The Quest for Transitivity, a Showcase of Fuzzy Relational Calculus

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Abstract. We present several relational frameworks for expressing similarities and preferences in a quantitative way. The main focus is on the occurrence of various types of transitivity in these frameworks. The first framework is that of fuzzy relations; the corresponding notion of transitivity is C-transitivity, with C a conjunctor. We discuss two approaches to the measurement of similarity of fuzzy sets: a logical approach based on biresidual operators and a cardinal approach based on fuzzy set cardinalities. The second framework is that of reciprocal relations; the corresponding notion of transitivity is cycle-transitivity. It plays a crucial role in the description of different types of transitivity arising in the comparison of (artificially coupled) random variables in terms of winning probabilities. It also embraces the study of mutual rank probability relations of partially ordered sets.

1 Introduction

Comparing objects in order to group together similar ones or distinguish better from worse is inherent to human activities in general and scientific disciplines in particular. In this overview paper, we present some relational frameworks that allow to express the results of such a comparison in a numerical way, typically by means of numbers in the unit interval. A first framework is that of fuzzy relations and we discuss how it can be used to develop cardinality-based, i.e. based on the counting of features, similarity measurement techniques. A second framework is that of reciprocal relations and we discuss how it can be used to develop methods for comparing random variables. Rationality considerations demand the presence of some kind of transitivity. We therefore review in detail the available notions of transitivity and point out where they occur.

This contribution is organised as follows. In Section 2, we present the two relational frameworks mentioned, the corresponding notions of transitivity and the connections between them. In Section 3, we explore the framework of fuzzy relations and its capacity for expressing the similarity of fuzzy sets. Section 4 is dedicated to the framework of reciprocal relations and its potential for the development of methods for the comparison of random variables. We wrap up in Section 5 with a short conclusion.

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2 Relational Frameworks and Their Transitivity

2.1 Fuzzy Relations

Transitivity is an essential property of relations. A (binary) *relation* R on a universe X (the *universe of discourse* or the *set of alternatives*) is called *transitive* if for any $(a,b,c) \in X^3$ it holds that $(a,b) \in R \wedge (b,c) \in R$ implies $(a,c) \in R$. Identifying R with its characteristic mapping, *i.e.* defining $R(a,b) = 1$ if $(a,b) \in R$, and $R(a,b) = 0$ if $(a,b) \notin R$, transitivity can be stated equivalently as $R(a,b) = 1 \wedge R(b,c) = 1$ implies $R(a,c) = 1$. Other equivalent formulations may be devised, such as

$$(R(a,b) \geq \alpha \wedge R(b,c) \geq \alpha) \Rightarrow R(a,c) \geq \alpha\,, \tag{1}$$

for any $\alpha \in]0,1]$. Transitivity can also be expressed in the following functional form

$$\min(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{2}$$

Note that on $\{0,1\}^2$ the minimum operation is nothing else but the Boolean conjunction.

A *fuzzy relation* R on X is an $X^2 \to [0,1]$ mapping that expresses the degree of relationship between elements of X: $R(a,b) = 0$ means a and b are not related at all, $R(a,b) = 1$ expresses full relationship, while $R(a,b) \in]0,1[$ indicates a partial degree of relationship only. In fuzzy set theory, formulation (2) has led to the popular notion of T-transitivity, where a t-norm is used to generalize Boolean conjunction. A binary operation $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if it is increasing in each variable, has neutral element 1 and is commutative and associative. The three main continuous t-norms are the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ (defined by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$). For an excellent monograph on t-norms and t-conorms, we refer to [39].

However, we prefer to work with a more general class of operations called conjunctors. A *conjunctor* is a binary operation $C : [0,1]^2 \to [0,1]$ that is increasing in each variable and coincides on $\{0,1\}^2$ with the Boolean conjunction.

Definition 1. *Let C be a conjunctor. A fuzzy relation R on X is called C-transitive if for any $(a,b,c) \in X^3$ it holds that*

$$C(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{3}$$

Interesting classes of conjunctors are the classes of semi-copulas, quasi-copulas, copulas and t-norms. *Semi-copulas* are nothing else but conjunctors with neutral element 1 [30]. Where t-norms have the additional properties of commutativity and associativity, quasi-copulas are 1-Lipschitz continuous [33,44]. A *quasi-copula* is a semi-copula that is *1-Lipschitz continuous*: for any $(x,y,u,v) \in [0,1]^4$ it holds that $|C(x,u) - C(y,v)| \leq |x-y| + |u-v|$. If instead of 1-Lipschitz continuity, C satisfies the *moderate growth property* (also called 2-monotonicity): for any

$(x, y, u, v) \in [0,1]^4$ such that $x \leq y$ and $u \leq v$ it holds that $C(x,v) + C(y,u) \leq C(x,u) + C(y,v)$, then C is called a *copula*.

Any copula is a quasi-copula, and therefore is 1-Lipschitz continuous; the converse is not true. It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-continuous. The t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are copulas as well. For any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$. For an excellent monograph on copulas, we refer to [44].

2.2 Reciprocal Relations

Another interesting class of $X^2 \to [0,1]$ mappings is the class of *reciprocal relations* Q (also called *ipsodual relations* or *probabilistic relations*) satisfying $Q(a,b) + Q(b,a) = 1$, for any $a, b \in X$. For such relations, it holds in particular that $Q(a,a) = 1/2$. Reciprocity is linked with completeness: let R be a complete ($\{0,1\}$-valued) relation on X, which means that $\max(R(a,b), R(b,a)) = 1$ for any $a, b \in X$, then R has an equivalent $\{0, 1/2, 1\}$-valued reciprocal representation Q given by $Q(a,b) = 1/2(1 + R(a,b) - R(b,a))$.

Stochastic Transitivity. Transitivity properties for reciprocal relations rather have the logical flavor of expression (1). There exist various kinds of stochastic transitivity for reciprocal relations [3,42]. For instance, a reciprocal relation Q on X is called *weakly stochastic transitive* if for any $(a,b,c) \in X^3$ it holds that $Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2$ implies $Q(a,c) \geq 1/2$, which corresponds to the choice of $\alpha = 1/2$ in (1). In [11], the following generalization of stochastic transitivity was proposed.

