Incentive Ratios of Fisher Markets

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Abstract. In a Fisher market, a market maker sells m items to n potential buyers. The buyers submit their utility functions and money endowments to the market maker, who, upon receiving submitted information, derives market equilibrium prices and allocations of its items. While agents may benefit by misreporting their private information, we show that the percentage of improvement by a unilateral strategic play, called incentive ratio, is rather limited—it is less than 2 for linear markets and at most $e^{1/e} \approx 1.445$ for Cobb-Douglas markets. We further prove that both ratios are tight.

1 Introduction

The Internet and world wide web have created a possibility for buyers and sellers to meet at a marketplace where pricing and allocations can be determined more efficiently and effectively than ever before. Market equilibrium, which ensures optimum fairness and efficiency, has become a paradigm for practical applications. It is well known that a market equilibrium always exists given mild assumptions on the utility functions of participating individuals [3].

However, there has been a major criticism on the market equilibrium in that it has not taken strategic behaviors of buyers and sellers into consideration: In a Fisher market, a market equilibrium price vector and associated allocations, computed in terms of utility functions and money endowments of participating individuals, may change even if one participant has a change in its utility function or endowment. Hence, an individual may misreport its private information to divert to a favorable outcome.

This phenomenon was first formally described by Adsul et al. [1] for linear and Chen et al. [5] for Leontief utility functions. Existence of such manipulations may impede potential uses of market equilibrium as a solution mechanism. To overcome such limitations, we explore the effect of a participant's incentive on the market equilibrium mechanism. We adopt the notion of *incentive ratio* [5] as the factor of the largest utility gains that a participant may achieve by behaving

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strategically in the full information setting (the formal definition is referred to Section 2). The ratio characterizes the extent to which utilities can be increased by manipulations of individuals. Similar ideas have been applied in auctions under the concept of approximate strategic-proofness such as in [19,14].

While the big space of manipulations suggest that one may substantially increase his utility by behaving strategically, surprisingly, it was shown in [5] that for any Leontief utility market, the incentive ratio is upper bounded by 2. In this paper, we further study the incentive ratios of two other important functions: linear and Cobb-Douglas utilities [2]. For both utility models, manipulations do help to improve one's obtained utility. The following example shows such a case in a Cobb-Douglas market (a similar example for linear markets can be found in [1]).

Example 1. In a Cobb-Douglas market, there are two items with unit supply each and two buyers with endowments $e_1 = \frac{1}{2}, e_2 = \frac{1}{2}$ and utility functions $u_1(x,y) = x^{\frac{1}{4}}y^{\frac{3}{4}}, u_2(x,y) = x^{\frac{3}{4}}y^{\frac{1}{4}}$, respectively. When both buyers bid their utility functions and endowments truthfully, the equilibrium price is $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$, and the equilibrium allocations are $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, \frac{1}{4})$; their utilities are $u_1 = u_2 = (\frac{1}{4})^{\frac{1}{4}}(\frac{3}{4})^{\frac{3}{4}}$. If buyer 1 strategically reports $u_1'(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$, then the equilibrium price is $\mathbf{p'} = (\frac{5}{8}, \frac{3}{8})$, and the equilibrium allocations are $(\frac{2}{5}, \frac{2}{3})$ and $(\frac{3}{5}, \frac{1}{3})$; their utilities are $u_1' = (\frac{2}{5})^{\frac{1}{4}}(\frac{2}{3})^{\frac{3}{4}}$ and $u_2' = (\frac{3}{5})^{\frac{3}{4}}(\frac{1}{3})^{\frac{1}{4}}$. Hence, $u_1' > u_1$ and the first buyer gets a strictly larger utility.

Our main results are the following, which bound the incentive ratios of linear and Cobb-Douglas markets.

Theorem. For any linear utility market, the incentive ratio is less than 2; and for any Cobb-Douglas utility market, the incentive ratio is at most $e^{1/e} \approx 1.445$. Both ratios are tight.

Our results give a further evidence for the solution concept of market equilibrium to be used in practical applications—while one may improve his utility by (complicated) manipulations, the increment is reasonably bounded by a small constant. Therefore, in a marketplace especially with incomplete information and a large number of participants, identifying a manipulation strategy is rather difficult and may be worthless. This echoes the results of, e.g., [18,13], saying that, in certain marketplaces, the fraction of participants with incentives to misreport their bids approaches zero as the market becomes large.

Our proof for the incentive ratio of linear markets is built on a reduction from fractional equilibrium allocations to an instance with integral equilibrium allocations, preserving the incentive ratio. We then, using the seminal Karush, Kuhn, Tucker (KKT) condition, show that if any participant is able to improve his utility by a factor of at least 2, everyone else can simultaneously obtain a utility increment. This, at a high level view, contradicts the market equilibrium condition of the original setting.

