Strong Connectivity of Sensor Networks with Double Antennae

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Abstract. Inspired by the well-known Dipole and Yagi antennae we introduce and study a new theoretical model of directional antennae that we call *double antennae*. Given a set P of n sensors in the plane equipped with double antennae of angle ϕ and with dipole-like and Yagi-like antenna propagation patterns, we study the *connectivity* and *stretch factor* problems, namely finding the minimum range such that double antennae of that range can be oriented so as to guarantee strong connectivity or stretch factor of the resulting network. We introduce the new concepts of $(2,\phi)$-connectivity and ϕ-angular range $r_\phi(P)$ and use it to characterize the optimality of our algorithms. We prove that $r_\phi(P)$ is a lower bound on the range required for strong connectivity and show how to compute $r_\phi(P)$ in time polynomial in n. We give algorithms for orienting the antennae so as to attain strong connectivity using optimal range when $\phi \geq 2\pi/3$, and algorithms approximating the range for $\phi \geq \pi/2$. For $\phi < \pi/3$, we show that the problem is NP-complete to approximate within a factor $\sqrt{3}$. For $\phi \geq \pi/2$, we give an algorithm to orient the antennae so that the resulting network has a stretch factor of at most 4 compared to the underlying unit disk graph.

Keywords: Connectivity, Double Antenna, Range, Stretch Factor, Unit Disk Graph.

1 Introduction

Directional antennae are versatile transceivers which are widely used in wireless communication. With proper design they are known to improve overall energy consump-

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tion [12], enhance network capacity [9,15], improve topology control [8], and offer the potential for mitigating various security threats [10], just to mention a few applications. The motivation for our present study comes from the work in [2] which introduced the network connectivity problem for directional sensors and provided several algorithms for analyzing angle-range tradeoffs.

Dipole antennae (or *dipoles, for short*) are well-known basic antennae that are commonly used in radio communication. At their simplest, they consist of two straight collinear conductors of equal length separated by a small gap. Moreover, the radiation pattern for such antennae–indicating the strength of the signal in a given direction–in the *xy*-plane is usually depicted by two equal size closed curves known as *lobes*. Figure 1 illustrates the variability of the strength of the signal depending on the direction of the beam (see [14]). When the two lobes are not identical with the apex of one of the two lobes being closer to the origin than the other, the resulting antenna radiation pattern corresponds to a *Yagi antenna*, thus indicating that the antenna's transmission range is longer in one direction versus its opposite.

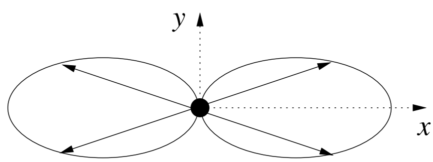


Fig. 1. Radiation pattern of a dipole antenna in the *xy*-plane

Motivated by the above, we introduce the following theoretical model of *Dipole-like* and *Yagi-like* antennae which we refer to as *double* antennae. These two concepts are captured in the following two geometric definitions.

Definition 1. *A* (ϕ, r)-double antenna *is an antenna with beamwidth or angle* ϕ *and radius* r *which can send and receive from either of the two sectors called beams depicted in Figure 2.*

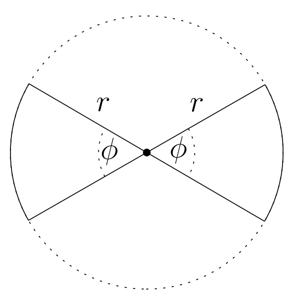


Fig. 2. Double antenna with beamwidth ϕ and range r

Definition 2. *More generally, a* (ϕ, r_1, r_2)-double *antenna is a double antenna with the range of one beam equal to* r_1 *and the range of the opposite beam* r_2*.*

Clearly, a (ϕ, r_1, r_1)-double antenna is also a $(\phi, \min\{r_1, r_2\})$-double antenna. Unless otherwise specified, in this paper, a double antenna refers to a (ϕ, r)-double antenna.

