i) Monoid* $\mathbf{D}_{3-0} = (D_{3-0}, \oplus, 0, U_3)$ *is a subsemigroup of* $\mathbf{D}_3$ *with neutral element* $0 = (0,0,0,0,0,0)$ *and with the only other idempotent* $0' = U_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$.
(ii) Subgroup of non-exclusive BBFs $G_3 = (\{(a,b,c,0,0,0) \,|\, a+b+c = 1, 0 < a,b,c\}, \oplus, "-", U_3)$ *and its subalgebras are subalgebras of* $\mathbf{D}_{3-0}$.
(iii) The sets of non-exclusive BFs $S_0, S_1, S_2, S_3, S_{1-2}, S_{1-3}, S_{2-3}$ *with the operation* $\oplus$ *and VBF* 0 *form commutative semigroups with neutral element* 0 *(monoids); they all are isomorphic*[7] *to the positive cone of the additive group of reals* $\mathbf{Re}_{\geq 0}$ *(to* $\mathbf{Re}^{+}_{\geq 0}$ *extended with* ∞ *in the case of* S*).*
(iv) Subsemigroups $\mathbf{D}_{1-2=3}$, $\mathbf{D}_{2-1=3}$ *and* $\mathbf{D}_{3-1=2}$ *(with their subalgebras* S_i*'s,* $G_{2=3}$*,*$G_{1=3}$ *and* $G_{1=2}$*) are subsemigroups (resp. subgroups in the case of* G_i*'s) of* $\mathbf{D}_{3-0}$ *(hence also of* $\mathbf{D}_3$*). Assuming validity of Hypothesis 1,* $\mathbf{D}_{1-2=3}$, $\mathbf{D}_{2-1=3}$ *and* $\mathbf{D}_{3-1=2}$ *are isomorphic to Dempster's semigroup* $\mathbf{D}_0$.
(v) Semigroups of non-exclusive BFs $(\{(a,b,0,0,0,0) \,|\, a+b < 1\}, \oplus)$, $(\{(a,0,c,0,0,0) \,|\, a+c < 1\}, \oplus)$, $(\{(0,b,c,0,0,0) \,|\, b+c < 1\}, \oplus)$, *are subsemigroups of* $\mathbf{D}_{3-0}$ *and all three are isomorphic to* $\mathbf{D}_0$ *without set of BBFs* G.
(vi) h *is homomorphism:* $(D_{3-0}, \oplus, 0, U_3) \longrightarrow (G_3, \oplus, "-", U_3)$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$*, i.e., the normalized plausibility of singletons probabilistic transformation.*

A generalization of the Hájek-Valdés operation "minus" $-$ and of homomorphism f from $\mathbf{D}_0$ to D_{3-0} is still under development.

5 Ideas for Future Research and Open Problems

The presented introductive study opens many interesting problems related to algebraic properties of belief functions on 3-element frame of discernment.

- Elaboration of the properties of D_{3-0} and related substructures required by investigation of conflicting BFs [12]:
  - a generalization of operation $-$ to D_{3-0} analogously to the operation $minus_{2=3}$ from $\mathbf{D}_{1-2=3}$;
  - and related issue: a generalization of the homomorphism f to D_{3-0}.
- The basic study of qBBFs should be supplemented by description of the extension D_{3-0} to D^{+}_{3-0} containing all quasi Bayesian BFs.
- Study of properties of general BFs, i.e. the semigroup $\mathbf{D}_3 = (D_3, \oplus, 0, U_3)$.

6 Conclusion

Dempster's semigroup of belief functions on 3-element frame of discernment was defined. Its substructures related to Bayesian and to quasi Bayesian belief functions were described and analyzed.

[7] o-isomorphic as in the case of $\mathbf{D}_0$ in fact, see Theorem 1. There is no ordering of elements of Ω_3, thus we are not interested in ordering of algebras S_i in this text.

A basis for a solution of the questions coming from research of conflicting belief functions (e.g. an existence of a generalisation of Hájek-Valdés operation "minus") was established.

