Chapter 5
Student OU-Type Processes

The classical Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$, starting from $x \in R^d$, is a solution of linear equation.

$$X_t = x + B_t - c \int_0^t X_s \mathrm{d}s, \quad t \geq 0, \tag{5.1}$$

where $c > 0$ and $\{B_t, t \geq 0\}$ is a standard d-dimensional Brownian motion. It is uniquely solved by

$$X_t = e^{-ct} x + \int_0^t e^{-c(t-s)} \mathrm{d}B_s, \quad t \geq 0,$$

where the last integral is a Wiener stochastic integral. We easily find that

$$\mathscr{L}(X_t) = G_{e^{-ct}x, \frac{1}{2c}(1-e^{-2ct})I_d} \Rightarrow G_{0, \frac{1}{2c}I_d},$$

as $t \to \infty$, where I_d is an identity $d \times d$ matrix.

If we replace $\{B_t, t \geq 0\}$ in (5.1) by an arbitrary Lévy process $\{Z_t, t \geq 0\}$ with the triplet (a, A, Π) of Lévy characteristics and the characteristic exponent $\varphi(z) = -\log Ee^{i\langle z, X_1 \rangle}$, the solution

$$X_t = e^{-ct} x + \int_0^t e^{-c(t-s)} \mathrm{d}Z_s, \quad t \geq 0 \tag{5.2}$$

is called the starting from $x \in R^d$ Ornstein–Uhlenbeck type process generated by (a, A, Π, c).

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The integral in (5.2) is defined analogously to the Wiener integral through converging in probability integral sums (see, e.g., [1]).

If we write

$$P_t(x, B) = \mathrm{P}\{X_t \in B\}, \quad x \in R^d, \quad B \in \mathscr{B}(R^d), \quad t \geq 0,$$

it can be proved (see [2]) that

$$\int_{R^d} e^{i\langle z,y\rangle} P_t(x, \mathrm{d}y) = \exp\left\{ i e^{-ct}\langle x, z\rangle - \int_0^t \varphi(e^{-cs}z)\mathrm{d}s \right\}, \quad x, z \in R^d, \quad t \geq 0,$$

implying that $P_t(x, \cdot)$ is an infinitely divisible probability measure with the triplet $(a_{t,x}, A_t, \Pi_t)$ of Lévy characteristics given by the formulas:

$$A_t = \int_0^t e^{-2cs}\mathrm{d}s A, \quad t \geq 0$$

$$\Pi_t(B) = \int_{R_0^d} \int_0^t 1_B(e^{-cs}y)\mathrm{d}s\,\Pi(\mathrm{d}y), \quad B \in \mathscr{B}(R_0^d), \quad t \geq 0,$$

and

$$\begin{aligned} a_{t,x} = e^{-ct}x + \int_0^t e^{-cs}\mathrm{d}s \\ + \int_{R_0^d} \int_0^t e^{-cs} y \left(1_{\{e^{-cs}|y|\leq 1\}} - 1_{\{|y|\leq 1\}}\right) \mathrm{d}s\,\Pi(\mathrm{d}y), \quad t \geq 0, \quad x \in R^d. \end{aligned}$$

Because

$$\begin{aligned} &\int_{R^d}\int_{R^d} e^{i\langle z,w\rangle} P_s(y, \mathrm{d}w) P_t(x, \mathrm{d}y) \\ &= \int_{R^d} \exp\left\{ i\langle y, e^{-cs}z\rangle - \int_0^s \varphi(e^{-cr}z)\mathrm{d}r \right\} P_t(x, \mathrm{d}y) \\ &= \exp\left\{ i\langle x, e^{-c(t+s)}z\rangle - \int_0^s \varphi(e^{-c(r+s)}z)\mathrm{d}r - \int_0^t \varphi(e^{-cr}z)\mathrm{d}r \right\} \end{aligned}$$

$$= \int_{R^d} e^{i\langle z,w\rangle} P_{t+s}(x, \mathrm{d}w),$$

$P_t(x, B), t \geq 0, x \in R^d, B \in \mathscr{B}(R^d)$, satisfies the Chapman–Kolmogorov identity

$$\int_{R^d} P_t(x, \mathrm{d}y) P_s(y, B) = P_{t+s}(x, B)$$

as the transition probability function of the time homogeneous Markov process X.