In this paper we are interested in the following two antenna orientation problems:

Problem 1 (Connectivity Problem with Double Antennae). Given a set P of n sensors in the plane each equipped with one double antenna with beamwidth $\phi \leq \pi$, determine the minimum antenna range, denoted by $\hat{r}_\phi(P)$, so that there exists an orientation of the antennae that induces a strongly connected transmission graph.

It is worth noting that for a sufficiently small angle ϕ, the problem is equivalent to the well-known bottleneck traveling salesman problem (BTSP) or Hamiltonian cycle that minimizes the longest edge. Therefore, a trivial upper-bound on the antenna range for $\phi \leq \pi$ of three times the optimal range can be computed by finding a Hamiltonian cycle with edge length bounded by three times the longest edge of the MST [3] [Problem C35.2-4] since the longest edge of the MST is also a lower bound for the orientation problem for any angle ϕ. However, for the BTSP a better analysis given in [13] shows that a 2-approximation can be obtained in polynomial time. Essentially, they proved that the lower bound for the BTSP is at least the longest edge of the 2-connected graph G that minimizes the longest edge. Thus, the 2-approximation is obtained easily since the square of any 2-connected graph is Hamiltonian [7].

Closely related to the orientation problem for attaining connectivity is the orientation problem to achieve constant stretch factor:

Problem 2 (Stretch Factor Problem with Double Antennae). Given a set P of n sensors in the plane each equipped with one double antenna with beamwidth $\phi \leq \pi$, determine the minimum antenna range so that there exists an orientation of the antennae that induces a c-directional spanner, where c is a constant.

1.1 Notation

We denote the Euclidean distance between points u and v by with $d(u,v)$. Let $UDG(P;r)$ denote the geometric graph (or straight line graph) such that P is the set of vertices and an edge $\{u,v\}$ exists if and only if $d(u,v) \leq r$. If r is normalized to be equal to 1 we simply denote the graph by $UDG(P)$. Throughout this paper the acronym UDG stands for Unit Disk Graph and the acronym MST for Euclidean Minimum Spanning Tree. Let $N_G(u)$ denote the set of neighbors of u.

Throughout this paper we assume that points are in general position, i.e., there do not exist three points that are collinear. It is well-known that the vertices of any MST have degree at most six since the angle that a vertex forms with two consecutive neighbors is at least $\pi/3$. However, when a vertex forms an angle of $\pi/3$ with every two consecutive neighbors, implies that at least three points are collinear. Hence, we assume that the max degree of the MST is five.

1.2 Related Work

The antenna orientation problem has been studied extensively since the problem was introduced by Caragianis et al [2]. for the single-directional antenna model. They proved that the connectivity problem for a single antenna (per sensor) is NP-complete with beamwidth less than $2\pi/3$ and gave an upper-bound on the range for a beamwidth greater than π which is tight when the angle is at least $8\pi/5$. In [6], the authors studied the connectivity problem when each sensor has k antennae. They proposed an algorithm for orienting the antennae so as to obtain a strongly connected graph with out-degree bounded by k and longest edge bounded by $2\sin(\frac{\pi}{k+1})$ times the optimal length required to attain connectivity. A useful survey of the connectivity problem is presented in [11].

In all the previous results, the lower bound on the antenna range was based on the longest edge of the MST. In this paper we will introduce the new concept of $(2,\phi)$-connectivity for a given angle ϕ so as to characterize the optimal lower bound required for connectivity.

The stretch factor problem for a single-directional antenna was studied for the first time in [4] for the particular cases of angles $\pi/2$ and $2\pi/3$. They proved that a range of 7 and 5 is always sufficient to create a 6-directional spanner and a 5-directional spanner (i.e., with stretch-factor 6 and 5), respectively. A more comprenhensive result is given in [1] where the authors gave an upper bound for all angles.

