Lyapunov Exponents for Discrete Time-Varying Systems

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Abstract. The purpose of this paper is to present some results on the effects of parametric perturbations on the Lyapunov exponents of discrete time-varying linear systems. We fix our attention on the greatest and smallest exponents.

Keywords: difference equation, stability, time-varying systems, Lyapunov exponents, linearization.

1 Introduction

In the laast decade there has been great interest of researchers from the theory of linear models in systems which combine logical switches and differential or difference equations. This interest is dictated first and foremost of the great utility of such models for modeling real-world objects. The usefulness of this growing demand depends on the methods of modeling, analysis and understanding of this structure. Although construction of a hybrid model is a relatively simple task, its analysis is already far from simplicity. During the analysis of hybrid systems many interesting and difficult mathematical problems arise. Many of them are associated with the dynamics, and in particular the stability of such models remains unsolved today.

Many properties of dynamic systems can be successfully characterized by certain numbers called numerical characteristics or characteristic exponents. These include: Lyapunov, Bohl, Perron, Izobov, Grobman exponents, generalized spectral radiuses. These numbers describe the different types of stability, growth of trajectories or sensitivity on parametric disturbances.

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Since the works of A. M. Lyapunov [40] and O. Perron [49, 50] the theory of Lyapunov exponents became the subject of intense research, as evidenced by the huge number of papers published on this subject.

This paper is devoted to the influence of parametric perturbations on the Lyapunov exponents of discrete linear systems with time varying coefficients. Different types of perturbations are considered, limited in terms of certain norms, tending to zero in a specified rate. For each of them we describe their impact on the value of Lyapunov exponents.

There are several monographs devoted in whole or in large part to the Lyapunov exponents, e.g. [4, 6, 12, 32, 41, 46]. But only in the last two a problem of parametric perturbations is discussed. Both of these items deals with continuous time systems, and in addition, they are available only in Russian.

2 Definitions of Characteristic Exponents and Basic Properties

In this section we will introduce the notation and basic properties of Lyapunov exponents of linear discrete time-varying systems:

$$x(n+1) = A(n)x(n), n \geq 0, \tag{1}$$

where $(A(n))_{n\in\mathbb{N}}$ is a bounded sequence invertible of s-by-s real matrices such that $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded.

The transition matrix of (1) is defined as

$$\mathscr{A}(m,k) = A(m-1)...A(k)$$

for $m > k$ and $\mathscr{A}(m,m) = I$, where I is the identity matrix. For a initial condition $x(0) = x_0 \in R^s$ the solution of (1) is denoted by $x(n,x_0)$, so

$$x(n,x_0) = \mathscr{A}(n,0)x_0.$$

Definition 1. Let $b = (b(n))_{n\in\mathbb{N}}$ be a sequence of real numbers. The number (or the symbol $\pm\infty$) defined as

$$\lambda(b) = \limsup_{n\to\infty} \frac{1}{n} \ln|b(n)|$$

is called the upper characteristic exponent or simply characteristic exponent of sequence $(b(n))_{n\in\mathbb{N}}$. For a sequence $v = (v(n))_{n\in\mathbb{N}}$ of vectors of normed space $(X, \|*\|)$ we define its characteristic exponent $\lambda(v)$ as a exponent of sequence $(\|v(n)\|)_{n\in\mathbb{N}}$.

It is easy to check that finite $\lambda(b)$ is a characteristic exponent of sequence of $b = (b(n))_{n\in\mathbb{N}}$ if, and only if, the following two conditions are simultaneously satisfied:

1. For any $\varepsilon > 0$ there exists constant D_ε such that for all $n \in \mathbb{N}$ the following inequality is satisfied

$$|b(n)| \leq D_\varepsilon \exp((\lambda(b)+\varepsilon)n);$$

2. For any $\varepsilon > 0$ the following equality is satisfied

$$\limsup_{n\to\infty} |b(n)| \exp((-\lambda(b)+\varepsilon)n) = \infty.$$

Fix an arbitrary norm $\|\cdot\|$ in R^s and denote induced operator norm by the same symbol. Denote

$$\sup_{n\in\mathbb{N}} \|A(n)\| = a, \;\; \sup_{n\in\mathbb{N}} \left\|A^{-1}(n)\right\| = a'. \tag{2}$$

Definition 2. For $x_0 \in R^s$, $x_0 \neq 0$ the Lyapunov exponent $\lambda_A(x_0)$ of (1) is defined as characteristic exponent of $(x(n,x_0))_{n\in\mathbb{N}}$ that is

$$\lambda_A(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n,x_0)\|.$$

We also define $\lambda_A(0) = -\infty$.