Definition 2. *Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,1/2) \leq 1/2$. A reciprocal relation Q on X is called g-stochastic transitive if for any $(a,b,c) \in X^3$ it holds that*

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq g(Q(a,b), Q(b,c))\,.$$

Note that the condition $g(1/2,1/2) \leq 1/2$ ensures that the reciprocal representation Q of any transitive complete relation R is always g-stochastic transitive. In other words, g-stochastic transitivity generalizes transitivity of complete relations. This definition includes the standard types of stochastic transitivity [42]:

(i) *strong* stochastic transitivity when $g = \max$;
(ii) *moderate* stochastic transitivity when $g = \min$;
(iii) *weak* stochastic transitivity when $g = 1/2$.

In [11], also a special type of stochastic transitivity has been introduced.

Definition 3. *Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,1/2) = 1/2$ and $g(1/2,1) = g(1,1/2) = 1$. A reciprocal relation Q on X is called g-isostochastic transitive if for any $(a,b,c) \in X^3$ it holds that*

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) = g(Q(a,b), Q(b,c))\,.$$

The conditions imposed upon g again ensure that g-isostochastic transitivity generalizes transitivity of complete relations. Note that for a given mapping g, the property of g-isostochastic transitivity is much more restrictive than the property of g-stochastic transitivity.

***FG*-Transitivity.** The framework of FG-transitivity, developed by Switalski [51,52], formally generalizes g-stochastic transitivity in the sense that $Q(a,c)$ is now bounded both from below and above by $[1/2,1]^2 \to [0,1]$ mappings.

Definition 4. *Let F and G be two $[1/2,1]^2 \to [0,1]$ mappings such that $F(1/2,1/2) \leq 1/2 \leq G(1/2,1/2)$, and $G(1/2,1) = G(1,1/2) = G(1,1) = 1$ and $F \leq G$. A reciprocal relation Q on X is called FG-transitive if for any $(a,b,c) \in X^3$ it holds that*

$$(Q(a,b) \geq 1/2 \;\wedge\; Q(b,c) \geq 1/2)$$
$$\Downarrow$$
$$F(Q(a,b),Q(b,c)) \leq Q(a,c) \leq G(Q(a,b),Q(b,c))\,.$$

Cycle-Transitivity. For a reciprocal relation Q, we define for all $(a,b,c) \in X^3$ the following quantities [11]:

$$\alpha_{abc} = \min(Q(a,b),Q(b,c),Q(c,a))\,,$$
$$\beta_{abc} = \text{median}(Q(a,b),Q(b,c),Q(c,a))\,,$$
$$\gamma_{abc} = \max(Q(a,b),Q(b,c),Q(c,a))\,.$$

Let us also denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\}$. A function $U : \Delta \to \mathbb{R}$ is called an *upper bound function* if it satisfies:

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;
(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma,1-\beta,1-\alpha) \geq 1\,. \tag{4}$$

The function $L : \Delta \to \mathbb{R}$ defined by $L(\alpha,\beta,\gamma) = 1-U(1-\gamma,1-\beta,1-\alpha)$ is called the *dual lower bound function* of the upper bound function U. Inequality (4) then simply expresses that $L \leq U$. Condition (i) again guarantees that cycle-transitivity generalizes transitivity of complete relations.

Definition 5. *A reciprocal relation Q on X is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in X^3$ it holds that*

$$L(\alpha_{abc},\beta_{abc},\gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc},\beta_{abc},\gamma_{abc})\,, \tag{5}$$

where L is the dual lower bound function of U.

Due to the built-in duality, it holds that if (5) is true for some (a, b, c), then this is also the case for any permutation of (a, b, c). In practice, it is therefore sufficient to check (5) for a single permutation of any $(a, b, c) \in X^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a, b, c) \in X^3$ (not being cyclic permutations of one another), e.g. (a, b, c) and (c, b, a). Hence, (5) can be replaced by

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) .$$

Note that a value of $U(\alpha, \beta, \gamma)$ equal to 2 is used to express that for the given values there is no restriction at all (as $\alpha + \beta + \gamma - 1$ is always bounded by 2).

Two upper bound functions U_1 and U_2 are called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that $\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$ is equivalent to $\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma)$.

If it happens that in (4) the equality holds for all $(\alpha, \beta, \gamma) \in \Delta$, then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, also (5) and (2.2) can only hold with equality. Furthermore, it then holds that $U(0, 0, 1) = 0$ and $U(0, 1, 1) = 1$.

Although C-transitivity is not intended to be applied to reciprocal relations, it can be cast quite nicely into the cycle-transitivity framework.

Proposition 1. [11] *Let C be a commutative conjunctor such that $C \leq T_{\mathbf{M}}$. A reciprocal relation Q on X is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha, \beta, \gamma) = \min(\alpha + \beta - C(\alpha, \beta), \beta + \gamma - C(\beta, \gamma), \gamma + \alpha - C(\gamma, \alpha)) .$$

Moreover, if C is 1-Lipschitz continuous, then U_C is given by

$$U_C(\alpha, \beta, \gamma) = \alpha + \beta - C(\alpha, \beta) .$$

Consider the three basic t-norms (copulas) $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$:

(i) For $C = T_{\mathbf{M}}$, we immediately obtain as upper bound function the median (the simplest self-dual upper bound function):

$$U_{T_{\mathbf{M}}}(\alpha, \beta, \gamma) = \beta .$$

(ii) For $C = T_{\mathbf{P}}$, we find

$$U_{T_{\mathbf{P}}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta .$$

(iii) For $C = T_{\mathbf{L}}$, we obtain

$$U_{T_{\mathbf{L}}}(\alpha, \beta, \gamma) = \begin{cases} \alpha + \beta & , \text{ if } \alpha + \beta < 1 , \\ 1 & , \text{ if } \alpha + \beta \geq 1 . \end{cases}$$

An equivalent upper bound function is given by $U'_{T_{\mathbf{L}}}(\alpha, \beta, \gamma) = 1$.

Cycle-transitivity also incorporates stochastic transitivity, although the latter fits more naturally in the FG-transitivity framework; in particular, isostochastic transitivity corresponds to cycle-transitivity w.r.t. particular self-dual upper bound functions [11]. We have shown that the cycle-transitivity and FG-transitivity frameworks cannot easily be translated into one another, which underlines that these are two essentially different approaches [6].

One particular form of stochastic transitivity deserves our attention. A probabilistic relation Q on X is called *partially stochastic transitive* [31] if for any $(a,b,c) \in X^3$ it holds that

$$(Q(a,b) > 1/2 \wedge Q(b,c) > 1/2) \Rightarrow Q(a,c) \geq \min(Q(a,b), Q(b,c)) .$$

Clearly, it is a slight weakening of moderate stochastic transitivity. Interestingly, also this type of transitivity can be expressed elegantly in the cycle-transitivity framework [24] by means of a simple upper bound function.

