Some Results on Subanalytic Variational Inclusions

Catherine Cabuzel and Alain Pietrus

Abstract. This chapter deals with variational inclusions of the form $0 \in f(x) + g(x) + F(x)$ where f is a locally Lipschitz and subanalytic function, g is a Lipschitz function, F is a set-valued map, acting all in $\mathbb{R}^n$ and n is a positive integer. The study of the previous variational inclusion depends on the properties of the function g. The behaviour as been examinated in different cases : when g is the null function, when g possesses divided differences and when g is not smooth and semismooth. We recall and give a summary of some known methods and the last section is very original and is unpublished. In this last section we combine a Newton type method (applied to f) with a secant type method (applied to g) and we obtain superlinear convergence to a solution of the variational inclusion. Our study in the present chapter is in the context of subanalytic functions, which are semismooth functions and the usual concept of derivative is replaced here by the the concept of Clarke's Jacobian.

1 Introduction

In this chapter, we present some methods for solving either variational inclusions of the type $0 \in f(x) + F(x)$ where f is a function and F is a set-valued map both defined on an open set of $\mathbb{R}^n$, or perturbed problems of the form $0 \in f(x) + g(x) + F(x)$ where g is the perturbation function.

Variational inclusions were introduced by Robinson [49, 50] at the end of the 70's as an abstract model for various problems encountered in fields such as mathematical programming, engineering [24], optimal control, economy

Catherine Cabuzel · Alain Pietrus
Laboratoire L.A.M.I.A - EA 4540, Université des Antilles et de la Guyane,
Département de Mathématiques et Informatique, Campus de Fouillole,
F–97159 Pointe–à–Pitre, France
e-mail: catherine.zebre@univ-ag.fr, apietrus@univ--ag.fr

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(Nash and Walras equilibrium), transport theory... More precisely, Robinson showed that complementary problems can be expressed by variational inclusions using the normal cone to a set. The important role played by variational inclusions in these topics led the researchers to improve the theory in relation with these inclusions; the high number of publications in the two last decades proves the interest showed in this area. A part of these works is devoted to the extension to variational inclusions of well-known classical algorithms existing for solving equations. Some of these works are used in this paper, the reader could also referred to ([1, 10, 11, 13, 14, 25, 26, 27, 28, 31]) for other developments.

Thanks to Rademacher's theorem [32], we know that Lipschitz functions are almost always differentiable, but these functions can be nonsmooth in some points. We can also underline the fact that in optimization the smoothness of applications can be broken (in some points) by some functions like the min or the max. Then it was quite natural to conceive algorithm taking into account such points. Let us point out the work of Benadada [4] in which the author gave a method for solving equations of the form $f(x) = 0$ when the function f is convex, non necessarily differentiable. Recently Bolte, Daniilidis and Lewis introduced an extension of Newton's method for subanalytic and locally Lipschitz functions [8]. The aim of this chapter is to present different recent algorithms to solve some variational inclusions in the case where the univoque part is subanalytic, not necessarily smooth. For all these algorithms we give some convergence results. Let us notice that Newton's method proposed in this paper is different from the one proposed by Benadada because we don't need convexity. All the methods introduced in this paper use the concept of regularity for set-valued maps (see [3, 23, 51, 52]).

For a better understanding, we organized the following development in three parts. After the presentation made currently, in section 2, we recall some results used further. The divided differences are given in Banach spaces, although they can be defined on more general spaces. Let us notice that continuity is sufficient to define divided differences and in the case where the function is Frechet differentiable, Byelostotskij [12] used in 1962 the following form:

$$[x_0, y_0, g] = \int_0^1 \nabla g(x_0 + t(y_0 - x_0))\mathrm{d}t.$$

Divided differences (see [2, 40]) have been used in various ways in numerical analysis, interpolation method (see Example 1), the solving of equations of the form $F(x) = 0$ by Hernandez and Rubio [33, 34, 35, 36] under Lipschitz condition (Definition 3) or ω-condition (Definition 4). In Example 2, the function used admits divided differences even when it is not differentiable. In our work, the divided differences are used in the secant-type method (Paragraph 4.2) where the perturbation function g admits first and second order divided differences.

Semialgebraic or semianalytic sets, and more generally, algebraic geometry, have been studied since the 50's. Important works were supervised by Lojasiewicz [43, 44, 45], Hironaka [37], Bochnak and al. [6], Bierstone and Milman [5]. Lojasiewicz was the first to establish the so-called Lojasiewicz inequality for a class of real analytic functions. This result was extended to C^1 subanalytic functions by Kurdyka and Parusinski in [41] and to C^1 nonsmmoth lower semicontinuous functions by Bolte, Daniilidis and Lewis in [7]. This inequality has also been used in order to obtain new results in partial differential equations by Huang [38] and Lojasiewicz [42] and in nonconvex optimization and in nonsmooth analysis. The Tarski-Seidenberg theorem ensures the stability of semialgebraic sets by projection and Gabrielov's theorem gives the stability of the complementary of subanalytic sets. These last properties explain the interest to use semialgebraic and subanalytic sets and functions. In Paragraph 2.2, we present some definitions, properties and examples of semianalytic and subanalytic sets and functions. We end this part with a short collection of standard properties in set–valued analysis. Now this theory is well elaborate and very complete books can be found on this topic. Some authors like Aubin and Frankowska [3], Dontchev [23], Mordukhovich [48], Rockafellar [52] gave very good contributions on this area.

In Section 3, we give a summary of a Newton type method introduced in [15]. This work was inspired by Dontchev's method [21]. Bolte, Daniilidis and Lewis in [8], extend the Newton method for solving classical equations for functions f which are subanalytic. Since almost all the methods used to approximate solutions of variational inclusions were made in the case where the function f is Frechet differentiable, here, following the work made in [8], we extend the result obtained by Dontchev in [21] to variational inclusions where the function is subanalytic non necessarily smooth. With the help of the Aubin property, we obtain a sequence which is superlinearly convergent to a solution of the variational inclusion.

The Section 4 is entirely devoted to the study of perturbed problems: we firstly examine a method which is inspired by a work of Geoffroy and Pietrus [29] where the perturbation function g is Lipschitz; an extended version of this work has been published in [16]. We show the linear convergence of the sequence obtained. To end this part, we focus on an original unpublished secant type method. This last method has been studied by Geoffroy and Pietrus [28] only in the case where the function f is smooth and here, our contribution is to extend to the context of nonsmooth subanalytic functions. It is easy to see that a variant of the previous method returns to a Newton-type method as in [15] under some regularity conditions on some set-valued map obtained after a modification of the original set-valued map.

2 Preliminary Results

In this section, we collect different results concerning divided differences, subanalytic functions and the continuity of set-valued maps. We use the following notations: in a metric space (Z, ρ), the distance from a point x to a set A is denoted $dist\,(x, A) = \inf\{\rho(x, y),\ y \in A\}$, the excess e from the set A to the set C is given by $e(A, C) = \sup\{dist\,(x, A),\ x \in C\}$; $B_r(x)$ stands for the closed ball centered at x with radius $r > 0$ and the norm is denoted by $\| \,.\, \|$. In two spaces X and Y, $\mathcal{L}(X, Y)$ denotes the space of linear operators actif from X into Y, $\Lambda : X \rightrightarrows Y$ denotes a set–valued map from X to the subsets of Y; its graph is defined by $graph\,\Lambda = \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and its inverse is defined by $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$.

2.1 Divided Differences

The concept of divided difference for an operator was used by authors in many works; one can also take advantage of the development made in [40] on this topic.