Observe that by the equivalence of all norms in R^s the above definition does not depend on the particular choice of the norm. The next Theorem contains some basic properties of Lyapunov exponents.

Theorem 1. *For the Lyapunov exponents of (1) the following properties hold:*

1. *if $x_0 \in R^s$ and $c \in R$, $c \neq 0$ then $\lambda_A(x_0) = \lambda_A(cx_0)$;*
2. *if $x_1, x_2 \in R^s$ then $\lambda_A(x_1+x_2) \leq \max\{\lambda_A(x_1), \lambda_A(x_2)\}$;*
3. *if $x_1, x_2 \in R^s$ and $\lambda_A(x_1) \neq \lambda_A(x_2)$ then*

$$\lambda_A(x_1+x_2) = \max\{\lambda_A(x_1), \lambda_A(x_2)\};$$

4. *if $x_1, ..., x_l \in R^s \setminus \{0\}$ and the numbers $\lambda_A(x_1), ..., \lambda_A(x_l)$ are distinct, then the vectors $x_1, ..., x_l$ are linearly independent;*
5. *if $x_1, ..., x_s$ is a basis or R^s then*

$$\limsup_{n\to\infty} \frac{1}{n} \ln|\det \mathscr{A}(n,0)| \leq \sum_{l=1}^{s} \lambda_A(x_l); \tag{3}$$

6. *if $x_0 \in R^s$ then $\lambda_A(x_0) \leq a$;*
7. *if $x_0 \in R^s$ then $\lambda_A(s) \leq \lambda_A(x_0)$, where $v = (v(n))_{n\in\mathbb{N}}$ is given by*

$$v(n) = \begin{cases} \sum_{l=0}^{n-1} x(l,x_0) \;\, if \;\, \lambda(x_0) \geq 0 \\ \sum_{l=n}^{\infty} x(l,x_0) \;\, if \;\, \lambda(x_0) < 0 \end{cases}.$$

The proof of points 1-4 is given in [6], Theorem 2.1, inequality (3), which is called Lyapunov inequality was shown in [22], point 6 is a obvious consequence of (2) and the definition of $\lambda_A(x_0)$, point 7 is proved in [16], Lemma 4. As a consequence of point 4 we see that the set $\{\lambda_A(x_0) : x_0 \in R^s \setminus \{0\}\}$ contains at most s elements, say $-\infty \leq \lambda_1(A) < \lambda_2(A) < ... < \lambda_r(A) < \infty$, and the set $\{\lambda_1, \lambda_2, ..., \lambda_r\}$ will be called the spectrum of (1). The greatest and the smallest exponent we will denote by $\lambda_g(A)$ and $\lambda_s(A)$, respectively. An immediate consequence of definition of operator norm is the following result, which express the greatest exponent just in terms of the matrices $(A(n))_{n\in\mathbb{N}}$.

Theorem 2. *The greatest exponent $\lambda_r(A)$ of (1) is given by the following formula*

$$\lambda_g(A) = \limsup_{n\to\infty} \frac{1}{n} \ln \|A(n-1)...A(0)\|. \tag{4}$$

Together with (1) we will consider the so-called dual or adjoint system

$$y(n+1) = B(n)y(n), n \geq 0, \tag{5}$$

where $B(n) = \left(A^T(n)\right)^{-1}$. The transition matrix of the dual system is given by

$$\mathscr{B}(m,k) = B(m-1)...B(k)$$

for $m > k$ and $\mathscr{B}(m,m) = I$.

Observe that an application of Lyapunov inequality (3) to the dual system leads to the following extension of point 6 of Theorem 1.

Lemma 1. *If $x_0 \in R^s \setminus \{0\}$, then $\lambda(x_0)$ is finite.*

Further extension of this result can be found in [37]. It demonstrates the following conditions weaker than (2) conditions

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|\mathscr{A}(n,0)\| < \infty$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|\mathscr{B}(n,0)\| < \infty$$

imply the finiteness of $\lambda(x_0)$ for $x_0 \in R^s \setminus \{0\}$.