References

1. Ayoun, A., Smets, P.: Data association in multi-target detection using the transferable belief model. Int. Journal of Intelligent Systems 16, 1167–1182 (2001)
2. Cobb, B.R., Shenoy, P.P.: A Comparison of Methods for Transforming Belief Function Models to Probability Models. In: Nielsen, T.D., Zhang, N.L. (eds.) ECSQARU 2003. LNCS (LNAI), vol. 2711, pp. 255–266. Springer, Heidelberg (2003)
3. Daniel, M.: More on Automorphisms of Dempster's Semigroup. In: Proceedings of the 3rd Workshop in Uncertainty in Expert Systems; WUPES 1994, pp. 54–69. University of Economics, Prague (1994)
4. Daniel, M.: Algebraic Structures Related to Dempster-Shafer Theory. In: Bouchon-Meunier, B., Yager, R.R., Zadeh, L.A. (eds.) IPMU 1994. LNCS, vol. 945, pp. 51–61. Springer, Heidelberg (1995)
5. Daniel, M.: Algebraic Properties of Structures Related to Dempster-Shafer Theory. In: Bouchon-Meunier, B., Yager, R.R., Zadeh, L.A. (eds.) Proceedings IPMU 1996, pp. 441–446. Universidad de Granada, Granada (1996)
6. Daniel, M.: Distribution of Contradictive Belief Masses in Combination of Belief Functions. In: Bouchon-Meunier, B., Yager, R.R., Zadeh, L.A. (eds.) Information, Uncertainty and Fusion, pp. 431–446. Kluwer Acad. Publ., Boston (2000)
7. Daniel, M.: Algebraic Structures Related to the Combination of Belief Functions. Scientiae Mathematicae Japonicae 60, 245–255 (2004); Sci. Math. Jap. Online 10
8. Daniel, M.: Probabilistic Transformations of Belief Functions. In: Godo, L. (ed.) ECSQARU 2005. LNCS (LNAI), vol. 3571, pp. 539–551. Springer, Heidelberg (2005)
9. Daniel, M.: Conflicts within and between Belief Functions. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) IPMU 2010. LNCS (LNAI), vol. 6178, pp. 696–705. Springer, Heidelberg (2010)
10. Daniel, M.: Non-conflicting and Conflicting Parts of Belief Functions. In: Coolen, F., de Cooman, G., Fetz, T., Oberguggenberger, M. (eds.) ISIPTA 2011; Proceedings of the 7th ISIPTA, pp. 149–158. Studia Universitätsverlag, Innsbruck (2011)
11. Daniel, M.: Conflicts of Belief Functions. Technical report V-1108, ICS AS CR, Prague (2011)
12. Daniel, M.: Morphisms of Dempster's Semigroup: A Revision and Interpretation. In: Barták, R. (ed.) Proc. of 14th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty CJS 2011, pp. 26–34. Matfyzpress, Prague (2011)
13. Daniel, M.: Introduction to an Algebra of Belief Functions on Three-element Frame of Discernment — a General Case (in preparation)
14. Haenni, R.: Aggregating Referee Scores: an Algebraic Approach. In: COMSOC 2008, 2nd International Workshop on Computational Social Choice, Liverpool, UK (2008)
15. Hájek, P., Havránek, T., Jiroušek, R.: Uncertain Information Processing in Expert Systems. CRC Press, Boca Raton (1992)
16. Hájek, P., Valdés, J.J.: Generalized algebraic foundations of uncertainty processing in rule-based expert systems (dempsteroids). Computers and Artificial Intelligence 10, 29–42 (1991)

17. Liu, W.: Analysing the degree of conflict among belief functions. Artificial Intelligence 170, 909–924 (2006)
18. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press, New Jersey (1976)
19. Smets, P.: Analyzing the combination of conflicting belief functions. Information Fusion 8, 387–412 (2007)
20. Valdés, J.J.: Algebraic and logical foundations of uncertainty processing in rule-based expert systems of Artificial Intelligence. PhD Thesis, Czechoslovak Academy of Sciences, Prague (1987)