1.3 Results of the Paper

We study the antenna orientation problems for connectivity and stretch-factor in the double antenna model. In Section 2 we introduce the new concepts of $(2,\phi)$-connectivity and ϕ-angular range $r_\phi(P)$ so as to characterize the optimality of our algorithms. Furthermore, we prove that $r_\phi(P)$ is the lower bound for the connectivity problem and show how to compute $r_\phi(P)$ in polynomial time. We prove tight bounds on the optimal angle necessary to cover all neighbours of a node in a MST in Section 3. Our results for the connectivity problem, including an NP-completeness proof when the beamwidth $\phi < \pi/3-\varepsilon$ and an optimal algorithm for the case $\phi > 2\pi/3$ are presented in Section 4. In Section 5 we give a linear time algorithm for the stretch factor problem that orients the antennae of beamwidth at least $\pi/2$ so as to obtain a 4-directional spanner. Finally, we conclude in Section 6 and present some open problems. Our main results, complexities, and resulting angle/range tradeoffs for n sensors in the plane are summarized in Table 1.

2 Lower Bounds

In this section we characterize the lower bounds for the orientation problem for connectivity with double antennae. Consider an antenna orientation for a given antenna beam width ϕ and optimal range $\hat{r}_\phi(P)$ on a set of points P that induces a strongly connected graph on P. Given $u \in P$, let $v \in P$ be a point not in one of the beams of u's antenna such that $d(u,v) \leq \hat{r}_\phi(P)$. The main observation is that a path between u and v must exist such that each edge in the path is of length at most $\hat{r}_\phi(P)$. This implies that the unit disk graph $UDG(P;\hat{r}_\phi(P)) \setminus \{\{u,v\}\}$ is connected. We use this observation to obtain a lower bound on $\hat{r}_\phi(P)$.

Table 1. Results for the double antenna connectivity and stretch-factor problems on n sensors in the plane and for antennae of beam width ϕ

Double Antenna Angle	Approximation Ratio	Complexity	Stretch Factor
$\frac{2\pi}{3} \leq \phi < \pi$	1	$O(n^2)$	-
$\frac{\pi}{2} \leq \phi \leq \frac{2\pi}{3}$	$\sqrt{3}$	$O(n \log n)$	-
$\frac{\pi}{2} \leq \phi < \pi$	$4\sin(\frac{\pi}{4}+\frac{\phi}{2})$	$O(n)$	4
$0 \leq \phi < \frac{\pi}{2}$	3	$O(n \log n)$	-
$\phi < \frac{\pi}{3} - \varepsilon$	$\sqrt{3} - \varepsilon$	NP-Complete	-

First we define a double antenna orientation "relative to" a given angle γ. Let γ be a given angle oriented with its right edge (in counterclockwise direction) on the axis as depicted in Figure 3; $u(\phi;\gamma)$ denotes the orientation of the double antenna of beamwidth ϕ at u starting from the right edge (in counterclockwise direction) of γ (see Figure 3). Given the graph $UDG(P;r)$, and an orientation γ, we define $E_\phi(P,r,u,\gamma)$ as the subset of edges incident to u which lie outside the two beams of $u(\phi;\gamma)$ (see Figure 3).

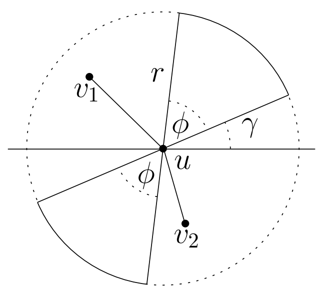


Fig. 3. $u(\phi;\gamma)$ denotes the double antenna with beam-width ϕ and orientation relative to γ. The edges $\{u,v_1\}$ and $\{u,v_2\}$ are in $E_\phi(P,r,u,\gamma)$.

Definition 3 (Set of Edges $E_\phi(P,r,u)$.). *Let $E_\phi(P,r,u)$ denote any set $E(P,r,u,\gamma)$, for $0 \leq \gamma \leq \pi$, such that the graph $UDG(P;r) \setminus E(P,r,u,\gamma)$ attains the minimum possible number of connected components.*

The following definition introduces the concept of $(2,\phi)$-connectivity of a UDG.