Denote by $\{\mu_1, \mu_2, ..., \mu_r\}$, $\mu_r < \mu_{r-1} < ... < \mu_1$ the spectrum of dual system. For each λ_i and μ_i we consider the following subspaces of R^s

$$E_i = \{v \in R^s : \lambda(v) \leq \lambda_i\}$$

and

$$F_i = \{v \in R^s : \mu(v) \leq \mu_i\},$$

and we set $E_0 = F_{s+1} = \{0\}$. The multiplicities n_i and m_i of Lyapunov exponent λ_i and μ_i are defined as $\dim E_i - \dim E_{i-1}$ and $\dim F_i - \dim F_{i+1}$, respectively for

$i = 1, ..., s$. If we have two bases $v_1, ..., v_s$ and $w_1, ..., w_s$ of R^s, then we will call them dual if $\langle v_i, w_j \rangle = \delta_{ij}$, where $< u, v >$ is the standard scalar product in R^s and δ_{ij} is the Kronecker symbol. For a base $V = \{v_1, ..., v_s\}$ of R^s we define the sum σ_V of Lyapunov exponents

$$\sigma_V = \sum_{i=1}^{s} \lambda_A(v_i).$$

The base $v_1, ..., v_s$ is called normal if for each $i = 1, ..., s$ there exists a basis of E_i composed of vectors $\{v_1, ..., v_s\}$. Formally, we should say that a basis is normal with respect to family E_i, $i = 1, ..., s$. It can be shown (see [7], remark after Theorem 1.2.5) that there always exist normal bases $v_1, ..., v_s$ and $w_1, ..., w_s$ (respectively of the families E_i and F_i) which are dual. It can be also shown (see [7], Theorem 1.2.3) that for the normal bases the sum σ_V of Lyapunov exponents is minimal and then, according to Lyapunov inequality (3), equal to

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\det \mathscr{A}(n, 0)| .$$

For a basis $v_1, ..., v_s$ matrix $\mathscr{V}(n)$, $n \in N$ whose columns are $x(n, v_1)$,..., $x(n, v_s)$ is called fundamental matrix of (1). For a fundamental matrix the kernel $\mathscr{G}(n, m) = \mathscr{V}(n)\mathscr{V}^{-1}(m)$, $n, m \in N$ is called a Green's matrix of (1). If the base is normal, then the fundamental and Green's matrices are called normal. In many our further consideration a crucial role will be played by the possibility of reduction of our system to an upper triangular one. It is guaranteed by the following theorem from [7], Theorem 7.

Theorem 3. *For each sequence $(A(n))_{n\in\mathbb{N}}$ there exists a sequence $(U(n))_{n\in\mathbb{N}}$ of orthogonal matrices such that $C_n = U_{n+1}^T A_n U_n$ is upper triangular.*

Together with (1) we consider the following perturbed system

$$z(n+1) = (A(n) + \Delta(n))\, z(n), \tag{6}$$

where $\Delta = (\Delta(n))_{n\in\mathbb{N}}$ is a sequence of s-by-s real matrices from a certain class $\mathfrak{M}$. Under the influence of the perturbation Δ, the characteristic exponents of (1) vary, in general, discontinuously. It is possible that a finite shift of the characteristic exponents of the original system (1) corresponds to an arbitrarily small $\sup \|\Delta(n)\|$. In particular, it is possible for an exponentially stable system to be perturbed by an exponentially decreasing perturbation and the resulting system is not stable. The quantities

$$\Lambda_u(\mathfrak{M}) = \sup\left\{\lambda_g(A+\Delta) : \Delta \in \mathfrak{M}\right\}$$
$$\Lambda_l(\mathfrak{M}) = \inf\left\{\lambda_g(A+\Delta) : \Delta \in \mathfrak{M}\right\}$$

are referred to as the maximal upper and minimal lower movability boundary of the higher exponent of (1) with perturbation in the class $\mathfrak{M}$. We may also consider similar quantities for the others elements of the spectrum.

