Max and Min Values of the Structural Similarity Function $S(x, a)$ on the L^2 Sphere $S_R(a), a \in \mathbb{R}^N$

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Abstract. Given a reference signal a, we analytically solve for the critical points that maximize/minimize the Structural Similarity function $S(x, a)$ while restricting ourselves to points x that lie on an L^2 sphere centered at a with fixed radius R. To do this, we employ the method of Lagrange multipliers and show that at least four (and as many as six) critical points exist, deriving the conditions that guarantee their existence.

1 Introduction

The original motivation for this problem came from a paper by Wang and Simoncelli [4] proposing a method of comparing two computational models M_1 and M_2 of a perceptually discernable quantity. First synthesize data that maximizes/minimizes M_1 while holding M_2 constant. Then reverse the roles and repeat. Subjective testing is then performed on these simulated data sets to determine which model is better.

The two image quality models examined in [4] were mean squared error (MSE) and the structural similarity (SSIM) index [2]. By holding MSE constant and varying SSIM, one is essentially travelling over an L^2 sphere of constant radius R and centered at a reference image I, searching for (critical) points at which the SSIM index between the image I' on the sphere and the image I at the center is maximized/minimized. Computationally, this can be performed using constrained gradient ascent/descent, as was done in [4]. (Expressions for the weighted SSIM indices (using local image patches) and their derivatives are presented in [4].)

The visual variation of images over such an L^2 sphere is illustrated very convincingly in the collection of *Einstein* images that appears in a paper by Wang and Bovik [3]. In this collection, the 8 bit-per-pixel *Einstein* test image is shown along with a number of distorted versions – some obtained by adding noise, some by blurring and some by shifting. The important fact is that the mean-squared error (MSE) of these distorted images – hence their L^2 distance from the original test image – is roughly the same. Visually, however, some images appear much closer to the undistorted image than others. This presentation illustrates very well the fact that MSE, an L^2-based distance, although convenient to employ,

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is not a good measure of visual quality. Indeed, the authors show in this figure that the SSIM index (and some variations) can assess the visual quality of the distorted images on the sphere: Images that are visually closer have higher SSIM values.

These studies lead naturally to the mathematical question of the maximum and minimum SSIM values – call them $S_{\max}$ and $S_{\min}$, respectively – on an L^2 sphere of radius R. One would expect that these extreme values will depend on R and, most probably, the variance of the reference image I at the center of the sphere. Unfortunately, a mathematical analysis of the problem in [4], which involves SSIM indices computed with local, hence overlapping, image patches (i.e., sliding windows), is intractable. Here we consider the following simplified version of the original problem which admits analytic solutions and gives insight into the general problem:

Given a point $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$, let $S_R(a)$ denote the L^2 sphere of radius R centered at a, i.e.,

$$S_R(a) = \{x \in \mathbb{R}^N \mid \|x - a\|_2 = R\}. \tag{1}$$

Find and classify the critical points of the SSIM function,

$$S(x, a) = \frac{4\bar{x}\bar{a}s_{xa}}{(\bar{x}^2 + \bar{a}^2)(s_x^2 + s_a^2)}, \tag{2}$$

on $S_R(a)$. Here,

$$\bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k, \quad s_{xa} = \frac{1}{N-1}\sum_{k=1}^{N}(x_k - \bar{x})(a_k - \bar{a}), \tag{3}$$

and the formula for $s_x^2 = s_{xx}$ follows.

Those familiar with SSIM will notice that we have set the so-called SSIM stability constants to zero, simplifying the form of the SSIM function and making possible an analytic solution of the problem.

A solution of this problem using Lagrange multipliers is outlined in the next section. (Complete details are to be found in [1].) In the final section, some examples of the absolute maximum and minimum SSIM values along with their associated images are presented.