Definition 4 ($(2,\phi)$-connectivity.). *Let P be a set of points in the plane. We say that for a given radius r, the graph $UDG(P;r)$ is $(2,\phi)$-connected if for any vertex $u \in P$, $UDG(P;r) \setminus E_\phi(P,r,u)$ is connected.*

Observe that when ϕ is sufficiently small so that each antenna can only cover one vertex in its beams, the concept of $(2,\phi)$-connectivity is equivalent to the well-known concept of 2-connectivity.

Definition 5 (ϕ-angular radius.). *We define the ϕ-angular radius as the minimum radius, denoted by $r_\phi(P)$, such that $UDG(P; r_\phi(P))$ is $(2,\phi)-$connected.*

Now we will prove that the ϕ-angular radius is a lower bound for the orientation problem with double antennae. Due to space constraints the proof is omitted.

Theorem 1. *For any set P of points, $r_\phi(P) \leq \hat{r}_\phi(P)$.*

The following theorem gives a simple algorithm to compute $r_\phi(P)$ in polynomial time.

Theorem 2. *Given a set P of n points in the plane in general position and an angle $\phi \geq 0$, there is an algorithm that computes $r_\phi(P)$ in $O(n^2)$ time.*

Proof. Let T be an MST on P and let r be the length of the longest edge in T. Let $S \subseteq P$ be such that for each vertex $u \in S$ the graph $T \setminus E_\phi(P,r,u)$ is not connected. For $u \in S$, let $r_\phi(P,u)$ be the minimum range such that $UDG(P; r_\phi(P,u)) \setminus E_\phi(P, r_\phi(P,u))$ is connected. Clearly, $r_\phi(P) = \max_{u \in S}(r_\phi(P,u))$. We will determine $r_\phi(P,u)$ independently for every vertex $u \in S$.

Consider a vertex $u \in S$ and let $G = T$. We add to G the shortest edge $\{v,w\}$ that connects two distinct components of $G \setminus E_\phi(P,r,u)$. Update r to be the longest edge in G, and repeat the above procedure until $G \setminus E_\phi(P,r,u)$ is connected. Since the removal of the longest edge of G will disconnect the graph $G \setminus E_\phi(P,r,u)$, it follows that $r_\phi(u)$ equals the length of the longest edge in G.

It remains to analyze the complexity of the algorithm. Let $u_0, u_1, \ldots, u_{d_G(u)}$ be the neighbors of u. To find $E_\phi(P,r,u)$ we check for each neighbor u_i of u which orientation $G \setminus E_\phi(P,r,u,\angle(u_i u u_0))$ leaves the minimum number of components. We will show that the degree of u never exceeds five. Since T is an MST the max degree of u is five. However, new edges can increase the degree of u. Assume that $\{u,v\}$ is added to G. Since $\{u,v\}$ is the smallest edge that connects two distinct components of $G \setminus E_\phi(P,r,u)$ the angle that $\{u,v\}$ forms with the neighbors of u is at least $\pi/3$. Therefore, the max degree of u is bounded by five since P is in general position.

Next we show that the algorithm can be implemented in $O(n^2)$ time. First consider the Delaunay Triangulation on P and sort the edges in a list L. Such a construction takes $O(n\log(n))$ time [5]. Further, L can be computed in $O(n\log(n))$ time since the number of edges is linear on the number of vertices. It is well-known that the Grabriel Graph on P is a subgraph of the Delaunay Triangulation on P, i.e., each edge $\{v,w\} \in L$, $D(v; d(v,w)) \cap D(w; d(w,v)) = \emptyset$ (where $D(x;r)$ denotes the open disk centered at x with radius r). Therefore, for a given u we can compute the shortest edge $\{u,v\}$ connecting two components in $G \setminus E_\phi(P,r,u)$ in $O(n)$ time since $\{u,v\} \in L \cup G^k(N_G(u))$ where $G^k(N_G(u))$ represents the complete graph of $N_G(u)$ and $|N_G(u)| \leq 5$. The theorem follows, since each vertex has at most 5 connected components and $|S| \leq |P|$.

