# Chapter 6
# Student Diffusion Processes

## 6.1 H-Diffusions

We shall consider the regular positive recurrent diffusion processes $X = \{X_t, t \geq 0\}$ on an open interval $(l, r) \subseteq R^1$ with the inaccessible end points and predetermined one-dimensional distributions (for used terminology see, e.g., [1, 2]).

Let $\tau_a = \inf\{t > 0 : X_t = a\}$, $a \in (l, r)$, and $s(x)$, $s \in (l, r)$ be the scale function for the process $X$, i.e. a strictly increasing continuous function such that for all $l < a \leq x \leq b < r$

$$\mathrm{P}^x\{\tau_a < \tau_b\} = \frac{s(b) - s(x)}{s(b) - s(a)},$$

where $\mathrm{P}^x$ denotes the underlying probability measure of the process given $X_0 = x$.

Let $m$ be the speed measure for the process $X$, characterized by the properties that $m(I) > 0$ for every non-empty subinterval $I$ of $(l, r)$ and for $l < a < x < b < r$

$$\mathrm{E}^x(\tau_a \wedge \tau_b) = \int\limits_{(a,b)} g_{s(a),s(b)}\left(s(x), s(y)\right) m(dy)$$

where

$$g_{a,b}(u, v) = \begin{cases} \dfrac{(b-u)(v-a)}{b-a}, & \text{if} \quad v \leq u, \\ \dfrac{(u-a)(b-v)}{b-a}, & \text{if} \quad u \leq v, \end{cases}$$

and the expectation $\mathrm{E}^x$ is taken with respect to the measure $\mathrm{P}^x$.

It is known (see [1–4]) that if $s(x) \to +\infty$, as $x \uparrow r$, $s(x) \to -\infty$, as $x \downarrow l$, and

$$|m| := m\left((l, r)\right) < \infty,$$

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then the diffusion $X$ is positive recurrent with the inaccessible end points. Moreover, if

$$\mathscr{L}(X_0) = \frac{m}{|m|},$$

the process $X$ will be strictly stationary and ergodic.

Let $\mathscr{G}(l, r)$ be a class of strictly positive differentiable functions $g(x)$, $x \in (l, r)$, such that for each $x \in (l, r)$ there exists $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \subset (l, r)$, satisfying

$$\int_{x-\varepsilon}^{x+\varepsilon} |g'(v)|\mathrm{d}v < \infty,$$

for some $x_0 \in (l, r)$, as $x \uparrow r$,

$$G(x) := \int_{x_0}^{x} g(v)dv \to +\infty,$$

and, as $x \downarrow l$, $G(x) \to -\infty$.

Let $h(x)$, $x \in (l, r)$ be a strictly positive measurable function such that

$$\int_{l}^{r} h(x)dx = 1. \tag{6.1}$$

Write $H(\mathrm{d}x) = h(x)\mathrm{d}x$,

$$a(x) = -\frac{1}{2}\frac{g'(x)}{h(x)g^2(x)}, \quad x \in (l, r), \tag{6.2}$$

and

$$\sigma^2(x) = (h(x)g(x))^{-1}, \quad x \in (l, r). \tag{6.3}$$

**Theorem 6.1** [5] *For each $g \in \mathscr{G}(l, r)$ and $h$, satisfying (6.1), there exists the unique weak solution for the stochastic differential equation*

$$\begin{cases} dX_t = a(X_t)dt + \sigma(X_t)dB_t, & t > 0 \\ \mathscr{L}(X_0) = H, \end{cases}$$

*which is a regular positive recurrent diffusion with the scale function*

$$s(x) = \int_{x_0}^{x} \frac{g(v)}{g(x_0)} dv, \quad x \in (l, r),$$

*and the speed measure* $m = g(x_0)H$.

*Here and below* $B = \{B_t, t \geq 0\}$ *is the standard univariate Brownian motion.*

The solution is a strictly stationary process with the one dimensional distribution $H$, called the H-diffusion (see [6]). The functions $g$ and $h$ are intrinsic characteristics of the H-diffusions, in terms of which their properties should be formulated.

*Example 6.2* Let $(l, r) = (0, 1)$,

$$\begin{aligned}
&h(x) = Cx^{\beta_1 - 1}(1-x)^{\beta_2 - 1}e^{\lambda x}, \quad x \in (0, 1),\\
&g(x) = \frac{1}{C\sigma^2}\left[x^{\alpha_1+\beta_1-1}(1-x)^{\alpha_2+\beta_2-1}e^{(\chi+\lambda)x}\right]^{-1},\\
&x \in (0, 1), \quad \alpha_1, \alpha_2, \lambda, \chi \in R^1, \quad \sigma^2 > 0, \quad \beta_1 > 0, \quad \beta_2 > 0.
\end{aligned}$$

Here and below $C$ is the norming constant. It is easy to check that $g \in \mathscr{G}(0, 1)$ if and only if $\alpha_1 + \beta_1 \geq 2$ and $\alpha_2 + \beta_2 \geq 2$.

