Chapter 9
The Operads As and A_∞

For me it all begins with Poincaré.
Jim Stasheff in "The pre-history of operads"
Contemp Math 202 (1997), 9–14

In this chapter, we first treat in detail the operad encoding the category of associative algebras along the lines of the preceding chapters. This is a particularly important example, because associative algebras are everywhere in mathematics, and because it will serve as a toy-model in the theory of operads.

In Sect. 9.1, we describe the nonsymmetric operad As and then the symmetric operad Ass, where the action of the symmetric group is taken into account. They both encode the category of associative algebras. Then we compute the Koszul dual cooperad $As^{¡}$. We show that As is a Koszul operad by analyzing in detail the Koszul complex. We show that the operadic homology of associative algebras is precisely Hochschild homology.

In Sect. 9.2, we proceed with the computation of the minimal model of As, that is $\Omega As^{¡}$. We show that this operad is exactly the operad A_∞, constructed by Jim Stasheff, encoding the category of "homotopy associative algebras", also known as A_∞-algebras. We describe this differential graded operad in terms of the Stasheff polytope (associahedron).

In Sect. 9.3, we study the bar–cobar construction on As, denoted $\Omega \mathrm{B} As$. We show that this ns operad can be understood in terms of a cubical decomposition of the Stasheff polytope (Boardman–Vogt W-construction). We compare $\Omega \mathrm{B} As$ and A_∞.

In Sect. 9.4, we deal with the *Homotopy Transfer Theorem*. In its simplest form it says that, starting with a dga algebra, any homotopy retract acquires a structure of A_∞-algebra. More generally A_∞-algebras are invariant under homotopy equivalence. These results are due to the work of V.K.A.M. Gugenheim, Hans Munkholm, Jim Stasheff, Tornike Kadeishvili, Alain Prouté, Serguei Merkulov, Martin Markl, Maxim Kontsevich and Yan Soibelman. In the next chapter this theorem is extended to any Koszul operad.

We make this chapter as self-contained as possible, so there is some redundancy with other parts of the book. We refer to Stasheff's paper [Sta10] for historical refer-

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ences on the subject, linking twisting morphisms, also known as twisting cochains, to A_∞-algebra structures, and giving tribute to N. Berikashvili, K.T. Chen and T. Kadeishvili.

9.1 Associative Algebras and the Operad *Ass*

We study the nonsymmetric operad As encoding the category of (nonunital) associative algebras. We show that its Koszul dual is itself: $As^! = As$ and that it is a Koszul nonsymmetric operad. We also study the associated symmetric operad, denoted by Ass.

9.1.1 Associative Algebra

By definition an *associative algebra* over $\mathbb{K}$ is a vector space A equipped with a binary operation

$$\mu : A \otimes A \to A, \qquad \mu(x, y) = xy$$

satisfying the *associativity relation*

$$(xy)z = x(yz)$$

for any $x, y, z \in A$. This relation may also be written as

$$\mu \circ (\mu, \mathrm{id}) = \mu \circ (\mathrm{id}, \mu),$$

and, in terms of partial compositions, as

$$\mu \circ_1 \mu = \mu \circ_2 \mu.$$

There is an obvious notion of morphism between associative algebras and we denote by As-alg the category of associative algebras.

Here we work in the monoidal category Vect of vector spaces over $\mathbb{K}$, but, because of the form of the relation, we could as well work in the monoidal category of sets, resp. topological sets, resp. simplicial sets. Then we would obtain the notion of monoid, resp. topological monoid, resp. simplicial monoid.

It is sometimes helpful to assume the existence of a unit, cf. Sect. 1.1.1, but here we work with nonunital associative algebras.

9.1.2 The Nonsymmetric Operad *As*

Since, in the definition, of an associative algebra, the generating operation μ does not satisfy any symmetry property, and since, in the associativity relation, the variables stay in the same order, the category of associative algebras can be encoded by a nonsymmetric operad, that we denote by As.

Let us denote by μ_n the n-ary operation defined as

$$\mu_n(x_1, \dots, x_n) := x_1 \dots x_n.$$

The space of n-ary operations As_n is one-dimensional spanned by μ_n, because the free associative algebra over V is $\overline{T}(V) = \bigoplus_n V^{\otimes n}$. Therefore $As_n = \mathbb{K}\mu_n$. Since $\dim As_n = 1$, the generating series of the ns operad As is

$$f^{As}(t) = \sum_{n \geq 1} t^n = \frac{t}{1-t}.$$

Classical Definition of As. Under the classical definition of a nonsymmetric operad the composition

$$\gamma : As_k \otimes As_{i_1} \otimes \cdots \otimes As_{i_k} \to As_{i_1+\cdots+i_k}$$

is simply given by the identification

$$\mathbb{K} \otimes \mathbb{K} \otimes \cdots \otimes \mathbb{K} \cong \mathbb{K}, \qquad \mu_k \otimes \mu_{i_1} \otimes \cdots \otimes \mu_{i_k} \mapsto \mu_{i_1+\cdots+i_k}.$$

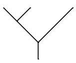
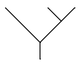
It simply follows from the composition of noncommutative polynomials.

Partial Definition of As. The partial composition is given by

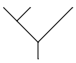
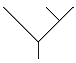
$$\mu_m \circ_i \mu_n = \mu_{m-1+n}$$

for any i because

$$x_1 \cdots x_{i-1}(x_i \cdots x_{m-1})x_m \cdots x_{m-1+n} = x_1 \cdots x_{m-1+n}.$$

Quadratic Presentation. The free ns operad on a binary operation $\mu = \mu_2$ is spanned by the planar binary trees (each internal vertex being labeled by μ): $\mathscr{T}(\mathbb{K}\mu)_n = \mathbb{K}[PBT_n]$ (cf. Sect. 5.9.5). The space $\mathscr{T}(\mathbb{K}\mu)_3 = \mathbb{K}[PBT_3]$ is 2-dimensional and spanned by the trees  and  corresponding to the operation $\mu \circ (\mu, \mathrm{id})$ and $\mu \circ (\mathrm{id}, \mu)$ respectively. The relator is the *associator*

$$as := -\mu \circ (\mu, \mathrm{id}) + \mu \circ (\mathrm{id}, \mu) \in \mathscr{T}(\mathbb{K}\mu)^{(2)},$$

i.e. $-$  $+$ . There is an identification

$$As = \mathscr{P}(\mathbb{K}\mu, \mathbb{K}as) = \mathscr{T}(\mathbb{K}\mu)/(as)$$

where (as) is the operadic ideal generated by the associator. In the quotient any tree with n leaves gives rise to the same element, that we have denoted by μ_n.

9.1.3 The Operad *Ass*

We denote by Ass the symmetric operad encoding the category of associative algebras. The categories of As-algebras and Ass-algebras are the same, that is the category of associative algebras, since the action of $\mathbb{S}_n$ on $Ass(n)$ is free (see below).

