# One-Way Finite Automata with Quantum and Classical States*

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**Abstract.** In this paper, we introduce and explore a new model of *quantum finite automata* (QFA). Namely, *one-way finite automata with quantum and classical states* (1QCFA), a one way version of *two-way finite automata with quantum and classical states* (2QCFA) introduced by Ambainis and Watrous in 2002 [3]. First, we prove that *coin-tossing one-way probabilistic finite automata* (coin-tossing 1PFA) [23] and *one-way quantum finite automata with control language* (1QFACL) [6] as well as several other models of QFA, can be simulated by 1QCFA. Afterwards, we explore several closure properties for the family of languages accepted by 1QCFA. Finally, the state complexity of 1QCFA is explored and the main succinctness result is derived. Namely, for any prime $m$ and any $\varepsilon_1 > 0$, there exists a language $L_m$ that cannot be recognized by any *measure-many one-way quantum finite automata* (MM-1QFA) [12] with bounded error $\frac{7}{9} + \epsilon_1$, and any 1PFA recognizing it has at last $m$ states, but $L_m$ can be recognized by a 1QCFA for any error bound $\epsilon > 0$ with $\mathbf{O}(\log m)$ quantum states and 12 classical states.

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## 1 Introduction

An important way to get a deeper insight into the power of various quantum resources and features for information processing is to explore power of various quantum variations of the basic models of classical automata. Of a special interest and importance is to do that for various quantum variations of classical finite automata because quantum resources are not cheap and quantum operations are not easy to implement. Attempts to find out how much one can do with very little of quantum resources and consequently with the most simple quantum variations of classical finite automata are therefore of particular interest. This paper is an attempt to contribute to such line of research.

There are two basic approaches how to introduce quantum features to classical models of finite automata. The first one is to consider quantum variants of the classical *one-way (deterministic) finite automata* (1FA or 1DFA) and the second one is to consider quantum variants of the classical *two-way finite automata* (2FA or 2DFA). Already the very first attempts to introduce such models, by Moore and Crutchfields [20] and Kondacs and Watrous [12] demonstrated that in spite of the fact that in the classical case, 1FA and 2FA have the same recognition power, this is not so for their quantum variations. Moreover, already the first important model of *two-way quantum finite automata* (2QFA), namely that introduced by Kondacs and Watrous, demonstrated that very natural quantum variants of 2FA are much too powerful - they can recognize even some non-context free languages and are actually not really finite in a strong sense. It started to be therefore of interest to introduce and explore some "less quantum" variations of 2FA and their power [1–6, 8, 14–19, 22, 26–30].

A very natural "hybrid" quantum variations of 2FA, namely, *two-way quantum automata with quantum and classical states* (2QCFA) were introduced by Ambainis and Watrous [3]. Using this model they were able to show in an elegant way that an addition of a single qubit to a classical model can enormously increase power of automata. A 2QCFA is essentially a classical 2FA augmented with a quantum memory of constant size (for states in a fixed Hilbert space) that does not depend on the size of the (classical) input. In spite of such a restriction, 2QCFA have been shown to be more powerful than *two-way probabilistic finite automata* (2PFA) [3].

Because of the simplicity, elegance and interesting properties of the 2QCFA model, as well as its natural character, it seems to be both useful and interesting to explore what such a new "hybrid" approach will provide in case of one-way finite automata and this we will do in this paper by introducing and exploring 1QCFA.

In the first part of the paper, 1QCFA are introduced formally and it is shown that they can be used to simulate a variety of other models of finite automata. Namely, 1DFA, coin-tossing 1PFA, measure-once 1QFA (MO-1QFA) [12], measure-many 1QFA (MM-1QFA) [12] and *one-way quantum finite automata with control language* (1QFACL) [6]. Of a special interest is the way how 1QCFA can simulate 1QFACL - an interesting model the behavior of which is, however, quite special. Our simulation of 1QFACL by 1QCFA allows to see behavior of 1QFACL in a quite transparent way. We also explore several closure properties of the family of languages accepted by 1QCFA. Finally, we derive a result concerning the state complexity of 1QCFA that also demonstrates a merit of this new model. Namely we show that for any prime $m$ and any $\varepsilon_1 > 0$, there exists a language $L_m$ than cannot be recognized by any MM-1QFA with bounded error $\frac{7}{9} + \epsilon_1$, and any 1PFA recognizing it has at last $m$ states, but $L_m$ can be recognized by a 1QCFA for any error bound $\epsilon > 0$ with $\mathbf{O}(\log m)$ quantum states and 12 classical states.

The rest of the paper is organized as follows. Definitions of all automata models explored in the paper are presented in Section 2. In Section 3 we show how several other models of finite automata can be simulated by 1QCFA. We also explore several closure properties of the family of languages accepted by 1QCFA in Section 4. In Section 5 the above mentioned succinctness result is proved and the last section contains just few concluding remarks.

## 2 Basic Models of Classical and Quantum Finite Automata

In the first part of this section we formally introduce those basic models of finite automata we will refer to in the rest of the paper and in the second part of this section, we formally introduce as a new model 1QCFA. Concerning the basics of quantum computation we refer the reader to [9, 21] and concerning the basic properties of the automata models introduced in the following we refer the reader to [9–11, 23, 25].

### 2.1 Basic Models of Classical and Quantum Finite Automata

In this subsection, we recall the definitions of DFA, 1PFA, MO-1QFA, MM-1QFA and 1QFACL.

**Definition 1.** *A deterministic finite automaton (DFA) $\mathcal{A}$ is specified by a 5-tuple*

$$\mathcal{A} = (S, \Sigma, \delta, s_0, S_{acc}), \tag{1}$$

*where:*

1. *$S$ is a finite set of classical states;*
2. *$\Sigma$ is a finite set of input symbols;*
3. *$s_0 \in S$ is the initial state of the machine;*
4. *$S_{acc} \subset S$ is the set of accepting states;*
5. *$\delta$ is the transition function:*

$$\delta : S \times \Sigma \to S. \tag{2}$$

Let $w = \sigma_1\sigma_2 \cdots \sigma_n$ be a string over the alphabet $\Sigma$. The automaton $\mathcal{A}$ accepts the string $w$ if a sequence of states, $r_0, r_1, \cdots, r_n$, exists in $S$ with the following conditions:

1. $r_0 = s_0$;
2. $r_{i+1} = \delta(r_i, \sigma_{i+1})$, for $i = 0, \cdots, n-1$;
3. $r_n \in S_{acc}$.

DFA recognize exactly the set of *regular languages* (RL).

