$[r,s,t]$ – Colouring of One Kind of Join Graphs

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Abstract. The concept of [r,s,t]-colourings was introduced by A. Kemnitz and M.Marangio in 2007 as follows: Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and $E(G)$. Given non-negative integers r, s and t, an [r,s,t]-colouring of a graph $G=(V(G),E(G))$ is a mapping c from $V(G)\cup E(G)$ to the colour set $\{0,1,2,\cdots,k-1\}$ such that $|c(v_i)-c(v_j)|\ge r$ for every two adjacent vertices v_i, v_j, $|c(e_i)-c(e_j)|\ge s$ for every two adjacent edges e_i, e_j, and $|c(v_i)-c(e_j)|\ge t$ for all pairs of incident vertices and edges, respectively. The [r,s,t]-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an [r,s,t]-colouring. In this paper, we determine the [r,s,t]-chromatic number for join graphs O_m+C_n.

Keywords: Empty graph, Cycle, Join graphs, [r, s, t]-colouring, [r,s,t]-chromatic number.

1 Introduction

The graphs we shall consider are finite, simple and undirected, unless stated otherwise, we follow the notations and terminologies in [1,2]. Let G be a graph. We denote its vertex set, edge set, minimum degree, maximum degree, order, size by $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$, $p(G)$ and $q(G)$, respectively, and vertex chromatic number, edge chromatic number and total chromatic number by $\chi(G)$, $\chi'(G)$ and $\chi_T(G)$ respectively. A (p,q) graph has order p and size q.

The join $G+H$ [3] of two disjoint graphs G and H is the graph having vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{xy|\ x\in V(G),\ y\in V(H)\}$.

Following the conception of vertex coloring, edge coloring and total coloring, in 2007, A. Kemnitz and M. Marangio[4] put forward the concept of [r,s,t]-coloring of graphs.

Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and $E(G)$. Given non-negative integers r, s and t, an [r,s,t]-colouring of a graph $G=(V(G),E(G))$ is a mapping c from $V(G)\cup E(G)$ to the colour set $\{0,1,2,\cdots,k-1\}$ such that $|c(v_i)-c(v_j)|\ge r$ for every two adjacent vertices v_i, v_j, $|c(e_i)-c(e_j)|\ge s$ for every two adjacent edges e_i, e_j, and $|c(v_i)-c(e_j)|\ge t$ for all pairs of incident vertices and

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edges, respectively. The [r,s,t]-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an [r, s, t]-colouring.

It is a generalization of classical vertex coloring,edge coloring and total coloring of a graph, which has broad significant applications.

In [4], A. Kemnitz and M. Marangio gave the [r,s,t]-chromatic number of general graphs of the boundary and some related properties, also discussed the [r, s, t]-chromatic number of the complete graph of order n. Other results on [r,s,t]-colouring are presented in [5, 6]. In our paper, we determine the [r,s,t]-chromatic number for join graphs $O_m + C_n$.

2 Useful Results

We begin by stating some useful results from the literature. A. Kemnitz and M. Marangio [4] have proved the following properties for $[r, s, t]-$colourings.

Lemma 1. If $H \subseteq G$ then

$$\chi_{r,s,t}(H) \le \chi_{r,s,t}(G)$$

Lemma 2. If $r' \le r, s' \le s, t' \le t$ then

$$\chi_{r',s',t'}(G) \le \chi_{r,s,t}(G)$$

Lemma 3. For the $[r, s, t]-$chromatic number of a graph G, there holds

$$\max\{ r(\chi(G)-1)+1, s(\chi'(G)-1)+1, t+1\} \le \chi_{r,s,t}(G) \le r(\chi(G)-1)+s(\chi'(G)-1)+t+1$$

According to the definition of chromatic number, chromatic index and join graphs, we can get the conclusion easily:

Lemma 4. For the join graphs $O_m + C_n$, there holds

(1) $$\chi(O_m + C_n) = \begin{cases} 3 & n \ is \ even; \\ 4 & n \ is \ odd. \end{cases}$$

(2) $$\chi'(O_m + C_n) = \begin{cases} \Delta + 1, & n = m+1; \\ \Delta, & others. \end{cases}$$

3 The Main Results

Corollary 1. For join graphs $O_m + C_n$, it holds

(1) If n is even and $n \ne m+1$, then

$$\max\{2r+1, s(\Delta-1)+1, t+1\} \le \chi_{r,s,t}(O_m + C_n) \le 2r + s(\Delta-1)+t+1;$$

(2) If n is even and $n = m+1$, then

$$\max\{ 2r+1, s\Delta+1, t+1 \} \le \chi_{r,s,t}(O_m + C_n) \le 2r + s\Delta + t + 1;$$

(3) If n is odd and $n \ne m+1$, then

$$\max\{ 3r+1, s(\Delta-1)+1, t+1 \} \le \chi_{r,s,t}(O_m + C_n) \le 3r + s(\Delta-1)+t+1;$$

(4) If n is odd and $n = m+1$, then

$$\max\{ 3r+1, s\Delta+1, t+1 \} \le \chi_{r,s,t}(O_m + C_n) \le 3r + s\Delta + t + 1.$$

Proof. By Lemma 3 and Lemma 4, the conclusion is obviously.

Theorem 1. Let O_m+C_n be a join graphs, n is even, if one of the following conditions holds, then, $\chi_{r,s,t}(O_m+C_n)=2r+1$.

(1) $n\geq m+2$, and $r\geq(n-1)s+2t$;

(2) $n=m+1$, and $r\geq(n+1)s+2t$;

(3) $n\leq m$, and $r\geq(m+1)s+2t$.

Proof. Let O_m is an empty graph and its vertex set $V(O_m)=\{v_1,v_2,\cdots,v_m\}$, cycle C_n vertex set $V(C_n)=\{u_1,u_2,\cdots,u_n\}$, edge set $E(C_n)=\{u_iu_{i+1}\mid i=1,2,\cdots,n; \textit{if } i+1>n \textit{ then } (i+1)\bmod n \textit{ take place it}\}$.

(1) If $n\geq m+2$, then $\Delta=\Delta(O_m+C_n)=\max\{n,m+2\}=n$, it holds $\chi'(O_m+C_n)=\Delta=n$ by Lemma 4. Let c be a coloring of O_m+C_n as follows:

Colour the vertices of join O_m+C_n with the $\chi(G)=3$ colours $0, r, 2r$ as follows,

$$c(v_i)=2r,\quad i=1,2,\cdots,m;$$
$$c(u_i)=\begin{cases}0, & i\equiv 1\bmod 2\\ r, & i\equiv 0\bmod 2\end{cases},\quad i=1,2,\cdots,n$$

and colour the edges of join O_m+C_n with the $\chi'(O_m+C_n)=\Delta=n$ colours to obtain a proper $[r,s,t]-$colouring of O_m+C_n, hence $\chi_{r,s,t}(O_m+C_n)\leq 2r+1$, by Corollary 1 (1) we have $\chi_{r,s,t}(O_m+C_n)\geq 2r+1$, then $\chi_{r,s,t}(O_m+C_n)=2r+1$.