The free associative algebra over the vector space V is known to be the reduced tensor module $\overline{T}(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ equipped with the concatenation product. It is called the *reduced tensor algebra*, cf. Sect. 1.1.3. So we have $Ass(V) = \overline{T}(V)$. If V is generated by the elements $x_1, \ldots, x_n$, then $\overline{T}(V)$ is the algebra of noncommutative polynomials in $x_1, \ldots, x_n$ modulo the constants: $\mathbb{K}\langle x_1, \ldots, x_n\rangle/\mathbb{K}1$. The composition γ on Ass, i.e. the map $\gamma(V) : \overline{T}(\overline{T}(V)) \to \overline{T}(V)$, is given by substitution of polynomials: if $P(X_1, \ldots, X_k)$ is a polynomial in the variables X_i and if each X_i is a polynomial in the variables x_j, then $P(X_1(x_1, \ldots, x_n), \ldots, X_k(x_1, \ldots, x_n))$ is a polynomial in the variables x_j called the composite. This composition is obviously associative.

From the polynomial description of the free associative algebra it follows that the space of n-ary operations is $Ass(n) \cong \mathbb{K}[\mathbb{S}_n]$ equipped with the right action given by multiplication in $\mathbb{S}_n$. The n-ary operation $\mu^\sigma \in Ass(n)$ corresponding to the permutation $\sigma \in \mathbb{S}_n$ is

$$\mu^\sigma(x_1, \ldots, x_n) := \mu_n(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) = x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}.$$

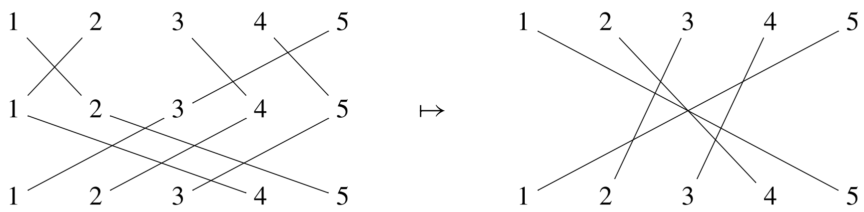
Hence $Ass(n)$ is the regular representation of $\mathbb{S}_n$:

$$Ass(n) \otimes_{\mathbb{S}_n} V^{\otimes n} = \mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_n} V^{\otimes n} = V^{\otimes n}.$$

The composition in the operad Ass is given by the composition of polynomials. It is induced by the maps

$$\gamma(i_1, \ldots, i_k) : \mathbb{S}_k \times \mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k} \to \mathbb{S}_{i_1+\cdots+i_k}$$

given by concatenation of the permutations and block permutation by the elements of $\mathbb{S}_k$. Here is an example with $k = 2$, $i_1 = 2$, $i_2 = 3$:



$$\big([21]; [21], [231]\big) \mapsto [54321].$$

The partial composition $\circ_i$ is easily deduced from the composition map γ in the polynomial framework. It simply consists in substituting the ith variable for a polynomial.

As a symmetric operad, *Ass* is presented as $Ass = \mathscr{T}(E_{Ass}, R_{Ass})$, where $E_{Ass} \cong \mathbb{K}[\mathbb{S}_2]$. We have denoted by μ the operation corresponding to $\mathrm{id}_{\mathbb{S}_2}$ and so the other linear generator is $\mu^{(12)}$. Under our convention, these two operations correspond to xy and yx respectively. The space of relations R_{Ass} is the sub-$\mathbb{S}_3$-module of $\mathscr{T}(E_{Ass})^{(2)}$ generated by $\mu \circ_1 \mu - \mu \circ_2 \mu$. It is clear that $\mathscr{T}(E_{Ass})^{(2)}$ is $2 \times 6 = 12$-dimensional spanned by the elements $(\mu \circ_1 \mu)^\sigma$, $(\mu \circ_2 \mu)^\sigma$, for $\sigma \in \mathbb{S}_3$ and that R_{Ass} is 6-dimensional spanned by the elements $(\mu \circ_1 \mu - \mu \circ_2 \mu)^\sigma$, for $\sigma \in \mathbb{S}_3$.

The characteristic of *Ass* in the algebra of symmetric functions is

$$F^{Ass} = 1 + p_1 + \cdots + p_1^n + \cdots = \frac{1}{1 - p_1}$$

where p_1 is the classical power symmetric function (cf. for instance [Mac95]).

9.1.4 Other Presentations of *Ass*

There are other presentations of the symmetric operad *Ass* which might be useful in certain problems. We will give only two of them. The second one has the advantage of showing that the Poisson operad is the limit, in a certain sense, of a family of operads all isomorphic to *Ass* (so *Pois* is a "tropical" version of *Ass*), see Sect. 13.3.4.

First, we take a generating operation μ_n of arity n for any $n \geq 2$. We take the following relations:

$$\mu_n \circ (\mu_{i_1}, \ldots, \mu_{i_n}) = \mu_{i_1 + \cdots + i_n}$$

for any tuples $(i_1, \ldots, i_n)$. Then obviously the associated operad is *As*. Considering the analogy with the presentation of groups, it is like presenting a group by taking its elements as generators and taking the table of multiplication as relations.

Here is another presentation in the symmetric framework.

Proposition 9.1.1 (Livernet–Loday, unpublished). *If* 2 *is invertible in* $\mathbb{K}$, *then the operad Ass admits the following presentation*:

- *a symmetric operation* $x \cdot y$ *and an antisymmetric operation* $[x, y]$ *as generating operations*,
- *the following relations*:

$$[x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y,$$
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = \big[y, [x, z]\big].$$

Proof. The equivalence between the two presentations is simply given by

$$x \cdot y = xy + yx,$$
$$[x, y] = xy - yx,$$

and, of course, $2xy = x \cdot y + [x, y]$. Observe that the Jacobi relation need not be put as an axiom in this presentation since it is a consequence of the second axiom and the commutativity property of the operation $x \cdot y$. □

For more information about the operation $x \cdot y$ see Sect. 13.10.

9.1.5 The Cooperad $As^¡$ and Koszul Duality

We compute the Koszul dual ns cooperad $As^¡$ of the ns operad As. By definition As is the quotient of the free ns operad generated by one binary operation μ, that is $\mathscr{T}(\mathbb{K}\mu)$, quotiented by the operadic ideal generated by the relator

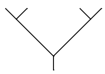
$$as := -\mu \circ (\mu, \mathrm{id}) + \mu \circ (\mathrm{id}, \mu) \in \mathscr{T}(\mathbb{K}\mu)_3.$$

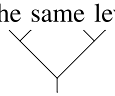
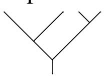
Let us denote by $\mu^c := s\mu$. So μ^c is a cooperation of arity 2 and degree 1. The cofree ns cooperad over $M = (0, 0, \mathbb{K}\mu^c, 0, \ldots)$, that is $\mathscr{T}^c(\mathbb{K}\mu^c)$, can be identified, as a graded vector space, with the vector space spanned by the planar binary trees, cf. Appendix C.1.1. The isomorphism

$$\psi : \mathbb{K}[PBT_n] \cong \mathscr{T}^c\big(\mathbb{K}\mu^c\big)_n$$

is given by

$\psi(|) := \mathrm{id}$, $\psi($  $) := \mu^c$, $\psi(r \vee s) := \big(\mu^c; \psi(r), \psi(s)\big)$.