In this case

$$\begin{aligned}
a(x) = \frac{\sigma^2}{2}\Big[&(\alpha_1 + \beta_1 - 1)x^{\alpha_1 - 1}(1-x)^{\alpha_2} - (\alpha_2 + \beta_2 - 1)\\
&\times x^{\alpha_1}(1-x)^{\alpha_2 - 1}(\lambda + \mu)x^{\alpha_1}(1-x)^{\alpha_2}\Big]e^{\chi x}, \quad x \in (0, 1),
\end{aligned}$$

and

$$\sigma^2(x) = \sigma^2 x^{\alpha_1}(1-x)^{\alpha_2}e^{\chi x}, \quad x \in (0, 1).$$

Taking $\alpha_1 = \alpha_2 = 1$, $\chi = 0$, we have the Wright–Fisher gene frequency model with mutation and selection in the population genetics (see, e.g., [1, 7]).

*Example 6.3* Let $(l, r) = (0, \infty)$,

$$\begin{aligned}
h(x) &= Cx^{\lambda - 1}\exp\left\{-\left(\chi x^{-\beta_1} + \psi x^{\beta_2}\right)\right\}, \quad x > 0\\
g(x) &= \frac{1}{C\sigma^2}x^{-(\lambda+\gamma)+1}\exp\left\{\chi x^{-\beta_1} + \psi x^{\beta_2}\right\}, \quad x > 0
\end{aligned}$$

where $\sigma^2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and either

(i) $\lambda, \gamma \in R^1$, $\chi > 0$, $\psi > 0$, or
(ii) $\chi = 0$, $\lambda > 0$, $\psi > 0$, $\lambda + \gamma > 2$, or
(iii) $\psi = 0$, $\lambda < 0$, $\chi > 0$, $\lambda + \gamma < 2$.

In all these cases $g \in \mathscr{G}(0, \infty)$,

$$a(x) = \frac{\sigma^2}{2}\left[(\gamma + \lambda - 1)x^{\gamma-1} + \chi\beta_1 x^{\gamma-\beta_1-1} - \psi\beta_2 x^{\gamma+\beta_2-1}\right], \quad x > 0$$

and

$$\sigma^2(x) = \sigma^2 x^\gamma, \quad x > 0.$$

If $\gamma = 2$, $\beta_2 = 1$, $\chi = 0$, $\lambda > 0$, we have that

$$a(x) = \frac{\sigma^2}{2}(\lambda + 1)x - \psi x^2, \quad x > 0,$$

$$\sigma^2(x) = \sigma^2 x^2$$

and

$$h(x) = Cx^{\lambda-1}e^{-\psi x},$$

giving us a diffusion version of the Pearl-Verhulst logistic population growth model (see [1]). This class of diffusions also contains the Cox–Ingersoll–Ross model for short interest rates in bond markets and its generalizations (see, e.g., [4, 8]).

*Example 6.4* Let $(l, r) = (-\infty, +\infty)$,

$$h(x) = C\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^\gamma \exp\left\{-\varkappa \arctan\left(\frac{x-\alpha}{\delta}\right)\right\}, \quad x \in R^1,$$

$$g(x) = \frac{\exp\left\{\varkappa \arctan\left(\frac{x-\alpha}{\delta}\right)\right\}}{C\sigma^2\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^{\lambda+\gamma}}, \quad x \in R^1 \quad \alpha, \lambda, \varkappa \in R^1, \quad \lambda < -\frac{1}{2},$$

$$\lambda + \gamma \le \frac{1}{2}, \quad \delta > 0, \quad \sigma^2 > 0.$$

In this case $g \in \mathscr{G}(-\infty, +\infty)$,

$$\mu(x) = \frac{\sigma^2}{\delta}\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^{\gamma-1}\left[(\lambda + \gamma)\left(\frac{x-\alpha}{\delta}\right) - \frac{\varkappa}{2}\right], \quad x \in R^1,$$

$$\sigma^2(x) = \sigma^2\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^\gamma, \quad x \in R^1.$$

Taking $\gamma = 1$, we have the Johannesma diffusion model for the stochastic activity of neurons (see [9–11]) and one of the Föllmer–Schweizer models for stock returns (see [12], also [13]). The stationary distribution is the skew Student's $t$-distribution with the skewness coefficient $\varkappa$. If $\varkappa = 0$, we arrive to the univariate Student's $t$-distribution.

## 6.2 Student Diffusions

**Definition 6.5** An H-diffusion process $X$ on $R^1$ is called a Student diffusion if $H = T_1(\nu, \sigma^2, \alpha)$, $\nu > 0, \sigma^2 > 0, \alpha \in R^1$.