(2) If $n=m+1$, then $\Delta=\Delta(O_m+C_n)=\max\{n,m+2\}=m+2=n+1$, it holds $\chi'(O_m+C_n)=\Delta+1=n+2$ by Lemma 4. Let c be a coloring of O_m+C_n as follows:

Firstly, colour the vertices of O_m+C_n with three colours $0, r, 2r$ as follows:

$$c(v_i)=2r,\quad i=1,2,\cdots,m;$$
$$c(u_i)=\begin{cases}0, & i\equiv 1\bmod 2\\ r, & i\equiv 0\bmod 2\end{cases},\quad i=1,2,\cdots,n$$

Secondly, colour the edges of O_m+C_n with the $n+2$ colours $t, t+s, t+2s,\cdots,t+ns, t+(n+1)s$ as follows,

$$c(v_iu_j)=t+[(i+j-2)\bmod n]s,\quad i=1,2,\cdots,m;\quad j=1,2,\cdots,n.$$

$c(u_iu_{i+1})=\begin{cases}t+ns & i\equiv 1\bmod 2\\ t+(n+1)s & i\equiv 0\bmod 2\end{cases}$, $i=1,2,\cdots,n$, if suffix $i+1>n$ then $(i+1)\bmod n$ takes place $i+1$.

Since $\chi'(O_m+C_n)=\Delta=n$, hence the edge colouring c is proper, so we abtain a proper $[r,s,t]-$colouring of O_m+C_n, hence $\chi_{r,s,t}(O_m+C_n)\leq 2r+1$. At the same time, it holds $\chi_{r,s,t}(O_m+C_n)\geq 2r+1$ by Corollary 1 (2), therefore $\chi_{r,s,t}(O_m+C_n)=2r+1$.

(3) If n is a even and $n \le m$, then $\Delta = \Delta(O_m + C_n) = \max\{n, m+2\} = m+2$, it holds $\chi'(O_m + C_n) = \Delta = m+2$ by Lemma 4. Let c be a coloring of $O_m + C_n$ as follows:

Firstly, colour the vertices of $O_m + C_n$ with three colours $0, r, 2r$ as follows:

$$c(v_i) = 2r, \quad i = 1,2,\cdots,m;$$
$$c(u_i) = \begin{cases} 0, & i \equiv 1 \bmod 2 \\ r, & i \equiv 0 \bmod 2 \end{cases}, \quad i = 1, 2, \cdots, n.$$

Secondly, colour the edges of $O_m + C_n$ with the $m+2$ colours $t, t+s, t+2s, \cdots, t+(m+1)s$, since $\chi'(O_m + C_n) = \Delta = m+2$, hence the edge colouring c is proper, then obtain a proper $[r,s,t]-$colouring of $O_m + C_n$, hence $\chi_{r,s,t}(O_m + C_n) \le 2r+1$. At the same time, it holds $\chi_{r,s,t}(O_m + C_n) \ge 2r+1$ by Corollary 1 (1), therefore $\chi_{r,s,t}(O_m + C_n) = 2r+1$.

Theorem 2. Let $O_m + C_n$ be a join graphs, n is odd, if one of the following conditions holds, then $\chi_{r,s,t}(O_m + C_n) = 3r+1$.

(1) $n \ge m+2$ and $r \ge (n-1)s + 2t$;

(2) $n = m+1$ and $r \ge (n+1)s + 2t$;

(3) $n \le m$ and $r \ge (m+1)s + 2t$.

Proof. Let empty graph O_m vertex set $V(O_m) = \{v_1, v_2, \cdots, v_m\}$, cycle C_n vertex set $V(C_n) = \{u_1, u_2, \cdots, u_n\}$, edge set $E(C_n) = \{u_i u_{i+1} \mid i = 1, 2, \cdots, n; \textit{if } i+1 > n \textit{ then } (i+1) \bmod n \textit{ take place it}\}$. If n is odd, by Lemma 4 we have $\chi(O_m + C_n) = 4$.