Definition 1. *An operator* $[x_0, y_0, g] \in \mathcal{L}(X, Y)$ *is called a* divided difference of first order *of the function* $g : X \to Y$ *at the points* x_0 *and* y_0 *if both following conditions are satisfied:*

- (a) $[x_0, y_0, g](y_0 - x_0) = g(y_0) - g(x_0)$ *for* $x_0 \neq y_0$*;*
- (b) *If* g *is Frechet differentiable at* $x_0 \in X$ *then we denote* $[x_0, x_0, g] = \nabla g(x_0)$*.*

Remark 1. The equality $[x, y, g] = [y, x, g]$ is generally false in infinite dimensional spaces when g is not Frechet differentiable, however it is true in the direction $y - x$. In other words, one has $[x, y, g](y - x) = [y, x, g](y - x)$ for all x and y in a Banach space X.

Definition 2. *An operator* $[x_0, y_0, z_0, g] \in \mathcal{L}(X, \mathcal{L}(X, Y))$ *is called a* divided difference of second order *of the function* $g : X \to Y$ *at the points* x_0, y_0, z_0 *if both following conditions are satisfied:*

- (a) $[x_0, y_0, z_0, g](z_0 - x_0) = [y_0, z_0, g] - [x_0, y_0, g]$ *for* x_0, y_0 *and* z_0 *distincts;*
- (b) *If* g *admits a second order Frechet derivative at* $x_0 \in X$ *then we denote* $[x_0, x_0, x_0, g] = \dfrac{\nabla^2 g(x_0)}{2}$*.*

The following examples in the real case shows the importance of this operator.

Example 1. Isaac Newton [1] constructed an interpolation polynomial not using the basis $\{1; x; x^2; ...; x^n\}$ or the Lagrange basis $\{L_i\}$, but the basic polynomials $\{\pi_i\}$ defined by

[1] Isaac Newton (1642-1727)

$$\pi_i(x) = \begin{cases} 1, & \text{if} \quad i = 0 \\ (x - x_0)(x - x_1)...(x - x_{i-1}), & \text{if} \quad 1 \leq i \leq n. \end{cases}$$

The interpolation polynomial P_n of a function fonction f at the points $x_0, x_1, x_2, ..., x_n$, is written $P_n(x) = a_0\,\pi_0(x) + a_1\,\pi_1(x) + ... + a_n\,\pi_n(x)$ where $P_n(x_j) = f(x_j)$ for $0 \leq j \leq n$, where $a_j = [x_0, x_1, ..., x_j, f]$. With this notation, the interpolation polynomial can be written $P_n(x) = [x_0, f]\,\pi_0(x) + [x_0, x_1, f]\,\pi_1(x) + ... + [x_0, x_1, ..., x_n, f]\,\pi_n(x)$ where $[x_0, x_1, ..., x_n, f]$ is defined by $[x_0, x_1, ..., x_n, f] = \dfrac{[x_1, ..., x_n, f] - [x_0, x_1, ..., x_{n-1}, f]}{x_n - x_0}$.
For example, considering the function f defined by $f(x) = 2^x$, the interpolation polynomial of f at the points $-2, -1, 0, 1, 2$ is
$P_4(x) \quad = \quad \dfrac{1}{4} + \dfrac{1}{4}(x+2) + \dfrac{1}{8}(x+2)(x+1) + \dfrac{1}{24}(x+2)(x+1)x\dfrac{1}{96}(x+2)$ $(x+1)x(x-1)$; thus we obtain $2^{\frac{1}{2}} = \sqrt{2} \approx P_4(\frac{1}{2}) = \frac{723}{512} \approx 1,4142$.

Example 2. [34] Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined for all $(x, y) \in \mathbb{R}^2$, $f(x, y) = y^2 + x - 7 + \dfrac{1}{9}|y|$. This function is not differentiable at every points of $\mathbb{R} \times \{0\}$, but its admits divided differences everywhere. One has

$$[u, v, f] = \Big([u, v, f]_{11}, [u, v, f]_{12}\Big)$$

where

$$[u, v, f]_{11} = \frac{f(u_1, v_2) - f(v_1, v_2)}{u_1 - v_1},\ u_1 \neq v_1.$$

$$[u, v, f]_{12} = \frac{f(u_1, u_2) - f(u_1, v_2)}{u_2 - v_2},\ u_2 \neq v_2.$$

with the previous definition, we obtain

$$[u, v, f] = \left(1, \frac{u_2^2 - v_2^2}{u_2 - v_2} + \frac{1}{9}\,\frac{|u_2| - |v_2|}{u_2 - v_2}\right)$$

Remark 2. There exists some links between differentiability and divided differences. One can show that if a function $f : X \to Y$ admits divided differences satisfying one of the following inequalities on an open set $\Omega \subset X$,

$$\|[x, y, f] - [x, z, f]\| \leq c_0\|y - z\|$$
$$\|[y, x, f] - [z, x, f]\| \leq c_1\|y - z\| \;\forall\, x, y, z\, \in \Omega$$

where c_0 and c_1 are positive constants, then f is Frechet differentiable on Ω and if both inequalities are verified, the Frechet derivative ∇f of f is Lipschitz on Ω with constant $c_0 + c_1$ (see [2]). Let us note c the Lipschitz constant of the first order divided differences on an open set Ω, then

$$\|\nabla f(x) - \nabla f(y)\| \leq c_2\|y - x\| \qquad \text{where}\, c_2 = 2c$$

and

$$\|[x,y,f]-\nabla f(z)\| \leq c(\|x-z\|+\|y-z\|), \qquad \forall\, x,y, \in \Omega.$$

Inversely, a function whose Frechet derivative is Lipschitz admits Lipschitz divided differences.

In the litterature, different methods use divided differences which satisfy Hölder or Lipschitz conditions, but sometimes, we can have more general conditions like ω-conditioning.

Definition 3. *A divided difference of a function g verifies a $(\nu - p)$-Hölder condition if there exists a constant ν such that for all $x, z, u, v \in \Omega \subset X$*

$$\|[x,z,g]-[u,v,g]\| \leq \nu(\|x-u\|^p + \|z-v\|^p) \text{ where } p \in [0,1] \tag{1}$$

When $p = 1$ condition (1) is a Lipschitz condition.

Definition 4. *A divided difference of a function g is ω-conditioned if it satisfies*

$$\|[x,z,g]-[u,v,g]\| \leq \omega(\|x-u\|, \|z-v\|) \tag{2}$$

with $x, z, u, v \in \Omega \subset X$ and $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function with respect to both variables.

Let us note that Hernandez and Rubio used condition (1) in [33] and condition (2) in [34, 35, 36] to solve nonlinear equations.

2.2 Semianalytic and Subanalytic Sets and Functions

The following definitions, properties and examples come from Dedieu's paper [19], but the reader can also see other publications on real algebraic geometry and specifically on semianalytic and subanalytic sets and functions. For example, one can find a full explanation of the mains properties in [5, 6, 8].

Definition 5. *A subset X of $\mathbb{R}^n$ is semianalytic if for each $a \in \mathbb{R}^n$ there exists a neighborhood U of a and real analytic functions $f_{i,j}$ on U such that*

$$X \cap U = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in U | f_{i,j}\, \varepsilon_{i,j}\, 0\}$$

where $\varepsilon_{i,j} \in \{<, >, =\}$.

Remark 3. One say that X is semialgebraic when $U = \mathbb{R}^n$ and the $f_{i,j}$ are polynomials.