From Theorem 6.1 it follows that for each $g \in \mathscr{G}(-\infty, \infty)$ there exists a Student diffusion. For example, taking $\lambda = -\frac{\nu+1}{2}$, $\gamma = 1$, $\varkappa = 0$, $\sigma^2 = \theta$, we find from Example 6.4 that the unique weak solution for the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = -\dfrac{\theta(\nu-1)}{2}\left(\dfrac{X_t-\alpha}{\delta}\right)\mathrm{d}t + \sqrt{\theta\left(1+\left(\dfrac{X_t-\alpha}{\delta}\right)^2\right)}\mathrm{d}B_t, \quad \theta > 0, \\ \mathscr{L}(X_0) = T_1(\nu, \delta^2\nu^{-1}, \alpha) \end{cases}$$

is a Student diffusion.

*Example 6.6* [8] The function $g(x) \equiv \sigma^{-2} > 0$, $x \in R^1$, belongs to $\mathscr{G}(-\infty, \infty)$. Thus for any strictly positive pdf $h(x)$, $x \in R^1$, the unique weak solution for the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \left(\sigma^2 h(X_t)\right)^{-\frac{1}{2}}\mathrm{d}B_t, \quad t > 0 \\ \mathscr{L}(X_0) = H, \end{cases}$$

is an H-diffusion.

If $\nu > 1$, as the unique weak solution for the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = -\theta\dfrac{X_t-\alpha}{\delta}\mathrm{d}t + \sqrt{\dfrac{2\theta\delta^2}{\nu-1}\left(1+\left(\dfrac{X_t-\alpha}{\delta}\right)^2\right)}\mathrm{d}B_t, \quad t > 0, \\ \mathscr{L}(X_0) = T_1(\nu, \delta^2\nu^{-1}, \alpha), \end{cases}$$

the Student diffusion is a member of the family of Kolmogorov–Pearson diffusions (see [14, 15]).

Now let us consider a Student diffusion $X = \{X_t, t \geq 0\}$, corresponding to the function $g \in \mathscr{G}(-\infty, \infty)$, and discuss the domain-of-attraction problem for the maximum values

$$M_T = \max_{0 \le t \le T} X_t, \quad T > 0,$$

using linear normalization.

We shall see that the problem for H-diffusion reduces to the classical extreme value theory and the criteria are expressed in the terms of functions g independently of the marginal distribution $H$.

**Definition 6.7** We say that an H-diffusion $X = \{X_t, t \ge 0\}$ belongs to the maximum domain of attraction of the nondegenerate distribution $Q$ ($X \in MDA_l(Q)$ for short) if there exist constants $a_T > 0$ and $b_T \in R^1$ such that, as $T \to \infty$,

$$\mathscr{L}\,(a_T(M_T - b_T)) \Rightarrow Q.$$

Define $\gamma_T$ from the equality $G(\gamma_T) = T$.

**Theorem 6.8** [6] *Let an* H*-diffusion X corresponds to the function* $g \in \mathscr{G}(l, r)$. *The following criteria hold true:*

*(i)* $X \in MDA_l(\Lambda)$ *if and only if there exists a function* $b(x) > 0$, $x \in (x_0, r)$, *such that, for each* $x \in R^1$,

$$\lim_{y \uparrow r} \frac{G(y)}{G(y + b(y)x)} = e^{-x};$$

*(ii)* $X \in MDA_l(\Phi_\gamma)$ *if and only if* $r = \infty$ *and, for each* $x > 0$,

$$\lim_{y \uparrow \infty} \frac{G(y)}{G(xy)} = x^{-\gamma}, \quad \gamma > 0;$$

*(iii)* $X \in MDA_l(\Psi_\gamma)$ *if and only if* $r < \infty$ *and, for each* $x > 0$

$$\lim_{y \downarrow 0} \frac{G(r - y)}{G(r - xy)} = x^{\gamma}, \quad \gamma > 0.$$

*Moreover, in the case (i)*

$$\int_{x_0}^{r} (G(v))^{-1}\, dv < \infty$$

*and we can take*

$$b(x) = G(x) \int_{x}^{r} (G(v))^{-1}\, dv,$$

$$a_T \sim \frac{1}{T} \left( \int_{\gamma_T}^{r} (G(v))^{-1}\, dv \right)^{-1},$$

$$b_T = \gamma_T + \chi_T,$$

*where* $\chi_T$ *are any constants such that* $a_T \chi_T \to 0$*, as* $T \to \infty$.

*In the case* (ii)

$$a_T \sim \gamma_T^{-1}, \quad b_T = 0$$

*and in the case* (iii)

$$a_T \sim (r - \gamma_T)^{-1}, \quad b_T = r.$$

*Proof* Under the assumptions of Theorem from Davis [16] (see also [17, 18]) we have that for any constants $u_T \uparrow \infty$, as $T \to \infty$.

$$\lim_{T\to\infty} \left| \mathrm{P}\{M_T \le u_T\} - F^T(u_T) \right| = 0,$$

where

$$F(x) = e^{-(G(x))^{-1}}, \quad x \in (l, r).$$

Let

$$\hat{F}(x) = \begin{cases} 0, & \text{for} \quad x < \hat{x}_0, \\ 1 - (G(x))^{-1}\, 1_{(\hat{x}_0, r)}, & \text{for} \quad x \ge \hat{x}_0, \end{cases}$$

where $G(\hat{x}_0) = 1$.

