Distributed Knowledge with Justifications

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Abstract. In this paper, we introduce justification counterparts of distributed knowledge logics. Our justification logics include explicit knowledge operators of the form $[\![t]\!]_i F$ and $[\![t]\!]_{\mathcal{D}} F$, which are interpreted respectively as "t is a justification that agent i accepts for F", and "t is a justification that all agents implicitly accept for F". We present Kripke style models and prove the completeness theorem. Finally, we give a semantical proof of the realization theorem.

1 Introduction

Justification logics (cf. [2]) are a new generation of epistemic logics in which the knowledge operators $\mathcal{K}_i F$ (agent i knows F) are replaced with evidence-based knowledge operators $[\![t]\!]_i F$ (agent i accepts t as an evidence for F), where t is a justification term. The first justification logic, Logic of Proofs **LP**, was introduced by Artemov in [1] as an one-agent justification counterpart of the epistemic modal logic **S4**. The exact correspondence between **LP** and **S4** is given by the *Realization Theorem*: all occurrences of knowledge operator $\mathcal{K}$ in a theorem of **S4** can be replaced by suitable terms to obtain a theorem of **LP**, and vise versa. Artemov used a cut-free sequent calculus of **S4** to give a syntactic proof of the realization theorem ([1]). A semantical proof of the realization theorem is presented by Fitting in [9].

Logic of proofs is a justification logic with a new operator $[\![\cdot]\!]$ for one agent. In [15] Yavorskaya (Sidon) studied two-agent justification logics that have interactions, e.g., evidences of one agent can be verified by the other agent, or evidences of one agent can be converted to evidences of the other agent. Renne introduced dynamic epistemic logics with justification, systems for multi-agent communication (see e.g. [13, 14]). Bucheli, Kuznets and Studer in [5] suggested an explicit evidence system with common knowledge, an attempt to find a justification counterpart of epistemic logics with common knowledge (although proving the realization theorem for this system is still an open problem). Dynamic justification logic of public announcements also studied in [4, 6]. None of the aforementioned papers deal with the notion of distributed knowledge.

In this paper, we study multi-agent evidence-based systems in a distributed environment. Distributed knowledge is the knowledge that is implicitly available in a group, and can be discovered through communication (cf. [8, 12]). We introduce an evidence-based knowledge operator for distributed knowledge $[\![t]\!]_{\mathcal{D}} F$,

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with the intuitive meaning "t is an evidence that all agents implicitly accept for F". In other words, $[\![t]\!]_{\mathcal{D}}F$ states that t is an evidence (or justification) that could be obtained for F if all agents pooled their knowledge (or justifications) together. To capture this notion, we present distributed knowledge logics with justifications $\mathbf{JK_n^D}$, $\mathbf{JT_n^D}$, $\mathbf{JS4_n^D}$, and $\mathbf{JS5_n^D}$. We establish basic properties of justification logics for our logics, and give two examples to show how these logics can be used to track evidences of distributed knowledge (more information on tracking evidences and its applications can be found in [3, 16, 17]).

We also present possible world semantics for these logics. In the present paper, we consider $[\![\cdot]\!]_{\mathcal{D}}$ as an agent, and give pseudo-Fitting models with additional accessibility relation $\mathcal{R}_{\mathcal{D}}$ and evidence function $\mathcal{E}_{\mathcal{D}}$ for distributed knowledge.

Finally, by proving the *Realization Theorem*, we show that our logics are the justification counterparts of the known distributed knowledge logics $\mathbf{K_n^D}$, $\mathbf{T_n^D}$, $\mathbf{S4_n^D}$, and $\mathbf{S5_n^D}$. There are several methods for proving the realization theorem, see e.g. [1, 9, 11]. We employ the technique of Fitting ([9]) to present a semantical proof of the realization theorem.[1]

2 Distributed Knowledge Logics

In this paper, we fix a set of n agents $G = \{1, 2, \ldots, n\}$. The language of distributed knowledge logics is obtained by adding the modal operators $\mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{D}$ to propositional logic. Hence, if A is a formula then $\mathcal{K}_i A$, for $i = 1, \ldots, n$, and $\mathcal{D}A$ are also formulas. The intended meaning of $\mathcal{K}_i A$ is "agent i knows A", and of $\mathcal{D}A$ is "A is distributed knowledge". Now, we recall the well known distributed knowledge logics (for more expositions see [8, 12]).

Definition 1. *The axioms of* $\mathbf{K_n^D}$ *are (where* $i = 1, \ldots, n$*):*

Taut. *Finite set of axioms for propositional logic,*
K. $\mathcal{K}_i(A \to B) \to (\mathcal{K}_i A \to \mathcal{K}_i B)$,
KD. $\mathcal{D}(A \to B) \to (\mathcal{D}A \to \mathcal{D}B)$,
$\mathbf{K_iD}$. $\mathcal{K}_i A \to \mathcal{D}A$.

The rules of inference are:

Modus Ponens: *from* A *and* $A \to B$, *infer* B,
Necessitation: *from* A *infer* $\mathcal{K}_i A$.

If the number of agents $n = 1$, *then we add the additional axiom:*

$\mathcal{D}A \to \mathcal{K}_1 A$.

Extensions of $\mathbf{K_n^D}$ *obtain by adding some axioms as follows:*

- $\mathbf{T_n^D} = \mathbf{K_n^D} + (\mathcal{K}_i A \to A) + (\mathcal{D}A \to A)$,
- $\mathbf{S4_n^D} = \mathbf{T_n^D} + (\mathcal{K}_i A \to \mathcal{K}_i\mathcal{K}_i A) + (\mathcal{D}A \to \mathcal{D}\mathcal{D}A)$,
- $\mathbf{S5_n^D} = \mathbf{S4_n^D} + (\neg\mathcal{K}_i A \to \mathcal{K}_i\neg\mathcal{K}_i A) + (\neg\mathcal{D}A \to \mathcal{D}\neg\mathcal{D}A)$.

[1] Since it seems the method used in the proof of the realization theorem in [10] is not correct, we use a different method in Section 5 to prove the realization theorem.

Note that the axioms $\mathcal{K}_i A \to A$ in $\mathbf{T_n^D}$ are redundant, since they follow from axioms $\mathcal{K}_i A \to \mathcal{D}A$ and $\mathcal{D}A \to A$.

In what follows, $\mathbf{L^D}$ is any of the logics $\mathbf{K_n^D}$, $\mathbf{T_n^D}$, $\mathbf{S4_n^D}$, or $\mathbf{S5_n^D}$. Next we recall Kripke models for the logics $\mathbf{L^D}$.

Definition 2. *A Kripke model $\mathcal{M}$ for $\mathbf{K_n^D}$ is a tuple $\mathcal{M} = (\mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \Vdash)$ where $\mathcal{W}$ is a non-empty set of worlds (or states), each $\mathcal{R}_i$ is a binary accessibility relation between worlds, and the forcing relation $\Vdash$ is a relation between pairs $(\mathcal{M}, w)$ and propositional letters, that can be extended to all formulas as follows:*

1. *$\Vdash$ respects classical Boolean connectives,*
2. *$(\mathcal{M}, w) \Vdash \mathcal{K}_i A$ iff for every $v \in \mathcal{W}$ with $w\mathcal{R}_i v$, $(\mathcal{M}, v) \Vdash A$,*
3. *$(\mathcal{M}, w) \Vdash \mathcal{D}A$ iff for every $v \in \mathcal{W}$ with $w\mathcal{R}_\mathcal{D} v$, $(\mathcal{M}, v) \Vdash A$, where $\mathcal{R}_\mathcal{D} = \bigcap_{i=1}^n \mathcal{R}_i$.*

For Kripke models of $\mathbf{T_n^D}$, $\mathbf{S4_n^D}$ and $\mathbf{S5_n^D}$ each $\mathcal{R}_i$ should be reflexive, reflexive and transitive and an equivalence relation, respectively.

Theorem 1. *([7]) $\mathbf{K_n^D}$, $\mathbf{T_n^D}$, $\mathbf{S4_n^D}$ and $\mathbf{S5_n^D}$ are sound and complete with respect to their models.*

3 Distributed Knowledge Logics with Justifications

In this section, we introduce distributed knowledge logics with justifications $\mathbf{JK_n^D}$, $\mathbf{JT_n^D}$, $\mathbf{JS4_n^D}$, and $\mathbf{JS5_n^D}$. In the rest of the paper, we extend our set of agents by the distributed knowledge operator $\mathcal{D}$, and denote by $*$ one of the agents in G or $\mathcal{D}$ (i.e. $* \in \{1, \ldots, n, \mathcal{D}\}$). Similar to the language used in [5] and [15], we define a set of terms as justifications for each $* \in \{1, \ldots, n, \mathcal{D}\}$. We start by defining the set of justification variables and constants:

$$Var^* = \{x_1^*, x_2^*, \ldots\} \qquad Cons^i = \{c_1^i, c_2^i, \ldots\}.$$

Now define the set of admissible terms Tm_* (for each $*$) as follows

1. $Var^* \subseteq Tm_*$,
2. $Cons^i \subseteq Tm_i$,
3. if $s, t \in Tm_*$, then $s +_* t$, $s \cdot_* t \in Tm_*$,
   for $\mathbf{JS4_n^D}$ and $\mathbf{JS5_n^D}$: if $t \in Tm_*$, then $!_* t \in Tm_*$,
   for $\mathbf{JS5_n^D}$: if $t \in Tm_*$, then $?_* t \in Tm_*$,
4. $Tm_i \subseteq Tm_\mathcal{D}$, for each $i \in G$.