Proposition 2. *Cycle-transitivity w.r.t. the upper bound function* U_{ps} *defined by*

$$U_{ps}(\alpha,\beta,\gamma) = \gamma$$

is equivalent to partial stochastic transitivity.

A Frequentist Interpretation. Finally, we provide an interesting interpretation of some important types of upper bound functions [23].

Definition 6. *Let* C *be a conjunctor and* Q *be a reciprocal relation on* X*. A permutation* $(a,b,c) \in X^3$ *is called a* C*-triplet if*

$$C(R(a,b), R(b,c)) \leq R(a,c) .$$

Let $\Delta_C(Q)$ denote the greatest number k such that any subset $\{a,b,c\} \subseteq X$ has k C-triplets. Obviously, Q is C-transitive if and only if $\Delta_C(Q) = 6$.

Proposition 3. *For any conjunctor* $C \leq T_{\mathbf{M}}$ *and any reciprocal relation* Q *on* X *it holds that* $3 \leq \Delta_C(Q) \leq 6$*. More specifically, it holds that*

(i) $\Delta_{T_{\mathbf{M}}}(Q) \in \{3,5,6\}$*;*
(ii) $\Delta_{T_{\mathbf{P}}}(Q) \in \{3,4,5,6\}$*;*
(iii) $\Delta_{T_{\mathbf{L}}}(Q) \in \{3,6\}$*.*

Proposition 4. *Let* C *be a commutative quasi-copula. A reciprocal relation* Q *on* X *is cycle-transitive w.r.t.*

(i) $U(\alpha,\beta,\gamma) = \beta + \gamma - C(\beta,\gamma)$ *if and only if* $\Delta_C(Q) \geq 4$*;*
(ii) $U(\alpha,\beta,\gamma) = \alpha + \gamma - C(\alpha,\gamma)$ *if and only if* $\Delta_C(Q) \geq 5$*;*
(iii) $U(\alpha,\beta,\gamma) = \alpha + \beta - C(\alpha,\beta)$ *if and only if* $\Delta_C(Q) = 6$*.*

Statement (iii) is nothing else but a rephrasing of Proposition 1. According the above proposition (statement (ii) applied to $C = T_{\mathbf{M}}$), partial stochastic transitivity of a reciprocal relation implies that it is 'at least 5/6' $T_{\mathbf{M}}$-transitive.

For ease of reference, we will refer to cycle-transitivity w.r.t. $U(\alpha,\beta,\gamma) = \beta + \gamma - C(\beta,\gamma)$ as weak C-transitivity, to cycle-transitivity w.r.t. $U(\alpha,\beta,\gamma) = \alpha + \gamma - C(\alpha,\gamma)$ as moderate C-transitivity, and to cycle-transitivity w.r.t. $U(\alpha,\beta,\gamma) = \alpha + \beta - C(\alpha,\beta)$ as (strong) C-transitivity.

3 Similarity of Fuzzy Sets

3.1 Basic Notions

Recall that an *equivalence relation* E on X is a reflexive, symmetric and transitive relation on X and that there exists a one-to-one correspondence between equivalence relations on X and partitions of X. In fuzzy set theory, the counterpart of an equivalence relation is a T-*equivalence*: given a t-norm T, a T-equivalence E on X is a fuzzy relation on X that is reflexive ($E(x,x) = 1$), symmetric ($E(x,y) = E(y,x)$) and T-transitive. A T-equivalence is called a T-equality if $E(x,y)$ implies $x = y$.

For the prototypical t-norms, it is interesting to note that (see e.g. [15,17]):

(i) A fuzzy relation E on X is a $T_{\mathbf{L}}$-equivalence if and only if $d = 1 - E$ is a pseudo-metric on X.
(ii) A fuzzy relation E on X is a $T_{\mathbf{P}}$-equivalence if and only if $d = -\log E$ is a pseudo-metric on X.
(iii) A fuzzy relation E on X is a $T_{\mathbf{M}}$-equivalence if and only if $d = 1 - E$ is a pseudo-ultra-metric on X. Another interesting characterization is that a fuzzy relation E on X is a $T_{\mathbf{M}}$-equivalence if and only if for any $\alpha \in [0,1]$ its α-cut $E_\alpha = \{(x,y) \in X^2 \mid E(x,y) \geq \alpha\}$ is an equivalence relation on X. The equivalence classes of E_α become smaller for increasing α leading to the concept of a partition tree (see e.g. [26]).

3.2 A Logical Approach

To any left-continuous t-norm T, there corresponds a residual implicator $I_T : [0,1]^2 \to [0,1]$ defined by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}\,,$$

which can be considered as a generalization of the Boolean implication. Note that $I_T(x,y) = 1$ if and only if $x \leq y$. In case $y < x$, one gets for the prototypical t-norms: $I_{\mathbf{M}}(x,y) = y$, $I_{\mathbf{P}}(x,y) = y/x$ and $I_{\mathbf{L}}(x,y) = \min(1-x+y, 1)$. An essential property of the residual implicator of a left-continuous t-norm is related to the classical syllogism:

$$T(I_T(x,y), I_T(y,z)) \leq I_T(x,z))\,,$$

for any $(x,y,z) \in [0,1]^3$. The residual implicator is the main constituent of the biresidual operator $\mathcal{E}_T : [0,1]^2 \to [0,1]$ defined by

$$\mathcal{E}_T(x,y) = \min(I_T(x,y), I_T(y,x)) = I_T(\max(x,y), \min(x,y))\,,$$

which can be considered as a generalization of the Boolean equivalence. Note that $\mathcal{E}_T(x,y) = 1$ if and only if $x = y$. In case $x \neq y$, one gets for the prototypical t-norms: $\mathcal{E}_{\mathbf{M}}(x,y) = \min(x,y)$, $\mathcal{E}_{\mathbf{P}}(x,y) = \min(x,y)/\max(x,y)$ and $\mathcal{E}_{\mathbf{L}}(x,y) = 1 - |x-y|$.