Because $1 - F(x) \sim 1 - \hat{F}(x)$, as $x \uparrow r$, the statement of Theorem 6.8, using the principle of equivalent tails, now follows from the classical extreme value theory (see, e.g., [19, 20]). □

Because, for $x \in (\hat{x}_0, r)$,

$$\hat{f}(x) := \hat{F}'(x) = \frac{g(x)}{2G^2(x)}$$

and

$$\hat{f}'(x) = \frac{1}{2}\frac{g'(x)}{G^2(x)} - \frac{g^2(x)}{G^3(x)},$$

we shall have the following analogue of classical von Mises theorem (see, [19–22]).

**Theorem 6.9** *[21] Let an* H*-diffusion $X$ correspond to the function $g \in \mathcal{G}(l, r)$. The following sufficient conditions are valid:*

*(i) if*

$$\lim_{x \uparrow r} \frac{g'(x)G(x)}{g^2(x)} = 1,$$

*then $X \in MDA_l(\Lambda)$;*

*(ii) if $r = \infty$ and*

$$\lim_{x \uparrow \infty} \frac{xg(x)}{G(x)} = \gamma > 0,$$

*then $X \in MDA_l(\Phi_\gamma)$;*

*(iii) if $r < \infty$ and*

$$\lim_{x \uparrow r} \frac{(r-x)g(x)}{G(x)} = \gamma > 0,$$

*then $X \in MDA_l(\Psi_\gamma)$.*

*Now the following Propositions are obvious.*

**Proposition 6.10** *Let a Student diffusion $X$ correspond to the function $g \in \mathcal{G}(-\infty, \infty)$.*

*There are two possibilities:*

*(1) $X \in MDA_l(\Lambda)$ if and only if there exists a function $b(x) > 0$, $x \in (x_0, \infty)$, such that, for each $x \in R^1$,*

$$\lim_{y \uparrow \infty} \frac{G(y)}{G\,(y + b(y)x)} = e^{-x},$$

*and*

*(2) $X \in MDA_l(\Phi_\gamma)$ if and only if, for each $x > 0$,*

$$\lim_{y \uparrow \infty} \frac{G(y)}{G(xy)} = x^{-\gamma}, \quad \gamma > 0.$$

*In the case (1) we can take*

$$b(x) = G(x) \left( \int_{\infty_T}^{\infty} (G(v))^{-1} \, dv \right)^{-1},$$

*and the norming constants*

$$a_T \sim \frac{1}{T} \left( \int_{\gamma_T}^{\infty} (G(v))^{-1} \, dv \right)^{-1},$$

$$b_T = \gamma_T + \chi_T,$$

*where $\chi_T$ are any constants such that $a_T \chi_T \to 0$, as $T \to \infty$.*

*In the case (2) the norming constants are $a_T \sim \gamma_T^{-1}$, $b_T = 0$.*

**Proposition 6.11** *Let a Student diffusion $X$ correspond to the function $g \in \mathcal{G}(-\infty, \infty)$.*

*Then, if*

$$\lim_{x \uparrow \infty} \frac{g'(x)G(x)}{g^2(x)} = 1,$$

*$X \in MDA_l(\Lambda)$,*
*and, if*

$$\lim_{x \uparrow \infty} \frac{xg(x)}{G(x)} = \gamma > 0,$$

*$X \in MDA_l(\Psi_\gamma)$.*

*Example 6.12* (continued Example 6.2) Let $\alpha_1 + \beta_1 > 2$. Using Theorem 6.9 (iii), because

$$\lim_{x \uparrow 1} \frac{(1-x)g(x)}{G(x)} = \alpha_1 + \beta_1 - 2,$$

$X \in MDA_l(\Psi_{\alpha_1+\beta_1-2})$.

*Example 6.13* (continued Example 6.3) In the both cases (i) and (ii)

$$\lim_{x \to \infty} \frac{g'(x)G(x)}{g^2(x)} = 1,$$

implying by Theorem 6.9 (i) that $X \in MDA_l(\Lambda)$.

In the case (iii), assuming that $\lambda + \gamma < 2$, we have that

$$\lim_{x \uparrow \infty} \frac{xg(x)}{G(x)} = 2 - \lambda - \gamma,$$

implying by Theorem 6.9 (ii) that $X \in MDA_l(\Phi_{2-\lambda-\gamma})$.

*Example 6.14* (continued Example 6.4) Assuming that $\lambda + \gamma < \frac{1}{2}$, we have that

$$\lim_{x \uparrow \infty} \frac{xg(x)}{G(x)} = 1 - 2(\lambda + \gamma),$$

implying by Theorem 6.9 (ii) that $X \in MDA_l(\Phi_{1-2(\lambda+\gamma)})$.

