New Results about Multi-band Uncertainty in Robust Optimization*

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Abstract. "The Price of Robustness" by Bertsimas and Sim [4] represented a breakthrough in the development of a tractable robust counterpart of Linear Programming Problems. However, the central modeling assumption that the deviation band of each uncertain parameter is single may be too limitative in practice: experience indeed suggests that the deviations distribute also internally to the single band, so that getting a higher resolution by partitioning the band into multiple sub-bands seems advisable.

In this work, we study the robust counterpart of a Linear Programming Problem with uncertain coefficient matrix, when a multi-band uncertainty set is considered. We first show that the robust counterpart corresponds to a compact LP formulation. Then we investigate the problem of separating cuts imposing robustness and we show that the separation can be efficiently operated by solving a min-cost flow problem. Finally, we test the performance of our new approach to Robust Optimization on realistic instances of a Wireless Network Design Problem subject to uncertainty.

Keywords: Robust Optimization, Multi-band Uncertainty, Compact Robust Counterpart, Cutting Planes, Network Design.

1 Introduction

A fundamental assumption in classical optimization is that all data are exact. However, many real-world problems involve data that are uncertain or not known with precision, because of erroneous measurements or adoptions of approximated numerical representations. If such uncertainty is neglected, optimal solutions

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computed for nominal data values may become costly or infeasible. As a consequence, including uncertainty in an optimization model is a critical issue when dealing with real-world problems.

During the last years, Robust Optimization (RO) has become a valid methodology to deal with optimization problems subject to uncertainty. A key concept of RO is to model uncertainty as hard constraints, that are added to the original formulation of the problem. This restricts the set of feasible solutions to robust solutions, i.e. solutions that are protected from deviations of the data. Such a robust approach is crucial when dealing with high risk events, such as aircraft scheduling [12], or sensor placement in contaminant warning systems for water distribution networks [15]. In such settings, standard approaches like deterministic optimization or Stochastic Programming fail to protect against severe deviations, leading to unpredictable consequences. For an exhaustive introduction to theory and applications of RO, we refer the reader to the book by Ben-Tal et al. [2] and to the recent survey by Bertsimas et al. [3].

An approach to model uncertain data that has attracted a lot of attention is the so called Γ-scenario set, introduced by Betsimas and Sim (BS) [4] and then adapted to several applications. The uncertainty model for a Linear Program (LP) considered in BS assumes that, for each coefficient a we are given a nominal value $\bar{a}$ and a maximum deviation d and that the actual value lies in the interval $[\bar{a} - d, \bar{a} + d]$. Moreover, a parameter Γ is introduced to represent the maximum number of coefficients that deviate from their nominal value. Hence, Γ controls the conservativeness of the robust model and its introduction comes from the natural observation that it is unlikely that all coefficients deviate from their nominal value at the same time. A central result presented in BS is that, under the previous characterization of the uncertainty set, the robust counterpart of an LP corresponds to a linear formulation. This counterpart has the desirable properties of being *purely linear* and, above all, *compact*, i.e. the number of variables and constraints is polynomial in the size of the input of the deterministic problem.

The use of a single deviation band may greatly limit the power of modeling uncertainty. This is particularly evident when the probability of deviation sensibly varies within the band: in this case, neglecting the inner-band behaviour and just considering the extreme values like in BS may lead to a rough estimate of the deviations and thus to unrealistic uncertainty set, which either overestimate or underestimate the overall deviation. Having a higher modeling resolution would therefore be very desirable. This can be accomplished by breaking the single band into multiple and narrower bands, each with its own Γ. Such model is particularly attractive when historical data about the deviations are available, a very common case in real-world problems. Thus, a multi-band uncertainty set can effectively approximate the shape of the distribution of deviations built on past observations, guaranteeing a much higher modeling power than BS.

This observation was first captured by Bienstock and taken into account to develop an RO framework for the special case of Portfolio Optimization [5]. Yet, no definition and intensive theoretical study of a more general multi-band model applicable in other contexts have been done. The main goal of this paper is to close such gap.