Of particular importance in this discussion is the fact that $\mathcal{E}_T$ is a T-equality on $[0,1]$. The biresidual operator obviously serves as a means for measuring equality of membership degrees. Any T-equality E on $[0,1]$ can be extended in a natural way to $\mathcal{F}(X)$, the class of fuzzy sets in X:

$$E'(A,B) = \inf_{x\in X} E(A(x),B(x))\,.$$

It then holds that E' is a T-equality on $\mathcal{F}(X)$ if and only if E is a T-equality on $[0,1]$. Starting from $\mathcal{E}_T$ we obtain the T-equality E^T. A second way of defining a T-equality on $\mathcal{F}(X)$ is by defining

$$E_T(A,B) = T(\inf_{x\in X} I_T(A(x),B(x)), \inf_{x\in X} I_T(B(x),A(x)))\,.$$

The underlying idea is that in order to measure equality of two (fuzzy) sets A and B, one should both measure inclusion of A in B, and of B in A. Note that in general $E_T \subseteq E^T$, while $E_{\mathbf{M}} = E^{\mathbf{M}}$. These T-equivalences can be used as a starting point for building metrics on $\mathcal{F}(X)$. The above ways of measuring equality of fuzzy sets are very strict in the sense that the "worst" element decides upon the value.

Without going into detail, it is worth mentioning that there exist an appropriate notion of fuzzy partition, called T-partition [16], so that there exists a one-to-one correspondence between T-equalities on X and T-partitions of X [17].

3.3 A Cardinal Approach

Classical Cardinality-Based Similarity Measures. A common recipe for comparing objects is to select an appropriate set of features and to construct for each object a binary vector encoding the presence (1) or absence (0) of each of these features. Such a binary vector can be formally identified with the corresponding set of present features. The degree of similarity of two objects is then often expressed in terms of the cardinalities of the latter sets. We focus our attention on a family of $[0,1]$-valued similarity measures that are rational expressions in the cardinalities of the sets involved [12]:

$$S(A,B) = \frac{x\,\alpha_{A,B} + t\,\omega_{A,B} + y\,\delta_{A,B} + z\,\nu_{A,B}}{x'\,\alpha_{A,B} + t'\,\omega_{A,B} + y'\,\delta_{A,B} + z'\,\nu_{A,B}}\,,$$

with $A, B \in \mathcal{P}(X)$ (the powerset of a finite universe X),

$$\begin{aligned}
\alpha_{A,B} &= \min(|A\setminus B|, |B\setminus A|)\,,\\
\omega_{A,B} &= \max(|A\setminus B|, |B\setminus A|)\,,\\
\delta_{A,B} &= |A\cap B|\,,\\
\nu_{A,B} &= |(A\cup B)^c|\,,
\end{aligned}$$

and $x, t, y, z, x', t', y', z' \in \{0,1\}$. Note that these similarity measures are symmetric, *i.e.* $S(A,B) = S(B,A)$ for any $A, B \in \mathcal{P}(X)$.

Reflexive similarity measures, *i.e.* $S(A,A) = 1$ for any $A \in \mathcal{P}(X)$, are characterized by $y = y'$ and $z = z'$. We restrict our attention to the (still large) subfamily obtained by putting also $t = x$ and $t' = x'$ [5,14], *i.e.*

$$S(A,B) = \frac{x \bigtriangleup_{A,B} + y\, \delta_{A,B} + z\, \nu_{A,B}}{x' \bigtriangleup_{A,B} + y\, \delta_{A,B} + z\, \nu_{A,B}}\,, \tag{6}$$

with $\bigtriangleup_{A,B} = |A \bigtriangleup B| = |A \setminus B| + |B \setminus A|$. On the other hand, we allow more freedom by letting the parameters x, y, z and x' take positive real values. Note that these parameters can always be scaled to the unit interval by dividing both numerator and denominator of (6) by the greatest among the parameters. In order to guarantee that $S(A,B) \in [0,1]$, we need to impose the restriction $0 \leq x \leq x'$. Since the case $x = x'$ leads to trivial measures taking value 1 only, we consider from here on $0 \leq x < x'$. The similarity measures gathered in Table 1 all belong to family (6); the corresponding parameter values are indicated in the table.

Table 1. Some well-known cardinality-based similarity measures

Measure	expression	x	x'	y	z	T
Jaccard [34]	$\frac{\|A\cap B\|}{\|A\cup B\|}$	0	1	0	1	$T_{\mathbf{L}}$
Simple Matching [50]	$1 - \frac{\|A\bigtriangleup B\|}{n}$	0	1	1	1	$T_{\mathbf{L}}$
Dice [29]	$\frac{2\|A\cap B\|}{\|A\bigtriangleup B\|+2\|A\cap B\|}$	0	1	2	0	–
Rogers and Tanimoto [46]	$\frac{n-\|A\bigtriangleup B\|}{n+\|A\bigtriangleup B\|}$	0	2	1	1	$T_{\mathbf{L}}$
Sneath and Sokal 1 [49]	$\frac{\|A\cap B\|}{\|A\cap B\|+2\|A\bigtriangleup B\|}$	0	2	1	0	$T_{\mathbf{L}}$
Sneath and Sokal 2 [49]	$1 - \frac{\|A\bigtriangleup B\|}{2n-\|A\bigtriangleup B\|}$	0	1	2	2	–

The $T_{\mathbf{L}}$- or $T_{\mathbf{P}}$-transitive members of family (6) are characterized in the following proposition.

Proposition 5. [14]

(i) *The $T_{\mathbf{L}}$-transitive members of family (6) are characterized by the necessary and sufficient condition* $x' \geq \max(y, z)$.
(ii) *The $T_{\mathbf{P}}$-transitive members of family (6) are characterized by the necessary and sufficient condition* $x\,x' \geq \max(y^2, z^2)$.

Fuzzy Cardinality-Based Similarity Measures. Often, the presence or absence of a feature is not clear-cut and is rather a matter of degree. Hence, if instead of binary vectors we have to compare vectors with components in the real unit interval $[0,1]$ (the higher the number, the more the feature is present), the need arises to generalize the aforementioned similarity measures. In fact, in the same way as binary vectors can be identified with ordinary subsets of a finite universe X, vectors with components in $[0,1]$ can be identified with fuzzy sets in X.