**Definition 2.** *A one-way probabilistic finite automata (1PFA) $\mathcal{A}$ is specified by a 5-tuple*

$$\mathcal{A} = (S, \Sigma, \delta, s_1, S_{acc}), \tag{3}$$

*where:*

1. *$S = \{s_1, s_2, \cdots, s_n\}$ is a finite set of classical states;*
2. *$\Sigma$ is a finite set of input symbols; $\Sigma$ is then extended to the tape symbol set $\Gamma = \Sigma \cup \{ ¢, \$\}$, where $¢ \notin \Sigma$ is called the left end-marker and $\$ \notin \Sigma$ is called the right end-marker;*
3. *$s_1 \in S$ is the initial state;*
4. *$S_{acc} \subset S$ is the set of accepting states;*
5. *$\delta$ is the transition function:*

$$\delta : S \times \Gamma \times S \to [0, 1]. \tag{4}$$

*For example, $\delta(s, \sigma, t)$ means that if $\mathcal{A}$ is in the state $s$ with the tape head scanning the symbol $\sigma$, then the automaton enters the state $t$ with probability $\delta(s, \sigma, t)$.*

*Note: A 1 PFA is a coin-tossing 1PFA if the range of its transition function $\delta$ is $\{0, 1/2, 1\}$. For any $s \in S$ and any $\sigma \in \Gamma$, $\delta(s, \sigma, t)$ is a so-called coin-tossing distribution[1] on $S$ such that $\sum_{t \in S} \delta(s, \sigma, t) = 1$. It is not hard to see that rational transition probabilities can be obtained by repeating coin-flip.*

For an input string $\omega = \sigma_1 \ldots \sigma_l$, the probability distribution on the states of $\mathcal{A}$ during its acceptance process can be traced using $n$-dimensional vectors. It is assumed that $\mathcal{A}$ starts to process the input word written on the input tape as $w = ¢\, \omega \$$ and let $v_0 = (1, 0, \ldots, 0)^T_{n \times 1}$ denote the initial probability distribution on states. If, during the acceptance process, the current probability distribution vector is $v$ and a tape symbol $\sigma$ is read, then the new state probability distribution vector will be, after the automaton step, $u = A_\sigma v$, where $A_\sigma$ is such a matrix that $A_\sigma(i, j) = \delta(s_j, \sigma, s_i)$. We then use $v_{|w|} = A_\$ A_{\sigma_l} \cdots A_{\sigma_1} A_{¢} v_0$ to denote the final probability distribution on states in case of the input $\omega$. The accepting probability of $\mathcal{A}$ with input $\omega$ is then

$$Pr[\mathcal{A} \text{ accepts } \omega] = \sum_{s_i \in S_{acc}} v_{|w|}(i), \tag{5}$$

where $v_{|w|}(i)$ denotes the $i$th entry of $v_{|w|}$.

[1] A coin-tossing distribution on a finite set $Q$ is a mapping $\phi$ from $Q$ to {0, 1/2, 1} such that $\sum_{q \in Q} \phi(q) = 1$.

**Definition 3.** *A measurement-once one-way quantum automaton (MO-1QFA) $\mathcal{A}$ is specified by a 5-tuple*

$$\mathcal{A} = (Q, \Sigma, \Theta, |q_0\rangle, Q_{acc}), \tag{6}$$

*where:*

1. *$Q$ is a finite set of quantum orthogonal states;*
2. *$\Sigma$ is a finite set of input symbols; $\Sigma$ is then extended to the tape symbol set $\Gamma = \Sigma \cup \{ ¢, \$\}$, where $¢ \notin \Sigma$ is called the left end-marker and $\$ \notin \Sigma$ is called the right end-marker;*
3. *$|q_0\rangle \in Q$ is the initial quantum state;*
4. *$Q_{acc} \subset Q$ is the set of accepting quantum states;*
5. *For each $\sigma \in \Gamma$, a unitary transformation $\Theta_\sigma$ is defined on the Hilbert space spanned by the states from $Q$.*

We describe the acceptance process of $\mathcal{A}$ for any given input string $\omega = \sigma_1 \cdots \sigma_l$ as follows. The automaton $\mathcal{A}$ states with the initial state $|q_0\rangle$, reading the left-marker ¢. Afterwards, the unitary transformation $\Theta_¢$ is applied on $|q_0\rangle$. After that, $\Theta_¢|q_0\rangle$ becomes the current state and the automaton reads $\sigma_1$. The process continues until $\mathcal{A}$ reads \$ and ends in the state $|\psi_\omega\rangle = \Theta_\$\Theta_{\sigma_l} \cdots \Theta_{\sigma_1}\Theta_¢|q_0\rangle$. Finally, a measurement is performed on $|\psi_\omega\rangle$ and the accepting probability of $\mathcal{A}$ on the input $\omega$ is equal to

$$Pr[\mathcal{A} \textit{ accepts } \omega] = \langle\psi_\omega|P_a|\psi_\omega\rangle = ||P_a|\psi_\omega\rangle||^2, \tag{7}$$

where $P_a = \sum_{q \in Q_{acc}} |q\rangle\langle q|$ is the projection onto the subspace spanned by $\{|q\rangle : |q\rangle \in Q_{acc}\}$.

**Definition 4.** *A measurement-many one-way quantum automaton (MM-1QFA) $\mathcal{A}$ is specified by a 6-tuple*

$$\mathcal{A} = (Q, \Sigma, \Theta, |q_0\rangle, Q_{acc}, Q_{rej}), \tag{8}$$

*where $Q$, $\Sigma$, $\Theta$, $|q_0\rangle$, $Q_{acc}$, and the tape symbol set $\Gamma$ are the same as those defined above in an MO-1QFA. $Q_{rej} \subset Q$ is the set of rejecting states.*

For any given input string $\omega = \sigma_1 \cdots \sigma_l$, the acceptance process is similar to that of MO-1QFA except that after every transition, MM-1QFA $\mathcal{A}$ measures its state with respect to the three subspaces that are spanned by the three subsets $Q_{acc}$, $Q_{rej}$ and $Q_{non}$, respectively, where $Q_{non} = Q \setminus (Q_{acc} \cup Q_{rej})$. In other words, the projective measurement consists of $\{P_a, P_r, P_n\}$, where $P_a = \sum_{q \in Q_{acc}} |q\rangle\langle q|$, $P_r = \sum_{q \in Q_{rej}} |q\rangle\langle q|$ and $P_n = \sum_{q \in Q_{non}} |q\rangle\langle q|$. The accepting and rejecting probability are given as follows (for convenience, we denote $\sigma_0 = ¢$ and $\sigma_{l+1} = \$$):

$$Pr[\mathcal{A} \textit{ accepts } \omega] = \sum_{k=0}^{l+1} ||P_a\Theta_{\sigma_k} \prod_{i=0}^{k-1} (P_n\Theta_{\sigma_i})|q_0\rangle||^2, \tag{9}$$

$$Pr[\mathcal{A}\ reject\ \omega] = \sum_{k=0}^{l+1} ||P_r \Theta_{\sigma_k} \prod_{i=0}^{k-1}(P_n \Theta_{\sigma_i})|q_0\rangle||^2. \tag{10}$$

An important convention: In this paper we define $\prod_{i=1}^{n} A_i = A_n A_{n-1} \cdots A_1$, instead of the usual one $A_1 A_2 \cdots A_n$.

