Chapter 2
Asymptotics

2.1 Asymptotic Behavior of Student's Pdf

Proposition 2.1 *For each* $x \in R^d$, *as* $\nu \to \infty$,

$$f_{\nu,\Sigma,a}(x) \to g_{a,\Sigma}(x). \tag{2.1}$$

Proof Let $a = 0$. Using the well-known formula that

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} e^{-z} z^z \left(1 + O\left(\frac{1}{z}\right)\right), \quad \text{as} \quad z \to \infty, \tag{2.2}$$

we find that, as $\nu \to \infty$,

$$\frac{\Gamma(\frac{\nu+d}{2})}{(\nu\pi)^{\frac{d}{2}}\Gamma(\frac{\nu}{2})} \sim \frac{\sqrt{\frac{4\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} (\frac{\nu+d}{2})^{\frac{\nu+d}{2}}}{(\nu\pi)^{\frac{d}{2}} \sqrt{\frac{4\pi}{\nu}} e^{-\frac{\nu}{2}} (\frac{\nu}{2})^{\frac{\nu}{2}}} \to \frac{1}{(2\pi)^{\frac{d}{2}}} \tag{2.3}$$

and, obviously,

$$\left(1 + \frac{\langle x\Sigma^{-1}, x\rangle}{\nu}\right)^{-\frac{\nu+d}{2}} \to e^{-\frac{1}{2}\langle x\Sigma^{-1}, x\rangle}. \tag{2.4}$$

Here and below "$\sim$" is the equivalence sign.

The statement (2.1) with $a = 0$ follows from (1.1), (2.2), (2.3) and (2.4).

Let now $a \neq 0$ and

$$y_\nu = \frac{2}{\nu + d}\left[\langle a\Sigma^{-1}, a\rangle(\nu + \langle x\Sigma^{-1}, x\rangle)\right]^{\frac{1}{2}}.$$

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Because, as $\nu \to \infty$, uniformly in y (see Appendix)

$$K_\nu(\nu y) \sim \sqrt{\frac{\pi}{2\nu}} \frac{\exp\{-\nu\sqrt{1+y^2}\}}{(1+y^2)^{\frac{1}{4}}} \left(\frac{y}{1+\sqrt{1+y^2}} \right)^{-\nu}$$

and

$$\sqrt{1+y_\nu^2} \sim 1 + \frac{1}{2} y_\nu^2,$$

we shall have that

$$\begin{aligned}
&K_{\frac{\nu+d}{2}}\left(\left[\langle a\Sigma^{-1}, a\rangle(\nu + \langle x\Sigma^{-1}, x\rangle) \right]^{\frac{1}{2}} \right) = K_{\frac{\nu+d}{2}}\left(\frac{\nu+d}{2} y_\nu \right) \\
&\sim \sqrt{\frac{\pi}{\nu+d}} \exp\left\{ -\frac{\nu+d}{2}\left(1 + \frac{1}{2}y_\nu^2\right) \right\} \left(\frac{y_\nu}{2+\frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\
&\sim \sqrt{\frac{\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} \exp\left\{ -\frac{1}{\nu+d} \langle a\Sigma^{-1}, a\rangle \left(\nu + \langle x\Sigma^{-1}, x\rangle\right) \right\} \left(\frac{y_\nu}{2+\frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}}.
\end{aligned} \tag{2.5}$$