In order to generalize a cardinality-based similarity measure to fuzzy sets, we clearly need fuzzification rules that define the cardinality of a fuzzy set and translate the classical set-theoretic operations to fuzzy sets. As to the first, we stick to the following simple way of defining the cardinality of a fuzzy set, also known as the sigma-count of A [55]: $|A| = \sum_{x\in X} A(x)$. As to the second, we define the intersection of two fuzzy sets A and B in X in a pointwise manner by $A \cap B(x) = C(A(x), B(x))$, for any $x \in X$, where C is a commutative conjunctor. In [14], we have argued that commutative quasi-copulas are the most appropriate conjunctors for our purpose. Commutative quasi-copulas not only allow to introduce set-theoretic operations on fuzzy sets, such as $A \setminus B(x) = A(x) - C(A(x), B(x))$ and $A \triangle B(x) = A(x) + B(x) - 2C(A(x), B(x))$, they also preserve classical identities on cardinalities, such as $|A \setminus B| = |A| - |A \cap B|$ and $|A \triangle B| = |A \setminus B| + |B \setminus A| = |A| + |B| - 2|A \cap B|$. These identities allow to rewrite and fuzzify family (6) as

$$S(A, B) = \frac{x(a + b - 2u) + yu + z(n - a - b + u)}{x'(a + b - 2u) + yu + z(n - a - b + u)}\,, \tag{7}$$

with $a = |A|$, $b = |B|$ and $u = |A \cap B|$.

Bell-Inequalities and Preservation of Transitivity. Studying the transitivity of (fuzzy) cardinality-based similarity measures inevitably leads to the verification of inequalities on (fuzzy) cardinalities. We have established several powerful meta-theorems that provide an efficient and intelligent way of verifying whether a classical inequality on cardinalities carries over to fuzzy cardinalities [13]. These meta-theorems state that certain classical inequalities are preserved under fuzzification when modelling fuzzy set intersection by means of a commutative conjunctor that fulfills a number of Bell-type inequalities.

In [35], we introduced the classical Bell inequalities in the context of fuzzy probability calculus and proved that the following Bell-type inequalities for commutative conjunctors are necessary and sufficient conditions for the corresponding Bell-type inequalities for fuzzy probabilities to hold. The Bell-type inequalities for a commutative conjunctor C read as follows:

$$\begin{aligned}
&B_1 : T_{\mathbf{L}}(p, q) \le C(p, q) \le T_{\mathbf{M}}(p, q)\\
&B_2 : 0 \le p - C(p, q) - C(p, r) + C(q, r)\\
&B_3 : p + q + r - C(p, q) - C(p, r) - C(q, r) \le 1
\end{aligned}$$

for any $p, q, r \in [0, 1]$. Inequality B_2 is fulfilled for any commutative quasi-copula, while inequality B_3 only holds for certain t-norms [36], including the members of the Frank t-norm/copula family $T^{\mathbf{F}}_{\lambda}$ with $\lambda \le 9 + 4\sqrt{5}$ [45]. Also note that inequality B_1 follows from inequality B_2.

Theorem 1. [13] *Consider a commutative conjunctor I that satisfies Bell inequalities B_2 and B_3. If for any ordinary subsets A, B and C of an arbitrary finite universe X it holds that*

$$\mathcal{H}(|A|, |B|, |C|, |A \cap B|, |A \cap C|, |B \cap C|, |X|) \ge 0\,,$$

where $\mathcal{H}$ denotes a continuous function which is homogeneous in its arguments, then it also holds for any fuzzy sets in an arbitrary finite universe Y.

If the function $\mathcal{H}$ does not depend explicitly upon $|X|$, then Bell inequality B_3 can be omitted. This meta-theorem allows us to identify conditions on the parameters of the members of family (7) leading to $T_{\mathbf{L}}$-transitive or $T_{\mathbf{P}}$-transitive fuzzy similarity measures. As our fuzzification is based on a commutative quasi-copula C, condition B_2 holds by default. The following proposition then is an immediate application.

Proposition 6. [13]

(i) *Consider a commutative quasi-copula C that satisfies B_3. The $T_{\mathbf{L}}$-transitive members of family (7) are characterized by $x' \geq \max(y, z)$.*
(ii) *The $T_{\mathbf{L}}$-transitive members of family (7) with $z = 0$ are characterized by $x' \geq y$.*
(iii) *Consider a commutative quasi-copula C that satisfies B_3. The $T_{\mathbf{P}}$-transitive members of family (7) are characterized by $x\,x' \geq \max(y^2, z^2)$.*
(iv) *The $T_{\mathbf{P}}$-transitive members of family (7) with $z = 0$ are characterized by $xx' \geq y^2$.*

However, as our meta-theorem is very general, it does not necessarily always provide the strongest results. For instance, tedious and lengthy direct proofs allow to eliminate condition B_3 from the previous theorem, leading to the following general result.

Proposition 7. [13] *Consider a commutative quasi-copula C.*

(i) *The $T_{\mathbf{L}}$-transitive members of family (7) are characterized by the necessary and sufficient condition $x' \geq \max(y, z)$.*
(ii) *The $T_{\mathbf{P}}$-transitive members of family (7) are characterized by the necessary and sufficient condition $x\,x' \geq \max(y^2, z^2)$.*

4 Comparison of Random Variables

4.1 Dice-Transitivity

Consider three dice A, B and C which, instead of the usual numbers, carry the following integers on their faces:

$$A = \{1, 3, 4, 15, 16, 17\}, \quad B = \{2, 10, 11, 12, 13, 14\}, \quad C = \{5, 6, 7, 8, 9, 18\}.$$

Denoting by $\mathcal{P}(X, Y)$ the probability that dice X wins from dice Y, we have $\mathcal{P}(A, B) = 20/36$, $\mathcal{P}(B, C) = 25/36$ and $\mathcal{P}(C, A) = 21/36$. It is natural to say that dice X is strictly preferred to dice Y if $\mathcal{P}(X, Y) > 1/2$, which reflects that dice X wins from dice Y in the long run (or that X statistically wins from Y, denoted $X >_s Y$). Note that $\mathcal{P}(Y, X) = 1 - \mathcal{P}(X, Y)$ which implies that the relation $>_s$ is asymmetric. In the above example, it holds that $A >_s B$, $B >_s C$

and $C >_s A$: the relation $>_s$ is not transitive and forms a cycle. In other words, if we interpret the probabilities $\mathcal{P}(X, Y)$ as constituents of a reciprocal relation on the set of alternatives $\{A, B, C\}$, then this reciprocal relation is even not weakly stochastic transitive.

