Set Partitions with No m-Nesting

Marni Mishna and Lily Yen

Abstract A partition of $\{1,\ldots,n\}$ has an m-nesting if it contains at least m disjoint blocks, and a subset of $2m$ points $i_1 < i_2 < \cdots < i_m < j_m < j_{m-1} < \cdots < j_1$, such that i_l and j_l are in the same block for all $1 \leq l \leq m$, but no other pairs are in the same block. In this note, we use generating trees to construct the class of partitions with no m-nesting, determine functional equations satisfied by the associated generating functions, and generate enumerative data for $m \geq 4$.

Keywords Set partition • Nesting • Pattern avoidance • Generating tree • Algebraic kernel method • Coefficient extraction • Enumeration

1 Introduction

Graphic representations of set partitions can contain various patterns and shapes. One particular pattern, known as an m-nesting, resembles a rainbow, for example. In this work we address the enumeration of set partitions that avoid m-nestings. These results are in the context of recent studies of other combinatorial objects that avoid similar or related patterns. We are particularly motivated by the study of protein folding [7] where such patterns arise in the molecular bonds and their presence has strong consequences on the geometry of the protein.

Our strategy parallels a recent generating tree approach used by Bousquet-Mélou to enumerate a family of pattern avoiding permutation classes [3]. A novel feature

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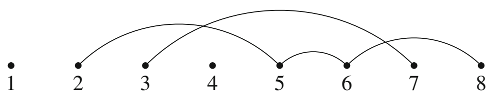
of this approach is that the length of the label in the generating tree is related to the length of the pattern avoided. Thus, the resulting expressions for generating functions are generic, and expressed in terms of m. The generating tree permits direct access to new enumerative data for set partitions avoiding m-nestings for some $m > 4$, and we present the equations as a starting point for further analysis.

1.1 Notation and Definitions

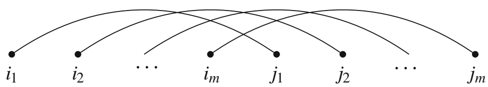
A set partition π of $[n] := \{1, 2, 3, \ldots, n\}$, denoted by $\pi \in \Pi_n$, is a collection of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. The number of set partitions of $[n]$ into k blocks is denoted $S(n, k)$, and is known as a Stirling number of the second kind. The total number of partitions of $[n]$ is the *Bell* number $B_n = \sum_k S(n, k)$. We represent π by a graph on the vertex set $[n]$ whose edge set consists of arcs connecting elements of each block in numerical order. Such an edge set is called the *standard representation* of the partition π, as seen in [6]. For example, the standard representation of

$$1|2\,5\,6\,8|3\,7|4$$

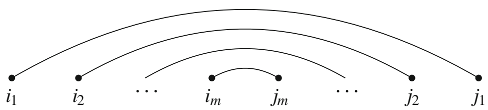
is given by the following graph with edge set $\{(2, 5), (5, 6), (6, 8), (3, 7)\}$:



With this representation, we can define two classes of patterns: crossings and nestings. An *m-crossing* of π is a collection of m edges $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$ such that $i_1 < i_2 < \cdots < i_m < j_1 < j_2 < \cdots < j_m$. Using the standard representation, an m-crossing is drawn as follows:



Similarly, we define an *m-nesting* of π to be a collection of m edges (i_1, j_1), $(i_2, j_2), \ldots, (i_m, j_m)$ such that $i_1 < i_2 < \cdots < i_m < j_m < j_{m-1} < \cdots < j_1$. This is drawn:



A partition is m-noncrossing if it contains no m-crossing, and it is said to be m-nonnesting if it contains no m-nesting.

1.2 Context and Plan

Chen, Deng, Du, Stanley and Yan in [6], and independently Krattenthaler in [8], gave a non-trivial bijective proof that m-noncrossing partitions of $[n]$ are equinumerous with m-nonnesting partitions of $[n]$, for all values of m and n. A straightforward bijection with Dyck paths illustrates that 2-noncrossing partitions (or simply, noncrossing partitions) are counted by Catalan numbers. Bousquet-Mélou and Xin in [4] showed that the sequence counting 3-noncrossing partitions is P-recursive, that is, satisfies a linear recurrence relation with polynomial coefficients. Indeed, they determined an explicit recursion, complete with solution and asymptotic analysis. They further conjectured that m-noncrossing partitions are not P-recursive for all $m \geq 4$. Certainly, the limit as m goes to infinity is not D-finite, since Bell numbers are well known not to be P-recursive because of the composed exponentials in the generating function $B(x) = e^{e^x - 1}$ (see Example 19 of [2]). If it turns out that m-noncrossing partitions do have a D-finite generating function, then we have a very interesting refinement of a non-D-finite class.