From (1.2) and (2.5) we elementarily find that

$$\begin{aligned}
f_{\nu,\Sigma,a}(x) &\sim \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \frac{2\exp\left\{\langle x\Sigma^{-1}, a\rangle\right\}}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} \left(\frac{\langle a\Sigma^{-1}, a\rangle}{\nu_+ \langle x\Sigma^{-1}, x} \right)^{\frac{\nu+d}{4}} \sqrt{\frac{\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} \\
&\quad \times \exp\left\{ -\frac{1}{\nu+d} \langle a\Sigma^{-1}, a\rangle \left(\nu + \langle x\Sigma^{-1}, x\rangle\right) \right\} \left(\frac{y_\nu}{2+\frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\
&\sim \frac{\exp\left\{\langle x\Sigma^{-1}, a\rangle\right\}}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} e^{-\langle a\Sigma^{-1}, a\rangle} e^{-\frac{d}{2}} \left(\frac{\nu + \langle x\Sigma^{-1}, x\rangle}{2+\frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\
&\sim \frac{\exp\left\{\langle x\Sigma^{-1}, a\rangle\right\}}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} e^{-\langle a\Sigma^{-1}, a\rangle} e^{-\frac{d}{2}} \\
&\quad \times \exp\left\{ -\frac{1}{2}(\langle x\Sigma^{-1}, x\rangle - d) \right\} \left(1 + \frac{1}{4}y_\nu^2\right)^{\frac{\nu+d}{2}}.
\end{aligned} \tag{2.6}$$

But

$$\left(1 + \frac{1}{4}y_\nu^2\right)^{\frac{\nu+d}{2}} = \left(1 + \frac{1}{(\nu+d)^2}\left[\langle a\Sigma^{-1}, a\rangle \left(\nu + \langle x\Sigma^{-1}, x\rangle\right) \right] \right)^{\frac{\nu+d}{2}}$$

$$\to \exp\left\{\frac{1}{2}\langle a\Sigma^{-1}, a\rangle\right\}. \tag{2.7}$$

Thus, (2.6) and (2.7) imply that, for each $x \in R^d$, as $\nu \to \infty$,

$$f_{\nu,\Sigma,a}(x) \to \frac{\exp\left\{\langle x\Sigma^{-1}, a\right\}}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2}\left(\langle a\Sigma^{-1}, a\rangle + \langle x\Sigma^{-1}, x\rangle\right)\right\} = g_{a,\Sigma}(x). \ \square$$

Proposition 2.2 *For each fixed* $x \epsilon R^d$ *and* $\nu > 0$*, as* $|a| \to 0$*,*

$$f_{\nu,\Sigma,a}(x) \to f_{\nu,\Sigma}(x).$$

Proof Indeed, as $|a| \to 0$,

$$K_{\frac{\nu+d}{2}}\left(\left[\langle a\Sigma^{-1}, a\rangle(\nu + \langle x\Sigma^{-1}, x\rangle)\right]^{\frac{1}{2}}\right)$$
$$\sim \Gamma\left(\frac{\nu+d}{2}\right) 2^{\frac{\nu+d}{2}-1}\left[\langle a\Sigma^{-1}, a\rangle(\nu + \langle x\Sigma^{-1}, x\rangle)\right]^{-\frac{\nu+d}{4}}$$

(see Appendix) and, having in mind formulas (1.1), (1.2),

$$f_{\nu,\Sigma,a}(x) \to \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \frac{2^{\frac{\nu+d}{2}}\Gamma(\frac{\nu+d}{2})}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} \left(\nu + \langle x\Sigma^{-1}, x\rangle\right)^{-\frac{\nu+d}{2}} = f_{\nu,\Sigma}(x). \qquad \square$$

Proposition 2.3 *(i) As* $|x| \to \infty$*,*

$$f_{\nu,\Sigma}(x) \sim c_{\nu,\Sigma}\left(\langle x\Sigma^{-1}, x\rangle\right)^{-\frac{\nu+d}{2}},$$

where

$$c_{\nu,\Sigma} = \frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\pi^{\frac{d}{2}}\Gamma(\frac{\nu}{2})\sqrt{|\Sigma|}}.$$

(ii) As $|x| \to \infty$*,* $a \neq 0$*,*

$$f_{\nu,\Sigma,a}(x) \sim c_{\nu,\Sigma,a}\left(\langle x\Sigma^{-1}, x\rangle\right)^{-\frac{\nu+d+1}{4}}$$

$$\times \exp\left\{-\left[\langle a\Sigma^{-1}, a\rangle\langle x\Sigma^{-1}, x\rangle\right]^{\frac{1}{2}} + \langle x\Sigma^{-1}, a\rangle\right\},$$

where

$$c_{\nu,\Sigma,a} = \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}\left(\langle a\Sigma^{-1}, a\rangle\right)^{\frac{\nu+d+1}{4}}}{\Gamma(\frac{\nu}{2})(2\pi)^{\frac{d-1}{2}}\sqrt{|\Sigma|}}.$$