For Cobb-Douglas utility markets, our proof lies on a different approach by revealing interconnections of the incentive ratios of markets with different sizes.

In particular, we prove that the incentive ratio is independent of the number of buyers, by showing a reduction from any n-buyer market to a 2-buyer market, and vice versa. This result implies that the size of a market is not a factor to affect the largest possible utility gain by manipulations. Given this property, we restrict on a market with 2 buyers to bound the incentive ratio.

1.1 Related Work

Eisenberg and Gale [11] introduced a convex program to capture market equilibria of Fisher markets with linear utilities. Their convex program can be solved in polynomial time using the ellipsoid algorithm [12] and interior point algorithm [21]. Devanur et al. [9] gave the first combinatorial polynomial time algorithm for computing a Fisher market equilibrium with linear utility functions. The first strongly polynomial time algorithm for this problem was recently given by Orlin [17]. For Cobb-Douglas markets, Eaves [10] gave the necessary and sufficient conditions for existence of a market equilibrium, and gave an algorithm to compute a market equilibrium in polynomial time. Other computational studies on different market equilibrium models and utilities can be found in, e.g., [7,6,4,12,8] and the references within.

The concept of incentive ratio, which quantifies the benefit of unilateral strategic plays from a single participant, is in spirit similar to approximate truthfulness [19,14]. In the study of incentive ratio, the focus is on classic market designs with a stable outcome. It makes no attempt to consider the mechanism itself, but rather, focuses on individual's strategic plays and measures his benefit due to the incentive incompatibility of the mechanism.

Organization. In Section 2, we define the market equilibrium mechanism model and the notion of incentive ratio. In Section 3 and 4, we consider linear and Cobb-Douglas utility markets and derive matching incentive ratios, respectively. We conclude our work in Section 5.

2 Preliminary

In a Fisher market M, there are a set of n buyers and a set of m divisible items of unit quantity each for sale. We denote by $[n] = \{1, 2, \cdots, n\}$ and $[m] = \{1, 2, \cdots, m\}$ the set of buyers and items, respectively. Each buyer i has an initial cash endowment $e_i > 0$, which is normalized to be $\sum_{i=1}^{n} e_i = 1$, and has a utility function $u_i(x_i)$, where $x_i = (x_{i1}, \ldots, x_{im}) \in [0,1]^m$ is an allocation vector denoting the amount that i receives from each item j.

An outcome of the market is a tuple $(\mathbf{p}, \mathbf{x})$, where $\mathbf{p} = (p_1, \ldots, p_m)$ is a price vector of all items and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an allocation vector. An outcome is called a *market equilibrium* if the following conditions hold: (i) All items are sold out, i.e., $\sum_{i=1}^{n} x_{ij} = 1$ for $j \in [m]$, and (ii) each buyer gets an allocation that maximizes its utility under the constraint $\sum_{j\in[m]} x_{ij} p_j \leq e_i$ for the given price vector. Such an equilibrium solution exists under a mild condition [3] on the utility functions.

One extensively studied class of utility functions is that of Constant Elasticity of Substitution (CES) functions [20]: For each i, its utility function $u_i(x_i) = \left(\sum_{j=1}^{m} \alpha_{ij} x_{ij}^{\rho}\right)^{\frac{1}{\rho}}$, where $-\infty < \rho < 1$ and $\rho \neq 0$, and $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im}) \geq 0$ is a given vector associated with buyer i. The CES utility functions allow us to model a wide range of realistic preferences of buyers, and have been shown to derive, in the limit, a number of special classes. In this paper, we will consider linear and Cobb-Douglas utility functions, which are derived when $\rho \to 1$ and $\rho \to 0$, respectively.

2.1 Incentive Ratio

Notice that a market equilibrium output crucially depends on the utility functions and endowments that buyers hold. This implies, in particular, that if a buyer manipulates his function or endowment, the outcome would be changed, and the buyer may possibly obtain a larger utility. This phenomenon has been observed for, e.g., linear [1] and Leontief functions [5]. One natural question is that how much such benefits can be obtained from manipulations with respect to the given market equilibrium rule (i.e., mechanism).