**Definition 5.** *A one-way quantum finite automata with control language (1QFACL) $\mathcal{A}$ is specified by as a 6-tuple*

$$\mathcal{A} = (Q, \Sigma, \Theta, |q_0\rangle, \mathcal{O}, \mathcal{L}), \tag{11}$$

*where:*

1. *$Q$, $\Sigma$, $\Theta$, $|q_0\rangle$ and the tape symbol set $\Gamma = \Sigma \cup \{ ¢, \$\}$ are the same as those defined above in an MO-1QFA;*
2. *$\mathcal{O}$ is an observable with the set of possible eigenvalues $\mathcal{C} = \{c_1, \cdots, c_s\}$ and the projector set $\{P(c_i) : i = 1, \cdots, s\}$ where $P(c_i)$ denotes the projector onto the eigenspace corresponding to $c_i$;*
3. *$\mathcal{L} \subset \mathcal{C}^*$ is a regular language (called here a control language).*

The input word $\omega = \sigma_1 \cdots \sigma_l$ to 1QFACL $\mathcal{A}$ is in the form: $w = ¢\omega\$$ (for convenience, we denote $\sigma_0 = ¢$ and $\sigma_{l+1} = \$$). Now, we define the behavior of $\mathcal{A}$ on the word $w$. The computation starts in the state $|q_0\rangle$, and then the transformations associated with symbols in the word $w$ are applied in succession. The transformation associated with any symbol $\sigma \in \Gamma$ consists of two steps:

1. Firstly, $\Theta_\sigma$ is applied to the current state $|\phi\rangle$ of $\mathcal{A}$, yielding the new state $|\phi'\rangle = \Theta_\sigma|\phi\rangle$.
2. Secondly, the observable $\mathcal{O}$ is measured on $|\phi'\rangle$. According to quantum mechanics principle, this measurement yields result $c_k$ with probability $p_k = ||P(c_k)|\phi'\rangle||^2$, and the state of $\mathcal{A}$ collapses to $P(c_k)|\phi'\rangle/\sqrt{p_k}$.

Thus, the computation on the word $w$ leads to a string $y_0 y_1 \ldots y_{l+1} \in \mathcal{C}^*$ with probability $p(y_0 y_1 \ldots y_{l+1}|\sigma_0\sigma_1 \ldots \sigma_{l+1})$ given by

$$p(y_0 y_1 \ldots y_{l+1}|\sigma_0\sigma_1 \ldots \sigma_{l+1}) = || \prod_{i=0}^{l+1}(P(y_i)\Theta_{\sigma_i})|q_0\rangle||^2. \tag{12}$$

A computation leading to a word $y \in \mathcal{C}^*$ is said to be accepted if $y \in \mathcal{L}$. Otherwise, it is rejected. Hence, the accepting probability of 1QFACL $\mathcal{A}$ is defined as:

$$Pr[\mathcal{A}\ accepts\ \omega] = \sum_{y_0 y_1 \cdots y_{l+1} \in \mathcal{L}} p(y_0 y_1 \ldots y_{l+1}|\sigma_0\sigma_1 \ldots \sigma_{l+1}) \tag{13}$$

### 2.2 Definition of 1QCFA

In this subsection we introduce 1QCFA and its acceptance process formally and in details.

2QCFA were first introduced by Ambainis and Watrous [3], and then studied by Qiu, Yakaryilmaz and etc. [24, 28, 32–34]. 1QCFA are the one-way version of 2QCFA. Informally, we describe a 1QCFA as a DFA which has access to a quantum memory of a constant size (dimension), upon which it performs quantum transformations and measurements. Given a finite set of quantum states $Q$, we denote by $\mathcal{H}(Q)$ the Hilbert space spanned by $Q$. Let $\mathcal{U}(\mathcal{H}(Q))$ and $\mathcal{O}(\mathcal{H}(Q))$ denote the sets of unitary operators and projective measurements over $\mathcal{H}(Q)$, respectively.

**Definition 6.** *A one-way finite automata with quantum and classical states (1QCFA) $\mathcal{A}$ is specified by a 10-tuple*

$$\mathcal{A} = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}, S_{rej}) \tag{14}$$

*where:*

1. *$Q$ is a finite set of quantum states;*
2. *$S$, $\Sigma$ and the tape symbol set $\Gamma = \Sigma \cup \{ ¢, \$\}$ are the same as those defined above in a 1PFA;*
3. *$|q_0\rangle \in Q$ is the initial quantum state;*
4. *$s_0 \in S$ is the initial classical state;*
5. *$S_{acc} \subset S$ and $S_{rej} \subset S$ are the sets of classical accepting and rejecting states, respectively;*
6. *$\Theta$ is the mapping:*
$$\Theta : S \times \Gamma \to \mathcal{U}(\mathcal{H}(Q)), \tag{15}$$
*assigning to each pair $(s, \gamma)$ a unitary transformation;*
7. *$\Delta$ is the mapping:*
$$\Delta : S \times \Gamma \to \mathcal{O}(\mathcal{H}(Q)), \tag{16}$$
*where each $\Delta(s, \gamma)$ corresponds to a projective measurement (a projective measurement will be taken each time a unitary transformation is applied; if we do not need a measurement, we denote that $\Delta(s, \gamma) = I$, and we assume the result of the measurement to be $\varepsilon$ with certainty);*
8. *$\delta$ is a special transition function of classical states. Let the results set of the measurement be $\mathcal{C} = \{c_1, c_2, \ldots, c_s\}$, then*
$$\delta : S \times \Gamma \times \mathcal{C} \to S, \tag{17}$$
*where $\delta(s, \gamma)(c_i) = s'$ means that if a tape symbol $\gamma \in \Gamma$ is being scanned and the projective measurement result is $c_i$, then the state $s$ is changed to $s'$.*