Proof (i) Obviously follows from (1.1).
(ii) Because, as $|x| \to \infty$,

$$K_{\frac{\nu+d}{2}}\left(\left[\langle a\Sigma^{-1}, a\rangle\left(\nu + \langle x\Sigma^{-1}, x\rangle\right)\right]^{\frac{1}{2}}\right)$$
$$\sim \sqrt{\frac{\pi}{2}}\left[\langle a\Sigma^{-1}, a\rangle\left(\nu + \langle x\Sigma^{-1}, x\rangle\right)\right]^{-\frac{1}{4}}$$
$$\times \exp\left\{-\left[\langle a\Sigma^{-1}, a\rangle\left(\nu + \langle x\Sigma^{-1}, x\rangle\right)\right]^{\frac{1}{2}}\right\},$$

from (1.2) we find that, as $|x| \to \infty$,

$$f_{\nu,\Sigma,a}(x) \sim \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}\left(\langle a\Sigma^{-1}, a\rangle\right)^{\frac{\nu+d-1}{4}}}{\Gamma(\frac{\nu}{2})(2\pi)^{\frac{d-1}{2}}\sqrt{|\Sigma|}} \frac{\exp\left\{\langle x\Sigma^{-1}, a\rangle\right\}}{\left(\nu + \langle x\Sigma^{-1}, x\rangle\right)^{\frac{\nu+d+1}{4}}}$$
$$\times \exp\left\{-\left[\langle a\Sigma^{-1}, a\rangle\left(\nu + \langle x\Sigma^{-1}, x\rangle\right)\right]^{\frac{1}{2}}\right\}$$
$$\sim c_{\nu,\Sigma,a}\left(\langle x\Sigma^{-1}, x\rangle\right)^{-\frac{\nu+d+1}{4}}$$
$$\times \exp\left\{-\left[\langle a\Sigma^{-1}, a\rangle\langle x\Sigma^{-1}, x\rangle\right]^{\frac{1}{2}} + \langle x\Sigma^{-1}, a\rangle\right\}.$$

□

Corollary 2.4 *Let d=1.*

(i) If $a > 0$, $x \to \infty$, then

$$f_{\nu,\sigma^2,a}(x) \sim \frac{1}{\sigma\Gamma(\frac{\nu}{2})}\left(\frac{\nu a}{2\sigma}\right)^{\frac{\nu}{2}} x^{-\frac{\nu}{2}-1}. \tag{2.8}$$

(ii) If $a > 0$, $x \to -\infty$, then

$$f_{\nu,\sigma^2,a}(x) \sim \frac{1}{\sigma\Gamma(\frac{\nu}{2})}\left(\frac{\nu a}{2\sigma}\right)^{\frac{\nu}{2}} |x|^{-\frac{\nu}{2}-1} \exp\left\{-\frac{2a|x|}{\sigma^2}\right\}. \tag{2.9}$$

(iii) If $a < 0$, $x \to \infty$, then

$$f_{\nu,\sigma^2,a}(x) \sim \frac{1}{\sigma\Gamma(\frac{\nu}{2})}\left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}} x^{-\frac{\nu}{2}-1} \exp\left\{-\frac{2|a|x}{\sigma^2}\right\}. \tag{2.10}$$

(iv) If $a < 0$, $x \to -\infty$, then

$$f_{\nu,\sigma^2,a}(x) \sim \frac{1}{\sigma\Gamma(\frac{\nu}{2})}\left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}} |x|^{-\frac{\nu}{2}-1}. \tag{2.11}$$

2.2 Asymptotic Distributions for Extremal and Record Values

Let now $d = 1$ and $\{X_n, n \geq 1\}$ a sequence of i.i.d. random variables with common Student's t-distribution function and let $M_n = \max_{1\leq j\leq n} X_j$.