To this end, Chen et al. [5] defined the notion of incentive ratio to characterize such utility improvements by manipulations. Formally, in a given market M, for each buyer $i \in [n]$, let $u_i(\cdot)$ be his true private utility function and U_i be the space of utility functions that i can feasibly report; note that $u_i \in U_i$. Define $U = U_1 \times U_2 \times \cdots \times U_n$ and $U_{-i} = U_1 \times \cdots \times U_{i-1} \times U_{i+1} \times \cdots \times U_n$. Another private information that every buyer holds is his endowment e_i. For a given input, a vector of utility functions $(u_1, \ldots, u_n) \in U$ and a vector of endowments $(e_1, \ldots, e_n)$, we denote by $x_i(u_1, \ldots, u_n; e_1, \ldots, e_n)$ the equilibrium allocation of buyer i. In the market equilibrium mechanism, a buyer i can report any utility function $u_i' \in U_i$ and endowment $e_i' \in \mathbb{R}^+$. The *incentive ratio* of buyer i in the market M is defined to be

$$\zeta_i^M = \max_{u_{-i} \in U_{-i}, e_{-i} \in \mathbb{R}^{+n-1}} \frac{\max_{u_i' \in U_i, e_i' \in \mathbb{R}^+} u_i(x_i(u_i', u_{-i}; e_i', e_{-i}))}{u_i(x_i(u_i, u_{-i}; e_i, e_{-i}))}.$$

In the above definition, the numerator is the largest possible utility of buyer i when he unilaterally changes his bid.[1,2] The incentive ratio of the market with respect to a given space of utility functions U is defined as $\zeta^M = \max_{i \in [n]} \zeta_i^M$. Incentive ratio quantifies the benefit of strategic behaviors of each individual buyer.

[1] Note that for some utility functions, an equilibrium allocation may not be unique. This may lead to different true utilities for a given manipulation bid. Our definition of incentive ratio is the strongest in the sense that it bounds the largest possible utility in all possible equilibrium allocations, which include, of course, the best possible allocation.

[2] Practically, a buyer can bid any endowment e_i'. However, reporting a larger budget results in a deficit in a resulting equilibrium, and thus, a negative utility. We therefore assume without loss of generality that $e_i' \leq e_i$ for all buyers.

For the considered linear and Cobb-Douglas (or any other CES) utilities, the true utility functions are characterized by the parameters $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where each $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})$. The definition of incentive ratio can be simplified as follows

$$\zeta^{\text{CES}} = \max_{i \in [n]} \max_{\alpha_{-i}, e_{-i}} \frac{\max_{\alpha'_i, e'_i} u_i(x_i(\alpha'_i, \alpha_{-i}; e'_i, e_{-i}))}{u_i(x_i(\alpha_i, \alpha_{-i}; e_i, e_{-i}))}.$$

3 Linear Utility Functions

In this section, we will consider incentive ratio for linear utility functions, i.e., $u_i(x_i) = \sum_{j \in [m]} \alpha_{ij} x_{ij}$. Our main result in this section is the following.

Theorem 1. *The incentive ratio of linear markets is*

$$\zeta^{\text{linear}} < 2.$$

Consider a given linear market M and an arbitrary input scenario (α, e) where every buyer i bids utility vector $\alpha_i = (\alpha_{ij})_{j \in [m]}$ and endowment e_i. Let $(\boldsymbol{p}, \boldsymbol{x})$ be a market equilibrium of the instance (α, e). Let $r_i = \frac{u_i(x_i)}{e_i}$ be the bang-per-buck of buyer i, where $x_i = (x_{ij})_{j \in [m]}$ is the allocation of buyer i in the equilibrium. The Karush, Kuhn, Tucker (KKT) condition [16] implies that for any item j, if $x_{ij} > 0$ then $\frac{\alpha_{ij}}{p_j} = r_i$, and if $x_{ij} = 0$ then $\frac{\alpha_{ij}}{p_j} \leq r_i$.

Consider any fixed buyer, say i^*, and a scenario when all other buyers keep their bids and i^* unilaterally changes his bid to $\alpha'_{i^*} = (\alpha'_{i^*j})_{j \in [m]}$ and e'_{i^*}. Denote the resulting instance by $(\alpha', e') = (\alpha'_{i^*}, \alpha_{-i^*}; e'_{i^*}, e_{-i^*})$ and its equilibrium by $(\boldsymbol{p}', \boldsymbol{x}')$. For each buyer $i \in [n]$, define

$$c_i = \frac{u_i(x'_i)}{u_i(x_i)} = \frac{\sum_{j \in [m]} \alpha_{ij} x'_{ij}}{\sum_{j \in [m]} \alpha_{ij} x_{ij}}$$

to be the factor of utility changes of the buyer. Note that c_{i^*} gives the factor of how much more utility that i^* can get by manipulation. For the new setting (α', e'), the utility of buyer i is changed by a factor of c_i; thus, his bang-per-buck is changed by a factor of c_i as well, i.e., becomes $r_i c_i$.

Lemma 1. $c_{i^*} < 2$.