2 The Solution

The Lagrangian function associated with this problem is given by

$$L(x) = S(x, a) + \lambda g(x), \tag{4}$$

where

$$g(x) = \sum_{k=1}^{N}(x_k - a_k)^2 - R^2 \tag{5}$$

represents the constraint and λ denotes the Lagrange multiplier. As usual, we impose the stationary constraints, $\frac{\partial L}{\partial x_p} = 0, \quad 1 \le p \le N$. The necessary partial derivatives are as follows (they will be useful again later),

$$\frac{\partial S}{\partial x_p} = \frac{4\bar{a}}{N(\bar{x}^2+\bar{a}^2)^2(s_x^2+s_a^2)^2}\left[s_{xa}(s_x^2+s_a^2)(\bar{a}^2-\bar{x}^2)\right. \\ \left. + \frac{N}{N-1}\bar{x}(\bar{x}^2+\bar{a}^2)(s_x^2+s_a^2)(a_p-\bar{a}) - \frac{2N}{N-1}\bar{x}s_{xa}(\bar{x}^2+\bar{a}^2)(x_p-\bar{x})\right]. \tag{6}$$

The Lagrangian stationarity constraints yield the equations

$$\frac{4\bar{a}}{N(\bar{x}^2+\bar{a}^2)^2(s_x^2+s_a^2)^2}\left[s_{xa}(s_x^2+s_a^2)(\bar{a}^2-\bar{x}^2) + \frac{N}{N-1}\bar{x}(\bar{x}^2+\bar{a}^2)(s_x^2+s_a^2)(a_p-\bar{a})\right. \\ \left. -\frac{2N}{N-1}\bar{x}s_{xa}(\bar{x}^2+\bar{a}^2)(x_p-\bar{x})\right] + 2\lambda(x_p-a_p) = 0, \quad 1 \le p \le N. \tag{7}$$

Summing up both sides of (7) for $1 \le p \le N$, yields the following equality,

$$\frac{4\bar{a}}{(\bar{x}^2+\bar{a}^2)^2(s_x^2+s_a^2)^2}s_{xa}(s_x^2+s_a^2)(\bar{a}^2-\bar{x}^2) + 2N\lambda(\bar{x}-\bar{a}) = 0. \tag{8}$$

Clearly, this equation is satisfied if $\bar{x} = \bar{a}$ but we must also examine the case $\bar{x} \neq \bar{a}$.

Case 1: $\bar{x} = \bar{a}$

After some simplification and manipulation, the equations in (7) become

$$\frac{1}{(N-1)(s_x^2+s_a^2)^2}\left[(s_x^2+s_a^2)(a_p-x_p) + (s_x^2+s_a^2-2s_{xa})(x_p-\bar{a})\right] \\ + \lambda(x_p-a_p) = 0, \qquad 1 \le p \le N. \tag{9}$$

It is easy to show that

$$s_x^2 + s_a^2 - 2s_{xa} = \frac{R^2}{N-1}. \tag{10}$$

Substituting this result into (9) yields a set of equations for $1 \le p \le N$ which, after a little algebra, become

$$\left[\frac{1}{(N-1)(s_x^2+s_a^2)} - \lambda\right](a_p-x_p) = -\frac{R^2}{(N-1)(s_x^2+s_a^2)^2}(x_p-\bar{a}). \tag{11}$$

If $x_p = \bar{a}$ for any $p \in \{1, 2, \cdots, N\}$, then there are two possibilities:

A. $x_p = a_p$, implying that $a_p = \bar{a}$. Note that the equations $x_p = a_p$ cannot be true for all $p \in \{1, 2, \cdots, N\}$ since this would imply that $x = a$, violating the condition that $x \in S_R(a)$.

B. At the extremum, the Lagrange multiplier λ satisfies the equation,

$$\lambda = \frac{1}{(N-1)(s_x^2 + s_a^2)}. \tag{12}$$

Sub-case B: If Eqn. (12) holds, then, from Eqn. (11), $x_p = \bar{a}$ for all $1 \le p \le N$. But this implies that

$$\sum_{k=1}^{N} (x_k - a_k)^2 = \sum_{k=1}^{N} (a_k - \bar{a})^2 = (N-1)s_a^2, \tag{13}$$

which is not necessarily equal to R^2. In fact, s_a^2 and R can be chosen independently. Hence x does not necessarily lie on $S_R(a)$, which violates the constraint.