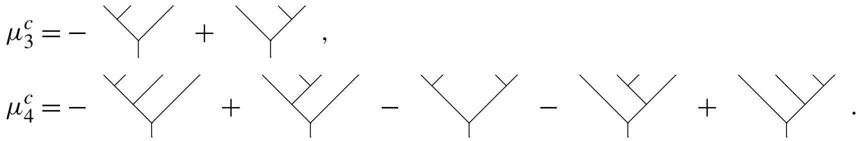
For instance we have $\psi($  $) = (\mu^c; \mu^c, \mu^c)$.

Since the generator μ^c has homological degree 1, there are signs involved in the explicitation of ψ, see Sect. 5.9.7. Let us consider the map $\varphi : \mathbb{S}_{n-1} \to PBT_n$ constructed in Appendix C.1.3 which follows from the identification of the symmetric group $\mathbb{S}_{n-1}$ with the set of planar binary trees with levels $\widetilde{PBT}_n$. We denote by $\tilde{t}$ the permutation in the pre-image of $t \in PBT_n$ which corresponds to the planar binary tree with *upward* levels. It means that, among the leveled trees representing t, we choose the tree whose levels of the vertices, which are at the same level in t, go upward when moving from left to right. For instance, if $t =$ , then $\tilde{t} =$  and the permutation is [231].

Observe that the element $\tilde{t}$ is easy to interpret in terms of the construction described in Sect. 5.5.5. We define

$\mu_1^c := |$, $\mu_2^c :=$ , and $\mu_n^c := -\sum_{t \in PBT_n} \mathrm{sgn}(\tilde{t})t$ for $n \geq 3$.

In low dimension we get



In this example, the involved permutations are [123], [213], [231], [312], [321]. The degree of μ_n^c is $n-1$.

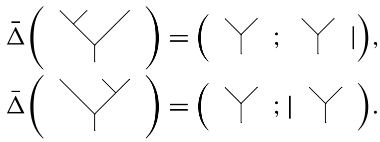
Lemma 9.1.2. *Let* Δ *be the decomposition map of the cofree ns cooperad* $\mathscr{T}^c(\mathbb{K}\mu^c)$. *We have*

$$\Delta\big(\mu_n^c\big) = \sum_{i_1+\cdots+i_k=n} (-1)^{\sum(i_j+1)(k-j)}\big(\mu_k^c;\mu_{i_1}^c,\ldots,\mu_{i_k}^c\big).$$

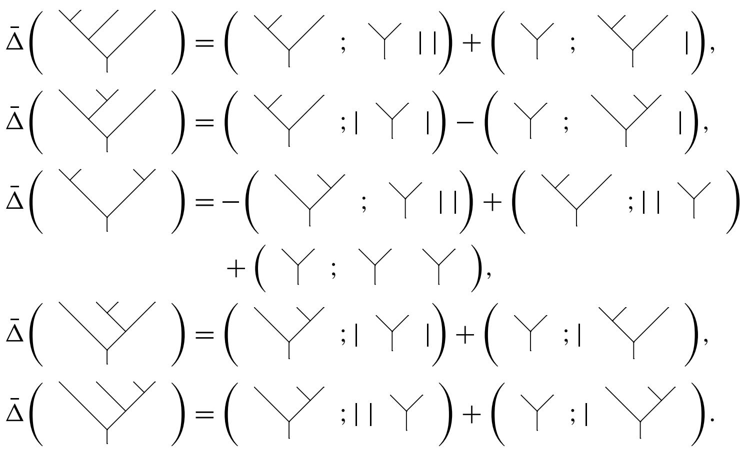
Proof. This is a tedious but straightforward computation. The necessity of the sign $\mathrm{sgn}(\tilde{t})$ in the definition of μ_n^c comes from the formula for Δ in the graded framework (cf. Sect. 5.8.6), which is

$$\Delta(r\vee s) = (r\vee s) + (-1)^{|t^{(2)}||s^{(1)}|}\big(t^{(1)}\vee s^{(1)};t^{(2)},s^{(2)}\big).$$

Here are examples of this computation in low dimension under the convention $\Delta(t) = (|;t) + \bar{\Delta}(t) + (t;|\cdots|)$:



As a consequence we get $\bar{\Delta}(\mu_3^c) = -(\mu_2^c;\mu_2^c,\mu_1^c) + (\mu_2^c;\mu_1^c,\mu_2^c)$.
Then



As a consequence we get

$$\bar{\Delta}(\mu_4^c) = (\mu_2^c; \mu_3^c, \mu_1^c) + (\mu_3^c; \mu_2^c, \mu_1^c, \mu_1^c) - (\mu_3^c; \mu_1^c, \mu_2^c, \mu_1^c) \\ + (\mu_2^c; \mu_1^c, \mu_3^c) + (\mu_3^c; \mu_1^c, \mu_1^c, \mu_2^c) - (\mu_2^c; \mu_2^c, \mu_2^c).$$ □

Proposition 9.1.3. *The ns cooperad $As^i \subset \mathcal{T}^c(\mathbb{K}\mu^c)$ is such that*

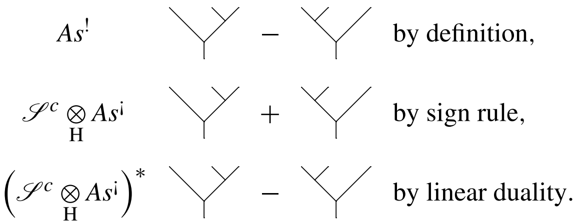
$$(As^i)_n = \mathbb{K}\mu_n^c.$$

Proof. By Lemma 9.1.2 the elements μ_n^c, $n \geq 1$, span a sub-cooperad of $\mathcal{T}^c(\mathbb{K}\mu^c)$. Since $\mu_3^c = as$, this sub-cooperad is universal among the sub-cooperads whose projection to the quotient space $\mathcal{T}^c(\mathbb{K}\mu^c)^{(2)}/\mathbb{K}\,as$ vanishes. Therefore this cooperad is $As^{¡}$. □

Proposition 9.1.4. *The quadratic ns operad As is self-dual for Koszul duality, that is*

$$As^! = As.$$

Proof. Recall from Sect. 7.7.1 that $As^! = (\mathcal{S}^c \underset{H}{\otimes} As^{¡})^*$. Denoting by  the generator and the cogenerator (depending on the context) the relation (or the corelation) can be written:



Hence $As^! = As$.

Of course we could as well apply directly Theorem 7.7.1 as follows. Since, in the presentation of As, the space of weight two operations in the free operad is of dimension 2 and the dimension of the space of relators R is of dimension 1, the orthogonal space $R^\perp$ is of dimension 1. Since R is orthogonal to itself for the quadratic form $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it follows that $R^\perp = R$ and therefore $As^! = As$. □

We now study the Koszulity of the ns operad As.

Theorem 9.1.5. *The ns operad As is a Koszul ns operad.*

Proof. We give here a proof based on the analysis of the Koszul complex. Let us describe the Koszul complex of the ns operad As following p. 70. We consider the arity n sub-chain complex of $As^{¡} \circ As$, that is $(As^{¡} \circ As)_n$. It is a finite complex of length n which reads:

$$As^{\text{¡}}{}_n \otimes As_1 \otimes \cdots \otimes As_1 \to \cdots \to \bigoplus As^{\text{¡}}{}_k \otimes As_{i_1} \otimes \cdots \otimes As_{i_k} \to \cdots \to As^{\text{¡}}{}_1 \otimes As_n$$

where $i_1 + \cdots + i_k = n$ and $i_j \geq 1$. Observe that each component $As^{\text{¡}}{}_k \otimes As_{i_1} \otimes \cdots \otimes As_{i_k}$ is one-dimensional.