The determination of the movability boundaries of the higher exponent under various perturbations is one of the main problem in the theory of Lyapunov exponents. This problem has been studied for continuous-time systems for many classes $\mathfrak{M}$. For example, upper bound for the higher exponent of (1) under small perturbations, the so-called central exponent $\Omega(A)$, was constructed in [12], p. 114. The attainability of this estimate was proved in [47] with the use of the classical rotation method. This problem was solved in [31] and [30] for linear systems with perturbations decreasing at infinity at various rates and in [5] for linear systems with perturbations determined by integral conditions. Later in [43] and [44] perturbations infinitesimal in mean with a weight function have been investigated. The recent monograph [32] is almost completely devoted to this problem.

3 Bounded Perturbation

In this chapter we will consider the perturbation set

$$\mathfrak{M}_q = \{\Delta = (\Delta(n))_{n\in\mathbb{N}} : \|\Delta\|_\infty < q\}, \tag{7}$$

where $\|\Delta\|_\infty = \sup \|\Delta(n)\|$. In the next paragraph we present a definition of stability of Lyapunov exponents and a sufficient condition for the stability. Next, we will present analytic formulas for maximal upper and minimal lower movability boundaries of the higher exponent of (1) with perturbation in the class $\mathfrak{M}_q$ in two-dimensional stationary case. After that, we will present basic facts about generalized spectral radius and finally, we will show how this tool may be used to determine $\Lambda_u(\mathfrak{M}_q)$ and $\Lambda_u(\mathfrak{M}_q)$ in case of stationary system.

We have the following definition.

Definition 3. The Lyapunov exponents of system (1) are called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\sup_{n\in N} \|\Delta(n)\| < \delta \tag{8}$$

implies the inequality

$$\left|\lambda_i'(A) - \lambda_i'(A+\Delta)\right| < \varepsilon, \, i = 1, ..., s.$$

To formulate our main results for a Green's matrix of (1) denote by $x_i(m,n)$ the i-th column of it and by μ_i the characteristic exponent of the sequence $(\|x_i(m,n)\|)_{m\in N}$, $i = 1,...,s$. The next theorem [18] constitutes discrete time version of Malkin [45] sufficient condition for continuity of Lyapunov exponents.

Theorem 4. *Suppose that for certain Green's $\mathcal{G}(m,n)$ matrix of (1) and any $\gamma > 0$ there exists $d > 0$ such that*

$$\|x_i(m,n)\| \le d \exp[(\mu_i + \gamma)(m-n)] \;\; \textit{for } m,n \in N, \, m \ge n, \, i = 1,...,s \tag{9}$$

and

$$\|x_i(m,n)\| \leq d\exp\left[(\mu_i-\gamma)(m-n)\right] \ \text{for}\ m,n\in N,\ n\geq m,\ i=1,...,s, \tag{10}$$

then the Lyapunov exponents of system (1) are stable.

Using Theorem 4 it can be shown that the Lyapunov exponents of time-invariant system are stable for invertible matrix system.

Theorem 5. *Lyapunov exponents of time-invariant system*

$$x(n+1) = Ax(n), \tag{11}$$

with invertible matrix A are stable.

Consider a time-invariant two-dimensional system

$$x(n+1) = Ax(n), n \geq 0, \tag{12}$$

where A is a two-by-two matrix in Jordan canonical form

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \tag{13}$$

or

$$A = \begin{bmatrix} a_1 & 0 \\ 1 & a_2 \end{bmatrix}, \tag{14}$$

where a_1, a_2 are positive. Together with (12) we consider the following disturbed system

$$y(n+1) = (A+Q(n))\,y(n), \tag{15}$$

where $(Q(n))_{n\in\mathbb{N}}$,

$$Q(n) = \begin{bmatrix} q_{11}(n) & q_{12}(n) \\ q_{21}(n) & q_{22}(n) \end{bmatrix}$$

is a sequence of two-by-two matrices such that

$$\left|q_{ij}(n)\right| \leq q \tag{16}$$

for all $i,j = 1,2$ and all $n = 0,1,\dots$.

In the considered case the set of Lyapunov exponents of system (15) contains at most 2 elements, say $\lambda_1(A+Q) \leq \lambda_2(A+Q)$. We will try to describe the influence of the perturbation $(Q(n))_{n\in\mathbb{N}}$, on the greatest Lyapunov exponent of (12). We will investigate the following counties $\lambda_2^{\min}(A,q) = \min\lambda_2(A+Q)$, $\lambda_2^{\max}(A,q) = \max\lambda_2(A+Q)$,where the maxima and minima are taken over all perturbation sequences $(Q(n))_{n\in\mathbb{N}}$ satisfying (16).