Sub-case A: Rearrange the equations in (11) for those values of p such that $a_p \neq x_p$ and call this set of p-values $\mathcal{P}_1$:

$$\frac{(N-1)(s_x^2 + s_a^2)^2}{R^2} \left[\frac{1}{(N-1)(s_x^2 + s_a^2)} - \lambda \right] = \frac{x_p - \bar{a}}{x_p - a_p}, \qquad p \in \mathcal{P}_1. \tag{14}$$

For each pair (a, R), the LHS of (14) is a constant at each extremum, independent of p. Denote this constant as

$$\beta = \beta(a, R) = \frac{x_p - \bar{a}}{x_p - a_p}, \qquad p \in \mathcal{P}_1. \tag{15}$$

It is now useful to examine the consequences of (15), noting that

$$x_p - \bar{a} = \beta(x_p - a_p), \qquad 1 \le p \le N, \tag{16}$$

since for $p \notin \mathcal{P}_1$, both sides of the equation are zero. First,

$$\sum_{k=1}^{N} (x_k - \bar{a})^2 = \beta^2 R^2 \quad \Longrightarrow \quad s_x^2 = \frac{1}{N-1}\beta^2 R^2, \tag{17}$$

The final result follows from $\bar{x} = \bar{a}$. Now assume $\beta \neq 0$. This is reasonable, since not all of the x_p are equal to $\bar{a}$. Then we can write

$$2s_{xa} = \frac{1}{N-1}\beta^2 R^2 - \frac{1}{N-1} R^2 + s_a^2. \tag{18}$$

If $\beta = 1$ then from (16), $a_p = \bar{a}$ and so $s_a^2 = 0$. Substituting Eqns. (18) and (17) into (2) yields the trivial result $S(x, a) = 0$. If $\beta \neq 1$ then, from Eqn. (10),

$$s_{xa} = \frac{\beta}{\beta - 1} s_a^2. \tag{19}$$

Returning to the structural similarity function in Eq. (2) and employing the results obtained so far, including $\bar{x} = \bar{a}$, we find that

$$S(x, a) = \frac{\beta s_a^2}{\beta s_a^2 + \frac{\beta - 1}{2} \frac{R^2}{N-1}}. \tag{20}$$

The combination of Eqs. (18) and (19) produces a cubic equation in β with roots $\beta = -1$ and

$$\beta = 1 \pm \frac{s_a}{R}\sqrt{(N-1)}. \tag{21}$$

It remains to find the values of $S(x, a)$ that correspond to each value of β. Substituting $\beta = -1$ into Eqn. (16) and rearranging yields the point

$$x = \frac{1}{2}(a - \bar{a}\underline{1}), \tag{22}$$

where $\underline{1}$ denotes the N-vector $(1, 1, \cdots, 1)$. Since this point does not lie on the sphere $S_R(a)$, the root $\beta = -1$ is rejected.

The other two roots in (21) yield the structural similarity values

$$S_{1,2} = \frac{a \pm c}{a \pm c + b}, \tag{23}$$

where

$$a = s_a^2, \qquad b = \frac{R^2}{2(N-1)}, \qquad c = s_a \frac{R}{\sqrt{N-1}}. \tag{24}$$

Since $a, b, c > 0$, it follows from simple algebra that $S_1 > S_2$.

Since the values of β corresponding to the extrema have been identified, the points $x \in \mathbb{R}^N$ at which each of the extrema occur can be computed from Eqn. (15). For the β value corresponding to each case, we can solve for x_p (details omitted) and find that the condition $\bar{x} = \bar{a}$ is satisfied. In vector format, the extremum point is given by

$$x = a \pm \frac{R}{s_a\sqrt{N-1}}\, a', \tag{25}$$

where $a' = a - \bar{a}\underline{1}$ denotes the zero-mean component of a. In fact,

$$x = a \pm R\hat{a}', \tag{26}$$

where $\hat{a}'$ is the unit vector in the direction of the zero-mean component a'. (This follows from $\|\hat{a}'\| = s_a\sqrt{N-1}$.) Thus, $x \in S_R(a)$.