Contributions and Outline. In this work, we study the robust counterpart of an LP with uncertain coefficient matrix, when a *multi-band uncertainty set* is considered. The main original contributions are:

- a compact formulation for the robust counterpart of an LP;
- an efficient method for the separation of robustness cuts (i.e., cuts that impose robustness), based on solving a min-cost flow instance;
- computational experiments comparing the performance of solving the compact formulation versus a cutting plane approach on realistic wireless network design instances.

In Section 2, we show that the robust counterpart of an LP under multi-band uncertainty corresponds to a compact Linear Programming formulation. We then proceed to study the separation problem of robustness cuts in Section 3. Finally, in Section 4, we test the performance of our new model and solution methods to Robust Optimization, to tackle the uncertainty affecting signal propagation in a set of realistic DVB-T instances of a wireless network design problem.

1.1 Model and Notation

We study the robust counterpart of Linear Programming Problems whose coefficient matrix is subject to uncertainty and the uncertainty set is modeled through multiple deviation bands. The deterministic Linear Program is of the form:

$$\begin{aligned} \max \sum_{j \in J} c_j \, x_j & & (LPP) \\ \sum_{j \in J} a_{ij} \, x_j \leq b_i & \qquad i \in I & \\ x_j \geq 0 & \qquad j \in J & \end{aligned}$$

where $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, n\}$ denote the set of constraint and variable indices, respectively. We assume that the value of each coefficient a_{ij} is uncertain and that such uncertainties are modeled through a set of scenarios $\mathcal{S}$. Each scenario $S \in \mathcal{S}$ defines a different coefficient matrix A^S. The robust counterpart of (LPP) thus corresponds to the following problem:

$$\begin{aligned} \max \sum_{j \in J} c_j \, x_j & \\ \sum_{j \in J} a_{ij}^S \, x_j \leq b_i & \qquad i \in I, S \in \mathcal{S} \\ x_j \geq 0 & \qquad j \in J. \end{aligned}$$

We note that uncertainty on the cost c and on the r.h.s. b can be included in a very straightforward way in the coefficient matrix, as explained in [3].

One of the purpose of this paper is to characterize the robust counterpart of (LPP) when the set of scenarios corresponds to what we call a *multi-band uncertainty set*. This set is denoted by $\mathcal{S}_M$ and generalizes the Bertsimas-Sim uncertainty model. Specifically, we assume that, for each coefficient a_{ij}, we are given a *nominal value* $\bar{a}_{ij}$ and maximum negative and positive deviations $d_{ij}^{K^-}, d_{ij}^{K^+}$ from $\bar{a}_{ij}$, such that the *actual value* a_{ij}^S lies in the interval $[\bar{a}_{ij} + d_{ij}^{K^-}, \bar{a}_{ij} + d_{ij}^{K^+}]$ for each scenario $S \in \mathcal{S}_M$. Moreover, we define a *system of deviation bands* by partitioning the single deviation band $[d_{ij}^{K^-}, d_{ij}^{K^+}]$ into K bands, defined on the basis of $K+1$ deviation values:

$$-\infty < d_{ij}^{K^-} < \cdots < d_{ij}^{-2} < d_{ij}^{-1} \; < \; d_{ij}^{0} = 0 \; < \; d_{ij}^{1} < d_{ij}^{2} < \cdots < d_{ij}^{K^+} < +\infty.$$

Through these deviation values, we define: 1) the zero-deviation band corresponding to the single value $d_{ij}^0 = 0$; 2) a set of positive deviation bands, such that each band $k \in \{1, \ldots, K^+\}$ corresponds to the range $(d_{ij}^{k-1}, d_{ij}^k]$; 3) a set of negative deviation bands, such that each band $k \in \{K^-, \ldots, -1\}$ corresponds to the range $[d_{ij}^k, d_{ij}^{k-1})$ (the interval of each band is thus closed on the endpoint with the higher absolute value). With a slight abuse of notation, in what follows we indicate a generic deviation band by $k \in K = \{K^-, \ldots, -1, 0, 1, \ldots, K^+\}$.