Indeed, each distributed justification logic includes those clauses in the construction of terms that contains the corresponding operator in its language. Note that by clause 4 there is no need to define variables $Var^\mathcal{D}$ for operator $\mathcal{D}$. However, since using variables in $Var^\mathcal{D}$ simplifies some arguments (see for instance Lemma 3) we keep it. In addition, as we will see from the formulation of our logics (see Definition 3), there is no need to define a set of justification constants $Cons^\mathcal{D}$ for $\mathcal{D}$.

Another different alternative is to define only one set of terms Tm that is admissible for all agents as well as for distributed knowledge operator (see e.g. [13, 14], in which Renne considers a set of terms for all agents' evidences). Nevertheless, using labels for justification variables and constants for each agent enables us to tracking evidences (see Example 1, and the discussion after it).

Formulas of the distributed knowledge logics with justifications are constructed as follows:

$$F := P \mid \bot \mid F \to F \mid [\![t]\!]_* F,$$

where P is a propositional variable and $t \in Tm_*$. The intended meaning of $[\![t]\!]_i F$ is "t is a justification that agent i accepts for F", and of $[\![t]\!]_{\mathcal{D}} F$ is "t is a justification that all agents implicitly accept for F". We begin by defining the language and axioms of the basic distributed knowledge logic with justifications.

Definition 3. *The language of* $\mathbf{JK}_\mathbf{n}^\mathbf{D}$ *contains only the operators* $\cdot_*$ *and* $+_*$*. The axioms of* $\mathbf{JK}_\mathbf{n}^\mathbf{D}$ *are:*

A0. *Finite set of axioms for propositional logic,*
A1. $[\![s]\!]_* A \vee [\![t]\!]_* A \to [\![s +_* t]\!]_* A$,
A2. $[\![s]\!]_* (A \to B) \to ([\![t]\!]_* A \to [\![s \cdot_* t]\!]_* B)$,
A3. $[\![t]\!]_i A \to [\![t]\!]_{\mathcal{D}} A$, *where* $t \in Tm_i$.

The rules of inference are:

R1. *Modus Ponens: from* A *and* $A \to B$, *infer* B,
R2. *Iterated Axiom Necessitation:* $\vdash [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A$, *where* A *is an axiom,* $c^{i_k}_{j_l}$*'s are justification constants and* $i_1, \ldots, i_m$ *are in* G.

If the number of agents $n = 1$, *then we add the additional axiom:*

A4. $[\![t]\!]_{\mathcal{D}} A \to [\![t]\!]_1 A$, *where* $t \in Tm_1$.

The justification system $\mathbf{JT}_\mathbf{n}^\mathbf{D}$ *is obtained from* $\mathbf{JK}_\mathbf{n}^\mathbf{D}$ *by adding the following axioms:*

A5. $[\![t]\!]_* A \to A$.

The justification system $\mathbf{JS4}_\mathbf{n}^\mathbf{D}$ *is obtained from* $\mathbf{JT}_\mathbf{n}^\mathbf{D}$ *by first extending the language with operators* $!_*$ *and then adding the following axioms:*

A6. $[\![t]\!]_* A \to [\![!_* t]\!]_* [\![t]\!]_* A$.

and replacing the rule **R2** *by the following simple one:*

R3. *Axiom Necessitation:* $\vdash [\![c^i]\!]_i A$, *where* A *is an axiom,* c^i *is a justification constant and* $i \in G$.

The justification system $\mathbf{JS5}_\mathbf{n}^\mathbf{D}$ *is obtained from* $\mathbf{JS4}_\mathbf{n}^\mathbf{D}$ *by first extending the language with operators* $?_*$ *and then adding the following axioms:*

A7. $\neg [\![t]\!]_* A \to [\![?_* t]\!]_* \neg [\![t]\!]_* A$.

Notice that, in the axioms **A1**, **A2**, **A6** and **A7** all occurrences of $*$ are the same agent. Moreover, axioms $[\![t]\!]_i A \to A$ in $\mathbf{JT_n^D}$ are redundant, since they can be obtained from axioms $[\![t]\!]_i A \to [\![t]\!]_{\mathcal{D}} A$ and $[\![t]\!]_{\mathcal{D}} A \to A$.

By $\mathbf{JL^D}$ we denote one of the logics $\mathbf{JK_n^D}$, $\mathbf{JT_n^D}$, $\mathbf{JS4_n^D}$, and $\mathbf{JS5_n^D}$. Following [15], we define constant specifications as follows:

Definition 4. *A Constant Specification $\mathcal{CS}$ for $\mathbf{JK_n^D}$ (or $\mathbf{JT_n^D}$) is a set of formulas of the form $[\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A$, where A is an axiom of $\mathbf{JK_n^D}$ (or $\mathbf{JT_n^D}$), $c^{i_l}_{j_l}$'s are justification constants and $i_1, \ldots, i_m$ are in G, and moreover it is downward closed:*

$$\text{if } [\![c^{i_m}_{j_m}]\!]_{i_m} [\![c^{i_{m-1}}_{j_{m-1}}]\!]_{i_{m-1}} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS}, \text{ then } [\![c^{i_{m-1}}_{j_{m-1}}]\!]_{i_{m-1}} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS}.$$

A constant specification $\mathcal{CS}$ is axiomatically appropriate if for each axiom A and $i \in G$ there is a constant $c^i \in Tm_i$ such that $[\![c^i]\!]_i A \in \mathcal{CS}$ and also $\mathcal{CS}$ is upward closed:

$$\text{if } [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS}, \text{ then } [\![c^{i_{m+1}}_{j_{m+1}}]\!]_{i_{m+1}} [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS},$$

for some $i_{m+1} \in G$ and constant $c^{i_{m+1}}_{j_{m+1}} \in Tm_{m+1}$.

Definition 5. *A Constant Specification $\mathcal{CS}$ for $\mathbf{JS4_n^D}$ (or $\mathbf{JS5_n^D}$) is a set of formulas of the form $[\![c^i]\!]_i A$, such that c^i is a justification constant in $Cons^i$, A is an axiom of $\mathbf{JS4_n^D}$ (or $\mathbf{JS5_n^D}$) and $i \in G$. A constant specification $\mathcal{CS}$ is axiomatically appropriate if for each axiom A and $i \in G$ there is a constant $c^i \in Tm_i$ such that*
$[\![c^i]\!]_i A \in \mathcal{CS}$.

Let $\mathbf{JL^D}(\mathcal{CS})$ be the fragment of $\mathbf{JL^D}$ where the (Iterated) Axiom Necessitation rule only produces formulas from the given $\mathcal{CS}$. Thus $\mathbf{JL^D}(\emptyset)$ is the fragment of $\mathbf{JL^D}$ without (Iterated) Axiom Necessitation rule. By $\mathbf{JL^D} \vdash F$ we mean $\mathbf{JL^D}(\mathcal{CS}) \vdash F$ for some constant specification $\mathcal{CS}$.

Definition 6. *A substitution σ is a mapping from $\bigcup_* Var^*$ to $\bigcup_* Tm_*$ such that each justification variable in Var^* maps to a term in Tm_*. The domain of σ is $dom(\sigma) := \{x \mid \sigma(x) \neq x\}$. The result of substitution σ on the term t and formula A is denoted by $t\sigma$ and $A\sigma$ respectively.*

Distributed knowledge logics with justifications $\mathbf{JL^D}$ enjoy the deduction theorem and substitution lemma (the proofs are standard and are omitted here).

Lemma 1. *Let $\mathcal{CS}$ be a constant specification.*

1. *Deduction Theorem for $\mathbf{JL^D}(\mathcal{CS})$: $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \to B$.*
2. *Substitution Lemma: (i) If $\Gamma \vdash A$ in $\mathbf{JL^D}(\mathcal{CS})$, then $\Gamma\sigma \vdash A\sigma$ in $\mathbf{JL^D}(\mathcal{CS}\sigma)$. (ii) If $\Gamma \vdash A$ in $\mathbf{JL^D}(\mathcal{CS})$, then $\Gamma(F/P) \vdash A(F/P)$ in $\mathbf{JL^D}(\mathcal{CS}')$, where $\mathcal{CS}' = \mathcal{CS}(F/P)$ and $A(F/P)$ denotes the result of simultaneously replacing all occurrences of propositional variable P by formula F in A.*

Distributed knowledge logics with justifications can internalize their own proofs. This is one of the fundamental properties of justification logics.

Lemma 2 (Internalization Lemma). *For each* $* \in \{1, \ldots, n, \mathcal{D}\}$*, the following statements hold:*

1. *If* $\mathbf{JL^D}(\mathcal{CS}) \vdash F$*, then* $\mathbf{JL^D}(\mathcal{CS}') \vdash [\![p]\!]_* F$*, for some term* p *in* Tm_* *and some* $\mathcal{CS}' \supseteq \mathcal{CS}$*.*
2. *Suppose* $\mathcal{CS}$ *is axiomatically appropriate. If* $\mathbf{JL^D}(\mathcal{CS}) \vdash F$*, then* $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![p]\!]_* F$*, for some term* p *in* Tm_**.*

Proof. By induction on the derivation of F. If F is an axiom, then using (Iterated) Axiom Necessitation rule $[\![c^i]\!]_i F$ is derivable in $\mathbf{JL^D}(\mathcal{CS}')$ for some $c^i \in Const^i$ and $\mathcal{CS}' = \mathcal{CS} \cup \{[\![c^i]\!]_i F\}$. If $\mathcal{CS}$ is axiomatically appropriate then there is a constant $c^i \in Tm_i$ such that $[\![c^i]\!]_i F \in \mathcal{CS}$, for each $i \in G$. Hence, $[\![c^i]\!]_i F$ is derivable in $\mathbf{JL^D_n}(\mathcal{CS})$, for each $i \in G$. Moreover, using axiom instance $[\![c^i]\!]_i F \to [\![c^i]\!]_{\mathcal{D}} F$, we can derive $[\![c^i]\!]_{\mathcal{D}} F$. If F is obtained by Modus Ponens from G and $G \to F$, then by the induction hypothesis, there are terms $t, s \in Tm_*$ such that $[\![t]\!]_* G$ and $[\![s]\!]_*(G \to F)$ are provable. By axiom **A2**, we derive $[\![s \cdot_* t]\!]_* F$.