(1) If $n \ge m+2$, then $\Delta = \Delta(O_m + C_n) = \max\{n, m+2\} = n$, by Lemma 4 we have $\chi'(O_m + C_n) = \Delta = n$. Let c be a coloring of $O_m + C_n$ as follows:

Firstly, colour the vertices of $O_m + C_n$ with four colours $0, r, 2r, 3r$ as follows:

$$c(v_i) = 3r, \quad i = 1,2,\cdots,m;$$
$$c(u_i) = \begin{cases} 0, & i \equiv 1 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ r, & i \equiv 0 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ 2r, & i = n \end{cases}$$

Secondly, colour the edges of $O_m + C_n$ with the n colours $t, t+s, t+2s, \cdots, t+(n-1)s$, since $\chi'(O_m + C_n) = \Delta = n$ to obtain a proper $[r,s,t]-$ colouring of $O_m + C_n$, hence $\chi_{r,s,t}(O_m + C_n) \le 2r+1$. And it holds $\chi_{r,s,t}(O_m + C_n) \ge 3r+1$ by Corollary 1 (3), then $\chi_{r,s,t}(O_m + C_n) = 3r+1$.

(2) If $n = m+1$ and n is odd, then $\Delta = \Delta(O_m + C_n) = \max\{n, m+2\} = m+2 = n+1$, by Lemma 4 we have $\chi'(O_m + C_n) = \Delta + 1 = n+2$. Let c be a coloring of $O_m + C_n$ as follows:

Firstly, colour the vertices of $O_m + C_n$ with four colours $0, r, 2r, 3r$ as follows:

$$c(v_i) = 3r, \quad i = 1,2,\cdots,m;$$
$$c(u_i) = \begin{cases} 0, & i \equiv 1 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ r, & i \equiv 0 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ 2r, & i = n \end{cases}$$

Secondly, colour the edges of $O_m + C_n$ with the $n+2$ colours $t, t+s, t+2s, \cdots, t+ns, t+(n+1)s$ to obtain a proper $[r,s,t]-$colouring of $O_m + C_n$, hence $\chi_{r,s,t}(O_m + C_n) \le 3r+1$, by Corollary 1 (4) we have $\chi_{r,s,t}(O_m + C_n) \ge 3r+1$, then $\chi_{r,s,t}(O_m + C_n) = 2r+1$.

(3) If n is odd and $n \le m$, then $\Delta = \Delta(O_m + C_n) = \max\{n, m+2\} = m+2$, by Lemma 4 we have $\chi'(O_m + C_n) = \Delta = m+2$, Let c be a coloring of $O_m + C_n$ as follows:

Firstly, to colour the vertices of $O_m + C_n$ with four colours $0, r, 2r, 3r$ as follows:

$$c(v_i) = 3r, \quad i = 1,2,\cdots,m;$$
$$c(u_i) = \begin{cases} 0, & i \equiv 1 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ r, & i \equiv 0 \bmod 2, \quad i = 1,2,\cdots,n-1, \\ 2r, & i = n \end{cases}$$

Secondly, to colour the edges of $O_m + C_n$ with the $m+2$ colours $t, t+s, t+2s, \cdots, t+(m+1)s$, since $\chi'(O_m + C_n) = \Delta = m+2$, hence the edge colouring c is proper, and $r-(t+(m+1)s) \ge t$, so c is a proper $[r,s,t]-$colouring, hence $\chi_{r,s,t}(O_m + C_n) \le 3r+1$, by Corollary 1 (3) we have $\chi_{r,s,t}(O_m + C_n) \ge 3r+1$, then $\chi_{r,s,t}(O_m + C_n) = 3r+1$.

Theorem 3. If $n(\ge 3)$ is even and $s \ge \max\{r, 2t\}$, then

$$\chi_{r,s,t}(O_m + C_n) = (\chi'(O_m + C_n) - 1)s + 1 \ .$$

Proof. If $n(\ge 3)$ is even, by Lemma 4 we have $\chi(O_m + C_n) = 3$, and C_n is a even cycle, then $n \ge 4$, hence

$$\chi'(O_m + C_n) - 1 \ge \Delta - 1 = \max\{n, m+2\} - 1 \ge 4 - 1 = 3$$

Let c be a coloring of $O_m + C_n$ as follows:

Firstly, colour the edges of $O_m + C_n$ with the $\chi'(O_m + C_n)$ colours $0, s, 2s, \cdots, (\chi'(O_m + C_n) - 1)s$, we can make c is proper edge colouring.