Additionally, for each band $k \in K$, we define a lower bound l_k and an upper bound u_k on the number of deviations that may fall in k, with l_k, u_k satisfying $0 \leq l_k \leq u_k \leq n$. In the case of band 0, we assume that $u_0 = n$, i.e. we do not limit the number of coefficients that take their nominal value. Furthermore, we assume that $\sum_{k \in K} l_k \leq n$ so that there always exists a feasible realization of the coefficient matrix. On the basis of these parameters, we formalize the set of scenarios $\mathcal{S}_M$: a scenario $S \in \mathcal{S}_M$ is feasible if and only if $a_{ij}^S \in [\bar{a}_{ij} + d_{ij}^{K^-}, \bar{a}_{ij} + d_{ij}^{K^+}]$ and $l_k \leq |\{j \in J \mid a_{ij}^S \text{ lies in band } k\}| \leq u_k$ for every $k \in K$, $i \in I$. In other words, we require that the deviations satisfy the system of multi-band uncertainty and thus the number of deviations falling in each band must satisfy the corresponding bounds.

We remark that, in order to avoid an overload of the notation, we assume that the number of bands K and the bounds l_k, u_k are the same for each constraint $i \in I$. Anyway, it is straightforward to modify all presented results to take into account different values of those parameters for each constraint. We now proceed to study the robust counterpart of (LPP) under multi-band uncertainty.

2 A Compact Robust LP Counterpart

The robust counterpart of an (LPP) under a multi-band uncertainty set defined by $\mathcal{S}_M$ can be equivalently written as:

$$\begin{aligned}
\max \; & \sum_{j \in J} c_j \, x_j \\
& \sum_{j \in J} \bar{a}_{ij} \, x_j + DEV_i(x, d) \leq b_i && i \in I \\
& x_j \geq 0 && j \in J
\end{aligned}$$

where $DEV_i(x, d)$ is the maximum overall deviation allowed by a system of deviation bands d for a feasible solution x when constraint i is considered. Note that we replace the actual value of a coefficient a_{ij} with the summation of the nominal value $\bar{a}_{ij}$ and a deviation d_{ij} falling in exactly one of the K bands. The computation of $DEV_i(x, d)$ corresponds to the optimal value of the following pure 0-1 Linear Program (note that in this case index i is fixed):

$$DEV_i(x,d) = \max \sum_{j \in J} \sum_{k \in K} d_{ij}^k \; x_j \; y_{ij}^k \qquad (DEV01)$$

$$l_k \;\leq\; \sum_{j \in J} y_{ij}^k \;\leq\; u_k \qquad k \in K \tag{1}$$

$$\sum_{k \in K} y_{ij}^k \leq 1 \qquad j \in J \tag{2}$$

$$y_{ij}^k \in \{0,1\} \qquad j \in J, k \in K. \tag{3}$$

The binary variables y_{ij}^k indicate if the deviation of a coefficient a_{ij} lies in band k. Constraints (2) ensure that each coefficient deviates in at most one band (actually these should be equality constraints, but, for assumption $u_0 = n$ made in Section 1.1, we can consider inequalities). Finally, constraints (1) impose the upper and lower bounds on the number of deviations falling in each band k. Thus, the optimal solution of (DEV01) defines a distribution of the coefficients among the bands that maximizes the deviation w.r.t. the nominal values, while respecting the bounds on the number of deviations of each band.

We now show that the polytope associated with the linear relaxation of (DEV01) is integral. The linear relaxation of (DEV01) is:

$$\max \sum_{j \in J} \sum_{k \in K} d_{ij}^k \; x_j \; y_{ij}^k \qquad \text{(DEV01-RELAX)}$$

$$l_k \leq \sum_{j \in J} y_{ij}^k \leq u_k \qquad k \in K \tag{4}$$

$$\sum_{k \in K} y_{ij}^k \leq 1 \qquad j \in J \tag{5}$$

$$y_{ij}^k \geq 0 \qquad j \in J, k \in K \tag{6}$$

where we dropped constraints $y_{ij}^k \leq 1$ since they are dominated by constraints (5).