Definition 6. *A subset X of $\mathbb{R}^n$ is subanalytic if each point $a \in \mathbb{R}^n$ admits a neighborhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set : there exists a semianalytic bounded set A in $\mathbb{R}^{n+p}$ such that $X \cap U = \Pi(A)$ where $\Pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the projection.*

Definition 7. *Let X be a subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}^m$ is semianalytic (resp. subanalytic) if its graph is semianalytic (resp. subanalytic).*

The class of semianalytic or algebraic sets is not stable under projection contrary to the class of semialgebraic or subanalytic sets (Tarsky-Seidenberg principle). Both the last class possesses interesting property of stability for elementary set operations (finite union, finite intersection, set difference); the closure, the interior and the connected components of a semianalytic set are semianalytic. The same properties hold for subanalytic sets. Unfortunately, the image of a bounded semianalytic set by a semianalytic function is not necessarily semianalytic (see [5]). Consequently, the class of semianalytic functions is not stable under algebraic operations (sum, product, composition see [43]), subanalytic functions have been introduced for this reason.

*Example 3. (*semianalytic sets and functions*).*

1. A semialgebraic set (resp. function) is semianalytic;
2. If f is a real semianalytic function, then the sets $\{f(x) \leq 0\}$, $\{f(x) < 0\}$ and $\{f(x) = 0\}$ are semianalytic;
3. A piecewise function defined on a semianalytic partition *(*finite union of points and open intervals, bounded or not*)* of $\mathbb{R}$ is semianalytic.

If X is a subanalytic and relatively compact set, the image of X by a subanalytic function is subanalytic (see [5]). Moreover if f and g are subanalytic continuous functions defined on a compact subanalytic set K then $f + g$ is subanalytic.

Example 4. There are interesting examples of subanalytic functions in relation with optimization.

1. If X is a closed subanalytic set of $\mathbb{R}^n$, the distance function $d(x, X) = \min_{y \in X} |x - y|$ is subanalytic.
2. The supremum of a finite family of subanalytic continuous functions is subanalytic.
3. Let X and T be subanalytic subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ where T is compact, if $f : X \times T \to \mathbb{R}$ is subanalytic and continuous then $g(x) = \min_{y \in T} f(x, y)$ is subanalytic.

For other examples and properties of semianalytic or subanalytic functions the reader can refer to [5] and [19].

In 1975, Clarke was the first to introduce the concept of generalized gradient (usually called now Clarke Jacobian) for a locally Lipschitz function. The generalized gradient is reduced to the gradient if the function is continuous differentiable, and to the subdifferential in convex analysis.

Thanks to [8], we know that every subanalytic locally Lipschitz function f admits directional derivatives (see [18]) which allow us to have estimates on the error occurred when the function is not Frechet differentiable, and moreover we have:

Proposition 1. [8] *If $f : X \subset \mathbb{R}^n \to \mathbb{R}^n$ is a subanalytic locally Lipschitz mapping then for all $x \in X$*

$$||f(x+d) - f(x) - f'(x;d)|| = o_x(||d||).$$

Definition 8. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function. The limiting Jacobian of f at $x \in \mathbb{R}^n$ is defined as*

$$\partial f(x) = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) : \exists u^k \in D; f'(u^k) \to A, k \to +\infty\}$$

where D denotes the points of differentiability of f.

Definition 9. *[18] The Clarke Jacobian of f at $x \in \mathbb{R}^n$ denoted $\partial^\circ f(x)$ is a nonempty subset of X^* (which is the topological dual of X) defined by*

$$\partial^\circ f(\overline{x}) = \{\xi \in X^* \mid f^\circ(\overline{x}, v) \geq < \xi, v > \ for\, all\, v \in X\}.$$

It is also the closed convex hull of $\partial f(x)$.

For all $\xi \in \partial^\circ f(\bar{x})$, we set

$$\|\xi\|_* = \sup_{\|v\| \leq 1} \{< \xi, v > \mid v \in X\}.$$

We have the following property:

Proposition 2. [18]
Let f be a Lipschitz function at $\overline{x} \in X$ with constant K.
(a) *$\partial^\circ f(\overline{x})$ is a convex compact subset of X^* and $\|\xi\|_* \leq K$ for all $\xi \in \partial^\circ f(\overline{x})$.*
(b) *For all $v \in X$, $f^\circ(\overline{x}, v) = max\{< \xi, v > \mid \xi \in \partial^\circ f(\overline{x})\}$.*

There is an important and useful result for studying convergence of sequences coming from classical method on subanalytic context. The result has been obtained by Bolte, Daniilidis and Lewis in [8].

Proposition 3. [8] *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz and subanalytic, there exists a positive rational number γ such that:*

$$\|f(y) - f(x) - \Delta(y)(y - x)\| \leq C_x \|y - x\|^{1+\gamma} \tag{3}$$

where y is close to x, $\Delta(y)$ is any element of $\partial^\circ f(y)$ and C_x is a positive constant.

Remark 4. The previous result is due to the fact that the subanalytic function $t \to o_x(t)$ in Proposition *1* admits a Puiseux development; so there exists a constant $c > 0$, a real number $\varepsilon > 0$ and a rational number $\gamma > 0$ such that $||f(x+d) - f(x) - f'(x;d)|| = c||d||^\gamma$ whenever $||d|| \leq \varepsilon$ (see [8]).

2.3 Pseudo-Lipschitz Maps

The pseudo-Lipschitz property, also called "Aubin property" or "Lipschitz-like property", has been introduced by J.-P. Aubin as a concept of continuity for set–valued maps. This property of F is equivalent to the metric regularity of F^{-1}. Characterizations of the pseudo-Lipschitz property are also obtained by Rockafellar [51, 52] using the Lipschitz continuity of the distance function $dist\,(y, F(x))$ around (x_0, y_0) and by Mordukhovich in [46, 47, 48] using the concept of coderivative of multifunctions. Lately, Dontchev, Quincampoix and Zlateva gave in [22] a derivative criterion of metric regularity of set–valued mappings based on some works of Aubin and co-authors. Pseudo-Lipschitz maps allow to treat ill-posed problems, when there is no uniqueness of solutions. More details, applications and other interesting results in relation with this concept can be found in [3, 20, 23].

Definition 10. *A set–valued map F is pseudo-Lipschitz around $(x_0, y_0) \in graph\,F$ with constant M if there exists constants a and b such that*

$$\sup_{z\in F(y')\cap B_a(y_0)} dist\,(z, F(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{4}$$

Using the excess, the inequality (4) can be replaced by the following

$$e(F(y') \cap B_a(y_0), F(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{5}$$

Furthermore, we use the concept of metric regularity which definition is:

Definition 11. *A set-valued map $F : X \rightrightarrows Y$ is metrically regular around $(x_0, y_0) \in graph\,F$ if there exists constants a, b and κ such that*

$$dist(x, F^{-1}(y)) \leq \kappa\, dist(y, F(x)), \quad \forall x \in B_a(x_0),\ y \in B_b(y_0). \tag{6}$$

The regularity modulus of F denoted by $RegF(x_0, y_0)$ is the infimum of all the values of κ for which (6) holds.

The result which follows is a generalization of a fixed point theorem in Ioffe-Tikhomirov [39] where in (b), the excess e is replaced by the Haussdorf distance. Its proof is given in [20] employing the standard iterative concept for contracting mapping.

Lemma 1. *Let (Z, ρ) be a complete metric space, let ϕ be a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*
(a) *$dist\,(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$,*
(b) *$e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda\, \rho(x_1, x_2)$, $\forall x_1, x_2 \in B_r(\eta_0)$,*
then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

Proposition 4. [48] *Let $F : X \rightrightarrows Y$ be a set-valued map and $(x_0, y_0) \in graph\,F$. F is metrically regular around (x_0, y_0) with constant κ if and only if F^{-1} is pseudo-Lipschitz around (y_0, x_0) with the same constant κ.*

In the continuation of this work, the distance ρ in Lemma 1 is replaced by the norm.