If $F = [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS}$, is obtained by the Iterated Axiom Necessitation rule **IAN** in $\mathbf{JK^D_n}$ or $\mathbf{JT^D_n}$, then using **IAN** we obtain $[\![c^i]\!]_i [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A$. If $\mathcal{CS}$ is axiomatically appropriate then it is upward closed, and therefore there is a constant $c^i \in Tm_i$ such that $[\![c^i]\!]_i [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A$ is in $\mathcal{CS}$, and hence is derivable in $\mathbf{JK^D_n}(\mathcal{CS})$ or $\mathbf{JT^D_n}(\mathcal{CS})$. Moreover, using axiom **A3**, we can derive $[\![c^i]\!]_{\mathcal{D}} [\![c^{i_m}_{j_m}]\!]_{i_m} \ldots [\![c^{i_1}_{j_1}]\!]_{i_1} A$.

If $F = [\![c^i]\!]_i A \in \mathcal{CS}$ is obtained by the Axiom Necessitation rule **AN** in $\mathbf{JS4^D_n}$ or $\mathbf{JS5^D_n}$, then use axiom **A6** to derive $[\![!_i\, c^i]\!]_i A$ in $\mathbf{JL^D_n}(\mathcal{CS})$. Moreover, using axiom **A3**, we can derive $[\![!_i\, c^i]\!]_{\mathcal{D}} A$. □

Lemma 3 (Lifting Lemma). *For each* $* \in \{1, \ldots, n, \mathcal{D}\}$*, the following statements are provable:*

1. *If* $[\![t_1]\!]_* A_1, \ldots, [\![t_m]\!]_* A_m, B_1, \ldots, B_l \vdash F$ *in* $\mathbf{JS4^D_n}(\mathcal{CS})$*, then*

$$[\![t_1]\!]_* A_1, \ldots, [\![t_m]\!]_* A_m, [\![x^*_1]\!]_* B_1, \ldots, [\![x^*_l]\!]_* B_l \vdash [\![p(\vec{t}, \vec{x})]\!]_* F \quad (\dagger)$$

   in $\mathbf{JS4^D_n}(\mathcal{CS}')$*, for some justification variables* x^*_i *(in* Var^**), term* $p(\vec{t}, \vec{x})$ *in* Tm_* *and* $\mathcal{CS}' \supseteq \mathcal{CS}$ *(all* $*$*'s in* $(\dagger)$ *stand for the same agent).*
2. *In part (1), if* $\mathcal{CS}$ *is axiomatically appropriate, then* $(\dagger)$ *is provable in* $\mathbf{JS4^D_n}(\mathcal{CS})$*.*

Proof. The proof is similar to the proof of Lemma 2, with two new cases. If $F =$
$[\![t_i]\!]_* A_i$, for some $1 \le i \le m$, then put $p(\vec{t}, \vec{x}) = !_* t_i$. If $F = B_i$, for some $1 \le i \le l$, then put $p(\vec{t}, \vec{x}) = x^*_i$. □

It is worth noting that the terms p and $p(\vec{t}, \vec{x})$ constructed, respectively, in the proof of lifting and internalization lemmas depends on the agent $*$.

Example 1. We prove that $\mathbf{JK_n^D}(\emptyset) \vdash [\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![s \cdot_{\mathcal{D}} t]\!]_{\mathcal{D}} B$, where $s \in Tm_i$ and $t \in Tm_j$. The proof is as follows:

1. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![s]\!]_i(A \to B)$, tautology in propositional logic
2. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![t]\!]_j A$, tautology in propositional logic
3. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![s]\!]_{\mathcal{D}}(A \to B)$, from 1 by reasoning in propositional logic and axiom **A3**
4. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![t]\!]_{\mathcal{D}} A$, from 2 by reasoning in propositional logic and axiom **A3**
5. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![s]\!]_{\mathcal{D}}(A \to B) \wedge [\![t]\!]_{\mathcal{D}} A$, from 3 and 4 by reasoning in propositional logic
6. $[\![s]\!]_i(A \to B) \wedge [\![t]\!]_j A \to [\![s \cdot_{\mathcal{D}} t]\!]_{\mathcal{D}} B$, from 5 by reasoning in propositional logic and axiom **A2**.

This is similar to the fact that $\mathbf{K_n^D} \vdash \mathcal{K}_i(A \to B) \wedge \mathcal{K}_j A \to \mathcal{D}B$. This theorem of $\mathbf{K_n^D}$ states that if agent i knows $A \to B$ and agent j knows A, then B is distributed knowledge, which means if all agents combine their knowledge together, they can infer B. But, in fact, to obtain knowledge of B we do not need the information of all agents other than agents i and j.

Distributed knowledge logics with justifications allow us to track evidences occur in $[\![\cdot]\!]_{\mathcal{D}}$. For instance, Example 1 shows that if s is an agent i's evidence for $A \to B$ and t is an agent j's evidence for A, then $s \cdot_{\mathcal{D}} t$ is an evidence for B that all agents can obtain whenever they combine their knowledge. Since $s \in Tm_i$ and $t \in Tm_j$, the term $s \cdot_{\mathcal{D}} t$ shows that in order to get knowledge of B and make a justification for it, we only require information of agents i and j, and particularly it determines which part of their knowledge is required.

Example 2. The rule

$$\frac{A_1 \wedge \ldots \wedge A_n \to B}{\mathcal{K}_1 A_1 \wedge \ldots \wedge \mathcal{K}_n A_n \to \mathcal{D}B}$$

is admissible in $\mathbf{L^D}$ (see, e.g., [12]). Likewise, we prove that the following rule is admissible in $\mathbf{JL_n^D}$:

$$\frac{A_1 \wedge \ldots \wedge A_n \to B}{[\![t_1]\!]_1 A_1 \wedge \ldots \wedge [\![t_n]\!]_n A_n \to [\![t]\!]_{\mathcal{D}} B}$$

for some term t in $Tm_{\mathcal{D}}$, where $t_i \in Tm_i$ for $i = 1, \ldots, n$. The proof is as follows:

1 . $A_1 \wedge \ldots \wedge A_n \to B$, hypothesis
2. $[\![t_1]\!]_1 A_1 \wedge \ldots \wedge [\![t_n]\!]_n A_n$, hypothesis
3.1. $[\![t_1]\!]_1 A_1$, from 2 by reasoning in propositional logic
3.2. $[\![t_2]\!]_2 A_2$, from 2 by reasoning in propositional logic
⋮
3.n. $[\![t_n]\!]_n A_n$, from 2 by reasoning in propositional logic
4.1. $[\![t_1]\!]_{\mathcal{D}} A_1$, from 3.1 by axiom **A3**

4.2. $[\![t_2]\!]_{\mathcal{D}}A_2$, from 3.2 by axiom **A3**

⋮

4.n. $[\![t_n]\!]_{\mathcal{D}}A_n$, from 3.n by axiom **A3**
5. $A_1 \to (A_2 \to \ldots \to (A_n \to B)\ldots)$, from 1 by reasoning in propositional logic
6. $[\![p]\!]_{\mathcal{D}}[A_1 \to (A_2 \to \ldots \to (A_n \to B)\ldots)]$, from 5 by Lemma 2
7.1. $[\![p \cdot_{\mathcal{D}} t_1]\!]_{\mathcal{D}}[A_2 \to (A_3 \to \ldots \to (A_n \to B)\ldots)]$, from 4.1 and 6 by axiom **A2**
7.2. $[\![p \cdot_{\mathcal{D}} t_1 \cdot_{\mathcal{D}} t_2]\!]_{\mathcal{D}}[A_3 \to (A_4 \to \ldots \to (A_n \to B)\ldots)]$, from 4.2 and 7.1 by axiom **A2**

⋮

7.n. $[\![p \cdot_{\mathcal{D}} t_1 \cdot_{\mathcal{D}} \ldots \cdot_{\mathcal{D}} t_n]\!]_{\mathcal{D}}B$, from 4.n and 7.(n-1) by axiom **A2**
8. $[\![t_1]\!]_1 A_1 \wedge \ldots \wedge [\![t_n]\!]_n A_n \to [\![t]\!]_{\mathcal{D}}B$, from 2 and 8 by the Deduction Theorem (Lemma 1), where $t = p \cdot_{\mathcal{D}} t_1 \cdot_{\mathcal{D}} \ldots \cdot_{\mathcal{D}} t_n$.

These two examples show that evidence-based distributed knowledge could be viewed as the knowledge the agents would have by pooling their individual justifications together.

4 Semantics

In this section, we consider $[\![\cdot]\!]_{\mathcal{D}}$ as an agent, rather than as explicit distributed knowledge, and give pseudo-Fitting models for all systems $\mathbf{JL^D}$. Fitting models first introduced by Fitting in [9] for **LP**.