Theorem 1. *The polytope described by the constraints of (DEV01-RELAX) is integral.*

Proof. We start by rewriting all the constraints of (DEV01-RELAX) into the form $\alpha^T y \leq \beta$ obtaining the following matrix form:

$$D_i\, y_i \;=\; \left(\begin{array}{c|c|c|c} -I & -I & \cdots & -I \\ \hline I & I & \cdots & I \\ \hline 1\cdots 1 & & & \\ & 1\cdots 1 & & \\ & & \ddots & \\ & & & 1\cdots 1 \end{array}\right) \begin{pmatrix} y_{i1}^{K^-} \\ \vdots \\ y_{i1}^{K^+} \\ \hline \vdots \\ y_{ij}^{k} \\ \vdots \\ \hline y_{in}^{K^-} \\ \vdots \\ y_{in}^{K^+} \end{pmatrix} \;\leq\; \begin{pmatrix} \vdots \\ -l_k \\ \vdots \\ \hline \vdots \\ u_k \\ \vdots \\ \hline \vdots \\ 1 \\ \vdots \end{pmatrix} \;=\; g_i.$$

Consider now the submatrix $\tilde{D}_i$ obtained from D_i by eliminating the top layer of blocks $(-I| - I| \cdots | - I)$. It is easy to verify that $\tilde{D}_i$ is the incidence matrix of a bipartite graph: the elements of the two disjoint set of nodes of the graph are in correspondence with the rows of the two distinct layers of blocks in $\tilde{D}_i$. Moreover, every column has exactly two elements that are not equal to zero, one in the upper layer and one in the lower layer. Being the incidence matrix of a bipartite graph, $\tilde{D}_i$ is a totally unimodular matrix [13].

In order to show that also the original matrix D_i is totally unimodular, we first need to recall the equivalence of the following three statements [13]: 1) A is a totally unimodular matrix; 2) a matrix obtained by duplicating rows of A is totally unimodular; 3) a matrix obtained by multiplying a row of A by -1 is totally unimodular. Since D_i can be obtained from $\tilde{D}_i$ by duplicating each row of the upper block, and multiplying each row of the duplicated block by -1, D_i is totally unimodular.

As D_i is totally unimodular and the vector g_i is integral, it is well-known that the polytope defined by $D_i y_i \leq g_i$ and $y_i \geq 0$ is integral, thus completing the proof. □

Since the polytope associated with (DEV01-RELAX) is integral, by strong duality we can use the dual problem of (DEV01-RELAX) to replace $\text{DEV}_i(x, d)$ in the robust counterpart of (LPP). The dual problem of (DEV01-RELAX) is:

$$\begin{aligned} \min \; & \sum_{k\in K} -l_k\, v_i^k + \sum_{k\in K} u_k\, w_i^k + \sum_{j\in J} z_i^j && \text{(DEV01-RELAX-DUAL)} \\ & -v_i^k + w_i^k + z_i^j \geq d_{ij}^k\, x_j && j \in J, k \in K \\ & v_i^k,\;\; w_i^k \geq 0 && k \in K \\ & z_i^j \geq 0 && j \in J \end{aligned}$$

where the dual variables v_i^k, w_i^k, z_i^j are respectively associated with the primal constraints (4, 5, 6) of (DEV01-RELAX) defined for constraint i. Replacing $\text{DEV}_i(x, d)$ by its dual yields the following compact linear robust counterpart of the original problem (LPP):

$$\max \sum_{j \in J} c_j\, x_j \tag{RLP}$$

$$\sum_{j \in J} \bar{a}_{ij}\, x_j - \sum_{k \in K} l_k\, v_i^k + \sum_{k \in K} u_k\, w_i^k + \sum_{j \in J} z_i^j \leq b_i \qquad i \in I$$

$$-v_i^k + w_i^k + z_i^j \geq d_{ij}^k\, x_j \qquad i \in I, j \in J, k \in K$$

$$v_i^k,\ w_i^k \geq 0 \qquad i \in I, k \in K$$

$$z_i^j \geq 0 \qquad i \in I, j \in J$$

$$x_j \geq 0 \qquad j \in J.$$

In comparison to (LPP), this compact formulation uses $2 \cdot K \cdot m + n \cdot m$ additional variables and includes $K \cdot n \cdot m$ additional constraints.