Given an input $\omega = \sigma_1 \cdots \sigma_l$, the word on the tape will be $w = ¢\,\omega\$$ (for convenience, we denote $\sigma_0 = ¢$ and $\sigma_{l+1} = \$$). Now, we define the behavior of 1QCFA $\mathcal{A}$ on the word $w$. The computation starts in the classical state $s_0$ and the quantum state $|q_0\rangle$, then the transformations associated with symbols in the word $\sigma_0\sigma_1 \cdots, \sigma_{l+1}$ are applied in succession. The transformation associated with a state $s \in S$ and a symbol $\sigma \in \Gamma$ consists of three steps:

1. Firstly, $\Theta(s,\sigma)$ is applied to the current quantum state $|\phi\rangle$, yielding the new state $|\phi'\rangle = \Theta(s,\sigma)|\phi\rangle$.
2. Secondly, the observable $\Delta(s,\sigma) = \mathcal{O}$ is measured on $|\phi'\rangle$. The set of possible results is $\mathcal{C} = \{c_1, \cdots, c_s\}$. According to such a quantum mechanics principle, such a measurement yields the classical outcome $c_k$ with probability $p_k = ||P(c_k)|\phi'\rangle||^2$, and the quantum state of $\mathcal{A}$ collapses to $P(c_k)|\phi'\rangle/\sqrt{p_k}$.
3. Thirdly, the current classical state $s$ will be changed to $\delta(s,\sigma)(c_k) = s'$.

An input word $\omega$ is assumed to be accepted (rejected) if and only if the classical state after scanning $\sigma_{l+1}$ is an accepting (rejecting) state. We assume that $\delta$ is well defined so that 1QCFA $\mathcal{A}$ always accepts or rejects at the end of the computation.

Let $L \subset \Sigma^*$ and $0 \le \epsilon < 1/2$, then 1QCFA $\mathcal{A}$ recognizes $L$ with bounded error $\epsilon$ if

1. For any $\omega \in L$, $Pr[\mathcal{A}\ accepts\ \omega] \ge 1 - \epsilon$, and
2. For any $\omega \notin L$, $Pr[\mathcal{A}\ rejects\ \omega] \ge 1 - \epsilon$.

## 3 Simulation of Other Models by 1QCFA

In this section, we prove that the following automata models can be simulated by 1QCFA: DFA, coin-tossing 1PFA, MO-1QFA, MM-1QFA and 1QFACL.

**Theorem 1.** *Any $n$ states DFA $\mathcal{A} = (S, \Sigma, \delta, s_0, S_{acc})$ can be simulated by a 1QCFA $\mathcal{A}' = (Q', S', \Sigma', \Theta', \Delta', \delta', |q_0\rangle', s_0', S'_{acc}, S'_{rej})$ with 1 quantum state and $n+1$ classical states.*

*Proof.* Actually, if we do not use the quantum component of 1QCFA, the automaton is reduced to a DFA. Let $Q' = \{|q_0\rangle'\}$, $S' = S \cup \{s_r\}$, $\Sigma' = \Sigma$, $s_0' = s_0$, $S'_{acc} = S_{acc}$ and $S'_{rej} = \{s_r\}$. For any $s \in S$ and any $\sigma \in \Sigma$, let $\Theta(s,\sigma) = I$, $\Delta'(s,\sigma) = I$, and the classical transition function $\delta'$ is defined as follows:

$$\delta'(s,\sigma)(c) = \begin{cases} s, & \sigma = ¢; \\ \delta(s,\sigma), & \sigma \in \Sigma, \\ s, & \sigma = \$, s \in S'_{acc}; \\ s_r, & \sigma = \$, s \notin S'_{acc}. \end{cases} \tag{18}$$

where $c$ is the measurement result.

**Theorem 2.** *Any $n$ states coin-tossing 1PFA $\mathcal{A}^1 = (S^1, \Sigma^1, \delta^1, s_1^1, S^1_{acc})$ can be simulated by a 1QCFA $\mathcal{A}^2 = (Q^2, S^2, \Sigma^2, \Theta^2, \Delta^2, \delta^2, |q_0\rangle^2, s_0^2, S^2_{acc}, S^2_{rej})$ with 2 quantum states and $n+1$ classical states.*

*Proof.* A coin-tossing 1PFA is essentially a DFA augmented with a fair coin-flip component. In every transition, coin-tossing 1PFA can use a fair coin-flip or not freely. Using the quantum component, a 1QCFA can simulate the fair coin-flip perfectly.

**Lemma 1.** *A fair coin-flip can be simulate by 1QCFA $\mathcal{A}$ with two quantum states, a unitary operation and a projective measurement.*

*Proof.* The automaton $\mathcal{A}$ simulates a coin-flip according to the following transition functions, with $|p_0\rangle$ as the starting quantum state. We use two orthogonal basis states $|p_0\rangle$ and $|p_1\rangle$. Let a projective measurement $M = \{P_0, P_1\}$ be defined by

$$P_0 = |p_0\rangle\langle p_0|, P_1 = |p_1\rangle\langle p_1|. \tag{19}$$

The results 0 and 1 represent the results of coin-flip "head" and "tail", respectively. The corresponding unitary operation will be

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{20}$$

This operator changes the state $|p_0\rangle$ or $|p_1\rangle$ to a superposition state $|\psi\rangle$ or $|\phi\rangle$, respectively, as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|p_0\rangle + |p_1\rangle), \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|p_0\rangle - |p_1\rangle). \tag{21}$$

When measuring $|\psi\rangle$ or $|\phi\rangle$ with $M$, we will get the result 0 or 1 with probability $\frac{1}{2}$, respectively. This is similar to a coin-flip process. If the result is 0, we simulate "head" result of the coin-flip; if the result is 1, we simulate "tail" result of the coin-flip. So the Lemma is proved.

If the current state of coin-tossing 1PFA $\mathcal{A}^1$ is $s$ and the scanning symbol is $\sigma \in \Sigma$, $\mathcal{A}^1$ makes a coin-flip. The current state of $\mathcal{A}^1$ will change to $t_1$ or $t_2$, in both cases with probability $\frac{1}{2}$. We use a 1QCFA $\mathcal{A}^2$ to simulate this step as follows:

1. Use the quantum component of 1QCFA $\mathcal{A}^2$ to simulate a fair coin-flip. We assume the outcome to be 0 or 1.
2. We define $\delta^2(s, \sigma)(0) = t_1$ and $\delta^2(s, \sigma)(1) = t_2$.

The other parts of the simulation are similar to the one described in the proof of Theorem 1.