Proposition 2.5 *(i) If pdf of $\mathscr{L}(X_1)$ is f_{ν,σ^2}, then, as $n \to \infty$,*

$$\mathscr{L}\left((K_1 n)^{-\frac{1}{\nu}} M_n\right) \Rightarrow \Phi_\nu,$$

where "⇒" means weak convergence of probability laws, Φ_ν is the Fréchet distribution

$$\Phi_\nu(x) = \begin{cases} \exp\left\{-x^{-\nu}\right\}, & \text{if} \quad x > 0 \\ 0, \quad \text{if} \quad x \leq 0, \end{cases}$$

and

$$K_1 = \frac{\Gamma(\frac{\nu+1}{2})\sigma^\nu}{\nu\sqrt{\pi}\Gamma(\frac{\nu}{2})}.$$

(ii) If pdf of $\mathscr{L}(X_1)$ is $f_{\nu,\sigma^2,a}$, $a > 0$, then, as $n \to \infty$,

$$\mathscr{L}\left((K_2 n)^{-\frac{2}{\nu}} M_n\right) \Rightarrow \Phi_{\frac{\nu}{2}},$$

where

$$K_2 = \frac{2(\frac{\nu a}{2\sigma})^{\frac{\nu}{2}}}{\nu\sigma\Gamma(\frac{\nu}{2})}.$$

(iii) If pdf of $\mathscr{L}(X_1)$ is f_{ν,a,σ^2}, $a < 0$, then, as $n \to \infty$,

$$\mathscr{L}\left(\frac{2|a|}{\sigma^2}M_n - \ln n - \left(\frac{\nu}{2}+1\right)\ln\ln n + \ln K_3\right) \Rightarrow \Lambda,$$

where Λ is the Gumbel distribution

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in R^1,$$

and

$$K_3 = \frac{\nu^{\frac{\nu}{2}}\sigma^{\frac{\nu}{2}+3}}{2^{\nu+2}\Gamma(\frac{\nu}{2})}.$$

Proof (i) From Proposition 2.3 (i) with $d = 1$ and the l'Hospital's rule we have, as $x \to \infty$,

$$\int_x^\infty f_{\nu,\sigma^2}(u)\mathrm{d}u \sim \frac{c_{\nu,\sigma}}{\nu\sigma}\left(\frac{x}{\sigma}\right)^{-\nu} = K_1 x^{-\nu}, \tag{2.12}$$

where

$$c_{\nu,\sigma} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)\sigma}.$$

The statement (i) is standard for Pareto-like distributions (see, e.g., [1, 2]).

(ii) From Corollary 2.4 (i) and the l'Hospital's rule we have that, as $x \to \infty$,

$$\int_x^\infty f_{\nu,\sigma^2,a}(u)\mathrm{d}u \sim K_2 x^{-\frac{\nu}{2}} \tag{2.13}$$

and the conclusion is analogs to (i).

(iii) From Corollary 2.4 (iii) and the l'Hospital's rule we find that, as $x \to \infty$,

$$\int_x^\infty f_{\nu,\sigma^2,a}(u)\mathrm{d}u \sim \frac{\sigma}{2|a|\Gamma(\frac{\nu}{2})}\left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}} x^{-\frac{\nu}{2}-1}\exp\left\{-\frac{2|a|x}{\sigma^2}\right\}. \tag{2.14}$$

The statement (iii) is standard for gamma-like distributions (see, e.g., [1, 2]).

□

Now let us recall main results on limit theorems for record values in the sequences of i.i.d. random variables $\{X_n, n \geq 1\}$ with a common continuous distribution function F which will be applied to the case of Student's t-distributions.

The record times are $L_1 = 1$, $L_{n+1} = \min\left\{k : k > n, X_k > X_{L_n}\right\}$ for $n = 1, 2, \ldots$, and the record values are $R_n = X_{L_n}$, $n = 1, 2, \ldots$. Let $W(x) = -\log(1 - F(x))$ be the integrated hazard function and the associate distribution function $A(x) = 1 - e^{-\sqrt{W(x)}}$, $x \in R^1$. Let $l_{a,b}(x) = ax + b$, $a > 0$, $b \in R^1$, be a group of affine homeomorphisms of R^1 with the composition law

$$l_{a_1,b_1} * l_{a_2,b_2} = l_{a_1 a_2, a_1 b_2 + b_1},$$

the unit element $l_{1,0}$ and the inverse $l_{a,b}^{-1} = l_{a^{-1}, a^{-1} b}$.