This example can be generalized as follows: we allow the dice to possess any number of faces (whether or not this can be materialized) and allow identical numbers on the faces of a single or multiple dice. In other words, a generalized dice can be identified with a multiset of integers. Given a collection of m such generalized dice, we can still build a reciprocal relation Q containing the *winning probabilities* for each pair of dice [28]. For any two such dice A and B, we define

$$Q(A, B) = \mathcal{P}\{A \text{ wins from } B\} + \frac{1}{2}\mathcal{P}\{A \text{ and } B \text{ end in a tie}\}.$$

The dice or integer multisets may be identified with independent discrete random variables that are uniformly distributed on these multisets (i.e. the probability of an integer is proportional to its number of occurrences); the reciprocal relation Q may be regarded as a quantitative description of the pairwise comparison of these random variables.

In the characterization of the transitivity of this reciprocal relation, a type of cycle-transitivity, which can neither be seen as a type of C-transitivity, nor as a type of FG-transitivity, has proven to play a predominant role. For obvious reasons, this new type of transitivity has been called dice-transitivity.

Definition 7. *Cycle-transitivity w.r.t. the upper bound function U_D defined by*

$$U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma,$$

is called dice-transitivity*.*

Dice-transitivity is nothing else but a synonym for weak $T_{\mathbf{P}}$-product transitivity. According to Proposition 4, dice-transitivity of a reciprocal relation implies that it is 'at least 4/6' $T_{\mathbf{P}}$-transitive. Dice-transitivity can be situated between $T_{\mathbf{L}}$-transitivity and $T_{\mathbf{P}}$-transitivity, and also between $T_{\mathbf{L}}$-transitivity and moderate stochastic transitivity.

Proposition 8. [28] *The reciprocal relation generated by a collection of generalized dice is dice-transitive.*

4.2 A Method for Comparing Random Variables

Many methods can be established for the comparison of the components (random variables, r.v.) of a random vector $(X_1, \ldots, X_n)$, as there exist many ways to extract useful information from the joint cumulative distribution function (c.d.f.) $F_{X_1,\ldots,X_n}$ that characterizes the random vector. A first simplification consists in comparing the r.v. two by two. It means that a method for comparing r.v. should only use the information contained in the bivariate c.d.f. F_{X_i,X_j}. Therefore, one can very well ignore the existence of a multivariate c.d.f. and just describe mutual dependencies between the r.v. by means of the bivariate c.d.f. Of course one should be aware that not all choices of bivariate c.d.f. are compatible with a multivariate c.d.f. The problem of characterizing those ensembles of bivariate c.d.f. that can be identified with the marginal bivariate c.d.f. of a single multivariate c.d.f., is known as the *compatibility problem* [44].

A second simplifying step often made is to bypass the information contained in the bivariate c.d.f. to devise a comparison method that entirely relies on the one-dimensional marginal c.d.f. In this case there is even not a compatibility problem, as for any set of univariate c.d.f. F_{X_i}, the product $F_{X_1} F_{X_2} \cdots F_{X_n}$ is a valid joint c.d.f., namely the one expressing the independence of the r.v. There are many ways to compare one-dimensional c.d.f., and by far the simplest one is the method that builds a partial order on the set of r.v. using the principle of first order stochastic dominance [40]. It states that a r.v. X is weakly preferred to a r.v. Y if for all $u \in \mathbb{R}$ it holds that $F_X(u) \leq F_Y(u)$. At the extreme end of the chain of simplifications, are the methods that compare r.v. by means of a characteristic or a function of some characteristics derived from the one-dimensional marginal c.d.f. The simplest example is the weak order induced by the expected values of the r.v.

Proceeding along the line of thought of the previous section, a random vector $(X_1, X_2, \ldots, X_m)$ generates a reciprocal relation by means of the following recipe.

Definition 8. *Given a random vector* $(X_1, X_2, \ldots, X_m)$*, the binary relation* Q *defined by*

$$Q(X_i, X_j) = \mathcal{P}\{X_i > X_j\} + \frac{1}{2}\,\mathcal{P}\{X_i = X_j\}$$

is a reciprocal relation.

For two discrete r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \sum_{k>l} p_{X_i,X_j}(k, l) + \frac{1}{2} \sum_{k} p_{X_i,X_j}(k, k)\,,$$

with p_{X_i,X_j} the joint probability mass function (p.m.f.) of (X_i, X_j). For two continuous r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as:

$$Q(X_i, X_j) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{x} f_{X_i,X_j}(x, y)\, dy\,,$$

with f_{X_i,X_j} the joint probability density function (p.d.f.) of (X_i, X_j).

For this pairwise comparison, one needs the two-dimensional marginal distributions. Sklar's theorem [44,48] tells us that if a joint cumulative distribution function F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C_{ij} such that for all x, y:

$$F_{X_i,X_j}(x, y) = C_{ij}(F_{X_i}(x), F_{X_j}(y))\,.$$

If X_i and X_j are continuous, then C_{ij} is unique; otherwise, C_{ij} is uniquely determined on $\mathrm{Ran}(F_{X_i}) \times \mathrm{Ran}(F_{X_j})$.

As the above comparison method takes into account the bivariate marginal c.d.f. it takes into account the dependence of the components of the random vector. The information contained in the reciprocal relation is therefore much richer than if, for instance, we would have based the comparison of X_i and X_j solely on their expected values. Despite the fact that the dependence structure is entirely captured by the multivariate c.d.f., the pairwise comparison is only apt to take into account pairwise dependence, as only bivariate c.d.f. are involved. Indeed, the bivariate c.d.f. do not fully disclose the dependence structure; the r.v. may even be pairwise independent while not mutually independent.

Since the copulas C_{ij} that couple the univariate marginal c.d.f. into the bivariate marginal c.d.f. can be different from one another, the analysis of the reciprocal relation and in particular the identification of its transitivity properties appear rather cumbersome. It is nonetheless possible to state in general, without making any assumptions on the bivariate c.d.f., that the probabilistic relation Q generated by an arbitrary random vector always shows some minimal form of transitivity.

Proposition 9. [7] *The reciprocal relation Q generated by a random vector is $T_{\mathbf{L}}$-transitive.*

4.3 Artificial Coupling of Random Variables

Our further interest is to study the situation where abstraction is made that the r.v. are components of a random vector, and all bivariate c.d.f. are enforced to depend in the same way upon the univariate c.d.f., in other words, we consider the situation of all copulas being the same, realizing that this might not be possible at all. In fact, this simplification is equivalent to considering instead of a random vector, a collection of r.v. and to artificially compare them, all in the same manner and based upon a same copula. The pairwise comparison then relies upon the knowledge of the one-dimensional marginal c.d.f. solely, as is the case in stochastic dominance methods. Our comparison method, however, is not equivalent to any known kind of stochastic dominance, but should rather be regarded as a graded variant of it (see also [8]).