**Theorem 3.** *Any $n$ quantum states MO-1QFA $\mathcal{A}^1 = (Q^1, \Sigma^1, \Theta^1, |q_0\rangle^1, Q^1_{acc})$ can be simulated by a 1QCFA $\mathcal{A}^2 = (Q^2, S^2, \Sigma^2, \Theta^2, \Delta^2, \delta^2, |q_0\rangle^2, s^2_0, S^2_{acc}, S^2_{rej})$ with $n$ quantum states and 3 classical states.*

*Proof.* We use the quantum component of 1QCFA to simulate the evolution of quantum states of MO-1QFA and use the classical states of 1QCFA to calculate the accepting probability. Let $Q^2 = Q^1$, $S^2 = \{s^2_0, s^2_a, s^2_r\}$, $\Sigma^2 = \Sigma^1$, $|q_0\rangle^2 = |q_0\rangle^1$, $S^2_{acc} = \{s^2_a\}$ and $S^2_{rej} = \{s^2_r\}$. For any current classical state $s$ and scanning symbol $\sigma$, the quantum transition function is defined to be

$$\Theta^2(s, \sigma) = \Theta^1(\sigma). \tag{22}$$

The measurement function is defined to be

$$\Delta^2(s,\sigma) = \begin{cases} I, & \sigma \neq \$; \\ \{P_a, P_r\}, & \sigma = \$. \end{cases} \tag{23}$$

where $P_a = \sum_{q\in Q_{acc}} |q\rangle\langle q|$, $P_r = I - P_a$. If we assume the outcome to be $c_a$ or $c_r$, then the classical transition function will be defined to be

$$\delta^2(s,\sigma)(c) = \begin{cases} s, & \sigma \neq \$; \\ s_a^2, & \sigma = \$, c = c_a; \\ s_r^2, & \sigma = \$, c = c_r. \end{cases} \tag{24}$$

**Theorem 4.** *Any $n$ quantum states MM-1QFA $\mathcal{A}^1 = (Q^1, \Sigma^1, \Theta^1, |q_0\rangle^1, Q^1_{acc}, Q^1_{rej})$ can be simulated by a 1QCFA $\mathcal{A}^2 = (Q^2, S^2, \Sigma^2, \Theta^2, \Delta^2, \delta^2, |q_0\rangle^2, s_0^2, S^2_{acc}, S^2_{rej})$ with $n$ quantum states and 3 classical states.*

*Proof.* We use the quantum component of 1QCFA to simulate both the evolution of quantum states of MM-1QFA and its projective measurements. We use the classical states of 1QCFA to calculate the accepting and rejecting probability. Let $Q^2 = Q^1$, $S^2 = \{s_0^2, s_a^2, s_r^2\}$, $\Sigma^2 = \Sigma^1$, $|q_0\rangle^2 = |q_0\rangle^1$, $S^2_{acc} = \{s_a^2\}$ and $S^2_{rej} = \{s_r^2\}$. For any current classical state $s$ and any scanning symbol $\sigma$, the quantum transition function is defined to be

$$\Theta^2(s,\sigma) = \Theta^1(\sigma). \tag{25}$$

The measurement function is defined to be

$$\Delta^2(s,\sigma) = \{P_a, P_r, P_n\}, \tag{26}$$

where $P_a = \sum_{q\in Q_{acc}} |q\rangle\langle q|$, $P_r = \sum_{q\in Q_{rej}} |q\rangle\langle q|$ and $P_n = \sum_{q\in Q_{non}} |q\rangle\langle q|$. If we assume the classical outcomes to be $c_a$, $c_r$ or $c_n$, then the classical transition function will be defined to be

$$\delta^2(s,\sigma)(c) = \begin{cases} s_a^2, & s = s_a^2; \\ s_r^2, & s = s_r^2; \\ s_a^2, & s = s_0^2, c = c_a; \\ s_r^2, & s = s_0^2, c = c_r; \\ s_0^2, & s = s_0^2, c = c_n, \sigma \neq \$; \\ s_r^2, & s = s_0^2, c = c_n, \sigma = \$. \end{cases} \tag{27}$$

Although 1QFACL can accept all regular languages, their behavior seems to be rather complicated. We prove that any 1QFACL can be simulated by a 1QCFA with an easy to understand behavior.

**Theorem 5.** *Any $n$ quantum states 1QFACL $\mathcal{A}^1 = (Q^1, \Sigma^1, \Theta^1, |q_0\rangle^1, \mathcal{O}^1, \mathcal{L}^1)$, whose control language $\mathcal{L}^1$ can be recognized by an $m$ states DFA $\mathcal{A} = (S, \Sigma, \delta, s_0, S_{acc})$, can be simulated by a 1QCFA $\mathcal{A}^2 = (Q^2, S^2, \Sigma^2, \Theta^2, \Delta^2, \delta^2, |q_0\rangle^2, s_0^2, S^2_{acc}, S^2_{rej})$ with $n$ quantum states and $m + 1$ classical states.*

*Proof.* We use the quantum component of 1QCFA to simulate the evolution of quantum states of 1QFACL and also its projective measurements. We use the classical states of 1QCFA to simulate DFA $\mathcal{L}^1$. Let $Q^2 = Q^1$, $S^2 = S \cup \{s_r\}$, $\Sigma^2 = \Sigma^1$, $s_0^2 = s_0$, $|q_0\rangle^2 = |q_0\rangle^1$, $S_{acc}^2 = S_{acc}$ and $S_{rej}^2 = \{s_r\}$. For any current classical state $s$ and any scanning symbol $\sigma$, the quantum transition function will be defined to be

$$\Theta^2(s, \sigma) = \Theta^1(\sigma). \tag{28}$$

The measurement function is defined to be

$$\Delta^2(s, \sigma) = \{P(c_i) : i = 1, \cdots, t\}, \tag{29}$$

where $P(c_i)$ denotes the projector onto the eigenspace corresponding to $c_i$. We assume that the set of possible classical outcomes is $\mathcal{C} = \{c_1, \cdots, c_t\}$, where $\mathcal{C} = \Sigma$, then the classical transition function will be defined to be

$$\delta^2(s, \sigma)(c) = \begin{cases} \delta(s, c), & \sigma \neq \$; \\ \delta(s, c), & \sigma = \$, \delta(s, c) \in S_{acc}; \\ s_r, & \sigma = \$, \delta(s, c) \notin S_{acc}. \end{cases} \tag{30}$$

## 4 Closure Properties of Languages Accepted by 1QCFA

For convenience, we denote by 1QCFA($\epsilon$) the classes of languages recognized by 1QCFA with bounded error $\epsilon$. Moreover, let $QS(\mathcal{A})$ and $CS(\mathcal{A})$ denote the numbers of quantum states and classical states of a 1QCFA $\mathcal{A}$. We start to consider the operation of complement.