3 Covering Neighbors in MST with Double Antennae

Given an MST of a set of points P and a vertex $u \in V$ of degree $k \leq 5$ we will characterize the beamwidth required by an antenna at u to cover all the neighbors of u. Recall that the MST on the set of points has maximum degree 5 and the angle between any two adjacent edges is at least $\pi/3$. Let $u_0, u_1, ..., u_{k-1}$ be the neighbors of u in G with corresponding angles $\alpha_i = \angle(u_i u u_{i+1})$ in counterclockwise order, where $k \leq 5$. We study separately the cases $k = 5,4,3,2$ and in each case, we give the value of α, the minimum beamwidth of double antenna that is required to ensure that all neighbours of u fall within one of the beams of the antenna. Clearly any beamwdith $\phi \geq \alpha$ is always sufficient to cover all neighbours. Due to space constraints the proofs of Lemmas 2,3 and 4 are omitted.

Lemma 1. *Let $k = 5$ and assume wlog that $\alpha_0 + \alpha_1$ is the smallest sum of two consecutive angles. Then, a double antenna of beamwidth $\alpha = \alpha_0 + \alpha_1$ is always sufficient and necessary to cover all five neighbors in the MST. Furthermore, $\alpha \in [\frac{2\pi}{3}, \frac{4\pi}{5}]$.*

Proof. Place the antenna as depicted in Figure 4 so that u_0 is on the edge of the antenna and u_1, u_2 are within the beam of the antenna. Since $\alpha_0, \alpha_1 \geq \pi/3$, we have $\alpha \geq 2\pi/3$. Therefore, the "dead" sectors of the antenna are of angle at most $\pi/3$. Since $\alpha_2, \alpha_4 \geq \pi/3$, neither u_3 nor u_4 can lie within the dead sectors of the antenna. Since three neighbors of u must be in the same side of the antenna beam, α is always necessary. Observe that $\frac{2\pi}{3} \leq \alpha \leq \frac{4\pi}{5}$ since all the angles are at least $\pi/3$.

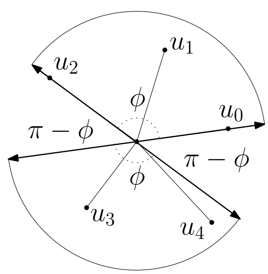


Fig. 4. Double antenna at u of degree 5

Lemma 2. *Let $k = 4$. Assume wlog that α_0 is the smallest angle and that $\alpha_1 \leq \alpha_3$. Then a double antenna of beamwidth α is always necessary and sufficient to cover all four neighbours in the MST, where $\alpha = \pi - \alpha_1$ if $\alpha_3 \geq \pi - \alpha_0$ and $\alpha = min(\alpha_2, \pi - \alpha_0)$ otherwise. Furthermore, $\alpha \in [\frac{\pi}{3}, \frac{2\pi}{3}]$.*

Lemma 3. *Let $k = 3$. Assume wlog that $\alpha_0 \leq \alpha_1 \leq \alpha_2$. Then a double antenna of beamwidth α is always sufficient and necessary to cover the three neighbors in the MST, where $\alpha = max\{\alpha_0, \pi - \alpha_1\}$. Furthermore, $\alpha \leq \frac{2\pi}{3}$.*

Lemma 4. *Let $k = 2$. Assume wlog that $\alpha_0 \leq \alpha_1$. Then a double antenna of beamwidth α is always sufficient and necessary to cover the two neighbors in the MST, where $\alpha = min\{\alpha_0, \pi - \alpha_0\}$. Furthermore, $\alpha \leq \frac{\pi}{2}$.*

4 Connectivity with Optimal Range

In this section we first show that the double antenna orientation problem is NP-complete for antenna angles less than $\frac{\pi}{3}$. In contrast, we will show that when the antenna beamwidth is sufficiently large, we can solve the orientation problem with optimal range. Due to space constraints the proofs of Theorems 3 and 4 are omitted.