Note that our discussions do not rely on any specific initial instance (α, e) and manipulation (α', e'). Thus, the above lemma immediately implies that the incentive ratio of buyer i^* is less than 2, i.e., $\zeta^M_{i^*} < 2$. The same argument holds for all other buyers. Therefore, $\zeta^{\text{linear}} < 2$ and Theorem 1 follows. In the remaining of this section, we will prove this lemma.

In our proof, we assume that all input utility coefficients and endowments are rational numbers; thus, the computed equilibrium is composed of rational numbers as well. To simplify the proof, we first reduce an equilibrium with fractional allocations to an instance with integral equilibrium allocations, preserving the factor c_i of utility gains.

Proposition 1. *In the given market M, there exist another two linear market instances, where one is derived from the other by one strategic play of a buyer, such that they admit $\{0,1\}$-integral equilibrium allocations and c_i remains unchanged for all buyers.*

By the above claim, in the following we assume without loss of generality that the two equilibrium allocations $\mathbf{x}$ and $\mathbf{x}'$ are $\{0,1\}$-integral. That is, for any $i \in [n]$ and $j \in [m]$, $x_{ij}, x'_{ij} \in \{0,1\}$. Let $S_i = \{j \in [m] \mid x_{ij} = 1\}$ and $S'_i = \{j \in [m] \mid x'_{ij} = 1\}$ be the sets of items allocated to buyer i in the two allocations, respectively.

Proposition 2. *For any buyer i, $\sum_{j \in S'_i} p_j \geq c_i e_i$.*

Proof. Since the allocations x_i and x'_i are integral, we have

$$c_i = \frac{u_i(x'_i)}{u_i(x_i)} = \frac{\sum_{j \in S'_i} \alpha_{ij}}{\sum_{j \in S_i} \alpha_{ij}}$$

For $j \in S_i$, we have $\frac{\alpha_{ij}}{p_j} = r_i$. Thus, $\sum_{j \in S_i} \alpha_{ij} = \sum_{j \in S_i} r_i p_j = r_i e_i$. For $j \in S'_i$, we have $\frac{\alpha_{ij}}{p_j} \leq r_i$. Thus, $\sum_{j \in S'_i} \alpha_{ij} \leq r_i \cdot \sum_{j \in S'_i} p_j$. Therefore,

$$\sum_{j \in S'_i} p_j \geq \frac{1}{r_i} \sum_{j \in S'_i} \alpha_{ij} = \frac{c_i}{r_i} \sum_{j \in S_i} \alpha_{ij} = \frac{c_i}{r_i} r_i e_i = c_i e_i.$$

The claim follows. □

Proposition 3. *Consider any buyer $i \neq i^*$ and any item $j \in [m]$. If $j \in S_i$, then $c_i \geq \frac{p_j}{p'_j}$; if $j \in S'_i$, then $\frac{p_j}{p'_j} \geq c_i$.*

Proof. Note that the bids of buyer i in the two scenarios (α, e) and (α', e') are the same. If $j \in S_i$, we have $\frac{\alpha_{ij}}{p_j} = r_i$ and $\frac{\alpha_{ij}}{p'_j} \leq r_i c_i$; therefore, $c_i \geq \frac{p_j}{p'_j}$. If $j \in S'_i$, we have $\frac{\alpha_{ij}}{p'_j} = r_i c_i$ and $\frac{\alpha_{ij}}{p_j} \leq r_i$; thus, $\frac{p_j}{p'_j} \geq c_i$. □

Finally, we are ready to prove Lemma 1.

Proof (of Lemma 1). Assume to the contrary that $c_{i^*} \geq 2$. By Proposition 2, $\sum_{j \in S'_{i^*}} p_j \geq c_{i^*} e_{i^*}$. Since $\sum_{j \in S_{i^*} \cap S'_{i^*}} p_j \leq \sum_{j \in S_{i^*}} p_j = e_{i^*}$, we have

$$\sum_{j \in S'_{i^*} \setminus S_{i^*}} p_j \geq (c_{i^*} - 1) \cdot e_{i^*}$$

Further, we have $\sum_{j \in S'_{i^*} \setminus S_{i^*}} p'_j \leq \sum_{j \in S'_{i^*}} p'_j = e'_{i^*}$. Hence,

$$\frac{\sum_{j \in S'_{i^*} \setminus S_{i^*}} p_j}{\sum_{j \in S'_{i^*} \setminus S_{i^*}} p'_j} \geq \frac{(c_{i^*} - 1) \cdot e_{i^*}}{e'_{i^*}} \triangleq \Delta$$

This implies there exists $j \in S'_{i^*} \setminus S_{i^*}$ such that $\frac{p_j}{p'_j} \geq \Delta$. Let R denote the set of all such items in $S'_{i^*} \setminus S_{i^*}$. From the above discussion, we have $R \neq \emptyset$.