The following Theorem contains analytic expressions for $\lambda_2^{\max}(A,q)$.

Theorem 6. *We have*

$$\lambda_2^{\max}(A,q) = \begin{cases} \ln\left[\frac{1}{2}\left(a_1+a_2+2q+\sqrt{(a_2-a_1)^2+4q^2}\right)\right] \\ \text{if } A \text{ is given by (13)} \\ \ln\left[\frac{1}{2}\left(a_1+a_2+2q+\sqrt{(a_2-a_1)^2+4q(1+q)}\right)\right] \\ \text{if } A \text{ is given by (14)} \end{cases}.$$

The problem of evaluating $\lambda_2^{\min}(A,q)$ seems to be much harder. Some particular cases are provided in the next Theorem.

Theorem 7. *If A is given by (13) and $4q \le |a_2 - a_1|$, then we have*

$$\lambda_2^{\min}(A,q) = \ln \frac{a_1+a_2+\sqrt{(|a_2-a_1|-4q)\,|a_2-a_1|}}{2}.$$

4 Generalized Spectral Radius and Subradius

Denote by $\rho(A)$ the spectral radius of a matrix A. Consider a nonempty set Σ of s-by-s matrices. For $m \ge 1$, Σ^m is the set of all products of matrices in Σ of length m,

$$\Sigma^m = \{A_1A_2...A_m : A_i \in \Sigma, i = 1,...,m\}.$$

Set

$$\overline{\alpha}_m = \sup_{A\in\Sigma^m} \|A\|,\ \underline{\alpha}_m = \inf_{A\in\Sigma^m} \|A\|,$$

$$\overline{\beta}_m = \sup_{A\in\Sigma^m} \rho(A),\ \underline{\beta}_m = \inf_{A\in\Sigma^m} \rho(A)$$

and define:

- the joint spectral subradius

$$\widehat{\rho}_*(\Sigma) = \inf_{m\ge 1} \underline{\alpha}_m^{1/m},$$

- the joint spectral radius

$$\widehat{\rho}(\Sigma) = \inf_{m\ge 1} \overline{\alpha}_m^{1/m},$$

-the generalized spectral subradius

$$\overline{\rho}_*(\Sigma) = \inf_{m\ge 1} \underline{\beta}_m^{1/m},$$

-the generalized spectral radius

$$\overline{\rho}(\Sigma) = \sup_{m\ge 1} \overline{\beta}_m^{1/m}.$$

The concepts of joint and generalized spectral radii were introduced in [53] and in [20] (see also [21]), respectively. Next in [8] and [23] two different proofs of the equality

$$\widehat{\rho}(\Sigma) = \overline{\rho}(\Sigma) \tag{17}$$

were given for the bounded set Σ. In [20] it was also shown that for bounded set Σ we have

$$\widehat{\rho}(\Sigma) = \lim_{m\to\infty} \overline{\alpha}_m^{1/m} = \limsup_{m\to\infty} \overline{\beta}_m^{1/m}. \tag{18}$$

Later for bounded set Σ we will denote the common value of $\widehat{\rho}(\Sigma)$ and $\overline{\rho}(\Sigma)$ by $\rho(\Sigma)$. The concepts of joint and the generalized spectral subradii were introduced in [26] to present conditions for Markov asymptotic stability of a discrete linear inclusion. In this paper it has been also shown that

$$\widehat{\rho}_*(\Sigma) = \overline{\rho}_*(\Sigma) \tag{19}$$

for finite Σ. In [9] these concepts have been related to the so called mortality problem. We say that the set of matrices Σ is mortal if the zero matrix can be expressed as the product of finitely many matrices from Σ. It appears that Σ is mortal if, and only if, $\widehat{\rho}_*(\Sigma) = 0$. Finally, in [13] inequality (19) was extended to the case of any nonempty set of matrices and it was shown that

$$\widehat{\rho}_*(\Sigma) = \lim_{m\to\infty} \underline{\alpha}_m^{1/m} = \liminf_{m\to\infty} \underline{\beta}_m^{1/m}. \tag{20}$$

Later for nonempty set Σ we will denote the common value of $\widehat{\rho}_*(\Sigma)$ and $\overline{\rho}_*(\Sigma)$ by $\rho_*(\Sigma)$. Because of the equalities (17) and (19) we can introduce the following definition.