Theorem 3. *For n sensors in the plane and $\phi < \frac{\pi}{3}$, it is NP-complete to approximate the optimal range $\hat{r}_\phi(P)$ to within a multiplicative factor of $\sqrt{3}$.*

Now we will show that when the antenna beamwidth is sufficiently large it is trivial to achieve optimal range as the next theorem shows.

Theorem 4. *Given an angle $\phi \geq \frac{4\pi}{5}$, there is an algorithm which for any set P of points in the plane in general position, orients double antennae of beamwidth ϕ using optimal antenna range so that the resulting graph is strongly connected.*

Next we will show how to achieve non-trivial optimal range with the use of the $(2,\phi)$-connectivity and ϕ-angular radius as a lower bound for an antenna beamwidth of at least $\frac{2\pi}{3}$.

Theorem 5. *Given an angle $\phi \geq \frac{2\pi}{3}$, there is an algorithm which for any set P of points in the plane in general position, orients double antennae of beamwidth ϕ using optimal antenna range so that the resulting graph is strongly connected. Furthermore, the algorithm can be implemented to run in $O(n^2)$ time.*

Proof. Let T be an Euclidean MST on P. Consider the set S of vertices $u \in T$ such that a double antenna of beamwidth ϕ cannot cover all the neighbors of u. From Lemmas 1-4, S consists only of vertices of degree five in T such that the angle that is formed with any three consecutive neighbors is greater than ϕ.

We will construct a strongly connected digraph such that every vertex in S has out-degree four and the angle that each vertex forms with two consecutive out-going edges is at least $\pi/3$. Furthermore, we will show that no new vertices of out-degree five will appear. Finally, all edges in the digraph will have length at most $r_\phi(P)$. Thus, the theorem follows from Lemmas 1-4 and Theorem 1.

Let $\overrightarrow{G}$ be the strongly connected directed graph obtained from T by replacing every edge in T by two opposing directed edges. Let G be the undirected graph of $\overrightarrow{G}$. We will include each vertex $u \in S$ in at least one cycle as follows:

Let u be any vertex in S and let $\{v,w\}$ be the shortest edge connecting two components of $G \setminus \{u\}$. Clearly, $d(v,w) \leq r_\phi(P,u)$ since at least one neighbor of u is not within its antenna beam of angle ϕ. Add $\{v,w\}$ to G to form a cycle C_u; see Figure 5. We "orient" C_u in $\overrightarrow{G}$ along any one direction (all the arcs in C_u in the opposite direction are removed). This process does not break the strong connectivity of $\overrightarrow{G}$. Finally, if $|C_u| > 3$, we remove from S every vertex that is in C_u. However, if $|C_u| = 3$, we only remove u from S. We repeat this process until S is empty.

Let $\mathcal{C}$ be the set of cycles that are formed with the addition to T of the new edges. We will prove that after adding each new cycle $C_u \in \mathcal{C}$ of hop-length greater than 3 the angle

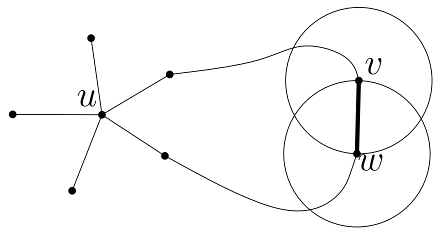


Fig. 5. The shortest edge $\{v,w\}$ connecting two components of $G \setminus \{u\}$ forms a cycle

that any two points form with a common neighbor in G is at least $\pi/3$. Indeed, since C_u is formed with the smallest edge $\{v,w\}$ connecting two components, $D(v;d(v,w)) \cap D(w;d(v,w))$ is empty. Therefore, the min angle that $\{v,w\}$ forms with the neighbors of v and w is at least $\pi/3$. Furthermore, since points are in general position the degree of the vertices in C_u is at most five. Therefore, the out-degree in $\overrightarrow{G}$ of every vertex in C_u is at most four. (At least one in-going edge in $\overrightarrow{G}$.)