Since m-noncrossing partitions of $[n]$ and m-nonnesting partitions of $[n]$ are equinumerous, we study m-nonnesting partitions in this paper and show how to generate the class using generating trees, and how to determine a recursion satisfied by the counting sequence for m-nonnesting partitions.

Our approach is an adaptation of Bousquet-Mélou's recent work on the enumeration of permutations with no long monotone subsequence in [3]. She combined the ideas of recursive construction for permutations via generating trees and the algebraic kernel method to determine and solve functional equations with multiple catalytic variables.

In Sect. 2, we employ Bousquet-Mélou's generating tree construction to find functional equations satisfied by the generating functions for set partitions with no m-nesting. The resulting equations, though similar to the equations arising in [3], have a key structural difference which resists a similar treatment of the algebraic kernel method followed by a constant term extraction as used by Bousquet-Mélou in [3]. However, the process does yield the result for nonnesting set partitions counted by the Catalan numbers. We refer interested readers to [9] for the processing of functional equations in the spirit of [3].

Using our constructions we generate new enumerative data for $m > 4$, discuss the limiting factors in data generation, and assess the current state of recurrences and explicit forms.

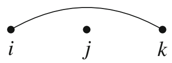
2 Generating Trees and Functional Equations

The generating tree construction for the class of m-nonnesting partitions is based on a standard generating tree description of partitions, and the constraint is incorporated using a vector labelling system. The generating tree construction has an immediate translation to a functional equation with m-variate series.

2.1 A Generating Tree for Set Partitions

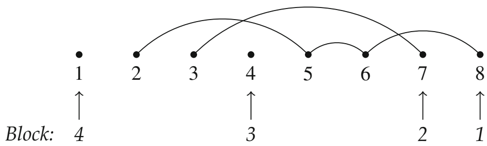
Let π be a set partition. Define $\mathrm{ne}(\pi)$ to be the maximal i such that π has an i-nesting, also called the *maximal nesting number* of π, and let $\Pi_n^{(m)}$ be the set of partitions of $[n]$ for $n \geq 0$ (where $n = 0$ means the empty partition) with $\mathrm{ne}(\pi) \leq m$, thus $(m+1)$-nonnesting. We define the union $\Pi^{(m)} = \cup_n \Pi_n^{(m)}$.

Note that an arc over a fixed point is not a 2-nesting, but a 1-nesting:



We next describe how to generate all set partitions via generating trees in the fashion of [2]. First, order the blocks of a given partition, π, by the maximal element of each block in descending order.

Example 1. The first block of 1|2 5 6 8|3 7|4 is 2 5 6 8; the second block is 3 7; the third block is singleton 4; and 1 is the last block. Using the standard representation,



we number the blocks in descending order (from the right to the left) according to the maximal element in each block (that is, the rightmost vertex of each block).

With the order of blocks thus defined, we warm up by generating all set partitions without nesting restriction first. Figure 1 contains the generating tree for all set partitions, in addition to the generating tree for the number of children of each node from the tree of set partitions to indicate how enumeration can be facilitated.

1. Begin with $\emptyset$ as the top node of the tree. It has only one child, so the corresponding node in the tree for the number of children is labelled 1.
2. To produce the $n+1$st level of nodes, take each set partition at the nth level, and either add $n+1$ as a singleton, or join $n+1$ to block j for each $1 \leq j \leq k$ if the set partition has k blocks.

Summarizing the description above in the notation of [2], we recall that the rewriting rule of a generating tree is denoted by:

$$[(s_0), \{(k) \rightarrow (e_{1,k})(e_{2,k}) \ldots (e_{k,k})\}],$$

where s_0 denotes the degree of the root, and for any node labelled k, that is, with k descendants, the label of each descendent is given by $(e_{j,k})$ for $1 \leq j \leq k$. Thus,

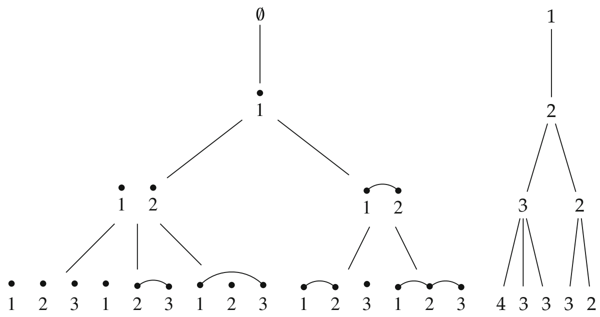


Fig. 1 Generating tree for set partitions and its corresponding generating tree of the number of children

the class of set partitions has a generating tree of labels given by $[(1) : (k) \to (k+1)(k)^{k-1}]$.