Definition 7. *A pseudo-Fitting model $\mathcal{M}$ for $\mathbf{JK_n^D}$ is a tuple*

$$\mathcal{M} = (\mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{R}_{\mathcal{D}}, \mathcal{E}_1, \ldots, \mathcal{E}_n, \mathcal{E}_{\mathcal{D}}, \Vdash_p)$$

(or $\mathcal{M} = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ for short) where $(\mathcal{W}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{R}_{\mathcal{D}}, \Vdash_p)$ is a Kripke model, in which $\mathcal{R}_{\mathcal{D}}$ is also a binary accessibility relation between worlds such that $\mathcal{R}_{\mathcal{D}} \subseteq \cap_{i=1}^{n} \mathcal{R}_i$. Admissible evidence functions $\mathcal{E}_*$ are mappings from the set of terms and formulas to the set of all worlds, i.e., $\mathcal{E}_*(t, A) \subseteq \mathcal{W}$, for any justification term t in Tm_* and formula A, and satisfying the following conditions. For all justification terms s and t and for all formulas A and B:

$\mathcal{E}1$. $\mathcal{E}_*(s, A) \cup \mathcal{E}_*(t, A) \subseteq \mathcal{E}_*(s +_* t, A)$,
$\mathcal{E}2$. $\mathcal{E}_*(s, A \to B) \cap \mathcal{E}_*(t, A) \subseteq \mathcal{E}_*(s \cdot_* t, B)$,
$\mathcal{E}3$. $\mathcal{E}_i(t, A) \subseteq \mathcal{E}_{\mathcal{D}}(t, A)$, *for each $i \in G$ and $t \in Tm_i$.*

If $n = 1$, then $\mathcal{R}_1 = \mathcal{R}_{\mathcal{D}}$ and evidence functions should also satisfy:

$\mathcal{E}4$. $\mathcal{E}_{\mathcal{D}}(t, A) \subseteq \mathcal{E}_1(t, A)$, *for each $t \in Tm_1$.*

The forcing relation $\Vdash_p$ is a relation between pairs $(\mathcal{M}, w)$ and propositional letters, that can be extended to all formulas as follows:

1. *$\Vdash_p$ respects classical Boolean connectives,*
2. *$(\mathcal{M}, w) \Vdash_p [\![t]\!]_* A$ iff $w \in \mathcal{E}_*(t, A)$ and for every $v \in \mathcal{W}$ with $w\mathcal{R}_* v$, $(\mathcal{M}, v) \Vdash_p A$.*

We say that A is true in a model $\mathcal{M}$ ($\mathcal{M} \Vdash_p A$) if it is true at each world of the model. For a set S of formulas, $\mathcal{M} \Vdash_p S$ if $\mathcal{M} \Vdash_p F$ for all formulas F in S. Given a constant specification $\mathcal{CS}$, a model $\mathcal{M}$ respects $\mathcal{CS}$ (or meets $\mathcal{CS}$) if $\mathcal{M} \Vdash_p \mathcal{CS}$. A set S of $\mathbf{JL^D}$-formulas is $\mathbf{JL^D}(\mathcal{CS})$-satisfiable if there is a model $\mathcal{M}$ for $\mathbf{JL^D}$ respecting $\mathcal{CS}$ and a world w in $\mathcal{M}$ such that $(\mathcal{M}, w) \Vdash A$ for all $A \in S$.

Pseudo–Fitting models for the other distributed justification logics have more restrictions on accessibility relations and evidence functions. For $\mathbf{JT_n^D}$ each $\mathcal{R}_*$ is reflexive. For $\mathbf{JS4_n^D}$ each $\mathcal{R}_*$ is reflexive and transitive and evidence functions should satisfy:

$\mathcal{E}5$. If $w \in \mathcal{E}_*(t, A)$ and $w\mathcal{R}_*v$, then $v \in \mathcal{E}_*(t, A)$,
$\mathcal{E}6$. $\mathcal{E}_*(t, A) \subseteq \mathcal{E}_*(!_*\, t, [\![t]\!]_* A)$,

For $\mathbf{JS5_n^D}$ each $\mathcal{R}_*$ is an equivalence relation and evidence functions should satisfy:

$\mathcal{E}7$. If $[\mathcal{E}_*(t, A)]^c \subseteq \mathcal{E}_*(?_*t, \neg[\![t]\!]_* A)$, where the superscript operation "c" on sets is the complement relative to the set of worlds $\mathcal{W}$,
$\mathcal{E}8$. If $w \in \mathcal{E}_*(t, A)$, then $(\mathcal{M}, w) \Vdash_p [\![t]\!]_* A$.

Next, we prove the completeness theorem for $\mathbf{JL^D}$. Since the proof is similar to the proof of the completeness theorem of justification logics in [2, 9], we omit the details.

Theorem 2 (Completeness). *For a given constant specification $\mathcal{CS}$, distributed justification logics $\mathbf{JL^D}(\mathcal{CS})$ are sound and complete with respect to their pseudo-Fitting models that respect $\mathcal{CS}$.*

Proof. Soundness is straightforward, as usual, by induction on derivations in $\mathbf{JL^D}(\mathcal{CS})$. Let us only check the validity of axiom **A7**, $\neg[\![t]\!]_* A \to [\![?_*t]\!]_* \neg [\![t]\!]_* A$, in a model of $\mathbf{JS5_n^D}$. Let $(\mathcal{M}, w) \Vdash_p \neg[\![t]\!]_* A$. By $\mathcal{E}8$, $w \notin \mathcal{E}_*(t, A)$. By $\mathcal{E}7$, $w \in \mathcal{E}_*(?_*t, \neg[\![t]\!]_* A)$, and by $\mathcal{E}8$ we have $(\mathcal{M}, w) \Vdash_p [\![?_*t]\!]_* \neg[\![t]\!]_* A$.

For completeness we first construct a canonical model $\mathcal{M} = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ as follows:

- $\mathcal{W}$ is the set of all maximally consistent sets in $\mathbf{JL^D}(\mathcal{CS})$,
- $\Gamma\mathcal{R}_*\Delta$ iff $\Gamma^{\sharp_*} \subseteq \Delta$, for $\Gamma, \Delta \in \mathcal{W}$,
- $\mathcal{E}_*(t, F) = \{\Gamma \in \mathcal{W} \,|\, [\![t]\!]_* F \in \Gamma\}$
- for each propositional letter P: $(\mathcal{M}, \Gamma) \Vdash_p P$ iff $P \in \Gamma$.

where P is a propositional variable and

$$\Gamma^{\sharp_*} = \{A \,|\, [\![t]\!]_* A \in \Gamma, \text{for some term}\, t \in Tm_*\}.$$

Forcing relation $\Vdash_p$ on arbitrary formulas is defined as in Definition 7.

Specially, for each $\mathbf{JL^D}$ the evidence function $\mathcal{E}_*$ in the canonical model $\mathcal{M}$ satisfies the corresponding properties $\mathcal{E}1 - \mathcal{E}8$ in the definition of pseudo–Fitting model. We only show the new property $\mathcal{E}3$ ($\mathcal{E}4$ is similar). Let $\Gamma \in \mathcal{E}_i(t, A)$.

Then $[\![t]\!]_i A \in \Gamma$. Since $[\![t]\!]_i A \to [\![t]\!]_{\mathcal{D}} A \in \Gamma$, we have $[\![t]\!]_{\mathcal{D}} A \in \Gamma$, and therefore $\Gamma \in \mathcal{E}_{\mathcal{D}}(t, A)$.

Let us now prove that $\mathcal{R}_{\mathcal{D}} \subseteq \cap_{i=1}^{n} \mathcal{R}_i$. Suppose $\Gamma \mathcal{R}_{\mathcal{D}} \Delta$ and $[\![t]\!]_i A \in \Gamma$, for an arbitrary $i \in G$. We have to show that $A \in \Delta$. Since $[\![t]\!]_i A \to [\![t]\!]_{\mathcal{D}} A \in \Gamma$, we have $[\![t]\!]_{\mathcal{D}} A \in \Gamma$, and therefore $A \in \Delta$. It is not difficult to verify that for $n = 1$ we have $\mathcal{R}_{\mathcal{D}} = \mathcal{R}_1$.

We now prove the Truth Lemma: for all formulas F we have

$$F \in \Gamma \quad \text{iff} \quad (\mathcal{M}, \Gamma) \Vdash_p F.$$

The proof is by induction on the complexity of F and is similar to that for justification logics in [2]. We only show the case when F is $[\![\, t \,]\!]_* G$.

If $[\![\, t \,]\!]_* G \in \Gamma$, then $\Gamma \in \mathcal{E}_*(t, G)$ by the definition of $\mathcal{E}_*$. In addition, for all $\Delta \in \mathcal{W}$ such that $\Gamma \mathcal{R}_* \Delta$, by the definition of $\mathcal{R}_*$, we have $G \in \Delta$. Hence, by the induction hypothesis, we obtain $(\mathcal{M}, \Delta) \Vdash_p G$. Thus $(\mathcal{M}, \Gamma) \Vdash_p [\![\, t \,]\!]_* G$.

If $[\![\, t \,]\!]_* G \notin \Gamma$, then $\Gamma \notin \mathcal{E}_*(t, G)$. Thus $(\mathcal{M}, \Gamma) \nVdash_p [\![\, t \,]\!]_* G$.