3 A Newton-Type Method

3.1 Description of the Method and Assumptions

Newton's method is known to be one of the most powerfull and usefull methods in optimization and related area of solving nonlinear equations and the number of publications developing either this method or some variants is impressive. In order to solve different problems in optimization for example, Newton's method is sometimes extended to the context of variational inclusions and the number of contributions in this area in the last decade is very important. Dontchev, with a partial linearization of the univoque part f of the variational inclusion, examined a newton-type method in [21] and he showed that when the Frechet derivative ∇f of f is Lipschitz, the sequence is quadratically convergent. Let us notice that interesting contributions have also been given by Bonnans in [9] in the case of semistable or hemistable solutions. The method we propose in this paragraph is inspired by Dontchev's work and we only give some elements of the whole work devoted to this subject in [15].

This study concerns variational inclusions of the type

$$0 \in f(x) + F(x) \tag{7}$$

where f is a function defined on $\mathbb{R}^n$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map.

To approximate x^* a solution of (7), we consider the method:

$$0 \in f(x_k)+\Delta f(x_k)(x_{k+1}-x_k)+F(x_{k+1}) \qquad \text{where} \quad \Delta f(x_k) \in \partial^\circ f(x_k) \tag{8}$$

and we prove both existence and convergence of the sequence (8) which is a Newton-type sequence, replacing in the classic sequence $\nabla f(x_k)$ by $\Delta f(x_k)$ where $\Delta f(x_k) \in \partial^\circ f(x_k)$.

We make the following assumptions on a neighborhood Ω of x^*:

- $(\mathcal{H}1)$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz subanalytic function;

Remark 5. From Proposition *2* and the previous assumption, it is easy to see that there exists $K_1 > 0$ such that for all $x \in \Omega$, $\forall \Delta f(x) \in \partial^\circ f(x)$, $|\Delta f(x)| \le K_1$.

- $(\mathcal{H}2)$ $\quad F$ is a set-valued map from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ with closed graph and for all $\Delta f(x^*) \in \partial^\circ f(x^*)$, the application $[f(x^*) + \Delta f(x^*)(. - x^*) + F(.)]^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ with constants a, b and modulus L which satisfies $2LK_1 < 1$.

We also define the following functions and set–valued maps:

$$\Lambda_k(x) = f(x_k) + \Delta f(x_k)(x - x_k), \tag{9}$$

$$\Lambda_{x^*}(x) = f(x^*) + \Delta f(x^*)(x - x^*), \tag{10}$$

$$Q(x) = \Lambda_{x^*}(x) + F(x) \tag{11}$$

And

$$\Psi_k(x) = Q^{-1}(\Lambda_{x^*}(x) - \Lambda_k(x)). \tag{12}$$

Let us note that x_1 is a solution of (8) for x_0 when x_1 is a fixed point for the set-valued map Ψ_0. So for the construction of the sequence (x_k), starting from an initial value x_0 in a neighborhood of a solution x^* of (7), by application of Lemma 1, we show that the map Ψ_0 possesses a fixed point x_1. By induction, from a current iterate x_k done by (8) and a function Ψ_k defined by (12), applying Lemma 1, we obtain next iterate x_{k+1} which is a fixed point of Ψ_k.

3.2 Convergence Results

The main result we obtained states as follows:

Theorem 1. *Let x^* a solution of (7), f a function which admits directional derivatives and satisfies $(\mathcal{H}1)$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ a set–valued map which satisfies $(\mathcal{H}2)$; there exists a positive constant C_* such that for all $C > \dfrac{L\,C_*}{1 - 2LK_1}$, one can find $\delta > 0$ such that for every starting point $x_0 \in B_\delta(x^*)$, there exists a sequence $(x_k)_{k\geq 0}$ defined by (7) which satisfies:*

$$\|x_{k+1} - x^*\| \leq C\|x_k - x^*\|^{1+\gamma}. \tag{13}$$

where γ is a rational positive number.

To prove Theorem 1, we firstly prove the existence of the iterate x_1 which is a fixed point of Ψ_0 (Proposition 5); then we justify that the same arguments hold for a current iterate x_k and a set-valued map Ψ_k, which complete the proof.

Proposition 5. *Under the assumptions of Theorem 1, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$ and $x_0 \neq x^*$, the map Ψ_0 admits a fixed point $x_1 \in B_\delta(x^*)$.*

Proof. In this part we just give the main ideas of the proof . We show that both assertions (a) and (b) of Lemma 1 are satisfied.
Proposition 3 furnishes constants C_x in inequality (3); C_* is the biggest value of these constants when $x \in \Omega$.
The assumption $(\mathcal{H}2)$ gives the constants a and b. Fix $\delta > 0$ such that

$$\delta < \min\left\{a;\ \sqrt[1+\gamma]{\frac{b}{2C_*}};\ \frac{b}{4K_1}\right\}. \tag{14}$$

As we generally don't know how to choose the initial value x_0 to ensure the convergence of the sequence, inequality (14) play an important role because it contains all the conditions to be sure that the process converges, (all iterates belong to a ball centered at x^* and each iterate is closer to the solution than the previous).

From the definition of the excess e, we have

$$dist(x^*, \Psi_0(x^*)) \leq e(Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}\{\Lambda_{x^*}(x^*) - \Lambda_0(x^*)\}).$$

Since $\|\Lambda_{x^*}(x^*) - \Lambda_0(x^*)\| = \|f(x_0) - f(x^*) - \Delta f(x_0)(x_0 - x^*)\|$,
$\|\Lambda_{x^*}(x^*) - \Lambda_0(x^*)\| \leq C_*\|x_0 - x^*\|^{1+\gamma}$ and with Proposition 3, thanks to the pseudo-lipschitzness of Q^{-1}, using (14), we know that $\Lambda_{x^*}(x^*) - \Lambda_0(x^*) \in B_b(0)$ and

$$dist(x^*, \Psi_0(x^*)) \leq L\,C_*\|x_0 - x^*\|^{1+\gamma}. \tag{15}$$

By setting $r = r_0 = C\|x_0 - x^*\|^{1+\gamma}$, since $C > \dfrac{L\,C_*}{1 - 2LK_1}$, one can find $\lambda \in]2LK_1, 1[$ such that $C(1-\lambda) > L\,C_*$ so that assertion (a) in Lemma 1 is satisfied and the above choice of r_0 implies that $r_0 < \delta < a$.

To show condition (b), we must check that when $x \in B_\delta(x^*)$, $\Lambda_{x^*}(x) - \Lambda_0(x) \in B_b(0)$.

Since

$$\|\Lambda_{x^*}(x) - \Lambda_0(x)\| \leq \|f(x_0) - f(x^*) - \Delta f(x_0)(x_0 - x^*)\| + \|(\Delta f(x_0) - \Delta f(x^*))(x - x^*)\|. \tag{16}$$

Using Proposition 3, inequality (16) we obtain

$$\|\Lambda_{x^*}(x) - \Lambda_0(x)\| \leq C_*\,\|x_0 - x^*\|^{1+\gamma} + 2K_1\|x - x^*\|$$

and with inequality (14) the result expected is given.
It follows that, for all $x', x'' \in B_{r_0}(x^*)$,

$$\begin{aligned} e(\Psi_0(x') \cap B_{r_0}(x^*), \Psi_0(x'')) \leq\ & L\|\Lambda_{x^*}(x') - \Lambda_0(x') - \Lambda_{x^*}(x'') + \Lambda_0(x'')\| \\ & -f(x^*) - \Delta f(x^*)(x'' - x^*) + f(x_0) + \Delta f(x_0)(x'' - x_0)\|. \end{aligned}$$

The fact that $\lambda \in]2LK_1, 1[$ shows that condition (b) of Lemma 1 is satisfied. Then there exists $x_1 \in B_{r_0}(x^*)$, fixed-point of Ψ_0, and x_1 verifies inequality (13).

