Chapter 7
Miscellanea

7.1 Mixed Moments of Student's *t*-Distributions

Let M_d be the Euclidean space of symmetric $d \times d$ matrices with the scalar product $\langle A_1, A_2\rangle := \text{tr}(A_1 A_2)$, $A_1, A_2 \in M_d$, $M_d^+ \subset M_d$ be the cone of nonnegative definite matrices and $\mathscr{P}(M_d^+)$ be a class of probability measures on M_d^+. Here $\text{tr}A$ denotes the trace of a matrix A.

The probability distribution of a d-dimensional random vector X is said to be the mixture of centered Gaussian distributions with the mixing distribution $U \in \mathscr{P}(M_d^+)$ (U-mixture for short) if, for all $z \in R^d$,

$$\mathrm{E}e^{i\langle z, X\rangle} = \int_{M_d^+} e^{-\frac{1}{2}\langle zA, z\rangle} U(\mathrm{d}A). \tag{7.1}$$

The distributional properties of such mixtures are well studied (see, e.g., [1, 2] and references therein).

Let $c_j = (c_{j_1}, \ldots, c_{j_d}) \in R^d$, $j = 1, 2, \ldots, 2n$. We shall derive formulas evaluating $\mathrm{E}\left(\prod_{j=1}^{2n} \langle c_j, X\rangle\right)$ for U-mixtures of Gaussian distributions, including Student's t-distribution.

Let Π_{2n} be the class of pairings σ on the set $I_{2n} = \{1, 2, \ldots, 2n\}$, i.e. the partitions of I_{2n} into n disjoint pairs, implying that

$$\text{card}\Pi_{2n} = \frac{(2n)!}{2^n n!}.$$

For each $\sigma \in \Pi_{2n}$, we define uniquely the subsets $I_{2n\backslash\sigma}$ and integers $\sigma(j)$, $j \in I_{2n\backslash\sigma}$, by the equality

$$\sigma = \left\{(j, \sigma(j)), j \in I_{2n\backslash\sigma}\right\}.$$

B. Grigelionis, *Student's* t*-Distribution and Related Stochastic Processes*,
SpringerBriefs in Statistics, DOI: 10.1007/978-3-642-31146-8_7,

If $U = \varepsilon_\Sigma$ is a Dirac measure with fixed $\Sigma \in M_d^+$, i.e. the Gaussian case, Isserlis theorem (in mathematical physics known as Wick theorem) says (see, e.g., [3–5]) that

$$\mathrm{E}\left[\prod_{j=1}^{2n} \langle c_j, X\rangle\right] = \sum_{\sigma\in\Pi_{2n}} \prod_{j\in I_{2n\setminus\sigma}} \langle c_j\Sigma, c_{\sigma(j)}\rangle := m_{2n}(c,\Sigma). \tag{7.2}$$

Write

$$\phi_U(\Theta) := \int_{M_d^+} e^{-\mathrm{tr}(A\Theta)} U(\mathrm{d}A), \quad \Theta \in M_d^+. \tag{7.3}$$

Theorem 7.1 *[6] The following statements hold:*

(i) The probability distribution of a d-dimensional random vector X is the U-mixture of centered Gaussian distributions if and only if

$$\mathrm{E}e^{i\langle z,X\rangle} = \phi_U\left(\frac{1}{2}z^T z\right), \tag{7.4}$$

where z^T is the transposed vector z.

(ii) If the probability distribution of X is the U-mixture of centered Gaussian distributions and, for $j = 1, 2, \ldots, 2n$,

$$\int_{M_d^+} \langle c_j A, c_j\rangle^n U(dA) < \infty, \tag{7.5}$$

then

$$\mathrm{E}\left[\prod_{j=1}^{2n} \langle c_j, X\rangle\right] = \sum_{\sigma\in\Pi_{2n}} \int_{M_d^+} m_{2n}^\sigma(c,A) U(dA), \tag{7.6}$$

where

$$m_{2n}^\sigma(c,A) = \prod_{j\in I_{2n\setminus\sigma}} \langle c_j A, c_{\sigma(j)}\rangle.$$

Proof (i) The statement follows from (7.1) and (7.3), because, obviously,

$$\mathrm{tr}\left((z^T z)A\right) = \langle zA, z\rangle.$$

(ii) Observe that card $I_{2n\setminus\sigma} = n$ and, for all $\sigma \in \Pi_{2n}$ and $A \in M_d^+$,

$$
\begin{aligned}
\prod_{j \in I_{2n\setminus\sigma}} \left|\langle c_j A, c_{\sigma(j)}\rangle\right|^n &\le n^{-n} \left(\sum_{j \in I_{2n\setminus\sigma}} \left|\langle c_j A, c_{\sigma(j)}\rangle\right| \right)^n \\
&\le n^{-1} \sum_{j \in I_{2n\setminus\sigma}} \left|\langle c_j A, c_{\sigma(j)}\rangle\right|^n \\
&\le \frac{2^{n-1}}{n} \sum_{j \in I_{2n\setminus\sigma}} \left[\langle c_j A, c_j\rangle^n + \langle c_{\sigma(j)} A, c_{\sigma(j)}\rangle^n\right] \\
&= \frac{2^{n-1}}{n} \sum_{j=1}^{2n} \langle c_j A, c_j\rangle^n. \qquad (7.7)
\end{aligned}
$$