Case 2: $\bar{x} \neq \bar{a}$

Returning to Eqn. (8), if $\bar{x} \neq \bar{a}$, then the factor $\bar{x} - \bar{a}$ may be divided out to obtain the result

$$\lambda = \frac{2\bar{a}s_{xa}(\bar{x} + \bar{a})}{N(\bar{x}^2 + \bar{a}^2)^2(s_x^2 + s_a^2)}. \tag{27}$$

Substituting this result into Eqn. (7) and rearranging yields the equation

$$\begin{aligned} &(x_p - \bar{x})\sigma_{xa}\left[(\sigma_x^2 + \sigma_a^2)(\bar{a} + \bar{x}) - \frac{2N}{N-1}\bar{x}(\bar{x}^2 + \bar{a}^2)\right] \\ &+(a_p - \bar{a})(\sigma_x^2 + \sigma_a^2)\left[\frac{N}{N-1}\bar{x}(\bar{x}^2 + \bar{a}^2) - \sigma_{xa}(\bar{x} + \bar{a})\right] = 0. \end{aligned} \tag{28}$$

As in Case 1, rearrange the equations in (28) for those values of p such that $a_p \neq \bar{a}$ and call this set of p-values $\mathcal{P}_2$. An analysis similar to that of Case 1 then shows that the following ratio is constant at each extremum:

$$\alpha = \alpha(a, R) = \frac{x_p - \bar{x}}{a_p - \bar{a}}, \qquad p \in \mathcal{P}_2. \tag{29}$$

The consequences of this relation will again be examined, noting that

$$x_p - \bar{x} = \alpha(a_p - \bar{a}), \qquad 1 \leq p \leq N, \tag{30}$$

since for $p \notin \mathcal{P}_2$, both sides of the equation are zero. First, squaring both sides and summing over $1 \leq p \leq N$ yields $s_x^2 = \alpha^2 s_a^2$. Furthermore, from the definition of s_{xa} it is found that $s_{xa} = \alpha s_a^2$.

In order to determine acceptable values for α, we return to Eqn. (6) which gives the components of the gradient vector $\mathbf{\nabla} S$. At a stationary point x, it is necessary that $\mathbf{\nabla} S(x)$ be a constant multiple of the outward normal vector $\hat{n}$ to the sphere S_R at x, given by

$$\hat{n} = \frac{1}{R}(x_1 - a_1, x_2 - a_2, \cdots, x_N - a_N) = \frac{1}{R}(x - a). \tag{31}$$

Now substitute the above results into Eqn. (6) to yield

$$\begin{aligned}\frac{\partial S}{\partial x_p} = \frac{4\bar{a}}{N(\bar{x}^2 + \bar{a}^2)^2 (s_x^2 + s_a^2)^2} & \left[\alpha s_a^4 (1 + \alpha^2)(\bar{a}^2 - \bar{x}^2)\right. \\ & \left. + \frac{N}{N-1}\bar{x}(\bar{x}^2 + \bar{a}^2)(1 - \alpha^2)(a_p - \bar{a})\right].\end{aligned} \tag{32}$$

In general, the only way that the gradient vector $\mathbf{\nabla} S$ can be a multiple of the normal vector $\hat{n}$ is when the final term in Eqn. (32) vanishes, i.e., when $\alpha = \pm 1$.

Sub-case 1: $\alpha = 1$. In this case, Eqn. (30) can be written as

$$x_p - a_p = \bar{x} - \bar{a}. \tag{33}$$

Furthermore, since x must lie on the sphere $S_R(a)$, squaring both sides of (33) and summing over $1 \leq p \leq N$ yields

$$\sum_{p=1}^{N} (x_p - a_p)^2 = R^2 = N(\bar{x} - \bar{a})^2. \tag{34}$$

This implies that

$$\bar{x} = \bar{a} \pm \frac{R}{\sqrt{N}}. \tag{35}$$

Substitution of this into Eqn. (33) yields the two critical points,

$$x_{1a,1b} = a \pm \frac{R}{\sqrt{N}}(1, 1, \cdots, 1) = a \pm R\hat{1}, \tag{36}$$

where $\hat{\underline{1}}$ is the unit vector in the direction $\underline{1} = (1, 1, \cdots, 1)$. In other words, the vector $x - a$ is perpendicular to the plane $\bar{x} = \bar{a}$.