*Example 6.15* (continued Example 6.6) Taking $x_0 = 0$, we find that $G(x) = \sigma^2 x$, $x \in R^1$, $\gamma_T = \sigma^{-2} T$ and

$$\frac{xg(x)}{G(x)} \equiv 1.$$

Thus, $X \in MDA_l(\Phi_1)$ and, as $T \to \infty$,

$$\mathscr{L}\left(\frac{\sigma^2}{T} M_T\right) \Rightarrow \Phi_1.$$

## 6.3 Point Measures of $\varepsilon$-Upcrossings for Student Diffusions

Let $\varepsilon > 0$ be fixed. The process $X = \{X_t, t \geq 0\}$ is said to have an $\varepsilon$-upcrossing of the level $u$ at $t_0$ if $X(t) < u$, for $t \in (t_0 - \varepsilon, t_0)$, and $X(t_0) = u$. Let $T > 0$ and $B \in \mathscr{B}((0, 1])$. Then

$$N_T(B) = \sharp\{\varepsilon - \text{crossings} \quad \text{of} \quad u_T \quad \text{by} \quad X \quad \text{on} \quad \text{the} \quad \text{set} \quad TB\}$$

is called the time normalized point measure of $\varepsilon$-upcrossings of the level $u_T$ by $X$.

The following statement is slightly weakened but essentially simplified version of the Borkovec and Klüppelberg result in [8] (for used terminology see, e.g., [23]).

**Theorem 6.16** [24] *Let an* H*-diffusion $X$ correspond to the function $g \in \mathscr{G}(l, r)$, pdf $h$ is continuous and there exists a constant $K$ such that, for all $x \in (l, r)$,*

$$\frac{h(x)G^2(x)\log(|G(x)| + 1)}{g(x)} \leq K. \tag{6.4}$$

*If $u_T \uparrow r$, as $T \to \infty$, and*

$$\lim_{T\to\infty} T^{-1}G(u_T) = (2\tau)^{-1}, \quad \tau > 0, \tag{6.5}$$

*then the point measure $N_T$ converges vaguely to the homogeneous Poisson point measure on $\mathscr{B}((0,1))$ with the intensity $\tau$, as $T \to \infty$.*

*Example 6.17* Let $(l, r) = (-\infty, \infty)$, $x_0 = 0$, $h(x)$, $x \in R^1$, be an arbitrary strictly positive continuous pdf, $g(x) \equiv \sigma^{-2} > 0$.

If there exists a constant $K$ such that, for all $x \in R^1$

$$x^2 \log(|x| + 1)\, h(x) \le K, \tag{6.6}$$

then the statement of Theorem 6.16 holds true with $\tau = \frac{\sigma^2}{2}$ and $u_T = T$.

Because for the skew Student's $t$-distribution (see Example 6.4 and [13])

$$h(x) = C_{\nu,\delta,\varkappa}\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}} \exp\left\{-\varkappa \arctan\left(\frac{x-\alpha}{\delta}\right)\right\}, \quad x \in R^1, \tag{6.7}$$

where

$$C_{\nu,\delta,\varkappa} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\delta\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \prod_{k=0}^{\infty}\left[1 + \frac{\varkappa^2}{(\nu+1+2k)^2}\right]^{-1},$$

we have that, as $|x| \to \infty$,

$$h(x) \sim C_{\nu,\delta,\varkappa}\delta^{\nu+1}|x|^{-(\nu+1)}. \tag{6.8}$$

In this case the assumption (6.4) is satisfied if and only if $\nu > 1$.

*Example 6.18* Let $X$ be a skew Student diffusion corresponding to the function

$$g(x) = \frac{\exp\left\{\varkappa \arctan\left(\frac{x-\alpha}{\delta}\right)\right\}}{C_{\nu,\delta,\varkappa}\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}+\gamma}}, \quad x \in R^1, \quad \alpha, \varkappa \in R^1, \quad \gamma \le 1 + \frac{\nu}{2}.$$

Having in mind (6.8), because, as $|x| \to \infty$,

$$g(x) \sim \frac{|x|^{\nu+1-2\gamma}}{C_{\nu,\delta,\varkappa}\delta^{\nu+1-2\gamma}}$$

and, using l'Hospital's rule,

$$G(x) \sim \frac{|x|^{\nu+2-2\gamma}}{C_{\nu,\delta,\varkappa}\delta^{\nu+1-2\gamma}(\nu+2-2\gamma)},$$

we find that the assumption (6.6) is satisfied if and only if $1 < \gamma \le 1 + \frac{\nu}{2}$.

If $1 < \gamma < 1 + \frac{\nu}{2}$, taking

$$u_T = \left(\frac{T}{2C_{\nu,\delta,\varkappa}\delta^{\nu+1-2\gamma}(\nu+2-2\gamma)}\right)^{\frac{1}{\nu+2-2\gamma}},$$

then the point measure $N_T$, as $T \to \infty$, converge vaguely to the Poisson measure with the intensity 1.