Using (7.5) and (7.7), we find that

$$
\begin{aligned}
\mathrm{E}\left[\prod_{j=1}^{2n} \langle c_j, X\rangle\right] &= \int_{M_d^+} m_{2n}(c, A) U(\mathrm{d}A) \\
&= \sum_{\sigma \in \Pi_{2n}} \int_{M_d^+} m_{2n}^{\sigma}(c, A) U(\mathrm{d}A).
\end{aligned}
$$

□

Taking (see also [7])

$$U = \mathscr{L}(Y\Sigma),$$

where $\Sigma \in M_d^+$ is fixed and

$$\mathscr{L}(Y) = GIG\left(-\frac{\nu}{2}, \nu, 0\right)$$

we have that

$$\phi_U(\Theta) = \frac{2\left(\frac{\nu}{2}\right)^{\frac{\nu}{4}} (\mathrm{tr}(\Sigma\Theta))^{\frac{\nu}{4}}}{\Gamma\left(\frac{\nu}{2}\right)} K_{\frac{\nu}{2}}\left(\sqrt{2\mathrm{tr}(\Sigma\Theta)}\right), \qquad (7.8)$$

$$\mathscr{L}(X) = T_d(\nu, \Sigma, 0) \qquad (7.9)$$

and, for $j = 1, 2, \ldots, 2n$

$$\int_{M_d^+} \langle c_j A, c_j\rangle^n U(\mathrm{d}A) = \begin{cases} \dfrac{\Gamma\left(\frac{\nu}{2} - n\right)}{\left(\dfrac{\nu}{2}\right)^{\frac{\nu}{2} - n}} \langle c_j \Sigma, c_j\rangle^n, & \text{if} \quad 2n < \nu, \\ \infty, & \text{if} \quad 2n \ge \nu. \end{cases}$$

Thus, for $2n < \nu$,

$$\int\limits_{R^d} \prod_{j=1}^{2n} \langle c_j, x\rangle T_d\left(\nu, \Sigma, 0\right)(\mathrm{d}x) = \frac{\Gamma\left(\frac{\nu}{2} - n\right)}{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}-n}} m_{2n}(c, \Sigma), \tag{7.10}$$

$$\int\limits_{R^d} \prod_{j=1}^{2n} \langle c_j, x\rangle T_d(\nu, \Sigma, \alpha)(\mathrm{d}x) = \int\limits_{R^d} \prod_{j=1}^{2n} \left[\langle c_j, y\rangle + \langle c_j, \alpha\rangle\right] T_d(\nu, \Sigma, 0)(\mathrm{d}y)$$

and because of anti-symmetry, for $2k + 1 < \nu$,

$$\int\limits_{R^d} \prod_{j=1}^{2k+1} \langle c_j, x\rangle T_d(\nu, \Sigma, 0)(\mathrm{d}x) = 0.$$

Remark 7.2 Let $\nu \geq d$ be an integer, $Y_1, \ldots, Y_\nu$ be i.i.d. d-dimensional centered Gaussian vectors with a covariance matrix Σ, $|\Sigma| > 0$, and $U = \mathscr{L}\left(\nu\Sigma_\nu^{-1}\right)$, where the matrix

$$W_\nu = \sum_{j=1}^{\nu} Y_j^T Y_j.$$

If $\nu \geq d$, the matrix W_ν is invertible with probability 1, because it is well known that the Wishart distribution

$$\mathscr{L}(W_\nu) := W_d(\Sigma, \nu)$$

has a density

$$W_d(\Sigma, \nu, A) = \begin{cases} \dfrac{|A|^{\frac{\nu-d-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\Sigma^{-1}A\right)\right\}}{\left(2^d|\Sigma|\right)^{\frac{\nu}{2}} \pi^{\frac{d(d-1)}{4}} \prod_{j=1}^{d} \Gamma\left(\frac{k-j+1}{2}\right)}, & \text{if} \quad |A| > 0, \\ 0, \quad \text{otherwise.} \end{cases}$$