Secondly, to colour the vertices of $O_m + C_n$ with three colours $t, t+s, t+2s$ as follows:

$$c(v_i) = t + 2s, \quad i = 1,2,\cdots,m;$$

$$c(u_i) = \begin{cases} t, & i \equiv 1 \bmod 2 \\ t+s, & i \equiv 0 \bmod 2 \end{cases}, \quad i = 1,2,\cdots,n.$$

Since $s \geq \max\{r, 2t\}$ we have $(t+s) - t = s \geq r$, $(t+2s) - t = 2s \geq r$, $(t+2s) - (t+s) = s \geq r$, $|ks - t| \geq t$, $|ks - (t+s)| \geq t$, $|ks - (t+ss)| \geq t$, which $k = 0, 1, 2, \cdots, \chi'(O_m + C_n)$, so c is a proper $[r,s,t]-$ colouring, hence $\chi_{r,s,t}(O_m + C_n) \leq (\chi'(O_m + C_n) - 1)s + 1$, by Lemma 3 we have $\chi_{r,s,t}(O_m + C_n) \geq (\chi'(O_m + C_n) - 1)s + 1$, then $\chi_{r,s,t}(O_m + C_n) = (\chi'(O_m + C_n) - 1)s + 1$.

Theorem 4. Let $O_m + C_n$ be a join graphs, if $s \geq r + 2t$, then

$$\chi_{r,s,t}(O_m + C_n) = (\chi'(O_m + C_n) - 1)s + 1 \cdot$$

Proof. Let c be a coloring of $O_m + C_n$ as follows:

Firstly, to colour the edges of $O_m + C_n$ with the $\chi'(O_m + C_n)$ colours $0, s, 2s, \cdots, (\chi'(O_m + C_n) - 1)s$, we can make c is proper edge colouring.

Secondly, to colour the vertices of $O_m + C_n$ with the following 4 colours selected as follows.

In the interval $(0, s)$ is to select $t, r+t$, and in the interval $(s, 2s)$ is to select $\max\{s+t, 2r+t\}$, $\max\{s+r+t, 3r+t\}$ these 4 colours, by Lemma 4 we have $\chi(O_m + C_n) \leq 4$, so we can make c is proper vertices colouring, c is a proper $[r,s,t]-$colouring, hence $\chi_{r,s,t}(O_m + C_n) \leq (\chi'(O_m + C_n) - 1)s + 1$, by Lemma 3 we have $\chi_{r,s,t}(O_m + C_n) \geq (\chi'(O_m + C_n) - 1)s + 1$, hence

$$\chi_{r,s,t}(O_m + C_n) = (\chi'(O_m + C_n) - 1)s + 1 \cdot$$

4 Summaries

In this paper, we give the exact values of the [r, s, t]- chromatic number of the joint graph $O_m + C_n$ which depends on the parities of m and n, if r, s, t meet certain conditions.

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References

[1] Bondy, J.A., Murty, U.S.R.: Graph theory. Springer, Berlin (2008)
[2] Chartrand, G., Lesniak, L.: Graphs and digraphs. Greg Hubit Bookworks, California (1979)
[3] Yap, H.P.: Total coloring of graphs. Lecture Notes in Mathematics, vol. 1623, pp. 1–11. Springer, Heidlberg (1996)
[4] Kemnitz, A., Marangio, M.: [r, s, t]- colorings of graphs. Discrete Math. 307, 199–207 (2007)
[5] Dekar, L., et al.: [r, s, t]-colorings of trees and bipartite graphs. Discrete Math. 310, 260–269 (2010)
[6] Xu, C., Ma, X., Hua, S.: [r, s, t]-coloring of $K_{n, n}$. J. Appl. Math. Comput. 31, 45–50 (2009)