Now suppose $\mathbf{JL^D}(\mathcal{CS}) \nvdash A$, then $\{\neg A\}$ is a $\mathbf{JL^D}(\mathcal{CS})$-consistent set. Extend it to a maximal consistent set Γ by standard Lindenbaum construction, then by truth lemma we have $(\mathcal{M}, \Gamma) \nVdash_p A$. □

Note that in the canonical model $\mathcal{M}$ we have $\cup_{i=0}^{n} \mathcal{E}_i(t, A) \subseteq \mathcal{E}_{\mathcal{D}}(t, A)$, for every term t and formula A.

Theorem 3 (Compactness). *For a given $\mathcal{CS}$ for $\mathbf{JL^D}$, a set of formulas S is $\mathbf{JL^D}(\mathcal{CS})$-satisfiable iff any finite subset of S is $\mathbf{JL^D}(\mathcal{CS})$-satisfiable.*

Proof. Suppose every finite subset of S is $\mathbf{JL^D}(\mathcal{CS})$-satisfiable. Clearly S is $\mathbf{JL^D}(\mathcal{CS})$-consistent. Extend S to a maximal consistent set Γ. Thus Γ is a world in the canonical model $\mathcal{M}$ of $\mathbf{JL^D}(\mathcal{CS})$. Since $S \subseteq \Gamma$, by the Truth Lemma, $(\mathcal{M}, \Gamma) \Vdash A$ for all $A \in S$. Therefore, S is satisfiable. □

One of the important properties of Fitting models is the fully explanatory property, which first proved by Fitting in [9] for models of the logic of proofs.

Definition 8. *A $\mathbf{JL^D}$-model $\mathcal{M}$ is a strong model if it has the* fully explanatory property*:*

1. *if for every v such that $w\mathcal{R}_* v$ we have $(\mathcal{M}, v) \Vdash_p A$, then for some term $t \in Tm_*$ we have $(\mathcal{M}, w) \Vdash_p [\![t]\!]_* A$, and*
2. *if for every v such that $w\mathcal{R}_1 v, \ldots, w\mathcal{R}_n v$ we have $(\mathcal{M}, v) \Vdash_p A$, then for some term $t \in Tm_{\mathcal{D}}$ we have $(\mathcal{M}, w) \Vdash_p [\![t]\!]_{\mathcal{D}} A$.*

It is worth noting that the term t introduced in the above definition depends on the formula A and world w. Moreover, the definition of the fully explanatory property of $\mathbf{JL^D}$-models is slightly different from that for one agent justification logics (see [2, 9]). In contrast to the one agent case, in Definition 8 we extended the fully explanatory property of models to multi-agent case in statement 1, and add the statement 2.

Theorem 4 (Strong Completeness). *For any axiomatically appropriate constant specification $\mathcal{CS}$, $\mathbf{JL^D}(\mathcal{CS})$ is sound and complete with respect to their strong models that respect $\mathcal{CS}$.*

Proof. It suffices to prove that, for any axiomatically appropriate constant specification $\mathcal{CS}$, the canonical model of $\mathbf{JL^D}(\mathcal{CS})$ satisfies the fully explanatory property. Let $\mathcal{M} = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ be the canonical model of $\mathbf{JL^D}(\mathcal{CS})$, and $\Gamma \in \mathcal{W}$.

(1) Suppose $* \in \{1, \ldots, n, \mathcal{D}\}$ and $(\mathcal{M}, \Delta) \Vdash_p A$ for every Δ such that $\Gamma \mathcal{R}_* \Delta$. Suppose towards a contradiction that there is no justification term $t \in Tm_*$ such that $(\mathcal{M}, \Gamma) \Vdash_p [\![t]\!]_* A$. Then, the set $\Gamma^{\sharp_*} \cup \{\neg A\}$ would have to be $\mathbf{JL^D}(\mathcal{CS})$-consistent. Indeed, otherwise

$$\mathbf{JL^D}(\mathcal{CS}) \vdash X_1 \to (X_2 \to \ldots \to (X_m \to A) \ldots),$$

for some $[\![t_1]\!]_* X_1, \ldots, [\![t_m]\!]_* X_m \in \Gamma$. Since the constant specification $\mathcal{CS}$ is axiomatically appropriate, by Lemma 2, we would obtain a term s in Tm_* such that

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s]\!]_* (X_1 \to (X_2 \to \ldots \to (X_m \to A) \ldots)).$$

By axiom **A2**,

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_1]\!]_* X_1 \to ([\![t_2]\!]_* X_2 \to \ldots \to ([\![t_m]\!]_* X_m \to [\![t]\!]_* A) \ldots).$$

where $t = s \cdot_* t_1 \cdot_* \ldots \cdot_* t_m$. Hence $[\![t]\!]_* A \in \Gamma$. Thus, by the Truth Lemma, $(\mathcal{M}, \Gamma) \Vdash_p [\![t]\!]_* A$, a contradiction. Now since $\Gamma^{\sharp_*} \cup \{\neg A\}$ is $\mathbf{JL^D}(\mathcal{CS})$-consistent, it could be extended to a maximal $\mathbf{JL^D}(\mathcal{CS})$-consistent set Δ. Since $\Gamma^{\sharp_*} \subseteq \Delta$, we have $\Gamma \mathcal{R}_* \Delta$. But since $A \notin \Delta$, by the Truth Lemma, $(\mathcal{M}, \Delta) \nVdash_p A$, which contradicts the assumption.

(2) Suppose for every $\Delta \in \mathcal{W}$ such that $\Gamma \mathcal{R}_1 \Delta, \ldots, \Gamma \mathcal{R}_n \Delta$ we have $(\mathcal{M}, \Delta) \Vdash_p A$, and $(\mathcal{M}, \Gamma) \nVdash_p [\![t]\!]_{\mathcal{D}} A$, for each $t \in Tm_{\mathcal{D}}$. We show that there is $\Delta \in \mathcal{W}$ with $\Gamma \mathcal{R}_{\mathcal{D}} \Delta$ such that $(\mathcal{M}, \Delta) \nVdash_p A$. We prove that $\Gamma^{\sharp_{\mathcal{D}}} \cup \{\neg A\}$ is consistent. Otherwise, for some $[\![t_1]\!]_{\mathcal{D}} X_1, \ldots, [\![t_m]\!]_{\mathcal{D}} X_m$ in Γ we have

$$\mathbf{JL^D}(\mathcal{CS}) \vdash X_1 \to (X_2 \to \cdots \to (X_m \to A) \cdots).$$

Since the constant specification $\mathcal{CS}$ is axiomatically appropriate, by Lemma 2, there is a term s in $Tm_{\mathcal{D}}$ such that

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s]\!]_{\mathcal{D}} (X_1 \to (X_2 \to \cdots \to (X_m \to A) \cdots)).$$

By axiom **A2**, we conclude that

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_1]\!]_{\mathcal{D}} X_1 \to ([\![t_2]\!]_{\mathcal{D}} X_2 \to \cdots \to ([\![t_m]\!]_{\mathcal{D}} X_m \to [\![t]\!]_{\mathcal{D}} A) \cdots)$$

where $t = s \cdot_{\mathcal{D}} t_1 \cdot_{\mathcal{D}} \cdots \cdot_{\mathcal{D}} t_m$. Hence, $[\![t]\!]_{\mathcal{D}} A \in \Gamma$, and by the Truth Lemma, $(\mathcal{M}, \Gamma) \Vdash_p [\![t]\!]_{\mathcal{D}} A$, which is a contradiction. Thus $\Gamma^{\sharp_{\mathcal{D}}} \cup \{\neg A\}$ is a consistent set. Now extend it to a maximal consistent set Δ. By the truth lemma $(\mathcal{M}, \Delta) \nVdash_p A$. On the other hand, it is obvious that $\Gamma \mathcal{R}_{\mathcal{D}} \Delta$, and since $\mathcal{R}_{\mathcal{D}} \subseteq \cap_{i=0}^{n} \mathcal{R}_i$, we have $\Gamma \mathcal{R}_i \Delta$, for each $i \in G$, which contradicts the assumption. □

5 Realization Theorem

In this section, we prove that each theorem of $\mathbf{JL^D}$ can be translated into a theorem of $\mathbf{L^D}$, and vise versa. First, we define a translation, called the forgetful projection, from formulas of $\mathbf{JL^D}$ to formulas of $\mathbf{L^D}$.

Definition 9. *For a $\mathbf{JL^D}$-formula F, the forgetful projection of F, denoted by F°, is defined inductively as follows:*

1. *For propositional letter P, $P^\circ = P$, and $\bot^\circ = \bot$,*
2. $(A \to B)^\circ = A^\circ \to B^\circ$,
3. $([\![t]\!]_i A)^\circ = \mathcal{K}_i A^\circ$,
4. $([\![t]\!]_\mathcal{D} A)^\circ = \mathcal{D} A^\circ$.

For a set S of justification formulas we let $S^\circ = \{F^\circ \,|\, F \in S\}$.