Coming back to the proof of Theorem 1, proceeding by induction, suppose that $x_k \in B_{r_{k-1}}(x^*)$, keeping $\eta_0 = x^*$ and $r_k = C\|x_k - x^*\|^{1+\gamma}$, we obtain the existence of a fixed-point $x_{k+1} \in B_{r_k}(x^*)$ for Ψ_k; that implies

$$\|x_{k+1} - x^*\| \leq C\|x_k - x^*\|^{1+\gamma}.$$

Then, the convergence of $(x_k)_{k\geq 0}$ to x^* is superlinear.

4 The Study of Perturbed Problems

In this part, we are concerned by problems of the form

$$0 \in f(x) + g(x) + F(x) \tag{17}$$

where f and g are functions defined on $\mathbb{R}^n$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map; in this model g stands for the perturbation function. Let us remark that the problem (17) can be considered as a perturbed problem associated to (7).

Various methods proposed to solve (17) use a combination of two different methods for f and g. They are chosen in relation with the properties of both functions. The function f is still subanalytic and locally lipschitz around x^* and in this part we develop two methods: the first is an iterative method where the function g is lipschitz and the second is a secant type method where the function g admits first and second order divided differences. The first method is the subject of a published paper (see [16]) and it follows a work done by Geoffroy et Piétrus [29] in the case where the function f is Frechet differentiable. The second method is an original unpublished method which mixes a Newton type method for f and a secant method for g.

Let us precise that these methods are derived from contributions of Catinas, Tetsuro and Xiaojun (see [17, 53, 54]) in the context of solving equations.

4.1 An Iterative Method in the Lipschitz Case

To solve variational inclusion(17), we introduce the sequence:

$$0 \in f(x_k) + g(x_k) + \Delta f(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \quad \text{with} \quad \Delta f(x_k) \in \partial^\circ f(x_k) \tag{18}$$

Let us note that if (x_k) converge to x^*, then x^* is solution of (17).

For all x, $y \in \mathbb{R}^n$, we set $A(x, y)$, the following set-valued map

$$A(x, y) = f(y) + \Delta f(y)(x - y) + g(y) + F(x). \tag{19}$$

For all $k \in \mathbb{N}$, the map $R_k : \mathbb{R}^n \to \mathbb{R}^n$ is defined by:

$$R_k(x) = f(x^*) + \Delta f(x^*)(x - x^*) + g(x^*) - f(x_k) - \Delta f(x_k)(x - x_k) - g(x_k) \tag{20}$$

and $\Psi_k : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by :

$$\Psi_k(x) = A(.,x^*)^{-1}[R_k(x)] \tag{21}$$

We make the assumptions valid in a neighborhood Ω de x^*:

- $(\mathcal{H}11)$ $f : \mathbb{R}^n \to \mathbb{R}^n$ is a subanalytic and locally Lipschitz function;
- $(\mathcal{H}12)$ g is a Lipschitz function with constant K_2;
- $(\mathcal{H}13)$ $A(.,x^*)^{-1}$ is pseudo-Lipschitz around $(0,x^*)$ with constant L and we have $2LK_1 < 1$ (where K_1 is the constant in Remark 5).

The main result obtained is:

Theorem 2. *Let x^* be a solution of* (17)*; suppose that assumptions* $(\mathcal{H}11)$-$(\mathcal{H}13)$ *are satisfied, then there exists a constant C_* such that for all $\dfrac{L(C_* + K_2)}{1 - 2LK_1} < C < 1$, one can find $\delta > 0$ such that for all initial value $x_0 \in B_\delta(x^*)$ $(x_0 \neq x^*)$, there exists a sequence $(x_k)_{k\geq 0}$ defined by* (20) *verifying:*

$$\|x_{k+1} - x^*\| \leq C\|x_k - x^*\|. \tag{22}$$

Remark 6. This theorem gives an order of convergence less interesting than Newton's method. Indeed we can observe that the lack of regularity of the perturbation function affects a lot the order of convergence.

To prove Theorem 2, we follow the same scheme used for the proof of Theorem 1 and firstly prove this result:

Proposition 6. *Under assumptions of Theorem* 2*, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$ $(x_0 \neq x^*)$, the set-valued map Ψ_0 admits a fixed point $x_1 \in B_\delta(x^*)$.*

Proof. The entire proof of the above theorem is given in [16], here we just give the important steps.
Assumption $(\mathcal{H}13)$ give the positive real numbers a, b and L such that:

$$e(A(.,x^*)^{-1}(y') \cap B_a(x^*), A(.,x^*)^{-1}(y'')) \leq L\|y' - y''\|, \quad \forall y',\, y'' \in B_b(0) \tag{23}$$

Let C_* be as defined in Paragraph 3.2. Fix $\delta > 0$ such that

$$\delta < \min\Big\{a,\, \frac{a}{C},\, \frac{b}{K_2 + 3C_*}\Big\}. \tag{24}$$

Let us apply Lemma 1.
The definition of the excess allow us to write:

$$dist(x^*, \Psi_0(x^*)) \leq e(A(.,x^*)^{-1}(0) \cap B_\delta(x^*), A(.,x^*)^{-1}[R_0(x^*)]). \tag{25}$$

For all $x_0 \neq x^*$ in $B_\delta(x^*)$, we have:

$$\begin{aligned}\|R_0(x^*)\| &= \|f(x^*) + g(x^*) - f(x_0) + \Delta f(x_0)(x_0 - x^*) - g(x_0)\| \\ &\leq \|f(x_0) - f(x^*) - \Delta f(x_0)(x_0 - x^*)\| + \|g(x^*) - g(x_0)\|.\end{aligned}$$

Thanks to Proposition 3 and assumption ($\mathcal{H}12$),

$$\|R_0(x^*)\| \leq C_* \|x_0 - x^*\|^{1+\gamma} + K_2 \|x^* - x_0\|. \tag{26}$$

For δ small enough,

$$\|R_0(x^*)\| \leq \delta(C_* + K_2).$$

and we obtain $\|R_0(x^*)\| < b$, with the use of (24).
From (25) and (26), we deduce

$$dist(x^*, \Psi_0(x^*)) \leq L\|R_0(x^*)\|$$

and

$$dist(x^*, \Psi_0(x^*)) \leq L(C_* + K_2)\|x^* - x_0\|. \tag{27}$$

Setting $r = r_0 = C\|x_0 - x^*\|$, since $1 > C > \dfrac{L(C_* + K_2)}{1 - 2LK_1}$, one can find $\lambda \in [2LK_1, 1[$ such that $C(1-\lambda) > L(C_* + K_2)$, and condition (a) in Lemma 1 is fulfilled and $r_0 < a$.

Let $x \in B_\delta(x^*)$ and denote by $R_0(x)$ the quantity $f(x^*) + \Delta f(x^*)(x - x^*) + g(x^*) - f(x_0) - \Delta f(x_0)(x - x_0) - g(x_0)$.

$$\begin{aligned}\|R_0(x)\| &\leq \|g(x^*) - g(x_0)\| + \|f(x_0) - f(x) - \Delta f(x_0)(x_0 - x)\| \\ &\quad + \|f(x^*) - f(x) - \Delta f(x^*)(x^* - x)\| \\ &\leq K_2\|x^* - x_0\| + C_*(\|x - x_0\|^{1+\gamma} + \|x - x^*\|^{1+\gamma}).\end{aligned}$$

for δ small enough, $\|R_0(x)\| \leq (K_2 + 3C_*)\delta$, which implies, with (24) that $\|R_0(x)\| < b$. One can deduce that for all $x \in B_\delta(x^*)$, $R_0(x) \in B_b(0)$; taking $x', x'' \in B_{r_0}(x^*)$, we have the following inequality satisfied by the excess:

$$e(\Psi_0(x') \cap B_{r_0}(x^*), \Psi_0(x'')) \leq e(\Psi_0(x') \cap B_\delta(x^*), \Psi_0(x''))$$

$$\begin{aligned}e(\Psi_0(x') \cap B_\delta(x^*), \Psi_0(x'')) &\leq L\,\|R_0(x') - R_0(x'')\| \\ &\leq L\,\|\Delta f(x^*)(x' - x'') - \Delta f(x_0)(x' - x'')\| \\ &\leq 2LK_1\,\|x' - x''\| \\ &\leq \lambda\,\|x' - x''\|.\end{aligned}$$

So condition (b) of Lemma 1 holds.

