A Certain Asymptotically Strict Pseudocontractive Mappings in the Intermediate Sense Semigroup

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Abstract. It is studied that the asymptotically kn-strict pseudocontractive mapping in the intermediate sense and proved the modified Mann iteration process with errors. Then the iteration converges strongly to a common fixed point p which is the nearest point to u in F. The results in this paper presented extend and improve the corresponding results of many authors.

Keywords: Asymptotically kn-strict pseudocontractive mapping in the intermediate sense, Mann iteration process with errors, Semigroup, Lipschitzian.

1 Introduction and Preliminaries

Throughout the paper, H be a real Hilbert space and R^+ denote the set of positive real number, with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$, respectively. C a closed convex subset of H. Let T: $C \to C$ be a mapping, we use $F(T)$ to denote the set of fixed point of the mapping T.

Definition 1.1 (1) The one-parameter family $\Im := \{T(t) : t \geq 0\}$ of mappings from C into itself is called a nonexpansive semigroup if the following conditions are satisfied:

(a) $T(0)x = x$ for each $x \in C$;

(b) $T(t+s)x = T(t)T(s)x$ for any $x \in C$ and $t, s \in R^+$;

(c) for any $x \in C$, the mapping $t \to T(t)x$ is continuous;

(d) for any $x, y \in C$, there exists $j(x+y) \in J(x+y)$ such that

$$\|T(t)x - T(t)y\| \leq \|x - y\| \tag{1.1}$$

for any $t \geq 0$.

(2) The one-parameter family $\Im$ of mappings from C into itself is called an asymptotically k_n – strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ semigroup and the conditions (a)-(c) and the following conditions (e) are satisfied:

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(e) there exists a $\{k_n\} \subset [0,1)$ with $\lim_{n\to\infty} \sup k_n < 1$, $\{\gamma_n\}$, $\{c_n\} \subset [0,1)$ with $\{k_n\} \subset [0,1)$ $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} c_n = 0$ such that

$$\left\|T^n(t)x - T^n(t)y\right\|^2 \le (1+\gamma_n)\|x-y\|^2 + k_n\left\|x - T^n(t)x - \left(y - T^n(t)y\right)\right\| + c_n \tag{1.2}$$

for all $x, y \in C$, $t \ge 0$ and $n \in N$.

Remark 1.2([3]). Let C be a nonempty subset of a Hilbert space H , $T: C \to C$ asymptotically k_n – strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then

$$\left\|T^n(t)x - T^n(t)y\right\| \le \frac{1}{1-k_n}\left(k_n\|x-y\| + \sqrt{(1+(1-k_n)\gamma_n)\|x-y\|^2 + (1-k_n)c_n}\right)$$

for all $x, y \in C$ and $n \in N$.

The convergence problems of the iterative sequences for nonexpansive semigroup to a common fixed point have been considered by some authors in the settings of Hilbert spaces [4, 7].

In 2009, Zhang, Yang, Li and Chen[7] introduce and study the strong theorem of nonexpansive semigroup the following iteration in Hilbert spaces:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \ge 0. \end{cases}$$

Recently, Zeng [7] study the strong convergence problem of the Ishikawa Iterative of error correction for the Lipschitzian and asymptotically pseudocontractive mappings in arbitrary real Banach space. Zhang [12], by using different methods, introduce and study the weak or strong convergence in general Banach spaces.

On the other hand, the class of asymptotically mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich[6] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu[2] and many others. Recently, D. R. Sahu, Hong-Kun Xu, Jen-Chih Yao [1] introduced the concept of asymptotically k – strict pseudocontractive mapping in the intermediate sense, Wu [3] study the mapping with k_n to extend the mapping.

Inspired by the authors above, the purpose of this paper is to introduce and study the strong converges problem of the following modified Mann iteration process with errors:

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n(t_m)x_n + \varepsilon_n, \quad \forall n \in \mathrm{N} \end{cases}$$

The following Lemmas are important to prove results in this paper.

Lemma 1.3([9]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \qquad n \ge 0.$$

Where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ or $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4([5]). Let $\{a_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality

$$a_{n+1} \le \beta_n a_n + \gamma_n, \qquad \forall n \in N.$$

If $\beta_n \ge 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.5([3]). Let C be a nonempty subset of a Hilbert space H , $T: C \to C$ a uniformly continuous asymptotically k_n – strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a bounded sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_{n+1}\| \to 0 \quad (n \to \infty)$.