3 Separation of Robustness Cuts

In this section, we consider the problem of testing whether a solution $x^* \in \mathbb{R}^n$ is robust feasible, i.e. $a_i^S x^* \leq b_i$ for every scenario $S \in \mathcal{S}_M$ and $i \in I$. This problems becomes important for adopting a cutting plane approach instead of directly solving the compact robust counterpart (RLP). This approach works as follows: start by solving the nominal problem (LPP) and then check if the optimal solution is robust. If not, generate a cut that imposes robustness (*robustness cut*) and add it to the problem. This initial step is then iterated as in a typical cutting plane method [13].

In the case of the Bertsimas-Sim model, the problem of separating a robustness cut is very simple [7]: given a solution x^*, for each constraint $i \in I$, the problem consists of sorting the deviations $d_{ij}^{K+} x_j^*$ in non-increasing order and choose the highest Γ_i deviations. If for some i the sum of these deviations exceeds $b_i - \sum_{j \in J} \bar{a}_{ij} x_j$ then we found a robustness cut to be added. Otherwise, x^* is robust.

In the case of multi-band uncertainty, this simple approach does not guarantee robustness of a computed solution. However, we prove that for a given solution $x^* \in \mathbb{R}^n$ and a constraint $i \in I$, checking the robust feasibility of x^* corresponds to solving a *min-cost flow problem* [1], whose instance is denoted by $(G, c)_x^i$ and defined as follows. G is a directed graph whose set of vertices V contains one vertex v_j for each variable index $j \in J$, one vertex w_k for each band $k \in K$ and two vertices s, t that are the source and the sink of the flow, i.e. $V = \bigcup_{j \in J} \{v_j\} \cup \bigcup_{k \in K} \{w_k\} \cup \{s, t\}$. The set of arcs A is the union of three sets A_1, A_2, A_3. A_1 contains one arc from s to every variable vertex v_j, i.e. $A_1 = \{(s, v_j) \mid j \in J\}$. A_2 contains one arc from every variable vertex v_j to every band vertex w_k, i.e. $A_2 = \{(v_j, w_k) \mid j \in J, k \in K\}$. Finally, A_3 contains one arc from every band vertex w_k to the sink t, i.e. $A_3 = \{(w_k, t) \mid k \in K\}$. By construction, $G(V, A)$ is bipartite and acyclic. Each arc $a \in A$ is associated to a triple (l_a, u_a, c_a), where l_a, u_a are lower and upper bounds on the flow that can be sent on a and c_a is the cost of sending one unit of flow on a. The values of the triples (l_a, u_a, c_a) are set in the following way: $(0,1,0)$ when $a \in A_1$; $(0,1,-d_{ij}^k x_j^*)$ when $a = (v_j, w_k) \in A_2$; $(l_k, u_k, 0)$ when $a = (w_k, t) \in A_3$. Finally, the amount of flow that must be sent trough the network from s to t is equal to n. The value of an (s,t)-flow is defined by $C(f) = \sum_{a \in A} c_a f_a$. An integral min-cost flow can be computed in polynomial time, using for example the successive shortest path algorithm [1].

We now prove that by solving the min-cost flow instance defined above, we obtain the maximum deviation for constraint i and solution x^*.

Lemma 1. *A solution $x^* \in \mathbb{R}^n$ is robust w.r.t. a multi-band scenario set $\mathcal{S}_M$ if and only if*

$$\bar{a}_i' x^* - C(f) \leq b_i$$

for every $i \in I$ and min-cost flow f of the instance $(G,c)_{x^*}^i$.