Here it is worthwhile to comment that *the critical points x in Eq. (36) represent constant greyscale shifts of the reference image a.*

The values of $S(x, a)$ at the critical points x_{1a} and x_{1b} are found to be

$$S_{1a,1b}(x, a) = \frac{2\bar{a}\left(\bar{a} \pm \frac{R}{\sqrt{N}}\right)}{2\bar{a}\left(\bar{a} \pm \frac{R}{\sqrt{N}}\right) + \frac{R^2}{N}}. \tag{37}$$

Sub-case 2: $\alpha = -1$. In this case, Eqn. (30) becomes

$$x_p - \bar{x} = \bar{a} - a_p. \tag{38}$$

Rewrite the above equation as

$$x_p - a_p = (\bar{x} - \bar{a}) + 2(\bar{a} - a_p). \tag{39}$$

Now square both sides and sum over the index $1 \leq p \leq N$:

$$R^2 = \sum_{p=1}^{N} (x_p - a_p)^2 = \quad \cdots \quad = N(\bar{x} - \bar{a})^2 + 4(N-1)s_a^2. \tag{40}$$

A slight rearrangement yields

$$\bar{x} = \bar{a} \pm \frac{1}{\sqrt{N}} \sqrt{R^2 - 4(N-1)s_a^2} \tag{41}$$

This result is feasible provided R and N are chosen so that

$$\Delta = R^2 - 4(N-1)s_a^2 \geq 0. \tag{42}$$

When $\Delta \geq 0$, substitution of (41) into (38) yields the following critical points,

$$x_{-1a,-1b} = 2\bar{a}\underline{1} - a \pm \sqrt{\Delta}\hat{\underline{1}}. \tag{43}$$

with corresponding values of $S(x, a)$ given by

$$S_{-1a} = -\frac{2\bar{a}\left(\bar{a} + \sqrt{\frac{\Delta}{N}}\right)}{2\bar{a}\left(\bar{a} + \sqrt{\frac{\Delta}{N}}\right) + \frac{\Delta}{N}}, \quad S_{-1b} = -\frac{2\bar{a}\left(\bar{a} - \sqrt{\frac{\Delta}{N}}\right)}{2\bar{a}\left(\bar{a} - \sqrt{\frac{\Delta}{N}}\right) + \frac{\Delta}{N}}. \tag{44}$$

The quantities S_{1a}, S_{1b}, S_{-1a}, and S_{-1b} have the same form that was encountered in the $\bar{x} = \bar{a}$ case. The same analysis shows that

$$\begin{aligned} S_{1a} &> S_{1b} \ \text{ and } \ S_{-1b} > S_{-1a} \quad \text{if} \quad \bar{a} > 0, \\ S_{1a} &< S_{1b} \ \text{ and } \ S_{-1b} < S_{-1a} \quad \text{if} \quad \bar{a} < 0. \end{aligned} \tag{45}$$

Once again, *the extremum points $x^{(1)}$ and $x^{(2)}$ in Eq. (43) represent constant greyscale shifts of the reference image a.*

3 Conclusions and Some Illustrative Results

Let us summarize our findings below, introducing appropriate notation to differentiate between the two cases. The following critical points and corresponding values of $S(x,a)$ were identified:

Case 1: $\bar{x} = \bar{a}$

$$S_\beta^{(1)} = \frac{s_a^2 + s_a \frac{R}{\sqrt{N-1}}}{s_a^2 + s_a \frac{R}{\sqrt{N-1}} + \frac{R^2}{2(N-1)}} \quad \text{at} \quad x = a + R\hat{a}'$$
$$S_\beta^{(2)} = \frac{s_a^2 - s_a \frac{R}{\sqrt{N-1}}}{s_a^2 - s_a \frac{R}{\sqrt{N-1}} + \frac{R^2}{2(N-1)}} \quad \text{at} \quad x = a - R\hat{a}' \tag{46}$$

Case 2: $\bar{x} \neq \bar{a}$

$$S_\alpha^{(1)} = \frac{\bar{a}\left(\bar{a} + \frac{R}{\sqrt{N}}\right)}{\bar{a}\left(\bar{a} + \frac{R}{\sqrt{N}}\right) + \frac{R^2}{2N}} \quad \text{at} \quad x = a + R\hat{\underline{1}}$$
$$S_\alpha^{(2)} = \frac{\bar{a}\left(\bar{a} - \frac{R}{\sqrt{N}}\right)}{\bar{a}\left(\bar{a} - \frac{R}{\sqrt{N}}\right) + \frac{R^2}{2N}} \quad \text{at} \quad x = a - R\hat{\underline{1}}.$$
$$S_\alpha^{(3)} = -\frac{\bar{a}\left(\bar{a} + \sqrt{\frac{\Delta}{N}}\right)}{\bar{a}\left(\bar{a} + \sqrt{\frac{\Delta}{N}}\right) + \frac{\Delta}{2N}} \quad \text{at} \quad x = 2\bar{a}\underline{1} - a + \sqrt{\Delta}\hat{\underline{1}},$$
$$S_\alpha^{(4)} = -\frac{\bar{a}\left(\bar{a} - \sqrt{\frac{\Delta}{N}}\right)}{\bar{a}\left(\bar{a} - \sqrt{\frac{\Delta}{N}}\right) + \frac{\Delta}{2N}} \quad \text{at} \quad x = 2\bar{a}\underline{1} - a - \sqrt{\Delta}\hat{\underline{1}}. \tag{47}$$