Because (see, e.g., [2, 8, 9])

$$\int\limits_{M_d^+} e^{-\frac{1}{2}\langle zA, z\rangle} U(\mathrm{d}A) = \int\limits_{R^d} e^{i\langle z, x\rangle} T_d(\nu, \Sigma, 0)(\mathrm{d}x)$$

$$= \mathrm{E}[e^{-\frac{1}{2}\langle z\Sigma, z\rangle Y}], \quad z \in R^d, \tag{7.11}$$

taking $z = tc$, $t \in R^1$, $c \in R^d$, we find that

$$\int\limits_{M_d^+} e^{-\frac{t^2}{2}\langle cA,c\rangle} U(\mathrm{d}A) = \mathrm{E}\left[e^{-\frac{t^2}{2}\langle c\Sigma,c\rangle Y}\right].$$

Thus, for all $c \in R^d$,

$$\mathscr{L}\left(\nu\langle cW^{-1}, c\rangle\right) = \mathscr{L}\left(\langle c\Sigma, c\rangle Y\right),$$

contradicting to the formula

$$\mathscr{L}\left(\langle cW_\nu^{-1}, c\rangle\right) = \mathscr{L}\left(\langle c\Sigma^{-1}, c\rangle \frac{1}{\chi^2_{\nu-d+1}}\right)$$

in [9].

Unfortunately, the last formula was used in [6], Example 3.

From (7.11) we easily find that

$$\int\limits_{R^d} e^{i\langle z,x\rangle} T_d(\nu, \Sigma, \alpha)(\mathrm{d}x) = \frac{e^{i\langle z,\alpha\rangle}}{2^{\frac{\nu}{2}-1}\Gamma\left(\frac{\nu}{2}\right)} \left(\nu\langle z\Sigma, z\rangle\right)^{\frac{\nu}{4}}$$
$$\times K_{\frac{\nu}{2}}\left(\sqrt{\nu\langle z\Sigma, z\rangle}\right), \quad z \in R^d,$$

(see [10, 11]).

7.2 Long-Range Dependent Stationary Student Processes

It is well known (see, e.g., [12]) that a real square integrable and continuous in quadratic mean stochastic process $X = \left\{X_t, t \in R^1\right\}$ is second order stationary if and only if it has the following spectral decomposition:

$$X_t = \alpha + \int\limits_{-\infty}^{\infty} \cos(\lambda t) v(\mathrm{d}\lambda) + \int\limits_{-\infty}^{\infty} \sin(\lambda t) w(\mathrm{d}\lambda), \quad t \in R^1,$$

where $\alpha = \mathrm{E}X_0$, $v(\mathrm{d}\lambda)$ and $w(\mathrm{d}\lambda)$ are mean 0 and square integrable real random measures such that, for each $A, A_1, A_2 \in \mathscr{B}(R^1)$,

$$\mathrm{E}\left[v(A_1)v(A_2)\right] = \mathrm{E}v^2(A_1 \cap A_2), \tag{7.12}$$

$$\mathrm{E}\,[w(A_1)w(A_2)] = \mathrm{E}w^2(A_1 \cap A_2), \tag{7.13}$$

$$\mathrm{E}\,[v(A_1)w(A_2)] = 0, \tag{7.14}$$

$$\tilde{F}(A) := \mathrm{E}v^2(A) = \mathrm{E}w^2(A). \tag{7.15}$$

The correlation function r satisfies

$$r(t) = \int_{-\infty}^{\infty} \cos(\lambda t) F(\mathrm{d}\lambda),$$

where

$$F(A) = \frac{\tilde{F}(A)}{\tilde{F}(R^1)}, \quad A \in \mathscr{B}(R^1).$$

Following [13], we shall construct a class of strictly stationary stochastic processes $X = \left\{X_t, t \in R^1\right\}$ such that

$$\mathscr{L}(X_t) \equiv T_1\left(\nu, \sigma^2, \alpha\right), \quad \nu > 2,$$

called the Student's stationary processes.

Recall the notion and some properties of the independently scattered random measures (i.s.r.m.) (see [13–15]).

Let $T \in \mathscr{B}(R^d)$, $\mathscr{S}$ be a σ-ring of subsets of T (i.e. countable unions of sets in $\mathscr{S}$ belong to $\mathscr{S}$ and, if $A, B \in \mathscr{S}$, $A \subset B$, then $B \backslash A \in \mathscr{S}$). The σ algebra generated by $\mathscr{S}$ is denoted $\sigma(\mathscr{S})$.

A collection of random variables $v = \{v(A), A \in \mathscr{S}\}$ defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is said to be an i.s.r.m. if, for every sequence $\{A_n, n \geq 1\}$ of disjoint sets in $\mathscr{S}$, the random variables $v(A_n)$, $n = 1, 2, \ldots$, are independent and

$$v\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} v(A_n) \quad a.s.,$$

whenever $\bigcup_{n=1}^{\infty} A_n \in \mathscr{S}$.