Now we consider a cycle C_u of hop-length three. Let $\{v,w\}$ be the shortest edge connecting two components of $G \setminus \{u\}$. Observe that both $D(u;d(u,v)) \cap D(v;d(u,v))$ and $D(u;d(u,w)) \cap D(w;d(u,w))$ are empty. However, $D(v;d(v,w)) \cap D(w;d(v,w))$ contains u. Therefore, the min angle that each edge incident to u,v and w forms with an edge of the triangle uvw is at least $\pi/3$. Thus, we reduce the out-degree of u to at most four since the points are in general position. Moreover, the out-degree of v and w remains the same. However, since v and w are not removed from S when C_u is created, they are included in distinct cycles provided that they are in S.

As in the proof of Theorem 2, we can show that the construction of $\overrightarrow{G}$ can be implemented in $O(n^2)$ time. Indeed, the edges to be added are always edges of either the Delaunay Triangulation on P or the closest neighbors of each vertex. Thus, the addition of each edge takes time $O(n)$. The theorem follows since $|S| = O(n)$ and the orientation of the antennae takes time $O(1)$.

For the next theorem we use the main result of [6][Theorem 1]. For convenience we state this theorem without proof.

Theorem 6 **(k-Antennae Orientation [6].).** *Consider a set S of n sensors in the plane and suppose each sensor has k, $1 \leq k \leq 5$, directional antennae with any angle $\phi \geq 0$. Then the antennae can be oriented at each sensor so that the resulting spanning graph is strongly connected and the range of each antenna is at most $2\sin\left(\frac{\pi}{k+1}\right)$ times the optimal. Moreover, given a MST on the set of points the spanner can be constructed with additional $O(n)$ overhead.*

The following theorem shows that for any angle $\phi \geq \pi/2$ we can always construct a strongly connected transmission network with longest edge bounded by $\sqrt{3}$ times the longest edge of the MST.

Theorem 7. *There is an algorithm which for any set of n points in the plane, orients double antennae of beamwidth $\frac{\pi}{2} \leq \phi \leq \frac{2\pi}{3}$ using range bounded by $\sqrt{3}$ times the optimal range so that the resulting graph is strongly connected.*

Proof. Let $\overrightarrow{G}$ be the strongly connected digraph with out-degree 2 (i.e., using $k = 2$ antennae) and range bounded by $\sqrt{3}$ obtained from Theorem 6. Since the out-degree of $\overrightarrow{G}$ is bounded by two, from Lemma 4 a double antenna with angle at most $\pi/2$ covers the two out-going edges. This completes the proof of the theorem.

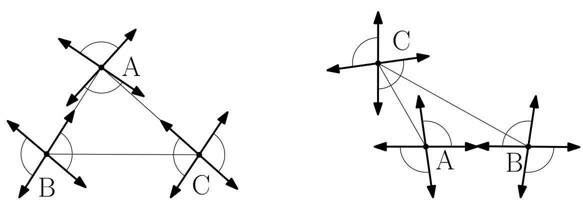
5 Stretch Factor

In this section, we consider the stretch factor problem, that is, finding an orientation of double antennae of minimum possible range that induces a c-directional spanner, for some constant c. That is, given a set P of points in the plane such that $UDG(P,1)$ is connected, we wish to replace omnidirectional antennae of range 1 with double antennae of angle ϕ and range r such that for any edge in $UDG(P,1)$, there is a path in the resulting strongly connected digraph of length at most c for some constant c. The basic idea is to partition the set of points into triples such that in each triple, there is at most one pair of vertices that is not connected in the UDG. For each triple $\{A,B,C\}$ we need to determine the antenna range r required so that there is an orientation of three directional antennae placed at A,B,C, respectively, so that every point within distance two of at least one of the points A,B,C is also within "directional antenna range" of radius r from at least one of these three points.

First we prove a basic lemma concerning double antenna orientation of three points A,B,C in the plane. The antenna orientation will depend on the largest angle, say $\alpha = \angle(BAC)$, that the three points form.