Lemma 4. *For any formula F of $\mathbf{JL^D}$, if $\mathbf{JL^D} \vdash F$ then $\mathbf{L^D} \vdash F^\circ$.*

Proof. By induction on a derivation of F in $\mathbf{JL^D}$. If F is an axiom of $\mathbf{JL^D}$, then it is easy to verify that F° is provable in $\mathbf{L^D}$. For instance, $([\![t]\!]_i A \to [\![t]\!]_\mathcal{D} A)^\circ = \mathcal{K}_i A^\circ \to \mathcal{D} A^\circ$, which is an instance of $\mathbf{K_iD}$ axiom. If F is obtained by Modus Ponens from G and $G \to F$, then by the induction hypothesis G° and $G^\circ \to F^\circ$ are provable in $\mathbf{L^D}$. Thus, F° is provable in $\mathbf{L^D}$. If $F = [\![c^{i_m}_{j_m}]\!]_{i_m} \dots [\![c^{i_1}_{j_1}]\!]_{i_1} A \in \mathcal{CS}$ is obtained by the Iterated Axiom Necessitation rule, then A° is provable in $\mathbf{L^D}$, since A is an axiom of $\mathbf{JL^D_n}$. Hence, by iterated applications of the Necessitation rule, we can derive $\mathcal{K}_{i_m} \dots \mathcal{K}_{i_1} A^\circ$. Likewise, If $F = [\![c^i]\!]_i A \in \mathcal{CS}$ is obtained by the Axiom Necessitation rule, then use the Necessitation rule to derive $\mathcal{K}_i A^\circ$. □

Definition 10. *Let A be a formula in the language of $\mathbf{L^D}$. A realization of the formula A is a $\mathbf{JL^D}$-formula A^r such that $(A^r)^\circ = A$.*

More precisely, a realization A^r is obtained by replacing each modality $\mathcal{K}_i$ in A by a term in Tm_i, and each modality $\mathcal{D}$ in A by a term in $Tm_\mathcal{D}$. A realization is called *normal* if all negative occurrences of modalities are replaced by distinct variables. In the rest of this section we will prove the following results:

$$\begin{array}{ll} \mathbf{JK^{D}_n}^\circ = \mathbf{K^D_n}, & \mathbf{JT^{D}_n}^\circ = \mathbf{T^D_n}, \\ \mathbf{JS4^{D}_n}^\circ = \mathbf{S4^D_n}, & \mathbf{JS5^{D}_n}^\circ = \mathbf{S5^D_n}. \end{array} \qquad (1)$$

The existence of an $\mathbf{JL^D}$-realization of any theorems of $\mathbf{L^D}$ can be established semantically by a method developed in [9].

Definition 11. *By $\mathbf{JL^{D-}}$ we mean the system $\mathbf{JL^D}$ in a language without operations $+_*$ and without axioms **A1**. Models of $\mathbf{JL^{D-}}$ are the same as for those of $\mathbf{JL^D}$ except that the evidence function is not required to satisfy the condition $\mathcal{E}1$.*

It is easy to verify that the internalization lemma holds for $\mathbf{JL^{D-}}$ and the fully explanatory property of the canonical model holds for $\mathbf{JL^{D-}}$-models (the canonical models of $\mathbf{JL^{D-}}$ are defined similar to the canonical models of $\mathbf{JL^D}$).

Let φ be a formula in the language of $\mathbf{L^D}$, fixed for the rest of this section. By subformula we mean subformula occurrence. The set of all subformulas, positive subformulas and negative subformulas of φ are denoted, respectively, by $Sub(\varphi)$, $Sub^+(\varphi)$ and $Sub^-(\varphi)$.

Definition 12. *Let $\mathcal{A}$ be any assignment of justification variables*

$$\bigcup_{*\in\{1,\dots,n,\mathcal{D}\}} Var^*$$

to negative subformulas of φ of the form $\mathcal{K}_i X$ or $\mathcal{D}X$ such that

- *If $\mathcal{A}(\mathcal{K}_i X) = x$, then $x \in Var^i$.*
- *If $\mathcal{A}(\mathcal{D}X) = x$, then $x \in Var^{\mathcal{D}}$.*

We define two mappings $w_{\mathcal{A}}$ and $v_{\mathcal{A}}$ of subformulas of φ to sets of formulas of $\mathbf{JL^D}$ and $\mathbf{JL^{D-}}$, respectively, as follows:

1. *$w_{\mathcal{A}}(P) = v_{\mathcal{A}}(P) = \{P\}$, where P is a propositional variable;*
   $w_{\mathcal{A}}(\bot) = v_{\mathcal{A}}(\bot) = \{\bot\}$.
2. *$w_{\mathcal{A}}(X \to Y) = \{X' \to Y' \,|\, X' \in w_{\mathcal{A}}(X), Y' \in w_{\mathcal{A}}(Y)\}$,*
   $v_{\mathcal{A}}(X \to Y) = \{X' \to Y' \,|\, X' \in v_{\mathcal{A}}(X), Y' \in v_{\mathcal{A}}(Y)\}$.
3. *If $\mathcal{K}_i X \in Sub^-(\varphi)$, then*
   $w_{\mathcal{A}}(\mathcal{K}_i X) = \{[\![x]\!]_i X' \,|\, \mathcal{A}(\mathcal{K}_i X) = x, x \in Var^i, X' \in w_{\mathcal{A}}(X)\}$,
   $v_{\mathcal{A}}(\mathcal{K}_i X) = \{[\![x]\!]_i X' \,|\, \mathcal{A}(\mathcal{K}_i X) = x, x \in Var^i, X' \in v_{\mathcal{A}}(X)\}$.
4. *If $\mathcal{K}_i X \in Sub^+(\varphi)$, then*
   $w_{\mathcal{A}}(\mathcal{K}_i X) = \{[\![t]\!]_i X' \,|\, t \in Tm_i, X' \in w_{\mathcal{A}}(X)\}$,
   $v_{\mathcal{A}}(\mathcal{K}_i X) = \{[\![t]\!]_i (X_1 \vee \ldots \vee X_m) \,|\, t \in Tm_i, X_1, \ldots, X_m \in v_{\mathcal{A}}(X)\}$.
5. *If $\mathcal{D}X \in Sub^-(\varphi)$, then*
   $w_{\mathcal{A}}(\mathcal{D}X) = \{[\![x]\!]_i X' \,|\, \mathcal{A}(\mathcal{D}X) = x, x \in Var^{\mathcal{D}}, X' \in w_{\mathcal{A}}(X)\}$,
   $v_{\mathcal{A}}(\mathcal{D}X) = \{[\![x]\!]_i X' \,|\, \mathcal{A}(\mathcal{D}X) = x, x \in Var^{\mathcal{D}}, X' \in v_{\mathcal{A}}(X)\}$.
6. *If $\mathcal{D}X \in Sub^+(\varphi)$, then*
   $w_{\mathcal{A}}(\mathcal{D}X) = \{[\![t]\!]_{\mathcal{D}} X' \,|\, t \in Tm_{\mathcal{D}}, X' \in w_{\mathcal{A}}(X)\}$,
   $v_{\mathcal{A}}(\mathcal{D}X) = \{[\![t]\!]_{\mathcal{D}} (X_1 \vee \ldots \vee X_m) \,|\, t \in Tm_{\mathcal{D}}, X_1, \ldots, X_m \in v_{\mathcal{A}}(X)\}$.

By $\neg v_{\mathcal{A}}(X)$ we mean $\{\neg X' \,|\, X' \in v_{\mathcal{A}}(X)\}$. It is assumed that $\mathcal{A}$ assigns different variables to different subformulas (this assumption is required in the proof of Lemma 6).

Let $\mathcal{M} = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ be the canonical model of $\mathbf{JL^{D-}}$. We may consider $\mathcal{M}$ as a model for $\mathbf{L^D}$, in which the accessibility relation $\mathcal{R}_{\mathcal{D}}$ and evidence functions $\mathcal{E}_*$ play no role and $\Vdash_p$ is defined as in Definition 2. In this case we write $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} A$ to denote that $\mathcal{M}$ is considered as a model of $\mathbf{L^D}$.

Lemma 5. *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathbf{JL^{D-}}$, and $\mathcal{M}$ be a canonical model for $\mathbf{JL^{D-}}$ that respects $\mathcal{CS}$. Then for each world Γ of the model:*

1. *If $\psi \in Sub^+(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p \neg v_{\mathcal{A}}(\psi)$, then $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \neg\psi$.*
2. *If $\psi \in Sub^-(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p v_{\mathcal{A}}(\psi)$, then $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \psi$.*

Proof. The proof is by induction on the complexity of ψ. The proof for propositional variables and the case for implication is similar to the proof of Proposition 7.7 in [9].

Suppose $\psi = \mathcal{K}_i X \in Sub^+(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p \neg v_{\mathcal{A}}(\psi)$. First we show that $\Gamma^{\sharp_i} \cup \neg v_{\mathcal{A}}(X)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent. Indeed, otherwise

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash Y_1 \to (Y_2 \to \ldots (Y_m \to X_1 \vee \ldots \vee X_k) \ldots)$$

for some $[\![t_1]\!]_i Y_1, \ldots, [\![t_m]\!]_i Y_m \in \Gamma$ and $X_1, \ldots, X_k \in v_{\mathcal{A}}(X)$. By the internalization lemma, since $\mathcal{CS}$ is axiomatically appropriate, there is a term $s \in Tm_i$ such that

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash [\![s]\!]_i [Y_1 \to (Y_2 \to \ldots (Y_m \to X_1 \vee \ldots \vee X_k) \ldots)]$$

By axiom **A2** and propositional reasoning, we have

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash [\![t_1]\!]_i Y_1 \wedge \ldots \wedge [\![t_m]\!]_i Y_m \to [\![t]\!]_i (X_1 \vee \ldots \vee X_k)$$

where $t = s \cdot_i t_1 \cdot_i \ldots \cdot_i t_m \in Tm_i$. Therefore, $(\mathcal{M}, \Gamma) \Vdash_p [\![t]\!]_i (X_1 \vee \ldots \vee X_k)$, which is impossible since $[\![t]\!]_i (X_1 \vee \ldots \vee X_k) \in v_{\mathcal{A}}(\psi)$. Hence, $\Gamma^{\sharp_i} \cup \neg v_{\mathcal{A}}(X)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent, and can be extended to a maximal $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent set Δ. Thus $\Gamma \mathcal{R}_i \Delta$ and $(\mathcal{M}, \Delta) \Vdash_p \neg v_{\mathcal{A}}(X)$. Since $X \in Sub^+(\varphi)$, by the induction hypothesis, $(\mathcal{M}, \Delta) \Vdash_{\mathbf{L^D}} \neg X$. Hence, $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \neg \psi$.