Let $v(A)$, $A \in \mathscr{S}$, be infinitely divisible,

$$\log \mathrm{E}e^{izv(A)} = izm_0(A) - \frac{1}{2}z^2 m_1(A) + \int_{R_0^+} \left(e^{izu} - 1 - iz\tau(u)\right) \Pi(A, \mathrm{d}u),$$

where m_0 is a signed measure, $\Pi(A, \mathrm{d}u)$ for fixed A is a measure on $\mathscr{B}(R_0^1)$ such that

$$\int_{R_0^1} \left(1 \wedge u^2\right) \Pi(A, \mathrm{d}u) < \infty;$$

$$\tau(u) = \begin{cases} u, & \text{if} \quad |u| \le 1, \\ \dfrac{u}{|u|}, & \text{if} \quad |u| > 1. \end{cases}$$

Assume now that $m_0 = m_1 = 0$ and

$$\Pi(A, \mathrm{d}u) = M(A)\Pi(\mathrm{d}u),$$

where $M(A)$ is some measure on T and $\Pi(\mathrm{d}u)$ is some Lévy measure on R_0^1.

Integration of functions on T with respect to ν is defined first for real simple functions $f = \sum_{j=1}^{n} x_j 1_{A_j}$, $A_j \in \mathscr{S}$, $j = 1, \ldots, n$, by

$$\int_A f(x)\nu(\mathrm{d}x) = \sum_{j=1}^{n} x_j \nu(A \cap A_j),$$

where A is any subset of T, for which $A \in \sigma(\mathscr{S})$ and $A \cap A_j \in \mathscr{S}$, $j = 1, \ldots, n$.

In general, a function $f\colon (T, \sigma(\mathscr{S})) \to \left(R^1, \mathscr{B}(R^1)\right)$ is said to be ν-integrable if there exists a sequence $\{f_n, n = 1, 2, \ldots\}$ of simple functions as above such that $f_n \to f$ M-a.e. and, for every $A \in \sigma(\mathscr{S})$, the sequence $\left\{\int_A f_n(x)\nu(\mathrm{d}x),\right.$ $\left. n = 1, 2, \ldots\right\}$ converges in probability, as $n \to \infty$. If f is ν-integrable, we write

$$\int_A f(x)\nu(\mathrm{d}x) = p - \lim_{n\to\infty} \int_A f_n(x)\nu(\mathrm{d}x).$$

The integrand $\int_A f(x)\nu(\mathrm{d}x)$ does not depend on the approximating sequence.

A function f on T is ν-integrable if and only if

$$\int_T Z_0\left(f(x)\right) M(\mathrm{d}x) < \infty$$

and

$$\int_T \left|Z\left(f(x)\right)\right| M(\mathrm{d}x) < \infty,$$

where

$$Z_0(y) = \int_{R_0^1} \left(1 \wedge (uy)^2\right) \Pi(\mathrm{d}u),$$

and

$$Z(y) = \int_{R_0^1} (\tau(uy) - y\tau(u))\, \Pi(\mathrm{d}u).$$

For such functions f

$$\log \mathrm{E} \exp \left\{ i\xi \int_A f(x)\nu(\mathrm{d}x) \right\} = \int_A \varkappa(\xi f(x))\, M(\mathrm{d}x),$$

where

$$\varkappa(\xi) = \int_{R_0^1} \left(e^{i\xi u} - 1 - i\xi\tau(u) \right) \Pi(\mathrm{d}u).$$

Let now $Y_t = \left(Y_t^1, Y_t^2\right)$, $t \geq 0$, be a bivariate Student-Lévy process such that

$$\mathscr{L}(Y_1) = T_2(\nu, \sigma^2 I_2, 0), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and F be an arbitrary probability distribution on R^1.

Let $T = R^1$, $\mathscr{S}$ be the σ-ring of subsets $A = \bigcup_{j=1}^{\infty} \left(a_j, b_j\right]$, where the intervals $\left(a_j, b_j\right]$, $j = 1, 2, \ldots$, are disjoint. Define i.m.r.m. v and w by the equalities:

$$v(A) = \sum_{j=1}^{\infty} \left(Y^1_{F(b_j)} - Y^1_{F(a_j)} \right)$$

and

$$w(A) = \sum_{j=1}^{\infty} \left(Y^2_{F(b_j)} - Y^2_{F(a_j)} \right), \quad A = \bigcup_{j=1}^{\infty} \left(a_j, b_j\right] \in \mathscr{S}.$$

Because, for $i = 1, 2,\ j = 1, 2, \ldots, \nu > 2$,

$$\mathrm{E}(Y^i_{F(b_j)} - Y^i_{F(a_j)}) = 0,$$

$$\mathrm{E}(Y^i_{F(b_j)} - Y^i_{F(a_j)})^2 = \frac{\sigma^2\nu}{\nu - 2} \left(F(b_j) - F(a_j) \right)$$

and

$$\sum_{j=1}^{\infty} \mathrm{E}(Y^i_{F(b_j)} - Y^i_{F(a_j)})^2 \leq \frac{\sigma^2\nu}{\nu - 2} < \infty,$$

the definition of v and w is correct.