Definition 4. For bounded set Σ we will denote the common value of $\widehat{\rho}(\Sigma)$ and $\overline{\rho}(\Sigma)$ by $\rho(\Sigma)$ and called it generalized spectral radius. For nonempty set Σ we will denote the common value of $\widehat{\rho}_*(\Sigma)$ and $\overline{\rho}_*(\Sigma)$ by $\rho_*(\Sigma)$ and called it generalized spectral subradius.

Denote by $\mathscr{D}(\Sigma)$ the set of all infinite sequences of elements of Σ. For fixed $d \in \mathscr{D}(\Sigma)$, $d = (A(1), A(2), ...)$ define $\Phi_d(m) = A(m-1)...A(1)A(0)$ and

$$\overline{\rho}(d) = \limsup_{m\to\infty} \|\Phi_d(m)\|^{\frac{1}{m}} .$$

From the definitions of $\rho(\Sigma)$, $\rho_*(\Sigma)$ and $\overline{\rho}(d)$ the following inequality follows

$$\rho_*(\Sigma) \leq \overline{\rho}(d) \leq \rho(\Sigma)$$

for bounded set $\sum$. Much deeper relations between these three quantities is given by the next Theorem.

Theorem 8. *For any nonempty set Σ we have*

$$\rho_*(\Sigma) = \inf_{d \in \mathscr{D}(\Sigma)} \overline{\rho}(d). \tag{21}$$

For any nonempty and bounded set $\sum$ we have

$$\rho(\Sigma) = \sup_{d \in \mathscr{D}(\Sigma)} \overline{\rho}(d) \tag{22}$$

and if Σ is in addition closed then the sup is max. Moreover, if the matrices in nonempty and bounded Σ are invertible, then for each $\gamma \in (\rho_*(\Sigma), \rho(\Sigma))$ there exists $d \in \mathscr{D}(\Sigma)$ such that $\overline{\rho}(d) = \gamma$.

Equalities (21) and (22) have been proved in [13] and [23], respectively. The attainability of sup in (22) has been established in [20] for finite Σ and next in [54] this result has been extended to the case of compact set. The last statement of the Theorem is shown in [15].

Unfortunately, if the family Σ is not just a single matrix, the computation of $\rho(\Sigma)$ and $\rho_*(\Sigma)$ are not easy tasks at all. The problem of numerical computation of $\rho(\Sigma)$ and $\rho_*(\Sigma)$ is discussed in [9]-[11], [24], [25] and [42]; see also the references therein.

5 Central Exponents: Definitions and Basic Properties

Using the concept of central exponents of families of sequences, we now define the concept of upper and lower central exponents of system (12).

Definition 5. The upper (lower) sequence of (1) is upper (lower) sequence of the family

$$\left\{ \left(\ln \frac{\|x(n+1,x_0)\|}{\|x(n,x_0)\|} \right)_{n \in \mathbb{N}} : x_0 \in R^s, \|x_0\| = 1 \right\}. \tag{23}$$

The set of all upper (lower) sequences of (1) will be denoted by $\mathscr{U}(A)$ ($\mathscr{L}(A)$). Analogically, upper (lower) central exponent of (1) is defined as upper (lower) central exponent of (23) and will be denoted as $\Omega(A)$ ($\omega(A)$).

Notice that the definition is correct. One may obtain the following characterization of upper and lower sequences in terms of transition matrix.

Theorem 9. *Bounded sequences $(r(n))_{n \in \mathbb{N}}$ and $(R(n))_{n \in \mathbb{N}}$ are lower and upper sequences for (1), respectively, if and only if, for any $\varepsilon > 0$ there exist constants $d_{r,\varepsilon}$ and $D_{R,\varepsilon}$ such that for all $m > k$ we have*

$$d_{R,\varepsilon} \exp\left(\sum_{i=k}^{m-1} (r(i) - \varepsilon) \right) \leq \|\mathscr{A}(m,k)\| \leq D_{R,\varepsilon} \exp\left(\sum_{i=k}^{m-1} (R(i) + \varepsilon) \right). \tag{24}$$

The lower central exponent does not require special consideration since the problem can be reduced to the investigation of upper central exponent for the adjoint system. This is the content of the next theorem.