Suppose $\psi = \mathcal{D} X \in Sub^+(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p \neg v_{\mathcal{A}}(\psi)$. First we show that $\Gamma^{\sharp_\mathcal{D}} \cup \neg v_{\mathcal{A}}(X)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent. Indeed, otherwise

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash Y_1 \to (Y_2 \to \ldots (Y_m \to X_1 \vee \ldots \vee X_k) \ldots)$$

for some $[\![t_1]\!]_\mathcal{D} Y_1, \ldots, [\![t_m]\!]_\mathcal{D} Y_m \in \Gamma$ and $X_1, \ldots, X_k \in v_{\mathcal{A}}(X)$. By internalization, there is a term $s \in Tm_\mathcal{D}$ such that

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash [\![s]\!]_\mathcal{D} [Y_1 \to (Y_2 \to \ldots (Y_m \to X_1 \vee \ldots \vee X_k) \ldots)]$$

By axiom **A2** and propositional reasoning, we have

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash [\![t_1]\!]_\mathcal{D} Y_1 \wedge \ldots \wedge [\![t_m]\!]_\mathcal{D} Y_m \to [\![t]\!]_\mathcal{D} (X_1 \vee \ldots \vee X_k)$$

where $t = s \cdot_\mathcal{D} t_1 \cdot_\mathcal{D} \ldots \cdot_\mathcal{D} t_m \in Tm_\mathcal{D}$. Therefore, $(\mathcal{M}, \Gamma) \Vdash_p [\![t]\!]_\mathcal{D} (X_1 \vee \ldots \vee X_k)$, which is impossible since $[\![t]\!]_\mathcal{D} (X_1 \vee \ldots \vee X_k) \in v_{\mathcal{A}}(\psi)$. Hence, $\Gamma^{\sharp_\mathcal{D}} \cup \neg v_{\mathcal{A}}(X)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent, and can be extended to a maximal $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent set Δ. Thus $\Gamma \mathcal{R}_\mathcal{D} \Delta$ and $(\mathcal{M}, \Delta) \Vdash_p \neg v_{\mathcal{A}}(X)$. Since $\mathcal{R}_\mathcal{D} \subseteq \bigcap_{i=1}^n \mathcal{R}_i$, we have $\Gamma \mathcal{R}_i \Delta$ for any $i \in G$. Since $X \in Sub^+(\varphi)$, by the induction hypothesis, $(\mathcal{M}, \Delta) \Vdash_{\mathbf{L^D}} \neg X$. Hence, $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \neg \psi$.

Suppose $\psi = \mathcal{K}_i X \in Sub^-(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p v_{\mathcal{A}}(\psi)$. Let X' be an arbitrary element of $v_{\mathcal{A}}(X)$. Then $[\![x]\!]_i X' \in v_{\mathcal{A}}(\psi)$, where $\mathcal{A}(\mathcal{K}_i X) = x$, and therefore $(\mathcal{M}, \Gamma) \Vdash_p [\![x]\!]_i X'$. Now for any world Δ such that $\Gamma \mathcal{R}_i \Delta$, $(\mathcal{M}, \Delta) \Vdash_p X'$. Thus $(\mathcal{M}, \Delta) \Vdash_p v_{\mathcal{A}}(X)$. Since $X \in Sub^-(\varphi)$, by the induction hypothesis, $(\mathcal{M}, \Delta) \Vdash_{\mathbf{L^D}} X$. Hence, $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \psi$.

Suppose $\psi = \mathcal{D}X \in Sub^-(\varphi)$ and $(\mathcal{M}, \Gamma) \Vdash_p v_\mathcal{A}(\psi)$. Let X' be an arbitrary element of $v_\mathcal{A}(X)$. Then $[\![x]\!]_\mathcal{D} X' \in v_\mathcal{A}(\psi)$, where $\mathcal{A}(\mathcal{D}X) = x$, and therefore $(\mathcal{M}, \Gamma) \Vdash_p [\![x]\!]_\mathcal{D} X'$. Now for any world Δ such that $\Gamma \mathcal{R}_\mathcal{D} \Delta$, we have $(\mathcal{M}, \Delta) \Vdash_p X'$. Thus $(\mathcal{M}, \Delta) \Vdash_p v_\mathcal{A}(X)$. Since $X \in Sub^-(\varphi)$, by the induction hypothesis, $(\mathcal{M}, \Delta) \Vdash_{\mathbf{L^D}} X$. Since $\mathcal{R}_\mathcal{D} \subseteq \bigcap_{i=1}^n \mathcal{R}_i$, we have $\Gamma \mathcal{R}_i \Delta$ for any $i \in G$. Hence, $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L^D}} \psi$. □

Corollary 1. *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathbf{JL^{D-}}$. If $\mathbf{L^D} \vdash \varphi$ then there are $\varphi_1, \ldots, \varphi_m \in v_\mathcal{A}(\varphi)$ such that*

$$\mathbf{JL^{D-}}(\mathcal{CS}) \vdash \varphi_1 \vee \ldots \vee \varphi_m.$$

Proof. Suppose towards a contradiction that $\mathbf{JL^{D-}}(\mathcal{CS}) \nvdash \varphi_1 \vee \ldots \vee \varphi_m$ for all $\varphi_1, \ldots, \varphi_m \in v_\mathcal{A}(\varphi)$. Thus $\neg v_\mathcal{A}(\varphi)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent. For otherwise there would be $\varphi_1, \ldots, \varphi_m \in v_\mathcal{A}(\varphi)$ such that $\mathbf{JL^{D-}}(\mathcal{CS}) \vdash \varphi_1 \vee \ldots \vee \varphi_m$, contrary to assumption. Since $\neg v_\mathcal{A}(\varphi)$ is $\mathbf{JL^{D-}}(\mathcal{CS})$-consistent, extend it to a maximal consistent set $\Gamma \in \mathcal{W}$. By Truth Lemma, $(\mathcal{M}, \Gamma) \Vdash_p \neg v_\mathcal{A}(\varphi)$. Hence, by Lemma 5, $(\mathcal{M}, \Gamma) \Vdash_{\mathbf{L}^D} \neg\varphi$, contra with the assumption $\mathbf{L^D} \vdash \varphi$ and Theorem 1. □

Lemma 6. *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathbf{JL^D}$. For every subformula ψ of φ and each $\psi_1, \ldots, \psi_m \in v_\mathcal{A}(\psi)$, there is a substitution σ and a formula $\psi' \in v_\mathcal{A}(\psi)$ such that:*

1. *If $\psi \in Sub^+(\varphi)$, then $\mathbf{JL^D}(\mathcal{CS}) \vdash (\psi_1 \vee \ldots \vee \psi_m)\sigma \rightarrow \psi'$.*
2. *If $\psi \in Sub^-(\varphi)$, then $\mathbf{JL^D}(\mathcal{CS}) \vdash \psi' \rightarrow (\psi_1 \wedge \ldots \wedge \psi_m)\sigma$.*

Proof. The proof is by induction on the complexity of ψ. The proof for propositional variables and the case for implication is similar to the proof of Proposition 7.8 in [9].

Suppose $\psi = \mathcal{K}_i X \in Sub^+(\varphi)$, and the result is known for X (which also occurs positively in φ). Let $\psi_1 = [\![t_1]\!]_i D_1, \ldots, \psi_m = [\![t_m]\!]_i D_m$ be in $v_\mathcal{A}(\psi)$, such that $t_1, \ldots, t_m \in Tm_i$ and $D_1, \ldots, D_m$ are disjunctions of formulas from $v_\mathcal{A}(X)$. Thus $D_1, \ldots, D_m \in v_\mathcal{A}(X)$. By the induction hypothesis, there is a substitution σ and $X' \in v_\mathcal{A}(X)$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash (D_1 \vee \ldots \vee D_m)\sigma \rightarrow X'$. Note that $(D_1 \vee \ldots \vee D_m)\sigma = D_1\sigma \vee \ldots \vee D_m\sigma$. Consequently, for each $j = 1, \ldots, m$, we have $\mathbf{JL^D}(\mathcal{CS}) \vdash D_j\sigma \rightarrow X'$. By the internalization lemma, there are terms $s_1, \ldots, s_m \in Tm_i$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s_j]\!]_i(D_j\sigma \rightarrow X')$, for each $j = 1, \ldots, m$. Hence by axiom **A2**

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_j\sigma]\!]_i D_j\sigma \rightarrow [\![s_j \cdot_i t_j\sigma]\!]_i X'.$$

Note that $[\![t_j\sigma]\!]_i D_j\sigma = ([\![t_j]\!]_i D_j)\sigma$. Let $t = s_1 \cdot_i t_1\sigma +_i \ldots +_i s_m \cdot_i t_m\sigma \in Tm_i$. By axiom **A1**,

$$\mathbf{JL^D}(\mathcal{CS}) \vdash ([\![t_j]\!]_i D_j)\sigma \rightarrow [\![t]\!]_i X',$$

for each $j = 1, \ldots, m$. Thus,

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \Big(\bigvee_{1 \le j \le m} [\![t_j]\!]_i D_j\Big)\sigma \rightarrow [\![t]\!]_i X'.$$