From (7.10) it follows that v and w satisfies (7.12)–(7.15) with

$$\tilde{F}(A) = \frac{\sigma^2 \nu}{\nu - 2} F(A), \quad A \in \mathscr{S}.$$

Thus, the process

$$X_t = \alpha + \int_{-\infty}^{\infty} \cos(ut) v(\mathrm{d}u) + \int_{-\infty}^{\infty} \sin(ut) w(\mathrm{d}u), \quad t \in R^1,$$

is well defined, strictly stationary,

$$\mathscr{L}(X_t) \equiv T_1(\nu, \sigma^2, \alpha)$$

and the correlation function r satisfies

$$r(t) = \int_{-\infty}^{\infty} \cos(ut) F(\mathrm{d}u), \quad t \in R^1.$$

Strict stationarity of X follows from the formula (see [13]):

$$\begin{aligned} \mathrm{E} e^{i \sum_{j=1}^{n} \eta_j X_{t_j}} = e^{i\alpha \sum_{j=1}^{n} \eta_j} & \\ \times \exp & \left\{ \int_{-\infty}^{\infty} \log \hat{h}_{\nu,\sigma} \left(\frac{1}{2} \sum_{j,k=1}^{n} \eta_j \eta_k \cos\big(u(t_j - t_k)\big) \right) F(\mathrm{d}u) \right\}, \\ & \eta_j, t_j \in R^1, \quad j = 1, \ldots, n, \end{aligned}$$

where

$$\begin{aligned} \hat{h}_{\nu,\sigma}(\theta) &:= \int_0^{\infty} e^{-\theta u} \frac{1}{\sigma^2} gig\left(\frac{u}{\sigma^2}; -\frac{\nu}{2}, \nu, 0\right) \mathrm{d}u \\ &= \frac{2}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\theta \sigma^2 \nu}{2}\right)^{\frac{\nu}{4}} K_{\frac{\nu}{2}}\left(\sqrt{2\sigma^2 \theta \nu}\right), \quad \theta > 0. \end{aligned}$$

As it was checked in [16], if

$$F(\mathrm{d}u) = f_{\beta,\gamma}(u)\mathrm{d}u, \quad 0 < \beta \le 1, \quad \gamma \in R^1,$$

where

$$f_{\beta,\gamma}(u) = \frac{1}{2}\left[f_{\beta,0}(u+\gamma) + f_{\beta,0}(u-\gamma)\right], \quad u \in R^1,$$

with

$$f_{\beta,0}(u) = \frac{2^{\frac{1-\beta}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\beta}{2}\right)} K_{1-\beta}\left(|u|\right)|u|^{\frac{(1-\beta)}{2}},$$

then

$$r(t) = \frac{\cos\gamma t}{(1+t^2)^{\frac{\beta}{2}}}, \quad t \in R^1,$$

and

$$\int_{-\infty}^{\infty} |r(t)|\,\mathrm{d}t = \infty,$$

implying long-range dependence of X (see also [17–20]).

Remark 7.3 Defining Student-Lamperti process $X^\star$ as (see [21])

$$X_t^\star = t^H X_{\log t}, \quad t > 0, \quad X_0^\star = 0, \quad H > 0.$$

we have that $X^\star$ is H-self-similar, i.e., for each $c > 0$, processes $\left\{X_{ct}^\star, t \geq 0\right\}$ and $\left\{c^H X_t^\star, t \geq 0\right\}$ have the same finite dimensional distributions, and (see [13])

$$\begin{aligned}\mathrm{E}e^{i\sum_{j=1}^{n}\eta_j X_{t_j}^\star} &= e^{i\alpha\sum_{j=1}^{n} t_j^H \eta_j} \\ &\quad\times \exp\left\{\int_{-\infty}^{\infty}\left[\log\hat{h}_{\nu,\sigma}\left(\frac{1}{2}\sum_{j,k=1}^{n}\eta_j\eta_k t_j^H t_k^H \cos\left(u\log\frac{t_j}{t_k}\right)\right)\right]F(\mathrm{d}u)\right\}, \\ &\quad t_j > 0, \quad \eta_j \in R^1, \quad j = 1,\ldots,n.\end{aligned}$$

In particular,

$$\mathrm{E}e^{i\eta X_t^\star} = e^{i\alpha t^H \eta}\hat{h}_{\nu,\sigma}\left(t^{2H}\frac{\eta^2}{2}\right), \quad t > 0, \quad \eta \in R^1,$$

and

$$\begin{aligned}\mathrm{E}e^{i\eta(X_t^\star - X_s^\star)} = e^{i\alpha(t^H - s^H)\eta}\exp\Bigg\{\int_{-\infty}^{\infty}\Bigg[\log\hat{h}_{\nu,\sigma}\Bigg(\frac{1}{2}\eta^2\Bigg(s^{2H} + t^{2H} \\ -2s^H t^H\cos\left(u\log\frac{t}{s}\right)\Bigg)\Bigg)\Bigg]F(\mathrm{d}u)\Bigg\}, \quad s,t > 0, \quad \eta \in R^1.\end{aligned}$$

7.3 Lévy Copulas

Considering the probability distributions F on R^d with the 1-dimensional Student's t marginals $F_{j,j} = 1, \ldots, d$, and having in mind their relationship with stochastic processes, we restricted ourselves to the cases when F is a mixture of the d-dimensional Gaussian distributions .