Theorem 10. *The lower central exponent ω of (1) is equal to the upper central exponent of the adjoint system (5), taken with the opposite sign.*

We have also the following formulas for upper and lower central exponent of (1) in terms of transition matrix.

Theorem 11. *There exist limits*

$$\lim_{N\to\infty}\frac{1}{N}\left(\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\ln\left\|\mathscr{A}(N+i,i)\right\|\right)$$

$$\lim_{N\to\infty}\frac{1}{N}\left(\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\ln\left\|\mathscr{A}\left((i+1)N,iN\right)\right\|\right),$$

and they are equal to

$$\Omega(A)=\inf_{N\in\mathbb{N}}\frac{1}{N}\left(\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\ln\left\|\mathscr{A}(N+i,i)\right\|\right)$$

$$=\inf_{N\in\mathbb{N}}\frac{1}{N}\left(\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\ln\left\|\mathscr{A}\left((i+1)N,iN\right)\right\|\right).$$

By Theorem 10 one can obtain analogical formulas for lower central exponent. Observe first that the orthogonality of matrices $U(n)$ in Theorem 3 leads to the following Theorem.

Theorem 12. *With the notation of Theorem 3 we have*

$$\mathscr{U}(A)=\mathscr{U}(C),\ \mathscr{L}(A)=\mathscr{L}(C).$$

In particular it implies that the central exponents of $(A(n))_{n\in\mathbb{N}}$ and $(C(n))_{n\in\mathbb{N}}$ are equal.

Consider now system (1) with matrices $A(n)$ being upper triangular with diagonal elements $a_{ii}(n)$. Denote $A_d(n)=diag\left[a_{ii}(n)\right]_{i=1,\dots,s}$ and by $a_{ij}(n)$ and $z_{ij}(n,k)$, the elements of $A(n)$ and $\mathscr{A}(n+1,k)$, respectively. By a straightforward calculation we have that for $n>k$

$$z_{ij}(n,k)=\begin{cases}\sum_{p=k}^{n-1}\sum_{l=i+1}^{j}a_{il}(n-p)z_{lj}(n-p-1,k)\prod_{q=n-p+1}^{n}a_{ii}(q)\ \text{for}\ i<j\\ \prod_{q=m}^{n}a_{ii}(q)\ \text{for}\ i=j\\ 0\ \text{for}\ i>j\end{cases}. \tag{25}$$

The next Theorem describes relation between upper, lower sequences of the original system and those with matrix coefficients A_d.

Theorem 13. *We have* $\mathscr{U}(A) = \mathscr{U}(A_d)$, $\mathscr{L}(A) = \mathscr{L}(A_d)$, $\Omega(A) = \Omega(A_d)$ *and* $\omega(A) = \omega(A_d)$.

If $(A(n))_{n\in\mathbb{N}}$ and $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ are bounded by a and a', respectively, then from definition of central and Lyapunov exponent is clear that

$$-\ln a' \leq \omega(A) \leq \lambda_s(A) \leq \lambda_g(A) \leq \Omega(A) \leq \ln a.$$

It is also not difficult to construct examples where $\Omega(A) < \ln a$, $\lambda_s(A) < \lambda_g(A)$ and $-\ln a' < \omega(A)$. There are known examples for which $\lambda_g(A) < \Omega(A)$.

Next theorem shows that under the condition that $\Delta(n) \to 0$, the central exponents of (1) and (6) coincide. Observe that invertibility of $A(n)$ implies that $A(n) + \Delta(n)$ are invertible starting from certain n_0. We will assume that they are invertible for all natural n and then the central exponents of (6) are well defined.

Theorem 14. *If* $\lim_{n\to\infty} \|\Delta(n)\| = 0$ *then* $\mathscr{L}(A) = \mathscr{L}(A+\Delta)$, $\mathscr{U}(A) = \mathscr{U}(A+\Delta)$ *and in particular* $\Omega(A) = \Omega(A+\Delta)$ *and* $\omega(A) = \omega(A+\Delta)$.

6 Regular Systems and Regularity Coefficients

In order to measure the irregularity of the system (1) some numerical characteristics, which are called coefficients of regularity are introduced. In this section we will consider three of them.

1. Lyapunov's coefficient of regularity ([40]) is defined as:

$$\sigma_L = \min \sigma_V - \liminf_{n\to\infty} \frac{1}{n} \ln |\det \mathscr{A}(n,0)|,$$

where minimum is taken over the set of all bases. In fact it is enough to take the minimum over the set of normal bases.