In order to describe the boundary map we need to compute $(\mathrm{Id} \circ_{(1)} \kappa)(\Delta_{(1)}(\mu_k^c))$. It is the coproduct of μ_k^c, but keeping on the right side only the terms which involve copies of id and one copy of μ^c (identified to μ under κ). So we get $\sum_j \pm\mu_{k-1}^c \circ (\mathrm{id}, \ldots, \mathrm{id}, \mu, \mathrm{id}, \ldots, \mathrm{id})$. Then we have to apply the associativity isomorphism to

$$\sum_j \pm\big(\mu_{k-1}^c \circ (\mathrm{id}, \ldots, \mathrm{id}, \mu, \mathrm{id}, \ldots, \mathrm{id})\big) \circ (\mu_{i_1}, \ldots, \mu_{i_k})$$

to get

$$\begin{aligned}&\sum_j \pm\mu_{k-1}^c \circ \big(\mathrm{id} \circ \mu_{i_1}, \ldots, \mathrm{id} \circ \mu_{i_{j-1}}, \mu \circ (\mu_{i_j}, \mu_{i_{j+1}}), \mathrm{id} \circ \mu_{i_{j+2}}, \ldots, \mathrm{id} \circ \mu_{i_k}\big)\\ &\quad = \sum_j \pm\mu_{k-1}^c \circ (\mu_{i_1}, \ldots, \mu_{i_{j-1}}, \mu_{i_j + i_{j+1}}, \mu_{i_{j+2}}, \ldots, \mu_{i_k}).\end{aligned}$$

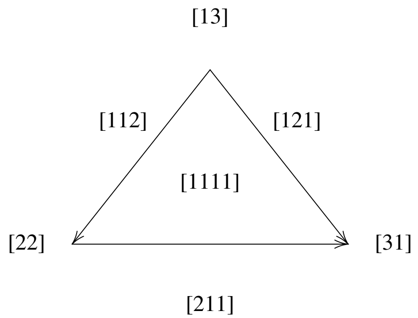
There is no ambiguity to denote the generator of $As^{\text{¡}}{}_k \otimes As_{i_1} \otimes \cdots \otimes As_{i_k}$ by $[i_1, \ldots, i_k]$, and we get

$$d\big([i_1, \ldots, i_k]\big) = \sum_j \pm[i_1, \ldots, i_j + i_{j+1}, \ldots, i_k].$$

The boundary map of this Koszul complex can be identified with the boundary map of the augmented chain complex (shifted by one) of the cellular simplex Δ^{n-2}:

$$C_{n-2}\big(\Delta^{n-2}\big) \to \cdots \to C_{k-2}\big(\Delta^{n-2}\big) \to \cdots \to C_0\big(\Delta^{n-2}\big) \to \mathbb{K}.$$

Compared to the classical way of indexing the vertices of the simplex Δ^{n-2} by integers $0, \ldots, n-1$, the vertex number i corresponds to the chain $[i+1, n-i-1]$ in $(As^{\text{¡}} \circ As)_n$. Here is the simplex Δ^2:

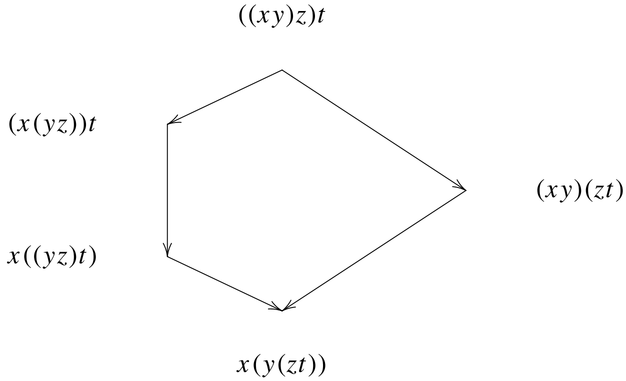


Since the simplex is contractible, its associated augmented chain complex is acyclic for any $n \geq 2$. For $n = 1$ the complex reduces to the space $As^{\text{¡}}{}_1 \otimes As_1 = \mathbb{K}$. □

REMARK. We could have also use the poset method (cf. Sect. 8.7), to prove the acyclicity of the Koszul complex. Or (see below), we could compute the Hochschild homology of the free algebra $\overline{T}(V)$ by providing an explicit homotopy.

Modulo all the apparatus, the shortest proof is the rewriting system method (cf. Sect. 8.1) since the only critical monomial is $((xy)z)t$ and the confluent property is immediate to verify:

- on one hand $((xy)z)t \mapsto (x(yz))t \mapsto x((yz)t) \mapsto x(y(zt))$ (left side of the pentagon),
- one the other hand $((xy)z)t \mapsto (xy)(zt) \mapsto x(y(zt))$ (right side of the pentagon).



9.1.6 Hochschild Homology of Associative Algebras

Since we know that $As^! = As$, we can describe explicitly the chain complex $C_\bullet^{As}(A)$, which gives the operadic homology of the associative algebra A, cf. Sect. 12.1.2. Let us introduce the boundary map

$$b' : A^{\otimes n} \to A^{\otimes n-1}$$

by the formula

$$b'(a_1, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, \dots, a_i a_{i+1}, \dots, a_n).$$

It is well known (and easy to check from the associativity of the product) that $(b')^2 = 0$. The resulting chain complex $(A^{\otimes\bullet}, b')$ is the *nonunital Hochschild complex*, up to a shift of degree.

Proposition 9.1.6. *The operadic chain complex of the associative algebra A is the nonunital Hochschild complex of A:*

$$C_{\bullet-1}^{As}(A) = \left(A^{\otimes\bullet}, b'\right).$$

Therefore the operadic homology of an associative algebra is the Hochschild homology up to a shift of degree.

Proof. By definition (Sect. 12.1.2) the complex $C_\bullet^{As}(A)$ is given by

$$\cdots \to As^{¡}{}_n \otimes A^{\otimes n} \to \cdots \to As^{¡}{}_1 \otimes A.$$

Since $As^{¡}{}_n$ is one-dimensional, we get $A^{\otimes n}$ in degree $n-1$. The boundary map is obtained as follows (cf. Sect. 6.6): we consider all the possibilities of "splitting" μ_n^c using one copy of μ^c on the right-hand side (infinitesimal decomposition map), that is

$$\Delta_{(1)}\big(\mu_n^c\big) = \sum \pm \mu_{n-1}^c \otimes \big(\mathrm{id}, \ldots, \mathrm{id}, \mu^c, \mathrm{id}, \ldots, \mathrm{id}\big)$$

and then we apply the element $(\mathrm{id}, \ldots, \mathrm{id}, \mu^c, \mathrm{id}, \ldots, \mathrm{id})$ to $(a_1, \ldots, a_n)$ after replacing the cooperation μ^c by the operation μ under κ. This gives precisely the boundary map b' since $\mu(a_i, a_{i+1}) = a_i a_{i+1}$. □

9.1.7 Homology and Cohomology with Coefficients

In the literature, homology and cohomology with coefficients appear most of the time for unital associative algebras. The comparison between unital and nonunital cases is not completely straightforward, see for instance [LQ84, Lod98] Sect. 1.4. We describe briefly the unital case.