We conclude the existence of $x_1 \in B_{r_0}(x^*)$, fixed point of Ψ_0, which verifies inequality (22).

To end the proof of Theorem 2, we proceed by induction, as in the previous case.

4.2 A Secant-Type Method

In this section, we propose a new method to approximate a solution of (17) when the perturbed function g admits first and second order divided differences; this method is inspired by a work of Geoffroy and Piétrus [28].

We associate to (17) the sequence

$$\begin{aligned} &0 \in f(x_k) + g(x_k) + (\Delta f(x_k) + [x_{k-1}, x_k, g])(x_{k+1} - x_k) + F(x_{k+1}) \quad (28)\\ &\Delta f(x_k) \in \partial^\circ f(x_k) \end{aligned}$$

If Δf is replaced by ∇f, we obtain the method introduced by Geoffroy and Piétrus in [28]. In their paper, they obtained a superlinear convergence of the method (28) when f is differentiable around x^*, g is differentiable at x^* and admits first and second order divided differences. The method (28) reduces to the Newton type method studied in [15] when f is subanalytic and $g = 0$. Let us notice that Catinas [17] has yet used this method for solving nonlinear equations and some recent contributions for variational inclusions on this topic have been given in [30]. We prove that the convergence of the sequence defined by (28) is superlinear, then we present two variants of this method.

4.2.1 Assumptions and Convergence Analysis

We make the following assumptions on a neighborhood $\Omega \subset \mathbb{R}^n$ of x^*:

- $(\mathcal{H}21)$ $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and subanalytic,
- $(\mathcal{H}22)$ g is differentiable at x^*,
- $(\mathcal{H}23)$ $\exists K_2 > 0$, $\forall x, y$ and $z \in \Omega$, $\|[x, y, z, g]\| \leq K_2$,
- $(\mathcal{H}24)$ $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map with closed graph, and $\forall \Delta f(x^*) \in \partial^\circ f(x^*)$, the set-valued map $[f(x^*)+g(.)+\Delta f(x^*)(.-x^*)+F(.)]$ is metrically regular around $(x^*, 0)$ with constant L such that $2L(2K_2 + K_1) < 1$ (where K_1 is the constant in Remark 5).

Remark 7. Using [20], we can show that the metric regularity of the set-valued map $[f(x^*) + g(.) + \Delta f(x^*)(. - x^*) + F(.)]$ is equivalent to the one of $[f(.) + g(.) + F(.)]$, but the constants of metric regularity are not the same.

We also define the the function Z_k and the set-valued maps P and Ψ_k by:

$$P(x) = f(x^*) + g(x) + \Delta f(x^*)(x - x^*) + F(x) \quad (29)$$

For all $k \geq 1$

$$\begin{aligned} Z_k(x) = &f(x^*) + g(x) + \Delta f(x^*)(x - x^*) - f(x_k) - g(x_k)\\ &-(\Delta f(x_k) + [x_{k-1}, x_k, g])(x - x_k) \end{aligned} \quad (30)$$

$$\Psi_k(x) = P^{-1}(Z_k(x)). \quad (31)$$

Now, we establish our principal result.

Theorem 3. *Let x^* be a solution of* (17)*, and suppose that* ($\mathcal{H}21$)-($\mathcal{H}24$) *are satisfied. Then there exists a positive constant C_* such that for all $C > \dfrac{L\,(C_* + K_2)}{1 - 2L(2K_2 + K_1)}$, one can find $\delta > 0$ such that for every starting points $x_0,\, x_1 \in B_\delta(x^*)$ (with $x_0 \neq x^*$, $x_1 \neq x^*$), there exists a sequence $(x_k)_{k\geq 0}$ defined by* (28) *which satisfies:*

$$\|x_{k+1} - x^* \| \leq C \parallel x_k - x^* \parallel \max\{\|x_k - x^*\|^\gamma, \|x_{k-1} - x^*\|\} \tag{32}$$

where γ is a rational positive number.

We firstly prove the two following lemma and proposition:

Lemma 2. *Under the assumptions of Theorem 3, the map P^{-1}, inverse of P given by* (29) *is pseudo-Lipschitz around* $(0, x^*)$.

Proof. Using assumption ($\mathcal{H}24$) and Proposition 4, since P is metrically regular around $(x^*, 0)$ with constant L, then P^{-1} is pseudo-Lipschitz around $(0, x^*)$ with the same constant.

Proposition 7. *Under the assumptions of Theorem 3, there exists $\delta > 0$ such that for all $x_0, x_1 \in B_\delta(x^*)$ (with $x_0 \neq x^*$ and $x_1 \neq x^*$), the map Ψ_1 admits a fixed point $x_2 \in B_\delta(x^*)$.*

Proof. For the proof of this proposition, we prove that both assertions (a) and (b) of Lemma 1 hold.

C_* is the constant C_x in Paragraph 3.2. Since P^{-1} is pseudo-Lipschitz around $(0, x^*)$, there exist constants a and b such that

$$e(P^{-1}(y') \cap B_a(x^*), P^{-1}(y'')) \leq L \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(0) \tag{33}$$

and choose $\delta > 0$ verifying

$$\delta < \min\left\{a,\ \sqrt{\frac{b}{8K_2}},\ \sqrt[1+\gamma]{\frac{b}{2C_*(1+2^{1+\gamma})}},\ \frac{1}{C},\ \sqrt[\gamma]{\frac{1}{C}}\right\}. \tag{34}$$

From the definition of the excess e, we have

$$dist(x^*, \Psi_1(x^*)) \leq e(P^{-1}(0) \cap B_\delta(x^*), \Psi_1(x^*)). \tag{35}$$

For all x_0, x_1 in $B_\delta(x^*)$ (such that $x_0 \neq x^*$ and $x_1 \neq x^*$), we have

$$\begin{aligned}\|Z_1(x^*)\| &= \|f(x^*) + g(x^*) - f(x_1) - g(x_1) - (\Delta f(x_1) + [x_0, x_1, g])(x^* - x_1)\| \\ &\leq \|f(x^*) - f(x_1) - \Delta f(x_1)(x^* - x_1)\| \\ &\quad + \|g(x^*) - g(x_1) - [x_0, x_1, g](x^* - x_1)\|.\end{aligned}$$

Using Definition 1,

$\|Z_1(x^*)\| \leq \|f(x_1) - f(x^*) - \Delta f(x_1)(x_1 - x^*)\| + \|([x_1, x^*, g] - [x_0, x_1, g])(x^* - x_1)\|.$
Therefore, with the help of Definition 2, Proposition 3 and assumption ($\mathcal{H}23$), we obtain

$$\|f(x_1) - f(x^*) - \Delta f(x_1)(x_1 - x^*)\| \leq C_*\|x_1 - x^*\|^{1+\gamma} \tag{36}$$

and,

$$\|([x_1, x^*, g] - [x_0, x_1, g])(x^* - x_1)\| \leq \|[x_0, x_1, x^*, g]\| \|x_1 - x^*\| \|x_0 - x^*\|. \tag{37}$$

Consequently,

$$\|Z_1(x^*)\| \leq C_*\|x_1 - x^*\|^{1+\gamma} + K_2\, \|x_1 - x^*\|\, \|x_0 - x^*\|$$

which implies, according to (34), $\|Z_1(x^*)\| < b$.