Consider the following iterative procedure:

1. Initialize: Let $A = \{i^*\}$ and $T = \{i \mid S_i \cap R \neq \emptyset\}$ (note that $T \neq \emptyset$).
2. Do the following until $T = \emptyset$:
   - Pick an arbitrary $i \in T$.
   - Let $T \leftarrow T \setminus \{i\}$ and $A \leftarrow A \cup \{i\}$.
   - Let $T \leftarrow T \cup \{k \notin A \cup T \mid S_k \cap (S'_i \setminus S_i) \neq \emptyset\}$.

Intuitively, in each iteration, we find all buyers that win some items from the set $S'_i \setminus S_i$ in the equilibrium allocation $\mathbf{x}$. Our main observation is that, for all buyers ever added to T in the procedure, their utility gains by manipulations are at least Δ. We prove this by induction on the iterations.

In the initialization step, this fact is true for all buyers in T: For any $i \in T$, by the definition of T and Proposition 3, there exists an item $j \in S_i \cap R$ such that $c_i \geq \frac{p_j}{p'_j}$. By the definition of R, $\frac{p_j}{p'_j} \geq \Delta$. Therefore, we have $c_i \geq \Delta$. Next, we consider the induction step. For any buyer k added to T during the procedure, since $S_k \cap (S'_i \setminus S_i) \neq \emptyset$, let j be an item in $S_k \cap (S'_i \setminus S_i)$. By Proposition 3, we have $c_k \geq \frac{p_j}{p'_j} \geq c_i$. Since i used to be in T, by induction hypothesis, $c_i \geq \Delta$; hence, $c_k \geq \Delta$. Therefore, for any buyer k ever added to T in the process, $c_k \geq \Delta$.

Note that the iterative procedure must terminate as every buyer can be added into T at most once. We consider the subset A at the end of the procedure, which includes all buyers ever added into T. Note that since the initial $T \neq \emptyset$, $A \setminus \{i^*\} \neq \emptyset$. By the rule of updating T, we know that all items in R and S'_i (for all $i \in A \setminus \{i^*\}$) are bought by buyers in the set A in the equilibrium allocation $\mathbf{x}$. That is,

$$\left(\bigcup_{i \in A \setminus \{i^*\}} S'_i \right) \cup R \cup (S'_{i^*} \cap S_{i^*}) \subseteq \bigcup_{i \in A} S_i.$$

Further, by the definition of R, we have $R \subseteq S'_{i^*}$, and thus,

$$\sum_{j \in (S'_{i^*} \setminus S_{i^*}) \setminus R} p_j < \Delta \cdot \sum_{j \in (S'_{i^*} \setminus S_{i^*}) \setminus R} p'_j \leq \Delta \cdot e'_{i^*} = (c_{i^*} - 1) \cdot e_{i^*}$$

Therefore,

$$\begin{aligned} \sum_{j \in \bigcup_{i \in A} S'_i} p_j &= \sum_{j \in (S'_{i^*} \setminus S_{i^*}) \setminus R} p_j + \sum_{j \in \left(\bigcup_{i \in A \setminus \{i^*\}} S'_i \right) \cup R \cup (S'_{i^*} \cap S_{i^*})} p_j \\ &< (c_{i^*} - 1) \cdot e_{i^*} + \sum_{j \in \left(\bigcup_{i \in A} S_i \right)} p_j \\ &\leq (c_{i^*} - 1) \cdot e_{i^*} + \sum_{i \in A} e_i \end{aligned} \quad (1)$$

On the other hand, we have

$$\begin{aligned}\sum_{j\in\bigcup_{i\in A}S'_i} p_j &= \sum_{i\in A}\sum_{j\in S'_i} p_j\\ &\geq \sum_{i\in A} c_i e_i \qquad \text{(Proposition 2)}\\ &\geq c_{i^*}e_{i^*} + \sum_{i\in A\setminus\{i^*\}} \Delta\cdot e_i \qquad (\forall\ i\in A\setminus\{i^*\},\, c_i\geq \Delta)\\ &= c_{i^*}e_{i^*} + \sum_{i\in A\setminus\{i^*\}} \frac{(c_{i^*}-1)\cdot e_{i^*}}{e'_{i^*}}\cdot e_i\\ &\geq c_{i^*}e_{i^*} + \sum_{i\in A\setminus\{i^*\}} (c_{i^*}-1)\cdot e_i \qquad (e_{i^*}\geq e'_{i^*})\end{aligned}$$

This contradicts formula (1), as $c_{i^*} \geq 2$ and $A \setminus \{i^*\}$ is nonempty. □

The proved ratio $\zeta^{\text{linear}} < 2$ is tight, as the following example shows.