2.2 A Vector Label to Track Nestings

The generating tree of set partitions generates all set partitions π graded by n, the size of π, but it does not keep track of nesting numbers. Also note that the number of children of π is one more than the number of blocks of π. Let us now address nestings.

Fix m. In order to keep track of nesting numbers, we need to define the *label* of $\pi \in \Pi^{(m)}$. To identify the position of a nesting, we consider the relative position of the smallest vertex incident to the nesting. Thus, the *rightmost* j-nesting is the set of j edges forming a j-nesting pattern such that its minimal incident vertex is greater than, or equal to the minimal vertex incident to all the other j-nestings. If one vertex is common to two j-nestings, we consider the second smallest incident vertex, and so on. Roughly, our labels keep track of the number of blocks to the right of a j-nesting that might potentially become a j-nesting based on how the next edge is added. Any edge added that affect nestings to the *left* of the right most j-nesting, will necessarily create a $j+1$ nesting because it will create an arc overtop of the rightmost j-nesting.

Definition 1. Define the label of a partition, $L(\pi) = (a_1(\pi), a_2(\pi), \ldots, a_m(\pi))$, or in short, $L(\pi) = (a_1, a_2, \ldots, a_m)$ as follows. For $1 \le j \le m$,

$$a_j(\pi) = \begin{cases} 1+ \text{ number of blocks in } \pi, & \text{if } \pi \text{ is } j\text{-nonnesting}, \\[2ex] \begin{array}{l} 1+ \text{ number of blocks ending to the right of} \\ \text{the smallest vertex in the rightmost } j\text{-nesting} \end{array} & \text{otherwise.} \end{cases}$$

Example 2. To continue the example, let $\pi = 1|2\,5\,6\,8|3\,7|4$ and suppose $m = 3$. Then $L(1|2\,5\,6\,8|3\,7|4) = (3, 4, 5)$ for the following reasons. The rightmost

1-nesting is the edge with largest vertex endpoint: $(6, 8)$. Hence, $a_1(\pi) = 3$ because blocks 1 and 2 end to the right of vertex 6. The rightmost 2-nesting is the set of edges $\{(5, 6), (3, 7)\}$ hence $a_2(\pi) = 4$ because 3 blocks end to the right of vertex 3. Finally, $a_3(\pi) = 5$ because the diagram has no 3-nesting, and is comprised of 4 blocks. Note that in this convention, the empty set partition has label $(1, 1, \dots, 1)$, since it has no nestings and no blocks.

A set partition in $\Pi^{(m)}$ always has a_m children. This is one more than the number of blocks, if there is no m-nesting (and hence there is no risk that adding an edge will create an $m + 1$-nesting). Otherwise, it indicates more than the number of blocks to which you can add an edge without creating an $m + 1$-nesting. The label of a set partition is sufficient to derive the label of each of its children, and this process is described in the next proposition. Also, remark that the label is a non-decreasing sequence, since the rightmost j-nesting either contains the rightmost $j - 1$ nesting or is to the left of it.

Proposition 1 (Labels of children). *Let π be in $\Pi_n^{(m)}$, the set of set partitions on $[n]$ avoiding $m+1$-nestings, and suppose the label of π is $L(\pi) = (a_1, a_2, \dots, a_m)$. Then, the labels of the a_m set partitions of $\Pi_{n+1}^{(m)}$ obtained by recursive construction via the generating tree are*

$$(a_1 + 1, a_2 + 1, \dots, a_m + 1) \qquad \textit{(Add } n + 1 \textit{ as a singleton to } \pi\textit{)}$$

and

$$\begin{array}{ll}
(2, a_2, a_3, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block 1)} \\
(3, a_2, a_3, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block 2)} \\
 & \vdots \\
(a_1, a_2, a_3, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1 - 1\textit{)} \\
(a_1+1, a_1+1, a_3, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1\textit{)} \\
(a_1+1, a_1+2, a_3, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_1+1\textit{)} \\
 & \vdots \\
(a_1+1, a_2+1, a_2+1, \dots, a_{m-1}, a_m) & \textit{(Add } n+1 \textit{ to block } a_2\textit{)} \\
 & \vdots \\
(a_1+1, a_2+1, a_3+1, \dots, a_{m-1}+1, a_{m-1}+1) & \textit{(Add } n+1 \textit{ to block } a_{m-1}\textit{)} \\
 & \vdots \\
(a_1+1, a_2+1, a_3+1, \dots, a_{m-1}+1, a_m) & \textit{(Add } n+1 \textit{ to block } a_m - 1\textit{)}
\end{array}$$