The domain of attraction problem for record values using linear normalization was solved by Resnick (see [3] also [4]). It was proved that the class of all possible non-degenerated weak limit laws Q such that for suitable constants $a_n > 0, b_n \in R^1$, as $n \to \infty$,

$$\mathscr{L}\left(l_{a_n,b_n}^{-1}(R_n)\right) \Rightarrow Q$$

coincide with the class of laws $\Phi\left(-\log(-\log G(\cdot))\right)$, where Φ is a standard normal distribution and G is a l-max stable law, i. e. a non-degenerated distribution on R^1 such that for any $n \geq 2$ there exist constants $a_n > 0$, $b_n \in R^1$ satisfying

$$G^n(x) = G\left(l_{a_n,b_n}(x)\right), \quad x \in R^1.$$

As in the classical extreme value theory this class can be factorized into three linear types, saying that probability distributions F_1 and F_2 are of the same linear type it there exist constants $a > 0$, $b \in R^1$ such that

$$F_1(x) = F_2\left(l_{a,b}(x)\right), \quad x \in R^1.$$

In the classical case these types are generated by the Fréchet distribution Φ_γ, the Gumbel distribution Λ and the Weibull distribution

$$\Psi_\gamma(x) = \begin{cases} 1, \quad \text{if} \quad x \geq 0, \\ \exp\left\{-(-x)^\gamma\right\}, \quad \text{if} \quad x < 0, \quad \gamma > 0, \end{cases}$$

which correspond to generators of three types of the limiting laws for $\mathscr{L}\left(l_{a_n,b_n}(R_n)\right)$:

$$\tilde{\Phi}_\gamma(x) = \begin{cases} 0, \quad \text{if} \quad x \leq 0, \\ \Phi(\log x^\gamma), \quad \text{if} \quad x > 0, \quad \gamma > 0, \end{cases}$$

$$\tilde{\Psi}_\gamma(x) = \begin{cases} \Phi(\log(-x)^\gamma), & \text{if} \quad x < 0, \\ 1, \quad \text{if} \quad x \ge 0, & \gamma > 0, \end{cases}$$

and the standard normal distribution $\Phi(x)$, $x \in R^1$.

We say that F belongs to the record domain of attraction under linear normalization of the non-degenerated distribution Q ($F \in \mathrm{RDA}_l(Q)$ for short) if there exist constants $a_n > 0$ and $b_n \in R^1$ such that $\mathscr{L}(l^{-1}_{a_n,b_n}(R_n)) \Rightarrow Q$, as $n \to \infty$.

Duality theorem of Resnick says that $F \in \mathrm{RDA}_l(\tilde{\Phi}_\gamma) \Leftrightarrow A \in \mathrm{MDA}_l(\Phi_{\frac{\gamma}{2}})$, $F \in \mathrm{RDA}_l(\tilde{\Psi}_\gamma) \Leftrightarrow A \in \mathrm{MDA}_l(\Psi_{\frac{\gamma}{2}})$ and $F \in \mathrm{RDA}_l(\Phi) \Leftrightarrow A \in \mathrm{MDA}_l(\Lambda)$, where $\mathrm{MDA}_l(Q)$ denotes the maximum domain of attraction under linear normalization of the non-degenerated distribution Q (see, e.g., [3]). As a corollary we find that in the case of heavy-tailed distributions F the record values cannot have non-degenerate limiting distributions if we use linear normalization. Indeed, for the Pareto-like distributions F, satisfying, as $x \to \infty$,

$$1 - F(x) \sim Kx^{-\delta}, \quad \delta > 0,$$

the associate distributions A satisfy, as $x \to \infty$,

$$1 - A(x) \sim e^{-\sqrt{\delta \log x}}.$$

In this case $A \bar{\in} \mathrm{MDA}_l(\Phi_{\frac{\gamma}{2}}) \cup \mathrm{MDA}_l(\Psi_{\frac{\gamma}{2}}) \cup \mathrm{MDA}_l(\Lambda)$. This fact is an argument to consider limit theorems for the record values using power normalization.