The case $C = T_{\mathbf{P}}$ generalizes Proposition 8, and applies in particular to a collection of independent r.v. where all copulas effectively equal $T_{\mathbf{P}}$.

Proposition 10. [27,28] *The reciprocal relation Q generated by a collection of r.v. pairwisely coupled by $T_{\mathbf{P}}$ is dice-transitive.*

Next, we discuss the case when using one of the extreme copulas to artificially couple the r.v. In case $C = T_{\mathbf{M}}$, the r.v. are coupled comonotonically. Note that this case is possible in reality. Comparing with Proposition 9, the following proposition expresses that this way of coupling does not lead to a gain in transitivity.

Proposition 11. [24,25] *The reciprocal relation Q generated by a collection of r.v. pairwisely coupled by $T_{\mathbf{M}}$ is $T_{\mathbf{L}}$-transitive.*

In case $C = T_{\mathbf{L}}$, the r.v. are coupled countermonotonically. This assumption can never represent a true dependence structure for more than two r.v., due to the compatibility problem.

Proposition 12. [24,25] *The reciprocal relation* Q *generated by a collection of r.v. pairwisely coupled by* $T_{\mathbf{L}}$ *is partially stochastic transitive.*

The proofs of these propositions were first given for discrete uniformly distributed r.v. [25,28]. It allowed for an interpretation of the values $Q(X_i, X_j)$ as winning probabilities in a hypothetical dice game, or equivalently, as a method for the pairwise comparison of ordered lists of numbers. Subsequently, we have shown that as far as transitivity is concerned, this situation is generic and therefore characterizes the type of transitivity observed in general [24,27].

The above results are special cases of a more general result [7,9].

Proposition 13. *Consider a Frank copula* $T^{\mathbf{F}}_{\lambda}$*, then the reciprocal relation* Q *generated by a collection of random variables pairwisely coupled by* $T^{\mathbf{F}}_{\lambda}$ *is cycle-transitive w.r.t. to the upper bound function* U^{λ} *defined by:*

$$U^{\lambda}(\alpha, \beta, \gamma) = \beta + \gamma - T^{\mathbf{F}}_{1/\lambda}(\beta, \gamma)\,.$$

4.4 Comparison of Special Independent Random Variables

Dice-transitivity is the generic type of transitivity shared by the reciprocal relations generated by a collection of independent r.v. If one considers independent r.v. with densities all belonging to one of the one-parameter families in Table 2, the corresponding reciprocal relation shows the corresponding type of cycle-transitivity listed in Table 3 [27].

Note that all upper bound functions in Table 3 are self-dual. More striking is that the two families of power-law distributions (one-parameter subfamilies of the two-parameter Beta and Pareto families) and the family of Gumbel distributions, all yield the same type of transitivity as exponential distributions, namely cycle-transitivity w.r.t. the self-dual upper bound function U_E defined by:

$$U_E(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma\,.$$

Table 2. Parametric families of continuous distributions

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, \infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, \infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R},\, \mu > 0$	$x \in]-\infty, \infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R},\, a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$(e^{-\lvert x-\lambda \rvert/\mu})/(2\mu)$	$\lambda \in \mathbb{R},\, \mu > 0$	$x \in]-\infty, \infty[$
Normal	$(e^{-(x-\lambda)^2/2\sigma^2})/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}, \sigma > 0$	$x \in]-\infty, \infty[$

Table 3. Cycle-transitivity for the continuous distributions in Table 1

Name	Upper bound function $U(\alpha,\beta,\gamma)$
Exponential Beta Pareto Gumbel	$\alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma$
Uniform	$\begin{cases} \beta+\gamma-1+\frac{1}{2}[\max(\sqrt{2(1-\beta)}+\sqrt{2(1-\gamma)}-1,0)]^2 & \beta \geq 1/2 \\ \alpha+\beta-\frac{1}{2}[\max(\sqrt{2\alpha}+\sqrt{2\beta}-1,0)]^2 & \beta < 1/2 \end{cases}$
Laplace	$\begin{cases} \beta+\gamma-1+f^{-1}(f(1-\beta)+f(1-\gamma)) & \beta \geq 1/2 \\ \alpha+\beta-f^{-1}(f(\alpha)+f(\beta)) & \beta < 1/2 \end{cases}$ with $f^{-1}(x)=\frac{1}{2}\left(1+\frac{x}{2}\right)e^{-x}$
Normal	$\begin{cases} \beta+\gamma-1+\Phi(\Phi^{-1}(1-\beta)+\Phi(1-\gamma)) & \beta \geq 1/2 \\ \alpha+\beta-\Phi(\Phi^{-1}(\alpha)+\Phi^{-1}(\beta)) & \beta < 1/2 \end{cases}$ with $\Phi(x)=(\sqrt{2\pi})^{-1}\int_{-\infty}^{x} e^{-t^2/2}dt$

Cycle-transitivity w.r.t. U_E can also be expressed as

$$\alpha_{abc}\beta_{abc}\gamma_{abc} = (1-\alpha_{abc})(1-\beta_{abc})(1-\gamma_{abc})\,,$$

which is equivalent to the notion of multiplicative transitivity [53]. A reciprocal relation Q on X is called *multiplicatively transitive* if for any $(a,b,c) \in X^3$ it holds that

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)}\,.$$

In the cases of the unimodal uniform, Gumbel, Laplace and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions. In [27], we have shown that the corresponding reciprocal relation is in that case moderately stochastic transitive.

4.5 Mutual Rank Transitivity in Posets

Partially ordered sets, posets for short, are witnessing an increasing interest in various fields of application. They allow for incomparability of elements and can be conveniently visualized by means of a Hasse diagram. Two such fields are environmetrics and chemometrics [1,2]. In these applications, most methods eventually require a linearization of the poset. A standard way of doing so is to rank the elements on the basis of their averaged ranks, i.e. their average position computed over all possible linear extensions of the poset. Although the computation of these averaged ranks has become feasible for posets of reasonable size [19], they suffer from a weak information content as they are based on marginal distributions only, as explained further. For this reason, interest is shifting to mutual rank probabilities instead.