It is known (see [2–6]) that, as $t \to \infty$, for each $x \in R^d$

$$P_t(x, \cdot) \Rightarrow \tilde{\mu}_c$$

if and only if

$$\int_{\{|y|>2\}} \log|y| \Pi(\mathrm{d}y) < \infty, \tag{5.3}$$

where the limit distribution $\tilde{\mu}_c$ satisfies

$$\int_{R^d} e^{i\langle z,y\rangle} \tilde{\mu}_c(\mathrm{d}y) = \exp\left\{ -\int_0^\infty \varphi(e^{-cs}z)\mathrm{d}s \right\}, \quad z \in R^d. \tag{5.4}$$

The distribution $\tilde{\mu}_c$ is self-decomposable with the triplet of Lévy characteristics $(\tilde{a}_c, \tilde{A}_c, \tilde{\Pi}_c)$, where

$$\tilde{a}_c = \frac{1}{c} a + \frac{1}{c} \int_{\{|y|>1\}} \frac{y}{|y|} \Pi(\mathrm{d}y),$$

$$\tilde{A}_c = \frac{1}{2c} A$$

and

$$\tilde{\Pi}_c(B) = \frac{1}{c} \int_{R^d} \int_0^\infty 1_B\left(e^{-s}y\right) \mathrm{d}s \, \Pi(\mathrm{d}y), \quad B \in \mathscr{B}(R_0^d).$$

There is one-to-one continuous in the topology of weak convergence correspondence between the class $ID_{\log}(R^d)$ of infinitely divisible distributions, satisfying the integrability assumption (5.3), and the class of self-decomposable distributions $L(R^d)$. It is given by the mapping

$$ID_{\log}(R^d) \ni \mu = \mathscr{L}(Z_1) \leftrightarrow \mathscr{L}\left(\int_0^\infty e^{-t}\mathrm{d}Z_t\right) = \tilde{\mu} \in L(R^d). \tag{5.5}$$

The correspondence (5.5) imply that for the triplet $(\tilde{a}, \tilde{A}, \tilde{\Pi})$ of Lévy characteristics for $\tilde{\mu}$ the following equalities hold true:

$$\tilde{a} = a + \int_{\{|y|>1\}} \frac{y}{|y|}\Pi(\mathrm{d}y)$$

$$\tilde{A} = \frac{1}{2}A$$

and

$$\tilde{\Pi}(B) = \int_{R^d}\int_0^\infty 1_B(e^{-s}y)\mathrm{d}s\,\Pi(\mathrm{d}y), \quad B \in \mathscr{B}(R_0^d).$$

Vice versa, if

$$\tilde{\Pi}(B) = \int_{S^{d-1}} \lambda(\mathrm{d}\xi)\int_0^\infty 1_B(r\xi)\frac{k_\xi(r)}{r}\mathrm{d}r, \quad B \in \mathscr{B}(R_0),$$

then

$$a = \tilde{a} - \int_{\{|y|>1\}} \frac{y}{|y|}\Pi(\mathrm{d}y),$$

$$A = 2\tilde{A} \tag{5.6}$$

and

$$\Pi(B) = -\int_{S^{d-1}} \lambda(\mathrm{d}\xi)\int_0^\infty 1_B(r\xi)\mathrm{d}k_\xi(r).$$

The process $\{Z_t, t \geq 0\}$, is called the background driving Lévy process (BDLP for short).

Definition 5.1 The subclass of the Ornstein–Uhlenbeck type processes, obtained by the correspondence (5.5) with the Student t-distribution $\tilde{\mu}$, is called the class of the Ornstein–Uhlenbeck type Student processes (Student OU-type processes for short).