Example 2. There are three items and two buyers with utilities and endowments: $u_1 = \left(\frac{1+\varepsilon}{2-2\varepsilon^2-\varepsilon^3}, \frac{1-\varepsilon-2\varepsilon^2-\varepsilon^3}{2-2\varepsilon^2-\varepsilon^3}, 0\right)$, $u_2 = \left(\varepsilon^2, \varepsilon, 1-\varepsilon-\varepsilon^2\right)$, $e_1 = \varepsilon+\varepsilon^2$, and $e_2 = 1-\varepsilon-\varepsilon^2$. When both buyers bid truthfully, the equilibrium price is $\mathbf{p} = (\varepsilon + \varepsilon^2, \frac{\varepsilon(1-\varepsilon-\varepsilon^2)}{1-\varepsilon^2}, \frac{(1-\varepsilon-\varepsilon^2)^2}{1-\varepsilon^2})$, and equilibrium allocations are $x_1 = (1,0,0)$ and $x_2 = (0,1,1)$; the utility of the first buyer is $u_1 = \frac{1+\varepsilon}{2-2\varepsilon^2-\varepsilon^3}$. When the first buyer bids $u'_1 = u_2$, the equilibrium price becomes $\mathbf{p}' = \left(\varepsilon^2, \varepsilon, 1-\varepsilon-\varepsilon^2\right)$, and the best equilibrium allocations are $x'_1 = (1,1,0)$ and $x'_2 = (0,0,1)$; the utility of the first buyer becomes $u'_1 = 1$. Thus, the utility gain is $\frac{u'_1}{u_1} = \frac{2-2\varepsilon^2-\varepsilon^3}{1+\varepsilon}$, which approaches to 2 when ε is arbitrarily small.

4 Cobb-Douglas Utility Functions

In a Cobb-Douglas market, buyers' utility functions are of the form $u_i(x_i) = \prod_{j\in[m]} x_{ij}^{\alpha_{ij}}$, where $\sum_{j=1}^{m}\alpha_{ij} = 1$, for all $i\in[n]$. To guarantee the existence of a market equilibrium, we assume that each item is desired by at least one buyer, i.e., $\alpha_{ij} > 0$ for some i. This, together with the fact that each buyer desires at least one item (followed by the fact that $\alpha_{ij} > 0$ for some j), implies that a market equilibrium always exists [15].

For a given Cobb-Douglas market with reported utilities $(\alpha_{ij})_{j\in[m]}$ and endowment e_i from each buyer i, market equilibrium prices and allocations are unique and can be computed by the following equations [10].

$$p_j = \sum_{i=1}^{n} e_i\alpha_{ij} \tag{2}$$

$$x_{ij} = \frac{e_i\alpha_{ij}}{\sum_{i=1}^{n} e_i\alpha_{ij}} \tag{3}$$

Based on these characterizations, in the following we will analyze the incentive ratio of Cobb-Douglas markets.

4.1 Manipulation on Endowments

Note that the private information of a buyer is composed of two parts: money endowment and utility function. In this section, we show that a buyer will never misreport his endowment.

Lemma 2. *In any Cobb-Douglas market, bidding endowments truthfully is a dominant strategy for all buyers.*

By the above result, in the following discussions, we assume that all buyers report their endowments truthfully, and will only consider their strategic behaviors on utility functions.

4.2 Reductions on Market Sizes

We first show that the incentive ratio of Cobb-Douglas markets is independent of the number of buyers. Let

$$\zeta(n) = \max\left\{\zeta^{M_n} \mid M_n \text{ is a Cobb-Douglas market with } n \text{ buyers}\right\}$$

be the largest incentive ratio of all markets with n buyers. Note that $\zeta(1) = 1$.

Theorem 2. *For Cobb-Douglas markets, incentive ratio is independent of the number of buyers, i.e.,* $\zeta(n) = \zeta(n')$ *for any* $n > n' \geq 2$.

The claim follows from the following two lemmas.

Lemma 3. *For any* n*-buyer market* M_n*, there is a 2-buyer market* M_2 *such that* $\zeta^{M_2} \geq \zeta^{M_n}$*. This implies that* $\zeta(2) \geq \zeta(n)$.

Proof. Consider a market M_n with n buyers; assume without loss of generality that the first buyer defines the maximal incentive ratio, i.e., $\zeta^{M_n} = \zeta_1^{M_n}$. Given M_n, we will construct a market M_2 with two buyers as follows.