In order to construct a homology or cohomology with coefficients one needs a notion of "representation" (the coefficients). In the associative case it is the notion of bimodule because of the following fact. For any (unital) algebra A and any abelian extension

$$0 \to M \to A' \to A \to 0$$

the space M is a (unital) bimodule over A. Recall that here A' is a (unital) algebra and the product of two elements of M is 0 (abelian hypothesis).

Let A be a unital associative algebra and M a unital A-bimodule. The Hochschild complex $C_\bullet(A, M)$ is defined as $C_n(A, M) := M \otimes A^{\otimes n}$ and the boundary map $b : C_n(A, M) \to C_{n-1}(A, M)$ is given by the formula $b = \sum_{i=0}^{i=n} (-1)^i d_i$, where

$$\begin{aligned}
d_0(m, a_1, \ldots, a_n) &:= (ma_1, a_2, \ldots, a_n),\\
d_i(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots a_i a_{i+1}, \ldots, a_n), \quad 1 \le i \le n-1,\\
d_n(m, a_1, \ldots, a_n) &:= (a_n m, a_1, \ldots, a_{n-1}).
\end{aligned}$$

The homology groups of $C_\bullet(A, M)$ are called the *Hochschild homology groups* of the unital algebra A with coefficients in the bimodule M.

The Hochschild complex of cochains $C^\bullet(A, M)$ is defined as $C^n(A, M) :=$-$\mathrm{Hom}(A^{\otimes n}, M)$ and the boundary map $b : C^n(A, M) \to C^{n+1}(A, M)$ is given by the formula $b = \sum_{i=0}^{i=n} (-1)^i d_i$, where

$$\begin{aligned} d_0(f)(a_1, \ldots, a_{n+1}) &:= a_1 f(a_2, \ldots, a_{n+1}), \\ d_i(f)(a_1, \ldots, a_{n+1}) &:= f(a_1, \ldots a_i a_{i+1}, \ldots, a_{n+1}), \quad 1 \le i \le n, \\ d_n(f)(a_1, \ldots, a_{n+1}) &:= f(a_1, \ldots, a_n) a_{n+1}. \end{aligned}$$

The homology groups of $C^\bullet(A, M)$ are called the *Hochschild cohomology groups* of the unital algebra A with coefficients in the bimodule M.

These groups appear as obstruction groups in many questions and there is an extensive literature about them (for a first approach see for instance [Lod98]). For instance there is a classification theorem for abelian extensions of A by M:

$$H^2(A, M) \cong \mathscr{E}xt(A, M).$$

Similarly there is a classification theorem for crossed modules of A by M:

$$H^3(A, M) \cong \mathscr{X}Mod(A, M).$$

Historically these complexes were constructed by hand by Hochschild. Of course they can be viewed as coming from the operad theory. The advantage is to produce similar complexes and theorems for Koszul operads without having to do ad hoc constructions and proofs in each case. This is the theme of Chap. 12.

9.1.8 Other Homology Theories for Associative Algebras

The operad As has more structure: it is a cyclic operad (cf. Sect. 13.14.6). As such there exists a homology theory for associative algebras which takes into account this extra structure, it is called *cyclic homology*. It was first devised by Alain Connes [Con85] and further studied in [LQ84] and [Tsy83] (see also the monograph [Lod98]), mainly for unital associative algebras. Cyclic homology is encoded by the cyclic category, denoted by ΔC. This notation accounts for the structure of this category which is made up of the simplicial category Δ and the cyclic groups. There is a similar category, where the cyclic groups are replaced by the symmetric groups: ΔS, cf. [FL91, Lod98]. It turns out that ΔS is precisely the category associated to the operad uAs encoding the category of unital associative algebras by the method in Sect. 5.4.1, as shown by Pirashvili in [Pir02a], cf. Example 2.

The operad As can also be considered as a permutad (cf. Sect. 13.14.7). We refer to [LR12] for more on this theme.

9.2 Associative Algebras Up to Homotopy

In the 1960s, Jim Stasheff introduced in [Sta63] the notion of A_∞-algebra, also called “associative algebra up to strong homotopy”. The idea is that associativity of the binary operation m_2 is satisfied only “up to higher homotopy”. It has a meaning algebraically whenever one works with a chain complex (A,d). It means that the associator of m_2 is not zero, but there exists a ternary operation m_3 (the homotopy) such that

$$m_2 \circ (m_2, \mathrm{id}) - m_2 \circ (\mathrm{id}, m_2) = d \circ m_3 + m_3 \circ \big((d, \mathrm{id}, \mathrm{id}) + (\mathrm{id}, d, \mathrm{id}) + (\mathrm{id}, \mathrm{id}, d)\big).$$

But then, mixing m_2 and m_3 leads to introduce a 4-ary operation m_4 satisfying some relation, and so on, and so on. The whole algebraic structure, discovered by Jim Stasheff, is encoded into the notion of A_∞-algebra, that we recall below. The relevant operad, denoted A_∞, can be described in terms of the Stasheff polytope.

On the other hand the operad theory gives an algorithm to construct explicitly the minimal model of As, which is the dgns operad $As_\infty := \Omega As^{¡}$. It gives rise to the notion of "homotopy associative algebra". It turns out that, not surprisingly, $A_\infty = As_\infty$.

9.2.1 A_∞-Algebra [Sta63]

We have seen in Sect. 2.2.1 that any associative algebra A gives rise to a dga coalgebra $(\overline{T}^c(sA), d)$, which is its bar construction. It is natural to look for a converse statement. Given a graded vector space A together with a codifferential m on the cofree coalgebra $\overline{T}^c(sA)$, what kind of structure do we have on A? The answer is given by the notion of A_∞-algebra. Here are the details.

By definition an A_∞-*algebra* is a graded vector space $A = \{A_k\}_{k\in\mathbb{Z}}$ equipped with a codifferential map $m : \overline{T}^c(sA) \to \overline{T}^c(sA)$ (so $|m| = -1$, $m \circ m = 0$) and m is a coderivation, cf. Sect. 1.2.7. Observe that, since $\overline{T}^c(sA)$ is cofree, the coderivation m is completely determined by its projection $proj \circ m : \overline{T}^c(sA) \to sA$, that is by a family of maps $A^{\otimes n} \to A, n \geq 1$.

To any A_∞-algebra (A,m), we associate an n-ary operation m_n on A as the following composite:

$$\begin{array}{ccc} (sa_1)\cdots(sa_n) & \qquad (sA)^{\otimes n} \xrightarrow{\;m_|\;} sA & \qquad sa \\ \uparrow & \cong\uparrow \qquad\qquad \downarrow\cong & \qquad \downarrow \\ a_1\cdots a_n & \qquad A^{\otimes n} \xrightarrow{\;m_n\;} A & \qquad a \end{array}$$

where $m_|$ is the restriction of m to $(sA)^{\otimes n}$ composed with the projection onto sA. The map m_n is of degree $n-2$ since the degrees of the involved maps in the composition are $n-1$ and -1 respectively.