Denoting

$$C(u_1, \ldots, u_d) := F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right), \quad u_j \in [0, 1], \quad j = 1, \ldots, d,$$

it is obvious that this function is the probability distribution function on the d-cube $[0,1]^d$ with uniform one-dimensional marginals, called the d-copula (see, e.g., [22]). Trivially,

$$F(x_1, \ldots, x_d) = C\left(F_1(x_1), \ldots, F_d(x_d)\right), \quad (x_1, \ldots, x_d) \in R^d. \tag{7.16}$$

Formula (7.16) with the arbitrary d-copula defines uniquely the probability distributions on R^d with the given Student's 1-dimensional marginals. These statements are very special cases of well known Sklar's theorem (see [23, 24]).

Thus, taking concrete d-copulas we shall obtain a wide class of multivariate generalizations of Student's t-distributions.

For instance, the Archimedean copulas have the from

$$C(u_1, \ldots, u_d) = \psi\left(\psi^{-1}(u_1) + \cdot + \psi^{-1}(u_d)\right), \quad u_j \in [0, 1], \quad j = 1, \ldots, d,$$

where ψ is a d-monotone function on $[0, \infty)$, i.e., for each $x \geq 0$ and $k = 0, 1, \ldots, d-2$,

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \psi(x) \geq 0,$$

$(-1)^{d-2}\psi^{(d-2)}(x)$, $x \geq 0$, is nonincreasing and convex function.

In particular, if

$$\psi(x) = (1 + x)^{-\frac{1}{\theta}}, \quad \theta \in (0, \infty), \quad x \geq 0,$$

we have the Clayton's copula

$$C(u_1, \ldots, u_d) = \left(\sum_{j=1}^{d} u_j^{-\theta} - d + 1\right)^{-\frac{1}{\theta}}, \quad u_j \in [0, 1], \quad j = 1, \ldots, d.$$

If $\phi(x) = \exp\left\{-x^{\frac{1}{\theta}}\right\}$, $\theta \geq 1$, $x \geq 0$, we obtain the Gumbel copula

$$C(u_1,\ldots,u_d)=\exp\left\{-\left(\sum_{j=1}^{d}\left(-\log u_j\right)^{\theta}\right)^{\frac{1}{\theta}}\right\},\quad u_j\in[0,1],\quad j=1,\ldots,d.$$

Unfortunately, it is difficult to describe if the copulation preserves such important for us properties of marginal distributions as infinite divisibility or self-decomposability.

A promising direction for future work is a notion of Lévy copulas and, analogously to the classical copulas, construction of new Lévy measures on R^d using marginal ones (see [25–28]). Following [28], we briefly describe an analogue of Sklar's theorem in this context.

Let $\bar{R}:=(-\infty,\infty]$. For $a,b\in\bar{R}^d$ we write $a\le b$, if $a_k\le b_k$, $k=1,\ldots,d$ and, in this case, denote

$$(a,b]:=(a_1,b_1]\times\ldots\times(a_d,b_d]\,.$$

Let $F:S\to\bar{R}$ for some subset $S\subset\bar{R}^d$. For $a,b\in S$ with $a\le b$ and $\overline{(a,b]}\subset S$, the F-volume of $(a,b]$ is defined by

$$V_F\left((a,b]\right):=\sum_{u\in\{a_1,b_1\}\times\cdots\times\{a_d,b_d\}}(-1)^{N(u)}F(u),$$

where $N(u):=\sharp\{k:u_k=a_k\}$.

A function $F:S\to\bar{R}$ is called d-increasing if $V_F\left((a,b]\right)\ge 0$ for all $a,b\in S$ with $a\le b$ and $\overline{(a,b]}\subset S$.