Proof. We show that for any flow f there exists a scenario $S_f \in \mathcal{S}_M$ with $(a_i^{S_f})' x^* = \overline{a}_i' x^* - C(f)$ and for every scenario $S \in \mathcal{S}_M$ there exists a flow f^S with $C(f^S) = \overline{a}_i' x^* - (a_i^S)' x^*$. Let $f : A \to \{0,1\}$ be a feasible flow in $(G,c)_{x^*}^i$. Then we obtain a feasible scenario $S \in \mathcal{S}_M$ by setting $a_{ij}^S = \bar{a}_{ij} + \sum_{k \in K} d_{ij}^k f_{jk}$, where f_{jk} denotes the flow on arc (v_j, w_k), i.e. $f_{jk} = f((v_j, w_k))$. Due to the flow conservation in every vertex v_j, there exists exactly one variable $f_{jk} = 1$, $j \in J$, $k \in K$. Furthermore, the amount of variables whose coefficients are in band $k \in K$ is at least l_k and at most u_k due to the upper and lower bounds on the arc (w_k, t). Hence, S_f is a feasible scenario and

$$\begin{aligned}
\sum_{j \in J} a_{ij}^{S_f} x_j^* &= \sum_{j \in J} \bar{a}_{ij} x_j^* + \sum_{j \in J} \sum_{k \in K} d_{ij}^k f_{ik} x_j^* \\
&= \sum_{j \in J} \bar{a}_{ij} x_j^* - C(f).
\end{aligned}$$

On the other hand, let $S \in \mathcal{S}_M$ be a feasible scenario. We set $f_{jk}^S = 1$ if and only if a_{ij}^S is in band $k \in K$. The flow on the other arcs is set in such a way that we preserve flow conservation in every vertex besides s and t. Then f^S is a feasible flow, since the lower and upper capacity bounds are satisfied due to the feasibility of S, and n units of flow are sent through the network. Furthermore,

$$\begin{aligned}
C(f^S) &= - \sum_{j \in J} \sum_{k \in K} d_{ij}^k x_j^* f_{jk} \\
&= \sum_{j \in J} \bar{a}_{ij} x_j^* - \sum_{j \in J} \sum_{k \in K} d_{ij}^k x_j^* f_{ik} - \sum_{j \in J} \bar{a}_{ij} x_j^* \\
&= (a_i^S)' x^* - \bar{a}_i' x_j^*.
\end{aligned}$$

This concludes the proof. □

According to this lemma, we can test the robustness of a solution $x^* \in \mathbb{R}^n$ by computing a min-cost flow f^i in $(G,c)_{x^*}^i$ for every $i \in I$. If $\bar{a}' x^* - C(f^i) \leq b_i$ for every i, then x^* is a robust solution. If x^* is not robust, there exists an index i such that $\bar{a}'x^* - C(f^i) > b_i$ and thus

$$\sum_{j \in J} \overline{a}_{ij} x_{ij} + \sum_{j \in J} \sum_{k \in K} d_{ij}^k f_{jk}^i x_{ij} \leq b_i \tag{7}$$

is valid for the polytope of the robust solutions and cuts off the solution x^*.

4 Computational Study

In this section, we test our new modeling and solution approaches to Robust Optimization on a set of realistic instances of the *Power Assignment Problem*, a problem arising in the design of wireless networks. In particular, we compare the efficiency of solving directly the compact formulation (RLP) with that of a cutting plane method based on the robustness cuts presented in Section 3. In the case of the Bertsimas-Sim model, such comparison led to contrasting conclusions (e.g., [7,8]).

The Power Assignment Problem. The *Power Assignment Problem* (PAP) is the problem of dimensioning the power emission of each transmitter in a wireless network, in order to provide service coverage to a number of user, while minimizing the overall power emission. The PAP is particularly important in the (re)optimization of networks that are updated to new generation digital transmission technologies. For a detailed introduction to the PAP and the general problem of designing wireless networks, we refer the reader to [11,6,10].