Therefore, letting $\psi' = [\![t]\!]_i X'$, we have

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \Big(\bigvee_{1\le j\le m} \psi_j\Big)\sigma \to \psi'.$$

Suppose $\psi = \mathcal{D}X \in Sub^+(\varphi)$, and the result is known for X (which also occurs positively in φ). Let $\psi_1 = [\![t_1]\!]_{\mathcal{D}}D_1, \ldots, \psi_m = [\![t_m]\!]_{\mathcal{D}}D_m$ be in $v_{\mathcal{A}}(\psi)$, such that $t_1, \ldots, t_m \in Tm_{\mathcal{D}}$ and $D_1, \ldots, D_m$ are disjunctions of formulas from $v_{\mathcal{A}}(X)$. By the induction hypothesis, there is a substitution σ and $X' \in v_{\mathcal{A}}(X)$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash (D_1 \vee \ldots \vee D_m)\sigma \to X'$. Consequently, for each $j = 1, \ldots, m$, we have $\mathbf{JL^D}(\mathcal{CS}) \vdash D_j\sigma \to X'$. By the internalization lemma, there are terms $s_1, \ldots, s_m \in Tm_{\mathcal{D}}$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s_j]\!]_{\mathcal{D}}(D_j\sigma \to X')$, for each $j = 1, \ldots, m$. Hence by axiom **A2**

$$\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_j\sigma]\!]_i D_j\sigma \to [\![s_j \cdot_{\mathcal{D}} t_j\sigma]\!]_{\mathcal{D}}X'.$$

Let $t = s_1 \cdot_{\mathcal{D}} t_1\sigma +_{\mathcal{D}} \ldots +_{\mathcal{D}} s_m \cdot_{\mathcal{D}} t_m\sigma$. By axiom **A1**,

$$\mathbf{JL^D}(\mathcal{CS}) \vdash ([\![t_j]\!]_{\mathcal{D}}D_j)\sigma \to [\![t]\!]_{\mathcal{D}}X',$$

for each $j = 1, \ldots, m$. Thus,

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \Big(\bigvee_{1\le j\le m} [\![t_j]\!]_{\mathcal{D}}D_j\Big)\sigma \to [\![t]\!]_{\mathcal{D}}X'.$$

Therefore, letting $\psi' = [\![t]\!]_{\mathcal{D}}X'$, we have

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \Big(\bigvee_{1\le j\le m} \psi_j\Big)\sigma \to \psi'.$$

Suppose $\psi = \mathcal{K}_iX \in Sub^-(\varphi)$, and the result is known for X (which also occurs negatively in φ). Let $\psi_1 = [\![x]\!]_iX_1, \ldots, \psi_m = [\![x]\!]_iX_m$ be in $v_{\mathcal{A}}(\psi)$, such that $\mathcal{A}(\mathcal{K}_iX) = x$ (where $x \in Var^i$), and $X_1, \ldots, X_m \in v_{\mathcal{A}}(X)$. By the induction hypothesis, there is a substitution σ and $X' \in w_{\mathcal{A}}(X)$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash X' \to (X_1 \wedge \ldots \wedge X_m)\sigma$. Since $\mathcal{A}$ assigns different variables to different subformulas, x does not occur in $X_1, \ldots, X_m$, and hence $x \notin dom(\sigma)$. It follows that, for each $j = 1, \ldots, m$, $\mathbf{JL^D}(\mathcal{CS}) \vdash X' \to X_j\sigma$. By the internalization lemma, there are terms $t_1, \ldots, t_m \in Tm_i$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_j]\!]_i(X' \to X_j\sigma)$, for each $j = 1, \ldots, m$. Thus $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s]\!]_i(X' \to X_j\sigma)$ for $s = t_1 +_i \ldots +_i t_m$. Therefore, for each $j = 1, \ldots, m$, $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![x]\!]_iX' \to [\![s \cdot_i x]\!]_i(X_j\sigma)$. Letting $\sigma' = \sigma \cup \{(x, s \cdot_i x)\}$ we have $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![x]\!]_iX' \to ([\![x]\!]_iX_j)\sigma'$, from which we get

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \psi' \to ([\![x]\!]_iX_1 \wedge \ldots \wedge [\![x]\!]_iX_m)\sigma'$$

for $\psi' = [\![x]\!]_iX'$.

Suppose $\psi = \mathcal{D}X \in Sub^-(\varphi)$, and the result is known for X (which also occurs negatively in φ). Let $\psi_1 = [\![x]\!]_{\mathcal{D}}X_1, \ldots, \psi_m = [\![x]\!]_{\mathcal{D}}X_m$ be in $v_{\mathcal{A}}(\psi)$, such that $\mathcal{A}(\mathcal{D}X) = x$ (where $x \in Var^{\mathcal{D}}$), and $X_1, \ldots, X_m \in v_{\mathcal{A}}(X)$. By the induction hypothesis, there is a substitution σ and $X' \in w_{\mathcal{A}}(X)$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash X' \to (X_1 \wedge \ldots \wedge X_m)\sigma$. Since $\mathcal{A}$ assigns different variables to different subformulas, x does not occur in $X_1, \ldots, X_m$, and hence $x \notin dom(\sigma)$. It follows that $\mathbf{JL^D}(\mathcal{CS}) \vdash X' \to X_j\sigma$, for each $j = 1, \ldots, m$. By the internalization lemma, there are terms $t_1, \ldots, t_m \in Tm_{\mathcal{D}}$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![t_j]\!]_{\mathcal{D}}(X' \to X_j\sigma)$, for each $j = 1, \ldots, m$. Thus $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![s]\!]_{\mathcal{D}}(X' \to X_j\sigma)$ for $s = t_1 +_{\mathcal{D}} \ldots +_{\mathcal{D}} t_m$. Therefore $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![x]\!]_{\mathcal{D}} X' \to [\![s \cdot_{\mathcal{D}} x]\!]_{\mathcal{D}}(X_j\sigma)$, for each $j = 1, \ldots, m$. Letting $\sigma' = \sigma \cup \{(x, s \cdot_{\mathcal{D}} x)\}$ we have $\mathbf{JL^D}(\mathcal{CS}) \vdash [\![x]\!]_{\mathcal{D}} X' \to ([\![x]\!]_{\mathcal{D}} X_j)\sigma'$, from which we get

$$\mathbf{JL^D}(\mathcal{CS}) \vdash \psi' \to ([\![x]\!]_{\mathcal{D}} X_1 \wedge \ldots \wedge [\![x]\!]_{\mathcal{D}} X_m)\sigma'$$

for $\psi' = [\![x]\!]_{\mathcal{D}} X'$. □

Corollary 2. *Let $\mathcal{CS}$ be an axiomatically appropriate constant specification for $\mathbf{JL^D}$. If $\mathbf{L^D} \vdash \varphi$ then there is a substitution σ and $\varphi' \in w_{\mathcal{A}}(\varphi)$ such that*

$$\mathbf{JL^D}(\mathcal{CS} \cup \mathcal{CS}\sigma) \vdash \varphi'.$$

Proof. Suppose $\mathbf{L^D} \vdash \varphi$. By Corollary 1, there are $\varphi_1, \ldots, \varphi_m \in v_{\mathcal{A}}(\varphi)$ such that $\mathbf{JL^{D-}}(\mathcal{CS}) \vdash \varphi_1 \vee \ldots \vee \varphi_m$. By Lemma 6, there is a substitution σ and a formula $\varphi' \in v_{\mathcal{A}}(\varphi)$ such that $\mathbf{JL^D}(\mathcal{CS}) \vdash (\varphi_1 \vee \ldots \vee \varphi_m)\sigma \to \varphi'$. By the substitution lemma, $\mathbf{JL^D}(\mathcal{CS}\sigma) \vdash (\varphi_1 \vee \ldots \vee \varphi_m)\sigma$, and therefore $\mathbf{JL^D}(\mathcal{CS} \cup \mathcal{CS}\sigma) \vdash \varphi'$. □

Our main theorem in this section is the realization theorem. In fact, we give a uniform realization theorem for all systems $\mathbf{JL^D_n}$.

Theorem 5 (Realization Theorem). $\mathbf{JL^{D\circ}} = \mathbf{L^D}$

Proof. One direction of the proof is done by Lemma 4. For the other direction suppose $\mathbf{L^D} \vdash \varphi$. By Corollary 2, there is a formula $\psi \in w_{\mathcal{A}}(\varphi)$ such that $\mathbf{JL^D} \vdash \psi$. Note that, by the definition of $w_{\mathcal{A}}$, ψ is a realization of φ, i.e. $\psi^\circ = \varphi$. □

6 Conclusions

In this paper we study logics of distributed knowledge with justifications. The advantage of this study is to incorporate the notion of evidence (or justification) into the distributed knowledge logics. For future work, it is natural to combine the justified distributed knowledge logic $\mathbf{JS4^D_n}$ with the explicit evidence system with common knowledge $\mathbf{LP^C_n}$ introduced in [5]. There remains also some questions: Are there Fitting models (that are pseudo-Fitting models without accessibility relation $\mathcal{R}_{\mathcal{D}}$) for $\mathbf{JL^D}$? Are $\mathbf{JL^D}$ conservative over multi-agent justification systems $\mathbf{JL_n}$ (the systems $\mathbf{JL^D}$ without distributed knowledge operator)? Are there cut-free tableau or Gentzen systems for $\mathbf{JL^D}$?

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