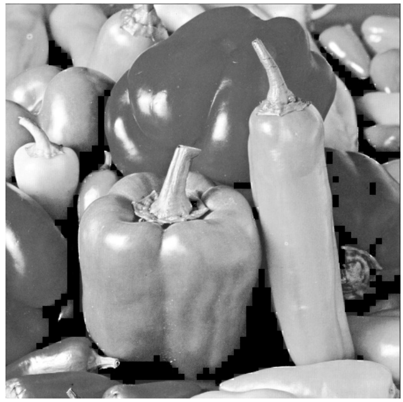
The last two critical points exist provided $\Delta = R^2 - 4(N-1)s_a^2 \geq 0$. *All extremum points for Case 2 represent constant greyscale shifts of the reference image a.*

These formulas have been verified numerically. It would be desirable to derive a condition that guarantees whether the global extrema for $S(x,a)$ occur on or off the plane $\bar{x} = \bar{a}$. Indeed, numerical results show that global maxima and minima may be obtained both on and off the plane. Unfortunately, actually classifying and comparing these critical points seems quite complicated, if indeed possible. However, we have shown that $S_\beta^{(1)} > S_\beta^{(2)}$. It is also true that

$$\begin{aligned} S_\alpha^{(1)} &> S_\alpha^{(2)} \text{ and } S_\alpha^{(4)} > S_\alpha^{(3)} \quad \text{if} \quad \bar{a} > 0, \quad \text{and} \\ S_\alpha^{(1)} &< S_\alpha^{(2)} \text{ and } S_\alpha^{(4)} < S_\alpha^{(3)} \quad \text{if} \quad \bar{a} < 0. \end{aligned} \tag{48}$$

Only the first set of relations holds for images with positive greyscale values.

(a) Best: SSIM = 0.9408



(b) Best: SSIM = 0.9121

(c) Original: SSIM = 1

(d) Original: SSIM = 1

(e) Worst: SSIM = -0.8466

(f) Worst: SSIM = -0.7783

Fig. 1. Best and worst 8×8-pixel block approximations to *Lena* and *Peppers* while each is constrained on an L^2 sphere of radius 300

We now present the results of some calculations involving the 512×512 pixel, 8 bit-per-pixel, *Lena* and *Peppers* test images. A local block-based approach has been taken. Let B_i denote an image subblock, considered as an N-vector. For each B_i, the best and worst approximations to B_i, according to SSIM, while being constrained on an L^2 sphere of radius R centered at B_i were computed. (This was done by evaluating all critical points and selecting the maximum and minimum values.) In the following calculations, 8×8 non-overlapping image subblocks were used and the radius $R = 300$. Fig. 1 shows the results of these experiments. The SSIM values reported in the figure are the averages of the $(64 \times 64 = 4096)$ non-overlapping block SSIM values.

It was found that the best approximations on the L^2 sphere almost always occur off the plane $\bar{x} = \bar{a}$ and correspond to the SSIM value $S_\alpha^{(1)}$, implying a constant greyscale shift given by Eq. (36). On the other hand, the worst approximations are usually obtained on the plane $\bar{x} = \bar{a}$. In fact, of all the eight test images used, *Peppers* was the only image where a best approximation occurred on the plane $\bar{x} = \bar{a}$ (note the dark black blocks in the shadows).

These findings are rather counterintuitive and represent, in our opinion, one of the most important contributions of this paper. One might think that the means of blocks need to be matched in order to optimize the SSIM. The above results show that, in fact, this is not the case – the decrease in SSIM caused by mismatched means can be overcompensated by an increase in the correlation term to produce a net increase in SSIM.

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