2. Perron's coefficient of regularity ([49]). Consider the values

$$\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_s \tag{26}$$

and

$$\mu'_s \leq \mu'_{s-1} \leq \ldots \leq \mu'_1 \tag{27}$$

of the Lyapunov exponents of (1) and (5), respectively, counted with their multiplicities. Then Perron's coefficient of regularity is defined as

$$\sigma_P = \max_{i=1,\ldots,s} \left(\lambda'_i + \mu'_i\right).$$

3. Grobman's coefficient of regularity ([12]). For a pair of dual bases $V = \{v_1, ..., v_s\}$ and $W = \{w_1, ..., w_s\}$ we define defect of dual bases

$$\gamma(V,W) = \max_{i=1,...,s} \left(\lambda(v_i) + \mu(w_i)\right).$$

Then Grobman's coefficient of regularity is defined as:

$$\sigma_G = \min \gamma(V,W), \tag{28}$$

where the minimum is taken over all pairs of dual bases.

We will also say about regularity of the sequence $(A(n))_{n\in\mathbb{N}}$ instead of regularity coefficient of (1).

The introduced coefficients σ_P, σ_L and σ_G are related by the following inequalities

$$0 \le \sigma_P \le \sigma_G \le s\sigma_P \tag{29}$$

and

$$0 \le \sigma_G \le \sigma_L \le s\sigma_G. \tag{30}$$

(see, [7] Theorem 1.2.6 for the proof of (29) and [16] Lemma 1 for the proof of (30)). It appears that considering regularity coefficients we may restrict ourselves to the uppertriangular system according to the following Theorem from [7].

Theorem 15. *If sequence $(C(n))_{n\in\mathbb{N}}$ is constructed for sequence $(A(n))_{n\in\mathbb{N}}$ according to Theorem 3, then the regularity coefficients σ_P, σ_L and σ_G are the same for $(A(n))_{n\in\mathbb{N}}$ and $(C(n))_{n\in\mathbb{N}}$.*

The next Theorem contains the main result of this section.

Theorem 16. *If*

$$\lambda(\Delta) < -\sigma_G, \tag{31}$$

then the spectra of (1) and (6) coincide.

As it is shown in the next theorem, in case of diagonal matrices $A(n)$, spectra of (1) and (6) coincide for perturbations with characteristic exponent equal to $-\sigma_G$.

Theorem 17. *If the matrices $A(n)$ are diagonal and*

$$\lambda(\Delta) \le -\sigma_G < 0, \tag{32}$$

then the spectra of (1) and (6) coincide.

Together with (1) consider the following non-homogeneous system

$$x(n+1) = A(n)x(n) + f(n), \tag{33}$$

where the sequence $f = (f(n))_{n\in\mathbb{N}}$ belongs to the class F consisting of all sequences g of s-dimensional vectors such that

$$-\infty < \lambda(g) < \infty.$$

For an initial condition x_0 the solution of (33) is denoted by $x(n,x_0)$ so

$$x(n,x_0,f) = \mathscr{A}(n,0)x_0 + \sum_{i=0}^{n-1} \mathscr{A}(n,i+1)f(i). \tag{34}$$

From this formula it follows that the set $\left\{\lambda\left((x(n,x_0,f))_{n\in\mathbb{N}}\right) : x_0 \in R^s, x_0 \neq 0\right\}$ contains at most $s+1$ elements. Denote by $\chi(A,f)$ the minimal characteristic exponents of solution of (33) that is

$$\chi(A,f) = \min_{x_0\neq 0} \lambda\left((x(n,x_0,f))_{n\in\mathbb{N}}\right). \tag{35}$$

We introduce a quantity $\sigma(A)$ which will measure the difference between Lyapunov exponents of the non-homogeneous system (33) and of f. Define $\sigma(A)$ by the following formula

$$\sigma(A) = \sup_{f\in F} (\chi(A,f) - \lambda(f)). \tag{36}$$

Finally, we have the following result.

Theorem 18. *The following inequality holds*

$$\frac{\sigma_G}{s} \leq \sigma(A) \leq \sigma_G. \tag{37}$$

In particular, system (1) is regular if, and only if, $\sigma(A) = 0$.

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