**Theorem 6.** *If $L \in 1QCFA(\epsilon)$, then also $L^c \in 1QCFA(\epsilon)$, where $L^c$ is the complement of $L$.*

*Proof.* Let a 1QCFA($\epsilon$) $\mathcal{A} = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}, S_{rej})$ accept $L$ with a bounded error $\epsilon$. We can construct the 1QCFA $\mathcal{A}^c$ only by exchanging the classical accepting and rejecting states in $\mathcal{A}$. That is, $\mathcal{A}^c = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}^c, S_{rej}^c)$, where $S_{acc}^c = S_{rej}$, $S_{rej}^c = S_{acc}$ and the other components remain the same as those defined in $\mathcal{A}$. Afterwards we have:

1. If $\omega \in L^c$, then $\omega \notin L$. Indeed, for an input $\omega$, $\mathcal{A}$ will enter a rejecting state with probability at least $1 - \epsilon$ at the end of the computation. With the same input $\omega$, $\mathcal{A}^c$ will enter an accepting state with probability at least $1 - \epsilon$ at the end of the computation. Hence, $\mathcal{A}^c$ accepts $\omega$ with the probability at least $1 - \epsilon$;
2. The case $\omega \notin L^c$ is treated in a symmetric way.

*Remark 1.* According to the construction given above, if $QS(\mathcal{A}) = n$, $CS(\mathcal{A}) = m$, then $QS(\mathcal{A}^c) = n$, $CS(\mathcal{A}^c) = m$.

**Theorem 7.** *If $L_1 \in 1QCFA(\epsilon_1)$ and $L_2 \in 1QCFA(\epsilon_2)$, then $L_1 \cap L_2 \in 1QCFA(\epsilon)$, where $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$.*

*Proof.* Let $\mathcal{A}^i = (Q^i, S^i, \Sigma^i, \Theta^i, \Delta^i, \delta^i, |q_0\rangle^i, s_0^i, S_{acc}^i, S_{rej}^i)$ be 1QCFA to recognize $L_i$ with bounded error $\epsilon_i$ $(i = 1, 2)$. We construct a 1QCFA $\mathcal{A} = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}, S_{rej})$ where:

1. $Q = Q^1 \otimes Q^2$,
2. $S = S^1 \times S^2$,
3. $\Sigma = \Sigma^1 \cap \Sigma^2$,
4. $s_0 = \langle s_0^1, s_0^2 \rangle$,
5. $|q_0\rangle = |q_0\rangle^1 \otimes |q_0\rangle^2$,
6. $S_{acc} = S_{acc}^1 \times S_{acc}^2$,
7. $S_{rej} = (S_{acc}^1 \times S_{rej}^2) \cup (S_{rej}^1 \times S_{acc}^2) \cup (S_{rej}^1 \times S_{rej}^2)$
8. For any classical state $s = \langle s^1, s^2 \rangle \in S$ and any $\sigma \in \Sigma$, the quantum transition function of $\mathcal{A}$ is defined to be
$$\Theta(s, \sigma) = \Theta(\langle s^1, s^2 \rangle, \sigma) = \Theta^1(s^1, \sigma) \otimes \Theta^2(s^2, \sigma). \tag{31}$$
9. For any classical state $s = \langle s^1, s^2 \rangle \in S$ and any $\sigma \in \Sigma$, the measurement function of $\mathcal{A}$ is defined to be
$$\Delta(s, \sigma) = \Delta(\langle s^1, s^2 \rangle, \sigma) = \Delta^1(s^1, \sigma) \otimes \Delta^2(s^2, \sigma). \tag{32}$$
As classical measurements outcomes are then tuples $c_{ij} = \langle c_i, c_j \rangle$.
10. For any classical state $s = \langle s^1, s^2 \rangle \in S$ and any $\sigma \in \Sigma$, the classical transition function of $\mathcal{A}$ is defined to be
$$\delta(s, \sigma)(c_{ij}) = \delta(\langle s^1, s^2 \rangle, \sigma)(\langle c_i, c_j \rangle) = \langle \delta^1(s^1, \sigma)(c_i), \delta^2(s^2, \sigma)(c_j) \rangle. \tag{33}$$

In terms of the 1QCFA $\mathcal{A}$ constructed above, for any $\omega \in \Sigma^*$, we have:

1. If $\omega \in L_1 \cap L_2$, then $\mathcal{A}$ will enter a state $\langle t_1, t_2 \rangle \in S_{acc}^1 \times S_{acc}^2$ at the end of the computation with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$. $\mathcal{A}$ accepts $\omega$ with the probability at least $(1 - \epsilon_1)(1 - \epsilon_2) = 1 - (\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2)$.
2. If $\omega \in L_1$ but $\omega \notin L_2$, then $\mathcal{A}$ will enter a state $\langle t_1, t_2 \rangle \in S_{acc}^1 \times S_{rej}^2$ at the end of the computation with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$. $\mathcal{A}$ rejects $\omega$ with the probability at least $1 - (\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2)$.
3. The case $\omega \notin L_1$ but $\omega \in L_2$ is symmetric to the previous one and therefore the same is the outcome.
4. If $\omega \notin L_1$ and $\omega \notin L_2$, then $\mathcal{A}$ will enter a state $\langle t_1, t_2 \rangle \in S_{rej}^1 \times S_{rej}^2$ at the end of the computation with probability at least $(1 - \epsilon_1)(1 - \epsilon_2)$. $\mathcal{A}$ rejects $\omega$ with the probability at least $1 - (\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2)$.

So $L_1 \cap L_2 \in 1QCFA(\epsilon)$.

*Remark 2.* According to the construction given above, let $QS(\mathcal{A}^1) = n_1$, $CS(\mathcal{A}^1) = m_1$, $QS(\mathcal{A}^2) = n_2$ and $CS(\mathcal{A}^2) = m_2$, then $QS(\mathcal{A}) = n_1 n_2$, $CS(\mathcal{A}) = m_1 m_2$.

A similar outcome holds for the union operation.