Definition 5.2 The subclass of the Ornstein–Uhlenbeck type processes, obtained by the correspondence (5.5) with the noncentral Student t-distributions $\tilde{\mu}$, satisfying self-decomposability condition (iii) of Proposition 4.6, is called the class of the noncentral Ornstein–Uhlenbeck type Student processes (noncentral Student OU-type processes for short).

We shall describe the BGDP, generating the Student OU-type processes.

Proposition 5.3 (i) *The Student OU-type processes are generated by the BDLP* $Z = \{Z_t, t \geq 0\}$ *with the triplets of Lévy characteristics* $(\gamma_0, 0, \Pi_0)$, *where*

$$\gamma_0 = \int\limits_{\{|x|\leq 1\}} x\pi_0(x)\mathrm{d}x + \alpha, \quad \alpha \in R^d,$$

$$\Pi_0(B) = \int\limits_B \pi_0(x)\mathrm{d}x, \quad B \in \mathscr{B}(R_0^d),$$

$$\pi_0(x) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(r^d l_0(r\xi)\right)|_{r\xi=x}$$

and

$$l_0(x) = \frac{\nu 2^{\frac{\mathrm{d}}{4}+1}\left(\langle x\Sigma^{-1}, x\rangle\right)^{-\frac{\mathrm{d}}{4}}}{\sqrt{|\Sigma|}(2\pi)^{\frac{\mathrm{d}}{2}}} \int\limits_0^\infty u^{\frac{\mathrm{d}}{4}} K_{\frac{\mathrm{d}}{2}}\left((2t\langle x\Sigma^{-1}, x\rangle)^{\frac{1}{2}}\right) g_{\frac{\nu}{2}}(2\nu t)\mathrm{d}t,$$

$\nu > 0$, Σ *is a symmetric positive definite* $d \times d$ *matrix.*

(ii) *The Student OU-type process* X, *generated by the BDLP* Z *with the triplet of Lévy characteristics* $(\gamma_0, 0, \Pi_0)$ *and* $\mathcal{L}(X_0) = T_d(\nu, \Sigma, \alpha)$ *is strictly stationary Markov process.*

Proof

(i) Follows directly from the Definition 5.1, the above stated properties of BGDP and the Proposition 4.5.
(ii) It is well-known property of time homogeneous Markov processes. □

Proposition 5.4

(i) *The noncentral Student OU-type processes are generated by the BDLP* $Z = \{Z_t, t \geq 0\}$ *with the triples of Lévy characteristics* $(\gamma_a, 0, \Pi_a)$, *where*

$$\gamma_a = \int\limits_{\{|x|\leq 1\}} x\pi_a(x)\mathrm{d}x + \alpha, \quad \alpha, a \in R^d,$$

$$\Pi_a(B) = \int\limits_B \pi_a(x)dx, \quad B \in \mathscr{B}(R_0^d),$$

$$\pi_a(x) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(r^d l_a(r\xi)\right)|_{r\xi=x}$$

and

$$l_a(x) = \frac{2\nu \exp\left\{\langle a\Sigma^{-1}, x\rangle\right\}}{\sqrt{|\Sigma|}(2\pi)^{\frac{d}{2}}\left(\langle x\Sigma^{-1}, x\rangle\right)^{\frac{d}{4}}} \int_0^\infty \left(\langle a\Sigma^{-1}, a\rangle + 2t\right)^{\frac{d}{4}}$$
$$\times K_{\frac{d}{2}}\left(\left((\langle a\Sigma^{-1}, a\rangle + 2t)\langle x\Sigma^{-1}, x\rangle\right)^{\frac{1}{2}}\right) g_{\frac{\nu}{2}}(2\nu t).$$

(ii) *The noncentral Student OU-type process $X^{(a)}$, generated by the BDLP Z with the triplet of Lévy characteristics $(\gamma_a, 0, \Pi_a)$ and $\mathcal{L}(X_0) = T_d(\nu, \Sigma, \alpha, a)$ is strictly stationary Markov process.*

Proof

(i) Follows directly from the Definition 5.2, the above stated properties of BDLP and the Proposition 4.5.
(ii) It is well-known property of time homogeneous Markov processes. □

References

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