With (33), we have

$$\begin{aligned} e(P^{-1}(0) \cap B_\delta(x^*), \Psi_1(x^*)) &= e(P^{-1}(0) \cap B_\delta(x^*), P^{-1}[Z_1(x^*)]) \\ &\leq L(C_*\|x_1 - x^*\|^{1+\gamma} + K_2\, \|x_1 - x^*\|\, \|x_0 - x^*\|) \end{aligned}$$

and, with (35), we obtain

$$dist(x^*, \Psi_1(x^*)) \leq L\,(C_* + K_2)\|x_1 - x^*\|\, \max\{\|x_1 - x^*\|^\gamma, \|x_0 - x^*\|\}. \tag{38}$$

By setting $\eta = x^*$ and $r = r_1 = C\|x_1 - x^*\|\, \max\{\|x_1 - x^*\|^\gamma, \|x_0 - x^*\|\}$, since $C > \dfrac{L\,(C_* + K_2)}{1 - 2L(2K_2 + K_1)}$, one can find $\lambda \in]2L(2K_2 + K_1), 1[$ such that $C(1 - \lambda) > L(C_* + K_2)$ so that the assertion (a) in Lemma 1 is satisfied, moreover, we have $r_1 < a$.

Let us show that condition (b) is also satisfied. For $x \in B_\delta(x^*)$, we have

$$\begin{aligned} \|Z_1(x)\| \leq\ & \|f(x^*) + g(x) + \Delta f(x^*)(x - x^*) - f(x_1) - g(x_1) \\ & - (\Delta f(x_1) + [x_0, x_1, g])(x - x_1)\| \\ \leq\ & \|-f(x) + f(x^*) + \Delta f(x^*)(x - x^*)\| + \|g(x) - g(x_1) - [x_0, x_1, g](x - x_1)\| \\ & + \|f(x) - f(x_1) - \Delta f(x_1)(x - x_1)\|. \end{aligned}$$

Thanks to Definition 2, Proposition 3 and ($\mathcal{H}23$), it follows

$$\begin{aligned} \|Z_1(x)\| &\leq C_*(\|x - x^*\|^{1+\gamma} + \|x - x_1\|^{1+\gamma}) \\ &\quad + \|[x_0, x_1, x, g]\|\, \|x - x_0\|\, \|x - x_1\| \\ &\leq C_*(\|x - x^*\|^{1+\gamma} + \|x - x_1\|^{1+\gamma}) + K_2\, \|x - x_0\|\, \|x - x_1\| \end{aligned}$$

which implies $\|Z_1(x)\| \leq C_*(1 + 2^{1+\gamma})\, \delta^{1+\gamma} + 4K_2\, \delta^2$. According to (34), $\|Z_1(x)\| < b$. We proved that if $x \in B_\delta(x^*)$, then $Z_1(x) \in B_b(0)$.

It follows that , for all $x', x'' \in B_{r_0}(x^*)$,

$$\begin{aligned}
e(\Psi_1(x') \cap B_{r_1}(x^*), \Psi_1(x'')) &\leq e(\Phi_1(x') \cap B_\delta(x^*), \Phi_1(x'')) \\
&\leq L\,\|Z_1(x') - Z_1(x'')\| \\
&\leq L\,\|g(x') - g(x'') - [x_0, x_1, g])(x' - x'')\| \\
&\quad + L\,\|(\Delta f(x^*) - \Delta f(x_1))(x' - x'')\| \\
&\leq L\,\|([x', x'', g] - [x_0, x_1, g])(x' - x'')\| \\
&\quad + L\,(\|\Delta f(x^*)\| + \|\Delta f(x_1)\|)\,\|x' - x''\| \\
&\leq L\,\|[x_1, x'', x', g](x' - x_1) + [x_0, x_1, x'', g](x'' - x_0)\|\|x' - x''\| \\
&\quad + L\,(\|\Delta f(x^*)\| + \|\Delta f(x_1)\|)\,\|x' - x''\| \\
&\leq 2L(2K_2\delta + K_1)\,\|x' - x''\|
\end{aligned}$$

and for δ small enough,

$$e(\Psi_1(x') \cap B_{r_1}(x^*), \Psi_1(x'')) \leq \lambda\|x' - x''\|.$$

Thus, the condition (b) of Lemma 1 is satisfied.

We conclude to the existence of $x_2 \in B_{r_1}(x^*)$, a fixed-point of Ψ_1 which satisfies inequality (32). Proceeding by induction, we suppose that $x_k \in B_{r_{k-1}}(x^*)$, keeping $\eta_0 = x^*$ and $r_k = C\|x_k - x^*\| \max\{\|x_k - x^*\|^\gamma, \|x_{k-1} - x^*\|\}$ and we obtain the existence of a fixed-point $x_{k+1} \in B_{r_k}(x^*)$ for Ψ_k so that x_{k+1} satisfies (32), that achieves the proof of Theorem 3.

4.2.2 Some Variants of the Secant Method

The first variant consists in replacing x_{k-1} by x_0 in (28). We obtain

$$\begin{aligned}
&0 \in f(x_k) + g(x_k) + (\Delta f(x_k) + [x_0, x_k, g])(x_{k+1} - x_k) + F(x_{k+1}) \quad (39)\\
&\Delta f(x_k) \in \partial^\circ f(x_k)
\end{aligned}$$

and we can show the following estimates :

$$\|x_{k+1} - x^* \| \leq C \| x_k - x^* \| \max\{\|x_k - x^*\|^\gamma, \|x_0 - x^*\|\} \qquad (40)$$

This *regula-falsi* type method is superlinearly convergent; but in this case the convergence of the sequence is slower than the convergence of the previous method because the upper bound in (40) involves x_k and x_0 instead of x_k and x_{k-1}.

For the second variant, we replace x_{k-1} by x_{k+1} in (28) and we obtain the sequence

$$\begin{aligned}
&0 \in f(x_k) + \Delta f(x_k)(x_{k+1} - x_k) + g(x_{k+1}) + F(x_{k+1}) \qquad (41)\\
&\Delta f(x_k) \in \partial^\circ f(x_k)
\end{aligned}$$

Setting $G = f + F$, this method is a Newton type one (see [15]) for solving the variational inclusion (7) where the set-valued map F is replaced by G. Let us note that in this case, assumption ($\mathcal{H}23$) is not necessary.

5 Conclusion

In this chapter, we gave different algorithms for variational inclusions in the context of subanalytic functions. Here, the main property used in order to obtain existence of convergent sequences is the metric regularity and a fixed point theorem for set-valued maps. The convergence obtained is local and we think that it may be possible to add another conditions to find a semilocal convergence. It is also possible to obtain similar results using properties more directly associated to the solutions and to obtain interesting numerical results in the case where the set-valed F is a cone, this is the aim of a forthcoming work.