- Input of M_n: n buyers $[n] = \{1, \dots, n\}$, each with an endowment e_i and a utility function $u_i = \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}}$.
- Construction of M_2: 2 buyers $1^*, 2^*$.
  - For 1^*, endowment $e_{1^*} = e_1$, utility function $u_{1^*} = \prod_{j=1}^{m} x_{1^*j}^{\alpha_{1^*j}}$, where $\alpha_{1^*j} = \alpha_{1j}$.
  - For 2^*, endowment $e_{2^*} = 1 - e_1$, utility function $u_{2^*} = \prod_{j=1}^{m} x_{2^*j}^{\alpha_{2^*j}}$, where $\alpha_{2^*j} = \sum_{i=2}^{n} \frac{e_i}{1-e_1}\alpha_{ij}$.

For the constructed M_2, we can easily verify it is a well defined Cobb-Douglas market, i.e., $e_{1^*} + e_{2^*} = 1$, $\sum_j \alpha_{1^*j} = 1$ and $\sum_j \alpha_{2^*j} = \sum_j \sum_{i=2}^{n} \frac{e_i}{1-e_1}\alpha_{ij} = 1$. The above reduction is based on unifying buyers $2, \dots, n$ in M_n into one buyer 2^* in M_2. We will prove that the incentive ratio of buyer 1^* in M_2 is the same as buyer 1 in M_n, i.e., $\zeta_{1^*}^{M_2} = \zeta_1^{M_n}$. This immediately implies that $\zeta^{M_2} \geq \zeta_{1^*}^{M_2} = \zeta_1^{M_n} = \zeta^{M_n}$, and thus, $\zeta(2) \geq \zeta(n)$.

Let $\mathbf{p} = (p_j)_{j\in[m]}$ and $\mathbf{p}^* = (p_j^*)_{j\in[m]}$ be the equilibrium prices of markets M_n and M_2 respectively. Further, let $x_1 = (x_{1j})_{j\in[m]}$ and $x_{1^*}^* = (x_{1^*j}^*)_{j\in[m]}$ denote the equilibrium allocations of buyer 1 in M_n and buyer 1^* in M_2, respectively. Since for any $j \in [m]$,

$$p_j = \sum_{i=1}^{n} e_i\alpha_{ij} = e_{1^*}\alpha_{1^*j} + (1-e_1)\sum_{i=2}^{n} \frac{e_i}{1-e_1}\alpha_{ij} = e_{1^*}\alpha_{1^*j} + e_{2^*}\alpha_{2^*j} = p_j^*,$$

we obtain

$$x_{1j} = \frac{e_1\alpha_{1j}}{\sum_{i=1}^{n} e_i\alpha_{ij}} = \frac{e_{1^*}\alpha_{1^*j}}{e_{1^*}\alpha_{1^*j} + e_{2^*}\alpha_{2^*j}} = x_{1^*j}^*.$$

Hence,

$$u_1(\mathbf{x}_1) = \prod_{j=1}^{m} x_{1j}^{\alpha_{1j}} = \prod_{j=1}^{m} (x_{1^*j}^*)^{\alpha_{1^*j}} = u_{1^*}(\mathbf{x}_{1^*}^*)$$

Denote by $\alpha_1' = (\alpha_{1j}')_{j\in[m]}$ the best response of buyer 1 in M_n. By the same argument as above, buyer 1^* can get the same utility in M_2 as buyer 1 in M_n by reporting $\alpha_1' = (\alpha_{1j}')_{j\in[m]}$. That is, $u_1' = u_{1^*}'$. This implies that $\zeta_{1^*}^{M_2} = \zeta_1^{M_n}$ and completes the proof of the claim. □

We can have a similar reduction from any 2-buyer market to an n-buyer market.

Lemma 4. *For any 2-buyer market M_2, there is an n-buyer market M_n such that $\zeta^{M_n} \geq \zeta^{M_2}$. This implies $\zeta(n) \geq \zeta(2)$.*

4.3 Incentive Ratio

Theorem 3. *The incentive ratio of Cobb-Douglas markets is*

$$\zeta^{\text{Cobb-Douglas}} \leq e^{1/e} \approx 1.445.$$

Proof. According to Theorem 2, it suffices to consider the case with 2 buyers. We consider two scenarios: For the fixed bid vector $(\alpha_{2j})_{j\in[m]}$ of buyer 2, buyer 1 bids $(\alpha_{1j})_{j\in[m]}$ and $(\alpha_{1j}')_{j\in[m]}$, respectively, with resulting equilibrium allocations $\mathbf{x}_1 = (x_{1j})_{j\in[m]}$ and $\mathbf{x}_1' = (x_{1j}')_{j\in[m]}$. Then,