The mutual rank probability relation is an intriguing object that can be associated with any finite poset. For any two elements of the poset, it expresses the probability that the first succeeds the second in a random linear extension of that poset. Its computation is feasible as well for posets of reasonable size [19,21], and approximation methods are available for more extensive posets [18]. However, exploiting the information contained in the mutual rank probability relation to come up with a ranking of the elements is not obvious. Simply ranking one element higher than another when the corresponding mutual rank probability is greater than 1/2 is not appropriate, as it is prone to generating cycles (called linear extension majority cycles in this context [22,38]). A solution to this problem requires a better understanding, preferably a characterization, of the transitivity of mutual rank probability relations, coined proportional probabilistic transitivity by Fishburn [32], and, for the sake of clarity, renamed mutual rank transitivity here. A weaker type of transitivity (called δ^*-transitivity, expression not shown here) has been identified by Kahn and Yu [37] and Yu [54]. We have identified a weaker type of transitivity, yet enabling us to position mutual rank transitivity within the cycle-transitivity framework.

Consider a finite poset $(P, \leq)$. The discrete random variable X_a denotes the position (rank) of an element $a \in P$ in a random linear extension of P. The *mutual rank probability* $p_{a>b}$ of two different elements $a, b \in P$ is defined as the fraction of linear extensions of P in which a succeeds b (a is ranked higher than b), i.e., $p_{a>b} = \text{Prob}\{X_a > X_b\}$. The $[0,1]$-valued relation $Q_P : P^2 \to [0,1]$ defined by $Q_P(a,b) = p_{a>b}$, for all $a, b \in P$ with $a \neq b$, and $Q_P(a,a) = 1/2$, for all $a \in P$, is a reciprocal relation. Note that in the way described above, with any finite poset $P = \{a_1, \ldots, a_n\}$ we associate a unique discrete random vector $X = (X_{a_1}, \ldots, X_{a_n})$ with joint distribution function $F_{X_{a_1},\ldots,X_{a_n}}$. The mutual rank probabilities $p_{a_i>a_j}$ are then computed from the bivariate marginal distributions $F_{X_{a_i},X_{a_j}}$.

Note that, despite the fact that the joint distribution function $F_{X_{a_1},\ldots,X_{a_n}}$ does not lend itself to an explicit expression, a fair amount of pairwise couplings are of a very simple type. If it holds that $a > b$, then a succeeds b in all linear extensions of P, whence X_a and X_b are comonotone. For pairs of incomparable elements, the bivariate couplings can vary from pair to pair. Certainly, these couplings cannot all be counter-monotone. Despite all this, it is possible to obtain transitivity results on mutual rank probability relations [10].

Definition 9. *The mutual rank probability relation Q_P associated with a finite poset $(P, \leq)$ is cycle-transitive w.r.t. the upper bound function U_{P} defined by*

$$U_{\mathrm{P}}(\alpha, \beta, \gamma) = \alpha + \gamma - \alpha\gamma \,.$$

Proposition 4 implies that the mutual rank probability relation of a poset it is 'at least 5/6' $T_{\mathbf{P}}$-transitive.

5 Conclusion

We have introduced the reader to two relational frameworks and the wide variety of types of transitivity they cover. When considering different types of transitivity, we can try to distinguish weaker or stronger types. Obviously, one type is called weaker than another, if it is implied by the latter. Hence, we can equip a collection of types of transitivity with this natural order relation and depict it graphically by means of a Hasse diagram.

The Hasse diagram containing all types of transitivity of reciprocal relations encountered in this contribution is shown in Figure 1. At the lower end of the diagram, $T_{\mathbf{M}}$-transitivity and multiplicative transitivity, two types of cycle-transitivity w.r.t. a self-dual upper bound function, are incomparable and can be considered as the strongest types of transitivity. At the upper end of the diagram, also $T_{\mathbf{L}}$-transitivity and weak stochastic transitivity are incomparable and can be considered as the weakest types of transitivity. Furthermore, note that the subchain consisting of partial stochastic transitivity, moderate product

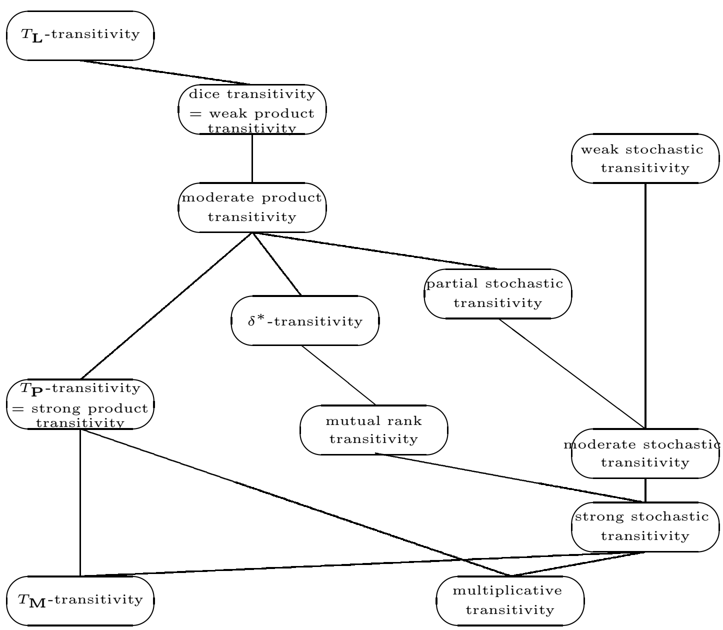

Fig. 1. Hasse diagram with different types of transitivity of reciprocal relations (weakest types at the top, strongest types at the bottom

transitivity and weak product transitivity, bridges the gap between g-stochastic transitivity and T-transitivity.

Anticipating on future work, in particular on applications, we can identify two important directions. The first direction concerns the use of fuzzy similarity measures. Moser [43] has shown recently that the T-equality E^T, with $T = T_{\mathbf{P}}$ or $T = T_{\mathbf{L}}$, is positive semi-definite. This question has not yet been addressed for the fuzzy cardinality-based similarity measures. Results of this type allow to bridge the gap between the fuzzy set community and the machine learning community, making some fuzzy similarity measures available as potential kernels for the popular kernel-based learning methods, either on their own or in combination with existing kernels (see e.g. [41] for an application of this type).

The second direction concerns the further exploitation of the results on the comparison of random variables. As mentioned, the approach followed here can be seen as a graded variant of the increasingly popular notion of stochastic dominance. Future research will have to clarify how these graded variants can be defuzzified in order to come up with meaningful partial orderings of random variables that are more informative than the classical notions of stochastic dominance. Some results into that direction can be found in [8,20].

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