References

1. Aragon, A.F.J., Dontchev, A.L., Gaydu, M., Geoffroy, M., Veliov, V.M.: Metric regularity of Newton's iteration. SIAM J. Control Optim. 49(2), 339–362 (2011)
2. Argyros, I.K.: Computational theory of the iterative methods. Studies in computational mathematics, vol. 15. Elsevier (2008)
3. Aubin, J.P., Frankowska, H.: Set valued–analysis. Birkhäuser, Boston (1990)
4. Benadada, Y.: Taux de convergence d'une généralisation de la méthode de Newton, http://www.infty08.net/art1.pdf
5. Bierstone, E., Milman, P.D.: Semianalytic and subanalytic sets. IHES. Publications Mathématiques 67, 5–42 (1988)
6. Bochnak, J., Coste, M., Roy, M.F.: Géométrie algébrique réelle. A series of modern surveys in mathematics, vol. 12. Springer (1986)
7. Bolte, J., Daniilidis, A., Lewis, A.S.: The Lojasiewicz inequality for nonsmooth subanalytic functions with application to subgradient dynamical systems. SIAM J. Optim. 17, 1205–1223 (2006)
8. Bolte, J., Daniilidis, A., Lewis, A.S.: Tame mapping are semismooth. Math. Programming (series B) 117, 5–19 (2009)
9. Bonnans, J.F.: Local analysis of Newton-type methods for variational inequalities and nonlinear programming. Appl. Math. Comp. 29(2), 161–186 (1994)
10. Burnet, S., Pietrus, A.: Local Analysis of a cubically convergent method for variational inclusions. Applicationes Mathematicae 38(2), 183–191 (2011)
11. Burnet, S., Jean-Alexis, C., Pietrus, A.: An iterative method for semistable solutions. RACSAM 105(1), 133–138 (2011)
12. Byelostotskij, A.J.: Some methods for the solution of functional equations (russian). Uspekhi Matem. Nauk. 17(5), 192–193 (1962)
13. Cabuzel, C.: A midpoint method for generalized equations under mild differentiability conditions. Acta Applicandae Mathematicae 116(3), 269–279 (2011)
14. Cabuzel, C., Piétrus, A.: Solving variational inclusions by a method obtained using a multipoint iteration formula. Revista Matematica Complutense 22 (1), 63–74 (2009)

15. Cabuzel, C., Piétrus, A.: Local convergence of Newton's method for subanalytic variational inclusions. Positivity 12, 525–533 (2008)
16. Cabuzel, C., Pietrus, A.: An iterative method for a perturbed subanalytic variational inclusions. Comptes Rendus de l'Académie Bulgare des Sciences 61(8), 973–978 (2008)
17. Catinas, E.: On some iterative methods for solving nonlinear equations. Rev. Anal. Numer. Theor. Approx. 23, 17–53 (1994)
18. Clarke, F.H.: Optimization and nonsmooth analysis. Society for industrial and applied mathematics (1990)
19. Dedieu, J.P.: Penalty functions in subanalytic optimization. Optimization 26, 27–32 (1992)
20. Dontchev, A.L., Hager, W.W.: An inverse function theorem for set–valued maps. Proc. Amer. Math. Soc. 121, 481–489 (1994)
21. Dontchev, A.L.: Local convergence of the Newton method for generalized equation. C.R.A.S. 322(1), 327–331 (1996)
22. Dontchev, A.L., Quincampoix, M., Zlateva, N.: Aubin Criterion for Metric Regularity. J. of Convex Analysis 13(2), 281–297 (2006)
23. Dontchev, A.L., Rockafellar, R.T.: Implicit functions and solution mapping, a view from variational analysis. Monographs in Mathematics (2009)
24. Ferris, M.C., Pang, J.S.: Engineering and economic applications of complementary problems. SIAM Rev. 39(4), 669–713 (1997)
25. Gaydu, M., Geoffroy, M.H., Jean-Alexis, C.: Metric subregularity of order q and solving of inclusions. Cent. Eur. J. Math. 9(1), 147–161 (2011)
26. Geoffroy, M.H., Hilout, S., Pietrus, A.: Acceleration of convergence in Dontchev's iterative method for solving variational inclusions. Serdica Math. J. 29/1, 45–54 (2003)
27. Geoffroy, M.H., Pascaline, G.: Generalized differentiation and fixed points sets behaviors with respect to Fisher convergence. J. of Math. Anal. Appl. 387, 464–474 (2012)
28. Geoffroy, M.H., Piétrus, A.: Local convergence of some iterative methods for generalized equations. J. Math. Anal. Appl. 290, 497–505 (2004)
29. Geoffroy, M.H., Pietrus, A.: An iterative method for perturbed generalized equations. Compte-rendu de l'Académie Bulgare des Sciences 57(11), 7–12 (2004)
30. Geoffroy, M.H., Piétrus, A.: A fast iterative scheme for variational inclusions. In: Discrete and Continuous Dynamical System-Supplement, pp. 250–258 (2009)
31. Geoffroy, M.H., Jean-Alexis, C., Piétrus, A.: A Hummel-Seebeck type method for variational inclusions. Optimization 58(4), 589–599 (2009)
32. Heinonen, J.: Lectures on Lipschtiz Analysis. In: Lectures at the 14th Jyväskylä Summer School (2004)
33. Hernandez, M.A., Rubio, M.J.: The secant method and divided differences Hölder continuous. Applied Mathematics and Computation 124, 139–149 (2001)
34. Hernandez, M.A., Rubio, M.J.: The secant method for nondifferentiable operators. Applied Mathematics Letters 15, 395–399 (2002)
35. Hernandez, M.A., Rubio, M.J.: Semilocal convergence of the secant method under mild convergence condition of differentiability. Comp. and Math. with Appl. 44, 277–285 (2002)
36. Hernandez, M.A., Rubio, M.J.: ω-conditioned divided differences to solve nonlinear equations. Monografias del Semin, Matem. Garcia de Galdeano 27, 323–330 (2003)

37. Hironaka, H.: Subanalytic sets. In: Number theory, Algebraic Geometry and Commutative Algebra, Tokyo Kinokuniya, pp. 453–493 (1973)
38. Huang, S.Z.: Gradient inequalities. With applications to asymptotic behaviour and stability of gradient-like systems. Mathematical Surveys and Monographs, vol. 126. American Mathematical Society, Providence (2006)
39. Ioffe, A.D., Tikhomirov, V.M.: Theory of extremal problems. North Holland, Amsterdam (1979)
40. Jeffreys, H., Jeffreys, B.S.: Methods of mathematical physics, 3rd edn. Cambridge University Press, England (1988)
41. Kurdyka, K., Parusinski, A.: w_f-stratification of subanalytic functions and the lojasiewicz inequality. C. R. Acad. Sci. Paris Sér. I Math. 318, 129–133 (1994)
42. Lojasiewicz, S.: Une propriété topologique des sous ensembles analytiques réels, Les équations aux dérivés partielles, pp. 87–89. Edition du centre national de la recherche scientifique, Paris (1963)
43. Lojasiewicz, S.: Ensembles semi-analytiques. IHES Mimeographed notes, Bures-sur-Yvettes (1964)
44. Lojasiewicz, S.: Triangulation of semianalytic sets. Ann. Scuola Norm. Sup. Pisa 3(18), 449–474 (1964)
45. Lojasiewicz, S.: Sur la géométrie semi et sous-analytique. Ann. Inst. Fourier 43, 1575–1595 (1993)
46. Mordukhovich, B.S.: Complete characterization of openess metric regularity and lipschitzian properties of multifunctions. Trans. Amer. Math. Soc. 340, 1–36 (1993)
47. Mordukhovich, B.S.: Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis. Trans. Amer. Math. Soc. 343, 609–657 (1994)
48. Mordukhovich, B.S.: Variational analysis and generalized differentiation I: Basic theory. A series of comprehensive studies in mathematics, vol. 330. Springer (2006)
49. Robinson, S.M.: Generalized equations and their solutions, part I: basic theory. Math. Programming Study 10, 128–141 (1979)
50. Robinson, S.M.: Generalized equations and their solutions, part II: application to nonlinear programming. Math. Programming Study 19, 200–221 (1982)
51. Rockafellar, R.T.: Lipschitzian properties of multifonctions. Non Linear Analysis 9, 867–885 (1984)
52. Rockafellar, R.T., Wets, R.J.B.: Variational analysis. A Series of Comprehensives Studies in Mathematics, vol. 317. Springer (1998)
53. Tetsuro, Y.: Numer. Func. Anal. Optimiz. 9(9&10), 987–994 (1987)
54. Xiaojun, C., Tetsuro, Y.: Numer. Funct. Anal. Optimiz. 10(1&2), 37–48 (1989)