*Example 6.19* (continued Example 6.3) In the case (i), using l'Hospital's rule, we have that, as $x \to \infty$,

$$G(x) \sim (\psi\beta_1)^{-1}x^{1-\beta_1}g(x) \tag{6.9}$$

and, as $x \to 0$,

$$G(x) \sim -(\chi\beta_2)^{-1}x^{1+\beta_2}g(x). \tag{6.10}$$

Thus, the assumption (6.4) is satisfied if and only if

$$2 - 2\beta_1 < \gamma < 2 + 2\beta_2$$

and (6.5) holds with $\tau = 1$ and

$$\begin{aligned} u_T = {} & \left(\frac{1}{\psi}\log T\right)^{\frac{1}{\beta_1}} + \frac{1}{\beta_1\psi}\left(\frac{1}{\psi}\log T\right)^{\frac{1}{\beta_1}-1} \\ & \times\left[\frac{\beta_1+\gamma+\lambda-2}{\beta_1}\log\left(\frac{1}{\psi}\log T\right) + \log\left(\beta_1\psi C\frac{\sigma^2}{2}\right)\right] \end{aligned} \tag{6.11}$$

Here we used formulas for asymptotic solutions of equations like $G(u_T) = T$ from [19], Table 3.4.4.

In the case (ii) we analogously find that, as $x \to \infty$, (6.10) holds, and, as $x \to 0$,

$$G(x) \sim \frac{x}{2-(\lambda+\gamma)}g(x), \tag{6.12}$$

implying that the assumption (6.4) is satisfied if and only if

$$2 < \gamma < 2\beta_2 + 2.$$

The equality (6.5) holds with $\tau = 1$ and $u_T$, defind by (6.11).

Finally, in the case (iii), as $x \to \infty$, it holds (6.12) and, as $x \to 0$, it holds (6.10), implying that the assumption (6.4) is satisfied if and only if

$$2 - 2\beta_1 < \gamma < 2.$$

The equality (6.5) holds with $\tau = 1$ and

$$u_T = \left[ (2 - \lambda - \gamma)(\frac{\sigma^2 T}{2}) \right]^{\frac{1}{2-\lambda-\gamma}}.$$

## 6.4 Kolmogorov–Pearson Diffusions

**Definition 6.20** An H-diffusion $X = \{X_t, t \geq 0\}$ in the interval $(l, r)$ is called the Kolmogorov–Pearson diffusion if it is a weak solution for the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \theta A(X_t)\mathrm{d}t + \sqrt{\theta B(X_t)}\mathrm{d}B_t, \quad t > 0, \quad \theta > 0, \\ \mathscr{L}(X_0) = H, \end{cases} \tag{6.13}$$

where

$$A(x) = p_0 + p_1 x, \quad x \in (l, r),$$

and

$$B(x) = q_0 + q_1 x + q_2 x^2 > 0, \quad x \in (l, r).$$

This class of diffusions was described by Kolmogorov in 1931 (see [25]). Ergodic distributions of these diffusions are contained in the family of Pearson distributions, satisfying the Pearson equation:

$$\frac{h'(x)}{h(x)} = \frac{2A(x) - B'(x)}{B(x)}, \quad x \in (l, r). \tag{6.14}$$

Last years this class of diffusions attracted attention of statisticians as a flexible and statistically tractable stochastic processes (see, e.g., [13, 26–32]).

Let $L^2((l, r); H)$ be a Hilbert space of equivalency classes of measurable functions $f : (l, r) \to R^1$ such that

$$||f||_H^2 := \int_l^r f^2(x)h(x)\mathrm{d}x < \infty$$

and $C^2((l,r))$ be a class of twice differentiable functions $f:(l,r)\to R^1$.

The generator

$$L=\frac{\theta}{2}B(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\theta A(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

of the Kolmogorov–Pearson diffusion $X$, satisfying (6.13), is a map

$$L:L^2((l,r);H)\cap C^2((l,r))\to L^2((l,r);H).$$

Let us recall the following classical results (see, e.g., [1, 33–35]).

Obviously, $L$ maps polynomials to polynomials. If, for all $n=0,1,\ldots$,

$$\int_l^r x^{2n}h(x)\mathrm{d}x<\infty,$$

there exists an orthonormal system of polynomials $\{P_n(x),x\in(l,r),n=0,1,\ldots\}$ such that

$$LP_n(x)+\lambda_nP_n(x)=0,\quad x\in(l,r),$$

where

$$\lambda_n=-n\theta\left(p_1+\frac{q_2}{2}(n+1)\right),\quad n=0,1,\ldots,\tag{6.15}$$

showing that the spectrum of—$L$ is discrete with the eigenvalues, given by (6.15), and the corresponding eigenfunctions $\{P_n(x),x\in(l,r),n=0,1,\ldots\}$, which under the additional assumption that

$$\lim_{x\to l-0}h(x)B(x)=\lim_{x\to r+0}h(x)B(x)=0\tag{6.16}$$

are given by the generalized Rodrigues formula:

$$P_n(x)=c_n\frac{\left[h(x)B^n(x)\right]^{(n)}}{h(x)},\quad x\in(l,r),\quad n=0,1,\ldots,\tag{6.17}$$

where

$$c_n^{-2}=\int_l^r\frac{\left(\left[h(x)B^n(x)\right]^{(n)}\right)^2}{h(x)}\mathrm{d}x.$$