Let

$$p_{\alpha,\beta}(x) = \alpha|x|^\beta \mathrm{sign} x, \quad \alpha > 0, \quad \beta > 0, \quad x \in R^1.$$

Observe that this class of functions form a group of homeomorphisms of R^1 with the composition law

$$p_{\alpha_1,\beta_1} * p_{\alpha_2,\beta_2} = p_{\alpha_1\alpha_2^{\beta_1},\beta_1\beta_2},$$

the unit element $p_{1,1}$ and the inverse

$$p^{-1}_{\alpha,\beta} = p_{\alpha^{-\beta^{-1}},\beta^{-1}}.$$

We say that F belongs to the record domain of attraction under power normalization of the non-degenerate distribution Q ($F \in \mathrm{RDA}_p(Q)$ for short) if there exist constants $\alpha_n > 0$, $\beta_n > 0$ such that, as $n \to \infty$, $\mathscr{L}\left(p^{-1}_{\alpha_n,\beta_n}(R_n)\right) \Rightarrow Q$.

A non-degenerate distribution function $\tilde{G}$ on R^1 is called p-max stable if for any $n \ge 2$ there exist constants $\tilde{\alpha}_n > 0$, $\tilde{\beta}_n > 0$ such that

$$\tilde{G}^n(x) = \tilde{G}(p_{\tilde{\alpha}_n,\tilde{\beta}_n}(x)), \quad x \in R^1.$$

Probability distributions F_1 and F_2 are of the same power type if there exist constants $\alpha > 0$, $\beta > 0$ such that $F_1(x) = F_2(p_{\alpha,\beta}(x))$, $x \in R^1$.

The class of non-degenerated limiting distributions for $\mathscr{L}(p^{-1}_{\alpha_n,\beta_n}(R_n))$, as $n \to \infty$, is equal to the class of law $\hat{\Phi}(-\log(-\log \hat{G}(\cdot)))$, where $\tilde{G}$ is a p-max stable law K, and is factorized to the six power types, generated by the distribution functions (see [5, 6]):

$$\hat{\Phi}_{1,\gamma}(x) = \begin{cases} 0, & \text{if} \quad x \leq 1, \\ \Phi(\gamma \log\log x), & \text{if} \quad x > 1, \quad \gamma > 0, \end{cases}$$

$$\hat{\Phi}_{2,\gamma}(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ \Phi(-\gamma \log|\log x|), & \text{if} \quad 0 < x < 1, \\ 1, & \text{if} \quad x \geq 1, \quad \gamma > 0, \end{cases}$$

$$\hat{\Phi}_{3,\gamma}(x) = \begin{cases} 0, & \text{if} \quad x \leq -1, \\ \Phi(-\gamma \log|\log|x||), & \text{if} \quad -1 < x < 0, \\ 1, & \text{if} \quad x \geq 0, \quad \gamma > 0, \end{cases}$$

$$\hat{\Phi}_{4,\gamma}(x) = \begin{cases} \Phi(-\gamma \log\log|x|), & \text{if} \quad x < -1, \\ 1, & \text{if} \quad x \geq -1, \quad \gamma > 0, \end{cases}$$

$$\hat{\Phi}_{5}(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ \Phi(\log x), & \text{if} \quad x > 0, \end{cases}$$

and

$$\hat{\Phi}_{6}(x) = \begin{cases} \Phi(-\log|x|), & \text{if} \quad x < 0, \\ 1, & \text{if} \quad x \geq 0. \end{cases}$$

There are the valid analog of Resnick's duality theorem and the principle of equivalent tails, which says that if continuous distribution functions F_1 and F_2 are such that $r(F_1) = r(F_2)$ and $1 - F_1(x) \sim 1 - F_2(x)$, as $x \uparrow r(F_1)$, then $F_1 \in \mathrm{RDA}_p(Q)$ if and only if $F_2 \in \mathrm{RDA}_p(Q)$ with the same normalizing constants, where $r(F) = \sup\{x : F(x) < 1\}$ and Q is a non-degenerate limiting distribution for record values using power normalization.