Lemma 5. *Consider three points A,B,C in the plane forming a triangle. Three identical double antennae of beamwidth $\phi \geq \pi/2$ can be oriented so as to cover the whole plane.*

Proof. We consider two cases depending on the size of α.



(a) $\alpha \leq \phi$. **(b)** $\alpha > \phi$.

Fig. 6. Orientation of three double antennae

Case $\alpha \leq \phi$. Without loss of generality asume that BC is horizontal and A is above BC. Orient the antennae as depicted in Figure 6a so that the antenna covers the triangle and the wedge of the antennae at B and C are on BC and CA respectively. Observe that the antennae cover the "whole plane" since each angle of the triangle is always covered.
Case $\alpha > \phi$. Without loss of generality asume that AB is the second smallest edge in the triangle, AB is horizontal and C is above AB. Orient the antennae as depicted in

Figure 6b so that the one antenna wedge of C is vertical and the wedge of the antennae at A and B are on AB. To prove that the orientation covers "the whole plane", observe that the antennae at A and B only leave a black (i.e., uncovered) corridor in the lower half-plane determined by AB. However, the antenna at C covers the black corridor. This completes the proof of the lemma.

We now consider double antennae of finite range. The following results hold for double antennae of range r, but can also be shown to hold for the weaker model of $(\phi, r, 2)$-double antennae. The proof of the following lemma is omitted due to space constraints.

Lemma 6. *Let A,B,C be three points such that $d(A,B) \leq 1$ and $d(A,C) \leq 1$. Assume $\frac{\pi}{2} \leq \phi \leq \pi$. We can orient three $(\phi, r, 2)$-double antennae (Yagi-like antennae) of beam width ϕ at A,B,C so that every point at distance at most two from one of these points is covered by one of the three antennae, where $r \leq 4\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$.*

Theorem 8. *Given $\frac{\pi}{2} \leq \phi < \pi$, there is an algorithm which for any connected $UDG(P)$ on a set P of points in the plane, orients $\left(\phi, 4\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right), 2\right)$-double antennae so that the resulting graph has stretch factor four. Furthermore, it can be done in linear time.*

Proof. Let $\mathcal{T}$ be any partition of the UDG with the maximal number of triples such that every triangle has two edges of length at most one. It is easy to see that such a partition can be constructed in linear time. For each triangle T in $\mathcal{T}$, we orient the antennae at T as shown in Lemma 5 with range $4\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$ and the antenna of each remaining sensor toward its nearest triangle. Observe that the closest triangle is at distance at most two. Let $\overrightarrow{G}$ be the strongly connected network induced by the antennae. We will prove that for each edge $\{u,v\} \in UDG(P)$, there is a directed path P from u to v and a directed path P' from v to u of hop-length no more than 4 hops. Let T and T' be in two different triangles in the partition $\mathcal{T}$.

- $u,v \in T$. Then $|P| \leq 2$ and $|P'| \leq 2$.
- $u \in T$ and $v \in T'$. Since $d(u,v) \leq 1$, v is in the coverage area of T. Therefore, u can reach v in at most three hops and $|P| \leq 3$. A similar argument shows that $|P'| \leq 3$.
- At least one of u or v is not in any triangle of $\mathcal{T}$. Assume without loss of generality that u is not in a triangle. Observe that there exists a triangle T at distance at most two from u. Otherwise, $\mathcal{T}$ is not maximal. Therefore, u can reach v through T in at most four hops, i.e., $|P| \leq 4$. Similarly, we can prove that $|P'| \leq 4$.

This completes the proof of the theorem.

6 Conclusion

In this paper we considered algorithms for orienting antennae with Dipole-like and Yagi-like antenna propagation patterns so as to attain optimal connectivity and stretch factor of the resulting directed network. It would be interesting to improve the bound in our connectivity results for the range $[\pi/3, \pi/2]$, and to prove better bounds for the stretch factor problem either in terms of range or in terms of stretch factor.

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