A classical LP formulation for the PAP can be defined by introducing the following elements: 1) a vector of non-negative continuous variables p that represent the power emissions of the transmitters; 2) a vector $P^{\max}$ of upper bounds on p that represent technology constraints on the maximum power emissions; 3) a matrix A of the coefficients that represent signal attenuation (*fading coefficients*) for each transmitter-user couple; 4) a vector of r.h.s. δ (signal-to-interference thresholds) that represent the minimum power values that guarantee service coverage. Under the objective of minimizing the overall power emission, the PAP can be written in the following matrix form:

$$\min\ \mathbf{1}'p \quad \text{s.t.} \quad Ap \geq \delta, \quad 0 \leq p \leq P^{\max} \qquad (PAP)$$

where exactly one constraint $a_i'p \geq \delta_i$ is introduced for each user i to represent the corresponding service coverage condition.

Each entry of matrix A is classically computed by a propagation model and takes into account many factors (e.g., distance between transmitter and receiver, terrain features). However, the exact propagation behavior of a signal cannot be evaluated and thus each fading coefficient is naturally subject to uncertainty. Neglecting such uncertainty may provide unpleasant surprises in the final coverage plan, where devices may turn out to be uncovered for bad deviations affecting the fading coefficients (this is particularly true in hard propagation scenarios, such as dense urban fabric). For a detailed presentation of the technical aspects of propagation, we refer the reader to [14].

Following the ITU recommendations (e.g., [9]), we assume that the fading coefficients are mutually independent random variables and that each variable is log-normally distributed. The adoption of the Bertsimas-Sim model would provide only a rough representation of the deviations associated with such distribution. We thus adopt the multi-band uncertainty model to obtain a more refined representation of the fading coefficient deviations. In what follows, we denote the Bertsimas-Sim and the multi-band uncertainty model by (BS) and (MB), respectively.

Computational Results. In this computational study, we consider realistic instances corresponding to region-wide networks that implement the Terrestrial Digital Video Broadcasting technology (DVB-T) [9] and were taken as reference for the design of the new Italian DVB-T national network. The uncertainty set is built taking into account the ITU recommendations [9] and discussions with our industrial partners in past projects about wireless network design. Specifically, we assume that each fading coefficient follows a log-normal distribution with mean provided by the propagation model and standard deviation equal to 5.5 dB [9]. In our test-bed, the (MB) uncertainty set of a generic fading coefficient a_{ij} is constituted by 3 negative and 3 positive deviations bands (i.e., $K = 6$). Each band has a width equal to the 5% of the nominal fading value $\bar{a}_{ij}$. Thus the maximum allowed deviation is +/- $0.15 \cdot \bar{a}_{ij}$. For each constraint i, the bounds l_k, u_k on the number of deviations are defined considering the cumulative distribution function of a log-normal random variable with standard deviation 5.5 dB. The (BS) uncertainty set of each constraint considers the same maximum deviation of (MB) and the maximum number of deviating coefficients is $\Gamma = \lceil 0.8 \cdot u^{\max} \rceil$, where $u^{\max} = \max\{u_k : k \in K \backslash \{0\}\}$. This technically reasonable assumption on Γ ensures that (BS) does not dominate (MB) a priori.