Then $\|x_n - T^n x_{n+1}\| \to 0$ *as* $n \to \infty$.

Lemma 1.6([7]). Let E be a arbitrary real Banach space, E^* is the dual space of E , $J: E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|x\| = \|f\| \right\}.$$

For all $x \in E$, we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Especially, if E be a Hilbert space, then $J = I$. Thus we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$$

for all $x, y \in E$.

Lemma 1.7([7]). Let H be a real Hilbert space, C a closed convex subset of H , for any $x \in H$ and $u \in C$, we have

(1) $\langle z - P_c x, P_c x - x\rangle \ge 0, \quad \forall z \in C$;

(2) $\langle z - u, u - x\rangle \ge 0, \quad \forall z \in C$, then $u = P_C x$,

where $P_c x$ is the nearest projection from H to C .

2 Main Result

Now, we give our main results in this paper.

Theorem 2.1. Let C be a nonempty closed convex subset of a Hilbert space H, $T:C\to C$ a continuous asymptotically k_n – strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $F(T)$ is closed and convex and $I-T$ is demiclosed at zero.

The proof can be found in [3].

By using Theorem 2.1, we have the following:

Theorem 2.2. Let C be a nonempty closed convex subset of a Hilbert space H, $T\in\Im$ of mapping from into C itself be a Lipschitzian and uniformly continuous asymptotically k_n – strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ semi-group. If $F(T(t))\neq\varnothing$, u is a given point in C.

Assume that $u_n\subset C$, $\{\alpha_n\}\subset(1/2,1)$ and $\{t_m\}$ is an increasing sequence in $[0,\infty)$. $\{x_n\}$ be defined:

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n(t_m)x_n+\varepsilon_n,\quad \forall n\in \mathrm{N}$$

If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\|\varepsilon_n\|<\infty$ and $\sum_{n=1}^{\infty}\alpha_n c_n<\infty$;

(ii) $\lim\limits_{m\to\infty}\sup\limits_{x\in C,s\in R^+}\|T(s)T(t_m)x-T(t_m)x\|=0.$

Then $\{x_n\}$ converges strongly to a common fixed point p which is the nearest point to u in F.

Proof: (1) Firstly, we prove that $\lim_{n\to\infty}\|x_n-p\|$ exists.

For any given $p\in F$, we have

$$\|x_{n+1}-p\|^2\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n c_n+2\left\langle(1-\alpha_n)(x_n-p)+\alpha_n\left(T^n(t_m)x_n-p\right),\varepsilon_n\right\rangle$$

$$+\alpha_n\left[(1+\gamma_n)\|x_n-p\|^2+k_n\|x_n-T^n(t_m)x_n\|^2\right]-\alpha_n(1-\alpha_n)\|T^n(t_m)x_n-x_n\|^2+\|\varepsilon_n\|^2. \tag{2.1}$$

And

$$2\left\langle(1-\alpha_n)(x_n-p)+\alpha_n\left(T^n(t_m)x_n-p\right),\varepsilon_n\right\rangle\le\left\|(1-\alpha_n)(x_n-p)+\alpha_n\left(T^n(t_m)x_n-p\right)\right\|^2+\|\varepsilon_n\|^2$$

$$\le(1+\gamma_n\alpha_n)\|x_n-p\|^2+\alpha_n c_n-\alpha_n(1-k_n-\alpha_n)\|T^n(t_m)x_n-x_n\|^2+\|\varepsilon_n\|^2. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\|x_{n+1}-p\|^2\le 2(1+\gamma_n)\|x_n-p\|^2+2\left(\alpha_n c_n+\|\varepsilon_n\|^2\right). \tag{2.3}$$

By condition (i) and Lemma 1.4, we know $\sum_{n=1}^{\infty}\|\varepsilon_n\|<\infty$, $\sum_{n=1}^{\infty}\alpha_n c_n<\infty$.

Thus $\lim_{n\to\infty}\|x_n-p\|$ exists, so $\{x_n\}$ be bounded.