Lemma 9.2.1. *An* A_∞-algebra *is equivalent to a graded vector space* $A = \{A_k\}_{k\in\mathbb{Z}}$ *equipped with an* n*-ary operation*

$$m_n : A^{\otimes n} \to A \quad \textit{of degree } n-2 \textit{ for all } n \geq 1,$$

which satisfy the following relations:

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{p+qr} m_k \circ_{p+1} m_q = 0, \quad n \geq 1,$$

where $k = p + 1 + r$.

Proof. First, recall that

$$m_k \circ_{p+1} m_q = m_k \circ \left(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}\right).$$

Let (A, m) be an A_∞-algebra as defined above. Since m is a coderivation, and since $\overline{T}^c(sA)$ is cofree over sA, by Proposition 1.2.2, m is completely determined by its composite

$$\overline{T}^c(sA) \xrightarrow{m} \overline{T}^c(sA) \xrightarrow{proj} sA.$$

The condition $m^2 = 0$ is equivalent to the vanishing of the composite

$$\overline{T}^c(sA) \xrightarrow{m} \overline{T}^c(sA) \xrightarrow{m} \overline{T}^c(sA) \twoheadrightarrow sA.$$

For each $n \geq 1$, its restriction to the n-tensors gives the relation $(rel)_n$. The signs come from the fact that we establish the formula on $A^{\otimes n}$ instead of $(sA)^{\otimes n}$. □

We sometimes write m_n^A in place of m_n if it is necessary to keep track of the underlying chain complex. Let us make the relation (rel_n) explicit for $n = 1, 2, 3$:

$$\begin{aligned}
&m_1 \circ m_1 = 0,\\
&m_1 \circ m_2 - m_2 \circ (m_1, \mathrm{id}) - m_2 \circ (\mathrm{id}, m_1) = 0,\\
&m_1 \circ m_3 + m_2 \circ (m_2, \mathrm{id}) - m_2 \circ (\mathrm{id}, m_2)\\
&\quad + m_3 \circ (m_1, \mathrm{id}, \mathrm{id}) + m_3 \circ (\mathrm{id}, m_1, \mathrm{id}) + m_3 \circ (\mathrm{id}, \mathrm{id}, m_1) = 0.
\end{aligned}$$

9.2.2 Homotopy and Operadic Homology of an A_∞-Algebra

Let A be an A_∞-algebra, e.g. a dga algebra. The relation $(rel)_1$ implies that (A, m_1) is a chain complex. We prefer to denote the differential m_1 by $-d$ and consider an A_∞-algebra as being a chain complex (A, d) equipped with higher operations, see below. The homology of the underlying chain complex (A, d) is called the *homotopy of the* A_∞*-algebra* A. The homology of the chain complex $(\overline{T}^c(sA), m^A)$ is called the *operadic homology* of the A_∞-algebra A. If A is a dga algebra, then the operadic homology is the *Hochschild homology* (of the nonunital dga algebra).

Proposition 9.2.2. *The homotopy of an* A_∞*-algebra is a graded associative algebra.*

Proof. This is an immediate consequence of the relations (rel_2) and (rel_3) recalled in the introduction of this section and in Sect. 9.2.4. □

A finer statement can be found in Corollary 9.4.5.

9.2.3 Examples

We already mentioned that a (nonunital) dga algebra (A, d) is an example of A_∞-algebra. Indeed it suffices to take $m_1 = -d$, $m_2 = \mu$ and $m_n = 0$ for $n \geq 3$. Observe that if A is a dga algebra, then $(T^c(s\bar{A}), m)$ is precisely the cobar construction on A.

When $A = C^\bullet_{\text{sing}}(X)$ is the singular cochain complex of a topological space X, it is endowed with an associative cup product. This associative algebra structure transfers to an A_∞-algebra structure on the singular cohomology $H^\bullet_{\text{sing}}(X)$. These operations were originally defined by Massey in [Mas58].

The singular chains on a based loop space ΩX of the connected topological space X is an A_∞-algebra, cf. [Sta63].

9.2.4 The Operad A_∞

Let A be an A_∞-algebra. For $n = 1$ the relation (rel_n) reads as follows:

$$m_1 \circ m_1 = 0.$$

So $d := m_1$ is a differential on A. The derivative (cf. p. 25) of the map m_n is

$$\partial(m_n) := dm_n - (-1)^{n-2} m_n\big((d, \mathrm{id}, \ldots, \mathrm{id}) + \cdots + (\mathrm{id}, \ldots, \mathrm{id}, d)\big).$$

Using this notation the relations (rel_n) become:

$$\begin{aligned}
&\big(rel'_2\big) : \partial(m_2) = 0,\\
&\big(rel'_3\big) : \partial(m_3) = -m_2 \circ (m_2, \mathrm{id}) + m_2 \circ (\mathrm{id}, m_2),\\
&\big(rel'_4\big) : \partial(m_4) = -m_2 \circ (m_3, \mathrm{id}) + m_3 \circ (m_2, \mathrm{id}, \mathrm{id}) - m_3 \circ (\mathrm{id}, m_2, \mathrm{id})\\
&\qquad\qquad + m_3 \circ (\mathrm{id}, \mathrm{id}, m_2) - m_2 \circ (\mathrm{id}, m_3).
\end{aligned}$$

More generally, for any $n \geq 2$, the relation (rel_n) can be written as:

$$(rel'_n) : \quad \partial(m_n) = - \sum_{\substack{n = p+q+r \\ k = p+1+r \\ k > 1, q > 1}} (-1)^{p+qr} m_k \circ \big(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}\big).$$

Therefore an A_∞-algebra can be seen as a chain complex (A, d) equipped with linear maps: $m_n : (A, d)^{\otimes n} \to (A, d)$, for $n \geq 2$, satisfying some relations. In other words, an A_∞-algebra is an algebra over some dgns operad, denoted A_∞. The generating operations of this operad correspond to the operations $m_n \in \mathrm{End}(A)$, $n \geq 2$, of degree $n-2$. So, the nonsymmetric operad A_∞ is free over these operations: $A_\infty = \mathscr{T}(\oplus_{n\geq 2}\mathbb{K}\mu_n)$. The differential structure is precisely given by the relations (rel'_n), $n \geq 2$.

Here we work in the homological framework (degree of d is -1), but one can of course define A_∞-algebras in the cohomological framework, see for instance [Kel01].

9.2.5 The Associahedron (Stasheff Polytope)

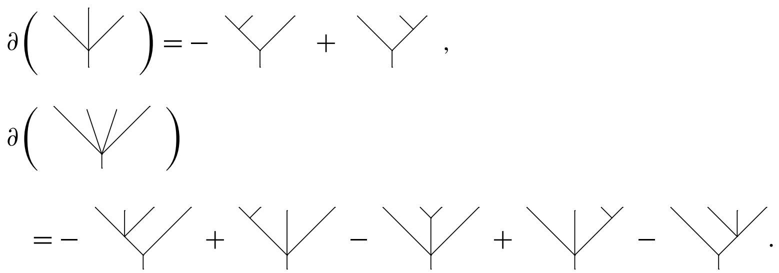
Let us recall from Appendix C that the associahedron $\mathscr{K}^n$ is a cell complex of dimension n homeomorphic to a ball, whose cells are in bijection with the planar trees with $n+2$ leaves. The chain complex associated to $\mathscr{K}^n$ is denoted by $C_\bullet(\mathscr{K}^n)$. A tree $t \in PT_{n+2,n-k+1}$ has $n+2$ leaves and $(n-k+1)$ internal vertices. It gives a chain in $C_k(\mathscr{K}^n)$. Since $\mathscr{K}^n$ is contractible, the homology of $C_\bullet(\mathscr{K}^n)$ is trivial except in dimension 0, where it is $\mathbb{K}$. We will identify the dg vector space of n-ary operations of the operad A_∞ (and of the operad As_∞) to $C_\bullet(\mathscr{K}^{n-2})$.