Definition 7.4 Let $F:\bar{R}^d\to\bar{R}$ be a d-increasing function such that $F(u_1,\ldots,u_d)=0$ if $u_i=0$ for at least one $i\in\{1,\ldots,d\}$. For any non-empty index set $I\subset\{1,\ldots,d\}$ the I-marginal of F is the function $F_I:\bar{R}^{|I|}\to\bar{R}$, defined by

$$F^I\left((u)i)_{i\in I}\right):=\lim_{a\to\infty}\sum_{(u_i)_{i\in I^c}\in\{-a,\infty\}^{|I^c|}}F(u_1,\ldots,u_d)\prod_{i\in I^c}\mathrm{sgn}u_i,$$

where $I^c=\{1,\ldots,d\}\backslash I$, $|I|:=\mathrm{card}I$, and

$$\mathrm{sgn}x=\begin{cases}1, & \text{if}\quad x\ge 0,\\ -1, & \text{if}\quad x<0.\end{cases}$$

Definition 7.5 A function $F:\bar{R}^d\to\bar{R}$ is called a Lévy copula if

1. $F(u_1,\ldots,u_d)\ne\infty$ for $(u_1,\ldots,u_d)\ne(\infty,\ldots,\infty)$,
2. $F(u_1,\ldots,u_d)=0$ if $u_i=0$ for at least one $i\in\{1,\ldots,d\}$,
3. F is d-increasing,
4. $F^{\{i\}}(u)=u$ for any $i\in\{1,\ldots,d\}$, $u\in R^1$.

Write

$$\mathscr{I}(x) := \begin{cases} (x, \infty), & \text{if} \quad x \le 0, \\ (-\infty, x], & \text{if} \quad x > 0. \end{cases}$$

Definition 7.6 Let $X = (X^1, \ldots, X^d)$ be an R^d-valued Lévy process with the Lévy measure Π. The tail integral of X is the function $V : \left(R^1 \backslash \{0\}\right)^d \to R^1$ defined by

$$V(x_1, \ldots, x_d) := \prod_{i=1}^{d} \operatorname{sgn}(x_i) \Pi\left(\mathscr{I}(x_1) \times \cdots \times \mathscr{I}(x_d)\right)$$

and, for any non-empty $I \subset \{1, \ldots, d\}$ the I-marginal tail integral V^I of X is the tail integral of the process $X^I := (X^i)_{i \in I}$.

We denote one-dimensional margins by $V_i := V^{\{i\}}$.

Observe, that marginal tail integrals $\{V^I : I \subset \{1, \ldots, d\}$ non-empty$\}$ are uniquely determined by Π. Conversely, Π is uniquely determined by the set of its marginal tail integral.

Relationship between Lévy copulas and Lévy processes are described by the following analogue of Sklar's theorem.

Theorem 7.7 [28]

1. *Let $X = \left(X^1, \ldots, X^d\right)$ be an R^d-valued Lévy process. Then there exists a Lévy copula F such that the tail integrals of X satisfy*

$$V\left((x_i)_{i \in I}\right) = F^I\left((V_i(x_i))_{i \in I}\right), \tag{7.17}$$

   for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in \left(R^1 \backslash \{0\}\right)^{|I|}$. The Lévy copula F is unique on $\operatorname{Ran} V_1 \times \cdots \times \operatorname{Ran} V_d$.

2. *Let F be a d-dimensional Lévy copula and V_i, $i = 1, \ldots, d$, be tail integrals of real-valued Lévy processes. Then there exists an R^d-valued Lévy process X whose components have tail integrals $V_1, \ldots, V_d$ and whose marginal tail integrals satisfy (7.17) for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in \left(R^1 \backslash \{0\}\right)^{|I|}$. The Lévy measure Π of X is uniquely determined by F and V_i, $i = 1, \ldots, d$.*

In the above formulation $\operatorname{Ran} V$ means the range of V. The reader is referred for proofs to [28].

An analogue of the Archimedean copulas is as follows (see [28]).

Let $\varphi : [-1, 1] \to [-\infty, \infty]$ be a strictly increasing continuous function with $\varphi(1) = \infty$, $\varphi(0) = 0$, and $\varphi(-1) = -\infty$, having derivatives of orders up to d on $(-1, 0)$ and $(0, 1)$, and, for any $k = 1, \ldots, d$, satisfying

$$\frac{\mathrm{d}^k \varphi(u)}{\mathrm{d}u^k} \geq 0, \quad u \in (0,1) \quad \text{and} \quad (-1)^k \frac{\mathrm{d}^k \varphi(u)}{\mathrm{d}u^k} \leq 0, \quad u \in (-1,0).$$

Let

$$\tilde{\varphi}(u) := 2^{d-2} \left(\varphi(u) - \varphi(-u)\right), \quad u \in [-1,1].$$

Then

$$F(u_1, \ldots, u_d) := \varphi\left(\prod_{i=1}^{d} \tilde{\varphi}^{-1}(u_i)\right)$$

defines a Lévy copula.

In particular, if

$$\varphi(x) := \eta \left(-\log|x|\right)^{-\frac{1}{\vartheta}} 1_{\{x>0\}} - (1-\eta)\left(-\log|x|\right)^{-\frac{1}{\vartheta}} 1_{\{x<0\}}$$

with $\vartheta > 0$ and $\eta \in (0,1)$, then

$$\tilde{\varphi}(x) = 2^{d-2} \left(-\log|x|\right)^{-\frac{1}{\vartheta}} \operatorname{sgn} x, \quad x \in -1, 1],$$

and

$$F(u_1, \ldots, u_d) = 2^{2-d} \left(\sum_{i=1}^{d} |u_i|^{-\vartheta}\right)^{-\frac{1}{\vartheta}} (\eta 1_{\{u_1 \ldots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \ldots u_d < 0\}}),$$

resembling the ordinary Clayton copulas.