The following analog of classical R. von Mises theorem [7] holds true.

Theorem 2.6 [8]. *Assume that the integrated hazard function $W(x)$ is differentiable in some neighborhood of $r(F)$. Then:*

(i) if $r(F) = \infty$ and

$$\lim_{x\to\infty} \frac{W'(x)x\log x}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then $F \in RDA_p(\hat{\Phi}_{1,\gamma})$;

(ii) if $0 < r(F) < \infty$ *and*

$$\lim_{x \uparrow r(F)} \frac{W'(x)x \log\left(\frac{r(F)}{x}\right)}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then $F \in RDA_p(\hat{\Phi}_{2,\gamma})$*;*

(iii) if $r(F) = 0$ *and*

$$\lim_{x \uparrow 0} \frac{W'(x)x \log|x|}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then $F \in RDA_p(\hat{\Phi}_{3,\gamma})$*;*

(iv) if $r(F) < 0$ *and*

$$\lim_{x \uparrow r(F)} \frac{W'(x)|x| \log\left(\frac{x}{r(F)}\right)}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then $F \in RDA_p(\hat{\Phi}_{4,\gamma})$*;*

(v) if W *is twice differentiable in some neighborhood of* $r(F)$ *and*

$$\lim_{x \uparrow r(F)} W(x)\left(\frac{W''(x)}{(W'(x))^2} + \frac{1}{xW'(x)}\right) = 0, \tag{2.15}$$

then for $0 < r(F) \leq \infty$ $F \in RDA_p(\hat{\Phi}_5)$ *and for* $r(F) \leq 0$ $F \in RDA_p(\hat{\Phi}_6)$*.*

Proposition 2.7

(i) If pdf of F *is* f_{ν,σ^2}*, then* $F \in RDA_p(\hat{\Phi}_5)$*.*
(ii) If pdf of F *is* $f_{\nu,\sigma^2,a}$*,* $a > 0$*, then* $F \in RDA_p(\hat{\Phi}_5)$*.*
(iii) If pdf of F *is* $f_{\nu,\sigma^2,a}$*,* $a < 0$*, then* $F \in RDA_l(\Phi)$*.*

Proof

(i) From the principle of equivalent tails and (2.12) it is enough to check (2.15) with $r(F) = \infty$ and the integrated hazard function

$$W(x) = \nu \ln x - \ln K_1.$$

Indeed,

$$\frac{W''(x)}{(W'(x))^2} + \frac{1}{xW'(x)} = \frac{-\frac{\nu}{x^2}}{\left(\frac{\nu}{x}\right)^2} + \frac{1}{\nu} \equiv 0.$$

(ii) From the principle of equivalent tails and (2.13) it is enough to check (2.15) with $r(F) = \infty$ and the integrated hazard function

$$W(x) = \frac{\nu}{2} \ln x - \ln K_2.$$

Again we find that

$$\frac{W''(x)}{(W'(x))^2} + \frac{1}{xW'(x)} = \frac{-\frac{\nu}{2x^2}}{\left(\frac{\nu}{2x}\right)^2} + \frac{2}{\nu} \equiv 0.$$

(iii) From (2.14) and the principle of equivalent tails it is enough to consider the integrated hazard function

$$W(x) = \left(\frac{\nu}{2} + 1\right) \ln x + \frac{2|a|}{\sigma^2} x - \ln K_3,$$

where

$$K_3 = \frac{\sigma}{2|a|\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}}.$$

The corresponding associated distribution

$$\begin{aligned} 1 - A(x) &= \exp\left\{-\sqrt{\left(\frac{\nu}{2} + 1\right) \ln x + \frac{2|a|}{\sigma^2} x - \ln K_3}\right\} \\ &\sim \exp\left\{-\sqrt{\frac{2|a|}{\sigma^2} x}\right\}, \quad \text{as} \quad x \to \infty. \end{aligned}$$

Using again the principle of equivalent tails, Resnick's duality theorem and criteria from the classical extreme value theory we easily find that $A \in \mathrm{MDA}_l(\Lambda)$ and thus $F \in \mathrm{RDA}_l(\Phi)$. □

References

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