(2) Next we show

$$\lim_{n\to\infty,m\to\infty}\left\|T\left(t_m\right)x_n-x_n\right\|=0.$$

From Remark 1.2, we easy to know

$$\left\|T^n\left(t_m\right)x_n-p\right\|\le\frac{1}{1-k_n}\left(k_n\left\|x_n-p\right\|+\sqrt{\left(1+\left(1-k_n\right)\gamma_n\right)\left\|x-y\right\|^2+\left(1-k_n\right)c_n}\right)\le \mathsf{M}$$

for some certain M.

By (2.1), we have

$$\begin{aligned}&\left\|x_{n+1}-p\right\|^2\le\left(1+\gamma_n\alpha_n\right)\left\|x_n-p\right\|^2+2\left\|\varepsilon_n\right\|\left[\left(1-\alpha_n\right)\left\|x_n-p\right\|+\alpha_n\left\|T^n\left(t_m\right)x_n-p\right\|\right]\\&-\left\|x_{n+1}-p\right\|^2+\alpha_nc_n+\left\|\varepsilon_n\right\|^2.\end{aligned}$$

Since $\left\{\left\|T^n\left(t_m\right)x_n-p\right\|\right\}$ be bounded and $\lim_{n\to\infty}\left\|x_{n+1}-p\right\|=\lim_{n\to\infty}\left\|x_n-p\right\|$ exists, and $\lim_{n\to\infty}\left\|\varepsilon_n\right\|=\lim_{n\to\infty}\gamma_n=0$.

We observe that

$$\lim_{n\to\infty,m\to\infty}\left\|T^n\left(t_m\right)x_n-x_n\right\|=0. \tag{2.4}$$

This implies that

$$\begin{aligned}&\left\|x_{n+1}-x_n\right\|=\left\|\left(1-\alpha_n\right)x_n+\alpha_nT^n\left(t_m\right)x_n+\varepsilon_n-x_n\right\|=\left\|\alpha_n\left(T^n\left(t_m\right)x_n-x_n\right)+\varepsilon_n\right\|\\&\le\alpha_n\left\|T^n\left(t_m\right)x_n-x_n\right\|+\left\|\varepsilon_n\right\|\\&\quad\to 0\end{aligned} \tag{2.5}$$

From (2.4), (2.5) and Lemma 1.5, we know that T is uniformly continuous, and by Theorem 2.1. Thus, we obtain that

$$\lim_{n\to\infty,m\to\infty}\left\|T\left(t_m\right)x_n-x_n\right\|=0.$$

(3) We prove that $\lim_{n\to\infty}\left\|T(t)x_n-x_n\right\|=0$, for all $t\ge 0$.

In fact, it follows from the condition (ii) and step (2) that, for any $t\ge 0$,

$$\begin{aligned}&\left\|T\left(t\right)x_{n_k}-x_{n_k}\right\|\le(1+L)\left\|x_{n_k}-T\left(t_{n_k}\right)x_{n_k}\right\|+\left\|T\left(s+t_{n_k}\right)x_{n_k}-T\left(t_{n_k}\right)x_{n_k}\right\|\\&\le(1+L)\left\|x_{n_k}-T\left(t_{n_k}\right)x_{n_k}\right\|\\&+\sup_{z\in\{x_n\},s\in R^+}\left\|T\left(s+t_{n_k}\right)z-T\left(t_{n_k}\right)z\right\|\to 0\quad (as\quad n_k\to\infty).\end{aligned} \tag{2.6}$$

Since $x_{n_k}\to p$ as $n_k\to\infty$ and $\lim_{n\to\infty}\left\|x_n-p\right\|$ exists, which implies that $x_n\to p\in F$ as $n\to\infty$, where p is any given point in F .

(4) Now we prove

$$\limsup_{n\to\infty}\left\langle x_n-P_Fu,u-P_Fu\right\rangle\le 0. \tag{2.7}$$

With the boundedness of $\left\{x_n\right\}$ and $\left\{T\left(t\right)x_n\right\}$, so $\left\langle T\left(t\right)x_n-P_Fu,u-P_Fu\right\rangle$ bounded.