If, for some integer $N$,

$$\int_l^r x^{2N} h(x)\mathrm{d}x < \infty, \tag{6.18}$$

but

$$\int_l^r |x|^{2N+1} h(x)\mathrm{d}x = \infty,$$

the spectrum of—$L$ consists of the continuous part and the finite number of discrete eigenvalues

$$\lambda_n = -n\theta\left(p_1 + \frac{q_2}{2}(n+1)\right), \quad n = 0, 1, \dots, N,$$

corresponding to the eigenfunctions $\{P_n(x), x \in (l, r), n = 0, 1, \dots, N\}$, defined by the formula (6.17).

Let

$$h_j = \int_l^r x^j h(x)\mathrm{d}x, \quad j = 0, 1, 2, \dots,$$

$$\Delta_n = \begin{vmatrix} 1 & h_1 & \dots & h_n \\ h_1 & h_2 & \dots & h_{n+1} \\ \dots & \dots & \dots & \dots \\ h_n & h_{n+1} & \dots & h_{2n} \end{vmatrix}, \quad \Delta_0 = 1,$$

and

$$Q_n(x) = \begin{vmatrix} 1 & h_1 & \dots & h_n \\ h_1 & h_2 & \dots & h_{n+1} \\ \dots & \dots & \dots & \dots \\ h_{n-1} & h_n & \dots & h_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \quad Q_0(x) \equiv 1.$$

Then

$$P_n(x) = \frac{Q_n(x)}{\sqrt{\Delta_{n-1}\Delta_n}}, \quad x \in (l, r), \quad n = 1, 2, \dots.$$

If $h$ is a pdf of the skew Student's $t$-distribution, given by (6.7), from Example 6.4 it follows that the corresponding H-diffusion is the Kolmogorov–Pearson diffusion with

$$A(x) = \frac{\theta}{\delta}\left[-\frac{\nu-1}{2}\left(\frac{x-\alpha}{\delta}\right) - \frac{\varkappa}{2}\right], \tag{6.19}$$

and

$$B(x) = \theta\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right), \quad x \in R^1, \quad \alpha, \varkappa \in R^1, \quad \nu, \delta > 0. \tag{6.20}$$

In this case from (6.8) it follows that (6.16) is satisfied if and only if $\nu > 1$, and (6.18) holds true with the largest integer $N$ satisfying $2N < \nu$ and denoted $N = \left\lfloor \frac{\nu}{2} \right\rfloor$. The discrete eigenvalues for the skew Student diffusion, defined by (6.19) and (6.20), are

$$\lambda_n = \frac{n\theta}{2\delta^2}(\nu - n), \quad n = 0, 1, \ldots, \left\lfloor \frac{\nu}{2} \right\rfloor.$$

The corresponding eigenfunctions are equal to

$$P_n(x) = c_n \frac{\left[h(x)\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right)^n\right]^{(n)}}{h(x)}, \quad n = 0, 1, \ldots, \left\lfloor \frac{\nu}{2} \right\rfloor \tag{6.21}$$

If $\varkappa = 0$, $h$ is the pdf of $T_1(\nu, \delta^2\nu^{-1}, \alpha)$. Following [30], polynomials (6.21) are called the Routh–Romanovsky polynomials (see [36, 37]).

If $\varkappa = \alpha = 0$, we have that, for $j < \nu$,

$$h_j^{(0)} := \begin{cases} \int\limits_{-\infty}^{\infty} x^j h(x)\mathrm{d}x = \dfrac{\delta^j}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\Gamma\left(\dfrac{j}{2} + \dfrac{1}{2}\right)\Gamma\left(\dfrac{\nu}{2} - \dfrac{j}{2}\right), & \text{if } j \text{ is even}, \\ 0, \quad \text{if } j \text{ is odd}, \end{cases}$$

and, for $\varkappa = 0$, $\alpha \neq 0$, $j < \nu$,

$$h_j^{(\alpha)} := \int\limits_{-\infty}^{\infty} x^j h(x)\mathrm{d}x = \sum_{k=0}^{j} \binom{j}{k} h_k^{(0)} \alpha^{j-k}.$$

We refer the reader to [30] (see also [9, 15]) where a version of the Student diffusion was considered with

$$A(x) = \frac{-x + \alpha}{\delta},$$

$$B(x) = \frac{2\delta^2}{\nu - 1}\left(1 + \left(\frac{x-\alpha}{\delta}\right)^2\right), \quad \alpha \in R^1, \quad \nu > 1, \quad \delta > 0,$$
$$\lambda_n = \frac{\theta}{\nu - 1} n(\nu - n), \quad n = 0, 1, \ldots, \left\lfloor \frac{\nu}{2} \right\rfloor$$

and the Routh–Romanovsky polynomials as corresponding eigenfunctions. Most important that in this paper the continuous part of spectrum is described in terms of the hypergeometric functions, obtained the spectral representation of transition probability density of $X$ and applied to the statistical inference of the model.