Proof. By careful inspection. □

Example 3. Consider the following partition from $\Pi_8^{(3)}$. The reader can refer to its arc diagram in Example 1 which shows that it is 3-nonnesting, thus also

4-nonnesting. The partition 1|2 5 6 8|3 7|4 with label $(3, 4, 5)$ has five children and their respective labels are:

π	$L(\pi)$
1|2 5 6 8|3 7|4|9	$(4, 5, 6)$
1|2 5 6 8 9|3 7|4	$(2, 4, 5)$
1|2 5 6 8|3 7 9|4	$(3, 4, 5)$
1|2 5 6 8|3 7|4 9	$(4, 4, 5)$
1 9|2 5 6 8|3 7|4	$(4, 5, 5)$

Example 4. As we mentioned before, 2-nonnesting set partitions are counted by Catalan numbers. The generating tree construction given in Proposition 1 restricted to this case is given by

$$[(1) : (k) \to (k + 1)(2)(3) \dots (k)],$$

which is the same construction for Catalan numbers given in [2]. The generating tree for 3-noncrossing partitions is given by

$$[(1, 1) : (i, j) \to (i + 1, j + 1)(2, j)(3, j) \cdots (i, j)(i + 1, i + 1)(i + 1, i + 2) \dots (i + 1, j)].$$

2.3 A Functional Equation for the Generating Function

The simple structure of the labels in Proposition 1 permits a direct translation from the generating tree to a functional equation.

Let us define $\tilde{F}(u_1, u_2, \dots, u_m; t)$ to be the ordinary generating function of partitions in $\Pi^{(m)}$ counted by the statistics $a_1, a_2, \dots, a_m$ and by size,

$$\begin{aligned}\tilde{F}(u_1, u_2, \dots, u_m; t) &:= \sum_{\pi \in \Pi^{(m)}} u_1^{a_1(\pi)} u_2^{a_2(\pi)} \dots u_m^{a_m(\pi)} t^{|\pi|} \\ &= \sum_{a_1, a_2, \dots, a_m} \tilde{F}_{\mathbf{a}}(t) u_1^{a_1} u_2^{a_2} \dots u_m^{a_m},\end{aligned}$$

where $\tilde{F}_{\mathbf{a}}(t)$ is the size generating function for the set partitions of $\Pi^{(m)}$ with the label $\mathbf{a} = (a_1, a_2, \dots, a_m)$. For example, when $m = 2$,

$$\tilde{F}(\mathbf{u}; t) = u_1 u_2 + u_1{}^2 u_2{}^2 t + \left(u_1{}^3 u_2{}^3 + u_1{}^2 u_2{}^2\right) t^2 + \left(u_1{}^4 u_2{}^4 + 2\, u_1{}^3 u_2{}^3 + u_1{}^2 u_2{}^2 + u_1{}^2 u_2{}^3\right) t^3 + \dots.$$

Proposition 1 implies

$$\tilde{F}(u_1,\ldots,u_m;t) = u_1u_2\ldots u_m + tu_1u_2\ldots u_m\tilde{F}(u_1,u_2,\ldots,u_m;t)$$
$$+t\sum_{a_1,a_2,\ldots,a_m}\tilde{F}_{\mathbf{a}}(t)u_2^{a_2}u_3^{a_3}\ldots u_m^{a_m}\sum_{\alpha=2}^{a_1}u_1^{\alpha}$$
$$+t\sum_{a_1,a_2,\ldots,a_m}\tilde{F}_{\mathbf{a}}(t)\sum_{j=2}^{m}\sum_{\alpha=a_{j-1}+1}^{a_j}u_1^{a_1+1}u_2^{a_2+1}\ldots u_{j-1}^{a_{j-1}+1}u_j^{\alpha}u_{j+1}^{a_{j+1}}\ldots u_m^{a_m}.$$

We can simplify the expression using the finite geometric series sum formula to rewrite this as the following expression.