Then we know $\limsup_{n\to\infty}\{\langle x_n - P_F u, u - P_F u\rangle\}$ exists. There exists a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$, and for a certain $q \in C$,

$$T(t)x_{n_j} \to q. \tag{2.8}$$

From step (3), we know that

$$q \in F := \bigcap_{t\geq 0} F\left(T(t)\right).$$

By Lemma 1.7 and (2.8)

$$\begin{aligned}&\limsup_{n\to\infty}\left\langle T(t)x_n - P_F u, u - P_F u\right\rangle = \lim_{j\to\infty}\left\langle T(t)x_{n_j} - P_F u, u - P_F u\right\rangle \\ &= \left\langle q - P_F u, u - P_F u\right\rangle \leq 0. \qquad (2.9)\end{aligned}$$

And from step (3), we have

$$\limsup_{n\to\infty}\left\langle x_n - P_F u, u - P_F u\right\rangle \leq 0. \tag{2.10}$$

(5) Finally, we prove $x_n \to P_F u \in F := \bigcap_{t\in R^+} F\left(T(t)\right)$, where $P_F u$ is the fixed point which is nearest to u in $F\left(T(t)\right)$.

In fact, by Lemma 1.6, we have

$$\begin{aligned}&\left\|x_{n+1} - P_F u\right\|^2 \leq \left(1-\alpha_n\right)^2 \left\|x_n - P_F u\right\|^2 + 2\alpha_n \left\langle T^n\left(t_m\right)x_n - P_F u, x_{n+1} - P_F u\right\rangle \\ &+2\left\langle \varepsilon_n, x_{n+1} - P_F u\right\rangle\end{aligned}$$

$$\begin{aligned}&=\left(1-\alpha_n\right)^2\left\|x_n - P_F u\right\|^2 + 2\alpha_n\left[\left\langle T^n\left(t_m\right)x_n - p, x_{n+1} - P_F u\right\rangle + \left\langle p - x_{n+1}, x_{n+1} - P_F u\right\rangle + \left\langle x_{n+1} - P_F u, x_{n+1} - P_F u\right\rangle\right] \\ &\qquad +2\left\langle \varepsilon_n, x_{n+1} - P_F u\right\rangle.\end{aligned}$$

Simplify the formula above

$$\begin{aligned}&\left\|x_{n+1} - P_F u\right\|^2 \leq \left(1 - \frac{\alpha_n^{\ 2}}{2\alpha_n - 1}\right)\left\|x_n - P_F u\right\|^2 \\ &+\frac{2\alpha_n}{1-2\alpha_n}\left[\left\langle T^n\left(t_m\right)x_n - x_n, x_{n+1} - P_F u\right\rangle + \left\langle x_n - x_{n+1}, x_{n+1} - P_F u\right\rangle\right] + \frac{2}{1-2\alpha_n}\left\langle \varepsilon_n, x_{n+1} - P_F u\right\rangle. (2.11)\end{aligned}$$

Let $a_{n+1} = \left\|x_{n+1} - P_F u\right\|^2$, $\gamma_n = \alpha_n^{\ 2}/2\alpha_n - 1$, $\delta_n = \frac{2\alpha_n}{1-2\alpha_n}\left[\left\langle T^n\left(t_m\right)x_n - x_n, x_{n+1} - P_F u\right\rangle + \left\langle x_n - x_{n+1}, x_{n+1} - P_F u\right\rangle\right] + \frac{2}{1-2\alpha_n}\left\langle \varepsilon_n, x_{n+1} - P_F u\right\rangle$.

By Lemma 1.3, we can obtain $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R.

Also, one hand, we can easy to find that $\{\gamma_n\}$ satisfy the condition $\sum_{n=1}^{\infty}\gamma_n = \infty$.

On the other hand, from (2.4), (2.5) and condition (i), we have

$$\limsup_{n\to\infty} \frac{2\alpha_n[\langle T^n(t_m)x_n - x_n, x_{n+1} - P_{Fu}\rangle + \langle x_n - x_{n+1}, x_{n+1} - P_{Fu}\rangle] + 2\langle \varepsilon_n, x_{n+1} - P_{Fu}\rangle}{-\alpha_n{}^2} \le 0.$$

So we obtain $\|x_n - P_F u\| \to 0$, namely

$$x_n \to P_F u \in F := \bigcap_{t\in R^+} F(T(t)).$$

This completes the proof.

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