The computational results are reported in Table 1. The tests were performed on a Windows machine with 1.80 GHz Intel Core 2 Duo processor and 2 GB RAM. All the formulations are implemented in C++ and solved by IBM ILOG Cplex 12.1, invoked by ILOG Concert Technology 2.9. We considered 15 instances of increasing size corresponding to realistic DVB-T networks. The first column of Table 1 indicates the ID of the instances. Columns $|I|, |J|$ indicate the number of variables and constraints of the problem, corresponding to the number of user devices and transmitters of the network, respectively. We remark that the coefficient matrices tend to be sparse, as only a (small) fraction of the transmitters is able to reach a user device with its signals. Columns $|I^+|, |J^+|$ indicate the number of additional variables and constraints needed in the compact robust counterpart (RLP). Columns PoR% report *the Price of Robustness* (PoR), i.e. the deterioration of the optimal value required to guarantee robustness. In particular, we consider the percentage increase of the robust optimal value w.r.t. the optimal value of the nominal problem, in the multi-band case (PoR% (MB)) and in the Bertsimas-Sim case (PoR% (BS)). Column $\Delta t\%$ reports the percentage increase of the time required to compute the robust optimal solution under (MB) by using the cutting plane method presented in Section 3 w.r.t. the time needed to solve the compact formulation (RLP). Finally, column Protect% is a measure of the protection offered by the robust optimal solution and is computed in the following way: for each instance, we generate 1000 realizations of the uncertain coefficient matrix and we then compute the percentage of realizations in which the robust optimal solution is feasible. This is done for both (MB) and (BS).

Looking at Table 1, the first evident thing is that the dimension of the compact robust counterpart under (MB) is much larger than that of the nominal problem. However, this is not an issue for Cplex, as all instances are solved within one hour and in most of the cases the direct solution of (RLP) takes less time than the cutting plane approach ($\Delta t\% < 0$). Anyway, for the instances of greater dimension the cutting plane approach becomes competitive and may even take less time ($\Delta t\% > 0$). Concerning the PoR, we note that under (MB) imposing robustness leads to a sensible increase in the overall power emission, that is anyway lower than that of (BS) in all but two cases. On the other hand, such increase of (MB) is compensated by a very good 90% protection on average. In the case of the PAP, (MB) thus seems convenient to model the log-normal uncertainty of fading coefficients, guaranteeing good protection at a reasonable price. Moreover, though (BS) offers higher protection for most instances, it is interesting to note that the increase of Protect% of (BS) w.r.t (MB) is lower than the corresponding increase of PoR% of (BS) w.r.t (MB).

Table 1. Overview of the computational results

ID	$\|I\|$	$\|J\|$	$\|I^+\|$	$\|J^+\|$	PoR% (MB)	PoR% (BS)	Δt%	Protect% (MB)	Protect% (BS)
D1	95	153	3519	10098	8.3	10.1	-18.7	88.20	92.53
D2	103	197	4728	14184	7.2	9.4	-19	91.35	92.47
D3	105	322	7406	21252	6.8	8.8	-16.9	93.12	96.40
D4	105	473	10406	28380	7.4	7.2	-15.1	92.08	91.42
D5	108	569	13087	37554	9.2	11.4	-13.6	89.23	90.29
D6	157	1088	27200	84864	6.6	9.1	-6.2	85.46	87.55
D7	165	1203	31278	101052	7.1	9.5	-4.9	87.91	89.16
D8	171	1262	32812	106008	8.7	10.8	-4.1	89.40	93.08
D9	178	1375	35750	115500	9.6	10.2	-2.8	90.11	91.90
D10	180	1448	39096	130320	7.9	9.6	-1.7	91.54	95.32
D11	180	1661	46058	159456	7.2	9.5	0.6	94.77	96.70
D12	181	1779	49812	170784	7.5	10.1	1.8	88.22	90.16
D13	183	1853	53737	189006	8.1	10.3	3.3	91.34	92.21
D14	183	1940	56260	197880	10.3	9.7	3.1	86.50	85.18
D15	185	2183	63307	222666	8.4	10.8	4.1	91.09	92.70

5 Conclusions and Future Work

In this work, we presented new theoretical results about multi-band uncertainty in Robust Optimization. Surprisingly, this natural extension of the classical single band model by Bertsimas and Sim has attracted very little attention and we have thus started to fill this theoretical gap. We showed that, under multi-band uncertainty, the robust counterpart of an LP is linear and compact and that the problem of separating a robustness cut can be formulated as a min-cost flow problem and thus be solved efficiently. Tests on realistic network design instances showed that our new approach performs very well, thus encouraging further investigations. Future research will focus on refining the cutting plane method and enlarging the computational experience to other relevant real-world problems.

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