Proposition 9.2.3. *The dg operad A_∞ encoding the category of A_∞-algebras is a dgns operad whose dg module of n-ary operations is the chain complex of the associahedron*:

$$(A_\infty)_n = C_\bullet\big(\mathscr{K}^{n-2}\big).$$

Proof. Since, as an operad, A_∞ is free with one generator in each arity $n \geq 2$, there is an isomorphism $(A_\infty)_n \cong \mathbb{K}[PT_n]$, where PT_n is the set of planar trees with n leaves. The generating operation m_n corresponds to the n-corolla under this isomorphism. On the other hand, the cells of the associahedron $\mathscr{K}^{n-2}$ are in bijection with PT_n, whence the identification of vector spaces $\mathbb{K}[PT_n] = C_\bullet(\mathscr{K}^{n-2})$. The boundary map of $(A_\infty)_n$ is given by formula (rel'_n). Once translated in terms of cells of the associahedron it gives precisely the boundary of the big cell of $\mathscr{K}^{n-2}$ since the facettes are labeled by the planar trees with two internal vertices. □

EXAMPLES. (rel'_3) and (rel'_4):



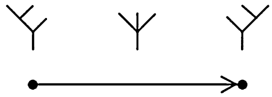


Fig. 9.1 Interval

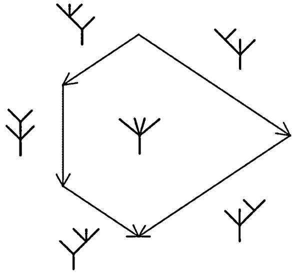


Fig. 9.2 Pentagon

These formulas are the algebraic translation of the cell boundaries shown in Figs. 9.1 and 9.2.

Proposition 9.2.4. *The operad $As_\infty := \Omega As^{¡}$ is a dgns operad whose space of n-ary operations is the chain complex of the associahedron*:

$$\left(\Omega As^{¡}\right)_n = (As_\infty)_n = C_\bullet\left(\mathscr{K}^{n-2}\right).$$

Proof. We work in the nonsymmetric operad context. As a ns operad $As_\infty = \mathscr{T}(s^{-1}\overline{As^{¡}})$ is free on $\overline{As^{¡}}$, that is free on the generating operations $\tilde{\mu}_n^c := s^{-1}\mu_n^c$, $n \geq 2$ of degree $n-2$. Hence, as in the previous case, there is an isomorphism of vector spaces $(As_\infty)_n \cong \mathbb{K}[PT_n] = C_\bullet(\mathscr{K}^{n-2})$. By definition of the cobar construction, the boundary map on $\mathscr{T}(s\overline{As^{¡}})_n$ is induced by the cooperad structure of $As^{¡}$, more precisely by the infinitesimal decomposition map $\Delta_{(1)}$, cf. Sect. 6.1.4. Let us show that this boundary map is precisely the boundary map of the chain complex $C_\bullet(\mathscr{K}^{n-2})$.

As a linear generator of $\mathscr{T}(s^{-1}\overline{As^{¡}})_n$ the element $\tilde{\mu}_n^c$ corresponds to the big cell (n-corolla) $t(n)$. The degree of $\tilde{\mu}_n^c$ is $n-2$. It is mapped to $\mu_n \in (A_\infty)_n$. The image of μ_n^c under the map $\Delta_{(1)}$ is

$$\Delta_{(1)}\left(\mu_n^c\right) = \sum_{p+1+r=k} (-1)^{(q+1)r} \mu_k^c \otimes \big(\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p}, \mu_q^c, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r}\big),$$

for $n = p+q+r$ by Lemma 9.1.2. Under the isomorphism $\mathscr{T}(s\overline{As^{¡}})_n \cong C_\bullet(\mathscr{K}^{n-2})$ this is precisely the boundary map of the associahedron. The evaluation of the boundary on the other cells follows from this case. □

Corollary 9.2.5. *The categories of A_∞-algebras and As_∞-algebras are the same*:

$$A_\infty = As_\infty := \Omega As^{¡}.$$

Proof. From the description of the operad A_∞ given in Proposition 9.2.3 and the description of the operad As_∞ given in Proposition 9.2.4 it is clear that we have a bijection

$$(A_\infty)_n = \mathbb{K}[PT_n] = (As_\infty)_n$$

which is compatible with the operad structure. Moreover both differentials coincide with the differential map in the chain complex of the associahedron, therefore the two dgns operads are identical. $\square$

9.2.6 Infinity-Morphism

Let A and B be two A_∞-algebras. An ∞-*morphism* $f : A \rightsquigarrow B$ (sometimes called A_∞-*morphism* in the literature) is by definition a morphism of dga coalgebras

$$f : \left(\overline{T}^c(sA), m^A\right) \to \left(\overline{T}^c(sB), m^B\right).$$

We adopt the notation $\rightsquigarrow$ to avoid confusion between morphism and ∞-morphism between two A_∞-algebras. Since $\overline{T}^c(sB)$ is cofree, an ∞-morphism is equivalent to a map $\overline{T}^c(sA) \to sB$, that is a family of maps of degree $n-1$

$$f_n : A^{\otimes n} \to B, \quad n \geq 1,$$

such that f_1 is a chain map, i.e. $d_B \circ f_1 = f_1 \circ d_A$ and such that

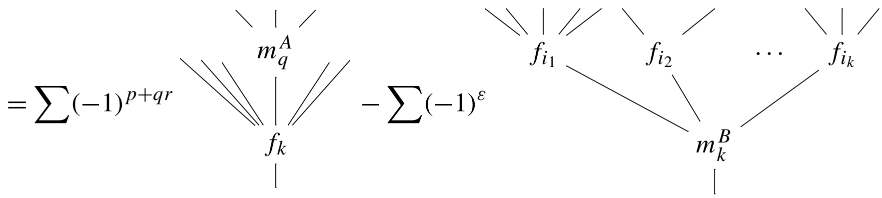
$$\sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{p+qr} f_k \circ \big(\underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{p}, m_q^A, \underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{r}\big)$$
$$- \sum_{\substack{k\geq 2 \\ i_1+\cdots+i_k=n}} (-1)^{\varepsilon} m_k^B \circ (f_{i_1}, \ldots, f_{i_k}) = \partial(f_n),$$

in $\mathrm{Hom}(A^{\otimes n}, B)$, for $n \geq 2$. The sign ε is given by the formula

$$\varepsilon = (k-1)(i_1-1) + (k-2)(i_2-1) + \cdots + 2(i_{k-2}-1) + (i_{k-1}-1).$$

Under the tree notation, this relation becomes

$\partial(f_n)$



From the definition, it follows that the composite of two ∞-morphisms is again an ∞-morphism. The category of A_∞-algebras equipped with the ∞-morphisms is denoted ∞-A_∞-alg. It is a good exercise to write down the explicit formulas for $(g \circ f)_n$ in terms of the f_i and g_i for $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$.