$$\zeta = \frac{u_1(\mathbf{x}_1')}{u_1(\mathbf{x}_1)} = \frac{\prod_{j=1}^{m} {x_{1j}'}^{\alpha_{1j}}}{\prod_{j=1}^{m} x_{1j}^{\alpha_{1j}}} = \frac{\prod_{j=1}^{m}\left(\frac{e_1\alpha_{1j}'}{e_1\alpha_{1j}'+e_2\alpha_{2j}}\right)^{\alpha_{1j}}}{\prod_{j=1}^{m}\left(\frac{e_1\alpha_{1j}}{e_1\alpha_{1j}+e_2\alpha_{2j}}\right)^{\alpha_{1j}}}$$
$$= \prod_{j=1}^{m}\left(\frac{\alpha_{1j}'(e_1\alpha_{1j}+e_2\alpha_{2j})}{\alpha_{1j}(e_1\alpha_{1j}'+e_2\alpha_{2j})}\right)^{\alpha_{1j}} = \prod_{j=1}^{m}\left(\frac{\alpha_{1j}'\alpha_{1j}+\frac{e_2}{e_1}\alpha_{2j}\alpha_{1j}'}{\alpha_{1j}'\alpha_{1j}+\frac{e_2}{e_1}\alpha_{2j}\alpha_{1j}}\right)^{\alpha_{1j}} \triangleq \prod_{j=1}^{m} R_j$$

where R_j, $j = 1, \ldots, m$, is the j-th term of the above formula.

For each item j, it is easy to see that $R_j > 1$ if and only if $\alpha'_{1j} > \alpha_{1j}$. Let $S = \{j \mid R_j > 1\}$, and $r_j = R_j^{1/\alpha_{1j}}$ for $j \in S$ (note that r_j is well defined for those items in S as $\alpha_{1j} > 0$). Therefore, $r_j > 1$ if and only if $R_j > 1$. Further, when $r_j > 1$, one can see that $\alpha'_{1j} \geq r_j \alpha_{1j}$. This implies that $\sum_{j \in S} r_j \alpha_{1j} \leq \sum_{j \in S} \alpha'_{1j} \leq 1$. Hence,

$$\zeta \leq \prod_{j \in S} r_j^{\alpha_{1j}} \leq \prod_{j \in S} e^{\frac{r_j \alpha_{1j}}{e}} \leq e^{\frac{\sum_{j \in S} r_j \alpha_{1j}}{e}} \leq e^{1/e}.$$

Note that the above second inequality follows from the fact that for any $x, y \geq 0$, $x^y \leq e^{xy/e}$, which can be verified easily. Therefore, the theorem follows. □

The upper bound established in the above claim is tight, which can be seen from the following example.

Example 3. There are 2 items and 2 buyers, with endowments e_1 and e_2, respectively. Let $n = \frac{e_2}{e_1}$, $\epsilon_1 = \frac{1}{n}$, and $\epsilon_2 = \frac{1}{n^3}$. The utilities vectors of the two buyers are defined as $u_1 = (\frac{1}{e}, 1 - \frac{1}{e})$ and $u_2 = (1 - \epsilon_2, \epsilon_2)$. When buyer 1 bids $u'_1 = (1 - \epsilon_1, \epsilon_1)$ instead of u_1, the incentive ratio is given by the following formula:

$$\left(\frac{\frac{1-\frac{1}{n}}{e} + n(1 - \frac{1}{n^3})(1 - \frac{1}{n})}{\frac{1-\frac{1}{n}}{e} + n(1 - \frac{1}{n^3})\frac{1}{e}} \right)^{\frac{1}{e}} \cdot \left(\frac{\frac{1}{n}(1 - \frac{1}{e}) + n \cdot \frac{1}{n^3} \cdot \frac{1}{n}}{\frac{1}{n}(1 - \frac{1}{e}) + n \cdot \frac{1}{n^3} \cdot (1 - \frac{1}{e})} \right)^{1-\frac{1}{e}}$$

When n goes to infinity, the limit of the left factor is $e^{1/e}$, and the limit of the right factor is 1. Therefore, the ratio approaches $e^{1/e}$.

5 Conclusions

We study two important class of utility functions in Fisher markets: linear and Cobb-Douglas utilities, and show that their incentive ratios are bounded by 2 and $e^{1/e}$, respectively. It is interesting to explore the incentive ratios of other CES functions. In particular, is the incentive ratio bounded by a constant for any CES function? Another interesting direction is to characterize Nash equilibria in the market mechanism. In the full version paper, we give a sufficient and necessary condition for truthful bidding being a Nash equilibrium in Cobb-Douglas markets. It is intriguing to have such characterizations for other CES functions.

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