The skew Student diffusion is known as the Johannesma diffusion model for the stochastic activity of neurons (see [9–11]) and as one of the Föllmer–Schweizer models for stock returns (see [12, 13]).

Classification of the Kolmogorov–Pearson diffusions to six types is given in [14, 15]. The characteristics of these types are the following:

(1)

$$A(x) = -x + \alpha, \quad B(x) \equiv 2, \quad (l, r) = (-\infty, \infty),$$
$$h(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2}, \quad x, \alpha \in R^1,$$
$$\lambda_n = n^2\theta, \quad n = 0, 1, \ldots$$

$\left\{P_n(x), x \in R^1, n = 0, 1, \ldots\right\}$ are the Hermite polynomials;

(2)

$$A(x) = -x + \alpha, \quad B(x) = 2x, \quad (l, r) = (0, \infty), \quad \alpha > 1,$$
$$h(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \quad x > 0,$$
$$\lambda_n = n\theta, \quad n = 0, 1, \ldots,$$

$\{P_n(x), x > 0, n = 0, 1, \ldots\}$ are the Laguerre polynomials;

(3)

$$A(x) = -x + \alpha, \quad B(x) = 2ax^2, \quad (l, r) = (0, \infty), \quad a > 0, \quad \alpha > 0,$$
$$h(x) = C_{a^{-1}+1,1,\frac{\alpha}{a}} \left(1 + x^2\right)^{-\frac{1}{2a}-1} \exp\left\{-\frac{\alpha}{a} \arctan(x - \alpha)\right\}, \quad x > 0$$
$$\lambda_n = n\theta\left(1 - a(n+1)\right), \quad n = 0, 1, \ldots, \left\lfloor \frac{1}{2} + \frac{1}{2a} \right\rfloor,$$

$\left\{P_n(x), x > 0, n = 0, 1, \ldots, \left\lfloor \frac{1}{2} + \frac{1}{2a} \right\rfloor\right\}$ are the Routh-Romanovsky polynomials;

(4)

$$
\begin{aligned}
A(x) &= -x+\alpha, \quad B(x) = 2ax^2, \quad (l,r) = (0,\infty), \quad a>0, \quad \alpha>0,\\
h(x) &= \frac{\left(\frac{\alpha}{a}\right)^{\frac{1}{a}+1}}{\Gamma\left(\frac{1}{a}+1\right)} x^{-\frac{1}{a}-2}\exp\left\{-\frac{\alpha}{ax}\right\}, \quad x>0,\\
\lambda_n &= n\theta\left(1-a(n+1)\right), \quad n=0,1,\ldots,\left\lfloor \frac{1}{2}+\frac{1}{2a}\right\rfloor,
\end{aligned}
$$

$\left\{P_n(x), x>0, n=0,1,\ldots,\left\lfloor \frac{1}{2}+\frac{1}{2a}\right\rfloor\right\}$ are the Bessel polynomials;

(5)

$$
\begin{aligned}
A(x) &= -x+\alpha, \quad B(x) = 2ax(x+1), \quad (l,r) = (0,\infty), \quad \alpha \ge a>0,\\
h(x) &= \frac{1}{B\left(\frac{\alpha}{a},\frac{1}{a}+1\right)} x^{\frac{\alpha}{a}-1}(1+x)^{-\frac{\alpha+1}{a}-1}, \quad x>0,\\
\lambda_n &= n\theta\left(1-a(n+1)\right), \quad n=0,1,\ldots,\left\lfloor \frac{1}{2}+\frac{1}{2a}\right\rfloor,
\end{aligned}
$$

$\left\{P_n(x), x>0, n=0,1,\ldots,\left\lfloor \frac{1}{2}+\frac{1}{2a}\right\rfloor\right\}$ are the Fisher–Snedocor polynomials;

(6)

$$
\begin{aligned}
A(x) &= -x+\alpha, \quad B(x) = 2ax(x-1), \quad (l,r) = (0,1), \quad -1<a<0,\\
&\quad 1+a \le \alpha \le -a,\\
h(x) &= \frac{1}{B\left(-\frac{\alpha}{a},-\frac{1-\alpha}{a}\right)} x^{-\frac{\alpha}{a}-1}(1-x)^{-\frac{\alpha+1}{a}-1}, \quad 0<x<1,\\
\lambda_n &= n\theta\left(1-2a(n+1)\right), \quad n=0,1,\ldots,
\end{aligned}
$$

$\{P_n(x), x\in(0,1), n=0,1,\ldots\}$ are Jacobi polynomials.

In the above formulas $B(z_1,z_2)$ means the Euler's beta function.

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