**Theorem 8.** *If $L_1 \in 1QCFA(\epsilon_1)$ and $L_2 \in 1QCFA(\epsilon_2)$, then $L_1 \cup L_2 \in 1QCFA(\epsilon)$, where $\epsilon = \epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$.*

*Proof.* Let $\mathcal{A}^i = (Q^i, S^i, \Sigma^i, \Theta^i, \Delta^i, \delta^i, |q_0\rangle^i, s_0^i, S_{acc}^i, S_{rej}^i)$ be 1QCFA to recognize $L_i$ with bounded error $\epsilon_i$ ($i = 1, 2$). The construction of the 1QCFA $\mathcal{A} = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0\rangle, s_0, S_{acc}, S_{rej})$ is the same as in the proof of Theorem 7 except for $S_{acc}$ and $S_{rej}$. We define $S_{acc} = (S_{acc}^1 \times S_{rej}^2) \cup (S_{rej}^1 \times S_{acc}^2) \cup (S_{acc}^1 \times S_{acc}^2)$ and $S_{rej} = S_{rej}^1 \times S_{rej}^2$. The rest of the proof is similar to the proof in Theorem 7.

*Remark 3.* In the last proof the set of input symbols was defined as $\Sigma = \Sigma^1 \cap \Sigma^2$. Actually, if we take $\Sigma = \Sigma^1 \cup \Sigma^2$, the theorem still holds. In that case, we extend $\Sigma^i$ to $\Sigma$ by adding a rejecting classical state $s_r^i$ to $\mathcal{A}^i$. For any classical state $s^i \in S^i$ and $\sigma^i \notin \Sigma^i$, the quantum transition function is defined to be $\Theta^i(s^i, \sigma^i) = I$, the measurement function is defined to be $\Delta^i(s^i, \sigma^i) = I$. We assume the measurement result to be $c$, then the classical transition function will be defined to be $\delta^i(s^i, \sigma^i)(c) = s_r^i$. For the new adding state $s_r^i$, we define the transition functions as follow: for any $\sigma \in \Sigma$, $\Theta^i(s_r^i, \sigma) = I$, $\Delta^i(s_r^i, \sigma) = I$, $\delta^i(s_r^i, \sigma)(c) = s_r^i$, where $c$ is the the measurement result.

## 5 Succinctness Results

State complexity and succinctness results are an important research area of classical automata theory, see [31], with a variety of applications. Once quantum versions of classical automata were introduced and explored, it started to be of large interest to find out through succinctness results a relation between the power of classical and quantum automata model. This has turned out to be an area of surprising outcomes that again indicated that relations between classical and corresponding quantum automata models is intriguing. For example, it has been shown, see [2, 4, 5, 13], that for some languages 1QFA require exponentially less states that classical 1FA, but for some other languages it can be in an opposite way.

Since 1QCFA can simulate both 1FA and 1QFA, and in this way they combine the advantages of both of these models, it is of interest to explore the relation between the state complexity of languages for the case that they are accepted by 1QCFA and MM-1QFA and this we will do in this section.

The main result we obtain when considering languages $L_m = \{a^*b^* \mid |a^*b^*| = km, k = 1, 2, \cdots\}$, where $m$ is a prime. For survey on the famous language $\{a^* \mid |a^*| = km, k = 1, 2, \cdots\}$, the reader may refer to [7].

Obviously, there exist a $2m + 2$ states DFA, depicted in Figure 1 that accepts $L_m$.

**Lemma 2.** *DFA $\mathcal{A}$ depicted in Figure 1 is minimal.*

*Proof.* We show that any two different state $s$ and $t$ are distinguishable (i.e., there exists a string $z$ such that exactly one of the following states $\widehat{\delta}(p, z)$[2] or $\widehat{\delta}(q, z)$ is an accepting state [31]).

[2] For any string $x \in \Sigma^*$ and any $\sigma \in \Sigma$, $\widehat{\delta}(s, \sigma x) = \widehat{\delta}(\delta(s, \sigma), x)$; if $|x| = 0$, $\widehat{\delta}(s, x) = s$ [11].

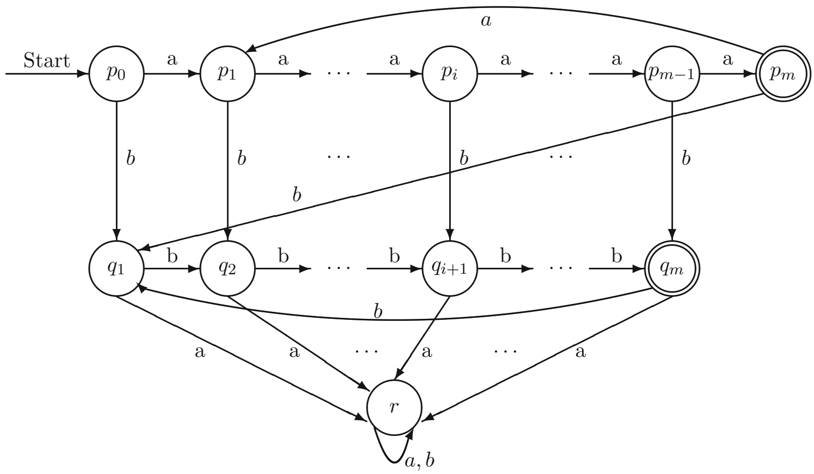


**Fig. 1.** DFA $\mathcal{A}$ recognizing $L_m$

1. For $0 \leq i \leq m$, $0 \leq j \leq m$ and $i \neq j$, we have $\widehat{\delta}(p_i, a^{m-i}) = p_m$ and $\widehat{\delta}(p_j, a^{m-i}) = p_k$, where $k \neq m$. Hence, $p_i$ and $p_j$ are distinguishable.
2. For $1 \leq i \leq m$, $1 \leq j \leq m$ and $i \neq j$, we have $\widehat{\delta}(q_i, b^{m-i}) = q_m$ and $\widehat{\delta}(q_j, b^{m-i}) = q_k$, where $k \neq m$. Hence, $q_i$ and $q_j$ are distinguishable.
3. For $0 \leq i \leq m$ and $1 \leq j \leq m$, we have $\widehat{\delta}(p_i, a^{m-i}) = p_m$ and $\widehat{\delta}(q_j, a^{m-i}) = r$. Hence, $p_i$ and $q_j$ are distinguishable.
4. Obviously, the state $r$ is distinguishable from any other state $s$.

Therefore, the Lemma has been proved.

**Lemma 3 ([2, 18]).** *For any prime $m$, any 1PFA recognizing $L_m$ with probability $1/2 + \epsilon$, for a fixed $\epsilon > 0$, has at least $m$ states.*

*Remark 4.* The proof can be obtained by an easy modification of the proof from the paper [2] where the state complexity of the language $L_p = \{a^i \mid i \text{ is divisible by } p\}$ is considered.