References

1. Barndorff-Niesen, O.E., Pedersen, J., Sato, K.: Multivariate subordination, self-decomposability and stability. Adv. Prob. **33**, 160–187 (2001)
2. Barndorff-Niesen, O.E., Perez-Abreu, V.: Extensions of type G and marginal infinite divisibility. Teor. Veroyatnost. Primenen. **47**(2), 301–319 (2002)
3. Isserlis, L.: On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. Biometrika **12**, 134–139 (1918)
4. Leonov, V.P., Shiryaev, A.N.: An a method of calculation of semi-invariants. Theory Probab. Appl. **4**, 319–329 (1959)
5. Simon, B.: The $P(\Phi)_2$ euclidean (quantum) field theory. Princeton University Press, New Jersy, (1974)
6. Grigelionis, B.: On the Wick theorem for mixtures of centered Gaussian distribution. Lith. Math. J. **49**(4), 372–380 (2009)
7. Vignat, C., Bhatnagar, S.: An extension of Wick's theorem. Stat. Probab. Lett. **78**(15), 2400–2403 (2008)
8. Johnson, N.L., Kotz, S.: Distributions in Statistics: continuous univariate distributions. Wiley, New York (1972)

9. Rao, C.R.: Linear statistical inference and its applications, Wiley, NewYork (1965)
10. Heyde, C.C., Leonenko, N.N.: Student processes, Adv. Appl. Prob. **37**, 342–365 (2005)
11. Joarder, A.H., Ali, M.M.: On the characteristic function of the multivariate t-distribution. Pak. J. Stat. **12**, 55–62 (1996)
12. Cramér, H., Leadbetter, M.R.: Stationary and Related Stochastic Processes. Wiley, New York (1967)
13. Barndorff-Niesen, O.E., Perez-Abreu, V.: Stationary and self-similar processes driven by Lévy processes. Stoch. Process. Appl. **84**, 357–369 (1999)
14. Kwapień, S., Woyozyński, N.A.: Random Series and Stochastic Integrals: single and multiple. Birkhäuser, Boston (1992)
15. Rajput, B.S., Rosinski, J.: Spectral representations of infinitely divisible processes. Probab. Theory Relat. Fields **82**, 451–487 (1989)
16. Anh, V.V., Knopova, V.P., Leonenko, N.N.: Continuous-time stochastic processes with cyclical long-range dependence. Austral. N. Z. J. Statist. **46** 275–296 (2004)
17. Anh, V.V., Heyde, C.C., Leonenko, N.N.: Dynamic models of long-memory processes driven by Lévy noise. J. Appl. Probab. **39**, 730–747 (2002)
18. Barndorff-Nielsen, O.E.: Superpositions of Ornstein-Uhlenbeck processes. Theory Probab. Appl. **45**(2), 175–194 (2001)
19. Barndorff-Niesen, O.E., Leonenko, N.N.: Spectral properties of superpositions of Ornstein-Uhlenbeck type processes. Method. Comput. Appl. Prob. **7**(3), 335–352 (2005)
20. Heyde, C.C., Yang, Y.: On defining long-range dependence. J. Appl. Prob. **34**, 939–944 (1997)
21. Lamperti, J.W.: Semi-stable stochastic processes. Trans. Am. Math. Soc. **104**, 62–78 (1962)
22. Nelsen, R.B.: An Introduction to Copulas. Springer, New York (1999)
23. Sklar, A.: Fonctions de répartition à n dimensions et leurs margés. Pub. Inst. Stat. Univ. Paris **8**, 229–231 (1959)
24. Sklar, A.: Random variables, distribution functions, and copulas-a personal look backward and forward. In: Distributions with fixed marginals and related topics. In: Rüschendorf, L., Schweizer, B., Taylor, M.D. (eds.), Institute of Mathematical Statistics, Hayward (1996)
25. Barndorff-Niesen, O.E., Lindner, A.M.: Lévy copulas: dynamics and transforms of upsilon type. Scand. J. Stat. **34**(2), 298–316 (2007)
26. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman & Hall/CRC, Boca Raton (2004)
27. Jaworski, P., Durante, F., Handle, F., Rychlik, T. (eds.): Copula theory and its applications. Lecture notes in statistics, Springer (2010)
28. Kallsen, J., Tankov, P.: Characterization of dependence of multidimensional Lévy processes using Lévy copulas. J. Multivar. Anal. **97**, 1551–1572 (2006)