Proposition 2. *The ordinary generating function of partitions in* $\Pi^{(m)}$ *counted by the statistics* a_1, a_2, …, a_m *and by size, denoted* $\tilde{F}(u_1,u_2,\ldots,u_m;t)$, *or simply* $\tilde{F}(\mathbf{u};t)$ *satisfies the following functional equation:*

$$\tilde{F}(\mathbf{u};t) = u_1\ldots u_m + tu_1u_2\ldots u_m\tilde{F}(\mathbf{u};t)$$
$$+tu_1\left(\frac{\tilde{F}(\mathbf{u};t) - u_1\tilde{F}(1,u_2,\ldots,u_m;t)}{u_1-1}\right)$$
$$+t\sum_{j=2}^{m}u_1u_2\ldots u_j\left(\frac{\tilde{F}(\mathbf{u};t)-\tilde{F}(u_1,\ldots,u_{j-2},u_{j-1}u_j,1,u_{j+1},\ldots,u_m;t)}{u_j-1}\right). \tag{1}$$

3 Computing Series Expansions

Notice that in Eq. (1), if one has a series expansion of $\bar{F}(\mathbf{u};t)$ correct up to t^k, then substituting this series into RHS of Eq. (1) yields the series expansion of $\bar{F}$ correct to t^{k+1} because the RHS of Eq. (1) contains a term free of t; otherwise, the degree of t is increased by 1. We have iterated Eq. (1) to get enumerative data for up to $m = 9$.

For 3-nonnesting set partitions, an average laptop running Maple 15 can produce 70 terms in a reasonable time (less than 24 h). For $m = 4$, only 38 terms; $m = 5$, 27 terms; $m = 6$, 20 terms; $m = 7$, 16 terms, $m = 8$, 12 terms; and finally $m = 9$, 12 terms. The limitation seems memory space due to the growing complication in the functional equation when m gets larger (Table 1).

4 Conclusion

The generating tree approach permits a direct translation to a functional equation involving an arbitrary number of catalytic variables satisfied by set partitions avoiding $m + 1$-nestings for any positive integer m. We avoid passing through

Table 1 Numbers of set partitions of n avoiding an $m + 1$-nesting. The OEIS numbers refer to entries in the Online Encyclopedia of Integer Sequences [1]. The entries in light grey emphasize that the restriction has no effect on set partitions of that size

		n														
m	OEIS #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	A000108	1	2	5	14	42	132	429	1,430	4,862	16,796	58,786	208,012	742,900	2,674,440	9,694,845
2	A108304	1	2	5	15	52	202	859	3,930	19,095	97,566	520,257	2,877,834	16,434,105	96,505,490	580,864,901
3	A108305	1	2	5	15	52	203	877	4,139	21,119	115,495	671,969	4,132,936	26,723,063	180,775,027	1,274,056,792
4	A192126	1	2	5	15	52	203	877	4,140	21,147	115,974	678,530	4,212,654	27,627,153	190,624,976	1,378,972,826
5	A192127	1	2	5	15	52	203	877	4,140	21,147	115,975	678,570	4,213,596	27,644,383	190,897,649	1,382,919,174
6	A192128	1	2	5	15	52	203	877	4,140	21,147	115,975	678,570	4,213,597	27,644,437	190,899,321	1,382,958,475

vacillating lattice walks or tableaux. The functional equation can be iterated to generate series data for $m + 1$-nonnesting set partitions, but ideally we would like to solve the equations, or find some other format from which more information can be obtained. For example, perhaps under further scrutiny one can decide if the generating functions are D-finite or not.

One possible route to a proof of non-D-finiteness is to use our expressions to determine bounds on the order and the coefficient degrees of the minimal differential equation satisfied by the generating function. Though a tantalizingly simple idea, the limitation is the lack of series data for large m.

The generating tree studied is for $m + 1$-nonnesting set partitions. The authors have tried to study a generating tree for $m + 1$-noncrossing set partitions in the hope of reproving the result of Chen et al. in [6] by tree isomorphism. However, the authors were unable to generate $m + 1$-noncrossing set partitions.

Finally, our generating tree approach is limited only to the non-enhanced case. For a more general treatment of the subject involving enhanced set partitions and permutations, both enhanced and non-enhanced, we refer the reader to [5] by Burrill, Elizalde, Mishna, and Yen.

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