**Lemma 4 ([2]).** *(Forbidden construction) Let $L$ be a regular language, and let $\mathcal{A}$ be its minimal DFA. Assume that there is a word $w$ such that $\mathcal{A}$ contains states $s, t$ (a forbidden construction) satisfying:*

*1. $s \neq t$,*
*2. $\widehat{\delta}(s, x) = t$,*
*3. $\widehat{\delta}(t, x) = t$ and*
*4. $t$ is neither "all-accepting" state, nor "all-rejecting" state (i.e., there exist strings $u$ and $v$ such that $\widehat{\delta}(t, u)$ is an accepting state and $\widehat{\delta}(t, v)$ is not an accepting state).*

*Then $L$ cannot be recognized by an MM-1QFA with bounded error $\frac{7}{9}+\epsilon$ for any fixed $\epsilon > 0$.*

**Theorem 9.** *For any fixed $\epsilon > 0$, $L_m$ cannot be recognized by an MM-1QFA with bounded error $\frac{7}{9}+\epsilon$.*

*Proof.* According to Lemma 4, we know that $L_m$ cannot be accepted by any MM-1QFA with bounded error $\frac{7}{9}+\epsilon$ since its minimal DFA (see Figure 1) contains the "Forbidden construction" of Lemma 4. For example, we can take $s = p_0$, $t = p_m$, $x = a^m$, then we have $\widehat{\delta}(p_0, a^m) = p_m$, $\widehat{\delta}(p_m, a^m) = p_m$, $\widehat{\delta}(p_m, b^m) = q_m$ and $\widehat{\delta}(p_m, ba) = r$.

Let $L_1 = \{a^*b^*\}$ and $L_2 = \{w \mid w \in \{a,b\}^*, |w| = km, k = 1, 2, \cdots\}$ where $m$ is a prime. So we have $L_m = L_1 \cap L_2$. We will show $L_1$ and $L_2$ can be recognized by 1QCFA.

**Lemma 5.** *The language $L_1$ can be recognized by a 1QCFA $\mathcal{A}^1$ with certainty with 1 quantum state and 4 classical states.*

*Proof.* $L_1$ can be accepted by a DFA $\mathcal{A}$ with 3 classical states (see Figure 2). According to Theorem 1, $\mathcal{A}$ can be simulated by a 1QCFA $\mathcal{A}^1$ with 1 quantum state and 4 classical states.

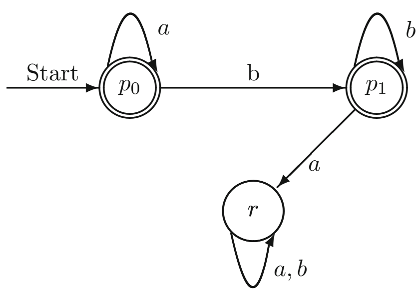


**Fig. 2.** A DFA recognizing the language $L_1$

**Lemma 6 ([2]).** *For any $\epsilon > 0$, there is an MM-1QFA $\mathcal{A}$ with $\mathbf{O}(\log m)$ quantum states recognizing $L_2$ with a bounded error $\epsilon$.*

**Lemma 7.** *For any $\epsilon > 0$, there is a 1QCFA $\mathcal{A}^2$ with $\mathbf{O}(\log m)$ quantum states and 3 classical states recognizing $L_2$ with a bounded error $\epsilon$.*

*Proof.* According to Lemma 6, there is an MM-1QFA $\mathcal{A}$ with $\mathbf{O}(\log m)$ quantum states recognizing $L_2$ with bounded error $\epsilon$. According to Theorem 4, an $\mathbf{O}(\log m)$ quantum states MM-1QFA $\mathcal{A}$ can be simulated by a 1QCFA with $\mathbf{O}(\log m)$ quantum states and 3 classical states.

**Theorem 10.** *For any $\epsilon > 0$, $L_m$ can be recognized by a 1QCFA with* $\mathbf{O}(\log m)$ *quantum states and 12 classical states with a bounded error $\epsilon$.*

*Proof.* $L_m = L_1 \cap L_2$. According to Lemma 5, the language $L_1$ can be recognized by 1QCFA $\mathcal{A}^1$ with 1 quantum state and 4 classical states with certainty (i.e., $\epsilon_1 = 0$). According to Lemma 7, for any $\epsilon > 0$, the language $L_2$ can be recognized by 1QCFA $\mathcal{A}^2$ with $\mathbf{O}(\log m)$ quantum states and 3 classical states with a bounded error $\epsilon$. According to Theorem 7, 1QCFA is closed under intersection. Hence, there is a 1QCFA $\mathcal{A}$ recognize $L_m$ with a bounded error $\epsilon$. Therefore $QS(\mathcal{A}^1) = 1$, $CS(\mathcal{A}^1) = 4$, $QS(\mathcal{A}^2) = \mathbf{O}(\log m)$ and $CS(\mathcal{A}^2) = 3$, so $QS(\mathcal{A}) = QS(\mathcal{A}^1) \times QS(\mathcal{A}^2) = \mathbf{O}(\log m)$, $CS(\mathcal{A}) = CS(\mathcal{A}^1) \times CS(\mathcal{A}^2) = 12$.

## 6 Conclusions

2QCFA were introduced by Ambainis and Watrous [3]. In this paper, we investigated the one-way version of 2QCFA, namely 1QCFA. Firstly, we gave a formal definition of 1QCFA. Secondly, we showed that DFA, coin-tossing 1PFA, MO-1QFA, MM-1QFA and 1QFACL can be simulated by 1QCFA. As we know, the behavior of 1QFACL seems to be rather complicated. However, when we used a 1QCFA to simulate a 1QFACL, the behavior of 1QCFA started to be seen as quite natural. Thirdly, we studied closure properties of languages accepted by 1QCFA, and we proved that the family of languages accepted by 1QCFA is closed under intersection, union, and complement. Fourthly, for any fixed $\epsilon_1 > 0$ and any prime $m$ we have showed that the language $L_m = \{a^*b^* \mid |a^*b^*| = km, k = 1, 2, \cdots\}$, cannot be recognized by any MM-1QFA with bounded error $\frac{7}{9} + \epsilon_1$, and any 1PFA recognizing it has at last $m$ states, but $L_m$ can be recognized by a 1QCFA for any error bound $\epsilon > 0$ with $\mathbf{O}(\log m)$ quantum states and 12 classical states. Thus, 1QCFA can make use of merits of both 1FA and 1QFA.

To conclude, we would like to propose some problems for further consideration.

1. How about the state complexity of 1QCFA compared with other 1QFA for recognizing the same languages, such as one-way quantum finite automata together with classical states in [26]?
2. Are 1QCFA closed under catenation and reversal?

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