An ∞-morphism f is called an *∞-quasi-isomorphism* when f_1 is a quasi-isomorphism. One can prove that ∞-quasi-isomorphisms are homotopy invertible. We refer to Chap. 10 for details on ∞-morphisms.

9.2.7 A_∞-Coalgebra

By definition a *conilpotent A_∞-coalgebra* in the monoidal category of sign-graded vector spaces is a graded vector space C equipped with a degree -1 square zero derivation on the tensor algebra:

$$\Delta : \overline{T}\big(s^{-1}C\big) \to \overline{T}\big(s^{-1}C\big).$$

Since the tensor algebra is free, the differential map Δ is completely determined by a family of maps

$$\Delta^n : C \to C^{\otimes n}$$

of degree $-n+2$ for all $n \geq 1$. These maps satisfy the following relations

$$\sum_{p+q+r=n} (-1)^{p+qr}\big(\mathrm{id}^{\otimes p} \otimes \Delta^q \otimes \mathrm{id}^{\otimes r}\big)\Delta^k = 0,$$

where $k = p+1+q$.

In the case where $\Delta^n = 0$ for $n > 2$, C is a noncounital dga coalgebra and $(T(s^{-1}C), \Delta)$ is its bar construction.

9.3 The Bar–Cobar Construction on *As*

In this section, we study the bar–cobar construction on the ns operad As, that is the dgns operad $\Omega \mathrm{B} As$. We give a geometric interpretation of the operad monomorphism $As_\infty = \Omega As^{¡} \rightarrowtail \Omega \mathrm{B} As$.

9.3.1 The NS Operad $\Omega \mathrm{B} As$

The operad $\Omega \mathrm{B} As$ is a dgns operad which is free as a ns operad. We will describe the chain complex $(\Omega \mathrm{B} As)_n$ in terms of the cubical realization $\mathscr{K}^{n-2}_{cub}$ of the associahedron (Boardman–Vogt W-construction, see also [BM03a]), as described in Appendix C.2.2.

Proposition 9.3.1. *The chain complex* $(\Omega \mathrm{B} As)_n$ *is precisely the chain complex of the cubical realization of the associahedron*:

$$(\Omega \mathrm{B} As)_n = C_\bullet\big(\mathscr{K}^{n-2}_{cub}\big).$$

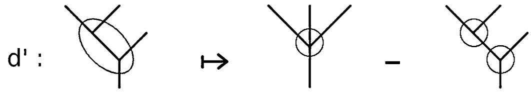


Fig. 9.3 The boundary map of $C_\bullet(\mathscr{K}^1_{cub})$

Proof. In Appendix C.2.2, it is shown that the chain complex $C_\bullet(\mathscr{K}^{n-2}_{cub})$ is spanned by the circled trees with n leaves:

$$C_\bullet(\mathscr{K}^{n-2}_{cub}) = \mathbb{K}[CPT_n].$$

We construct a map $(\Omega \mathrm{B} As)_n \to \mathbb{K}[CPT_n]$ as follows. Since $\Omega \mathrm{B} As = \mathscr{T}(s^{-1} \times \overline{\mathrm{B} As}) \cong \mathscr{T}(s^{-1}\overline{\mathscr{T}^c}(sAs))$, a linear generator is a planar tree whose vertices are labeled by linear generators of $\overline{\mathrm{B} As}$. Since $\overline{\mathrm{B} As}$ is spanned by planar trees, we get trees of trees, that is circled trees. The trees inside the circles are elements of $\overline{\mathrm{B} As}$.

We need now to show that the differential structure of $\Omega \mathrm{B} As$ corresponds exactly to the boundary map of the chain complex $C_\bullet(\mathscr{K}^{n-2}_{cub})$. By definition (cf. Sect. 6.5), the boundary map in $\Omega \mathrm{B} As$ has two components: d_1 coming from the boundary map of $\mathrm{B} As$ and d_2 coming from the cooperad structure of $\mathrm{B} As$. Since $\mathrm{B} As$ is dual to $\Omega As^{¡}$, it follows that $(\mathrm{B} As)_n$ can be identified with the chain complex $C_\bullet(\mathscr{K}^{n-2})$.

On the geometric side, the boundary map of $C_\bullet(\mathscr{K}^{n-2}_{cub})$ is made up of two components d'_1 and d'_2. Let us focus on the top-cells of $C_\bullet(\mathscr{K}^{n-2}_{cub})$, which are encoded by planar binary trees with one circle. Its boundary is made up of two kinds of elements: those which have only one circle (and one edge less), and those which have two circles (and the same number of edges). The elements of the first type account for d_1 and the elements of the second type account for d_2. Under the identification of linear generators we verify that $d_1 = d'_1$ and $d_2 = d'_2$. An example is given in Fig. 9.3. □

Proposition 9.3.2. *There is a sequence of dgns operad morphisms*

$$As_\infty = \Omega As^{¡} \xrightarrow{\sim} \Omega \mathrm{B} As \xrightarrow{\sim} As$$

which are quasi-isomorphisms. In arity n they are given by the quasi-isomorphisms

$$C_\bullet(\mathscr{K}^{n-2}) \xrightarrow{\sim} C_\bullet(\mathscr{K}^{n-2}_{cub}) \xrightarrow{\sim} \mathbb{K}\mu_n$$

where the first one is given by the cellular homeomorphism $\mathscr{K}^{n-2} \to \mathscr{K}^{n-2}_{cub}$ and the second one is the augmentation map.

Proof. The augmentation map $\Omega As^{¡} \to As$ has a dual which is $As^{¡} \to \mathrm{B} As$. Taking the cobar construction gives $\Omega As^{¡} \to \Omega \mathrm{B} As$. Under the identification with the chain complex of the two cellular decompositions of the associahedron, we get, in arity $n+2$, a chain map $C_\bullet(\mathscr{K}^n) \to C_\bullet(\mathscr{K}^n_{cub})$. It is given by the identification of $\mathscr{K}^n$ with itself, i.e. the big cell of $\mathscr{K}^n$ is sent to the sum of the top-cells $\mathscr{K}^n_{cub}$.

The augmentation map $\Omega As^{¡} \to As$ is obtained by taking the homology. Since the associahedron is a convex polytope, it is homeomorphic to a ball and so its homology is the homology of a point. In degree 0, each vertex (i.e. each planar binary tree) is sent to μ_n, the generator of As_n. $\square$

9.4 Homotopy Transfer Theorem for the Operad *As*

The notion of associative algebra is not stable under homotopy equivalence in general. Indeed, if V is a chain complex homotopy equivalent to a dga algebra A, the product induced on V is not necessarily associative. However the algebra structure on A can be transferred to an A_∞-algebra structure on V. We will see that, more generally, an A_∞-algebra structure on A can be transferred to an A_∞-algebra structure on V. So the category of A_∞-algebras is stable under homotopy equivalence. An interesting particular case is when V is the homotopy of A, that is $V = H_\bullet(A, d)$. Then, we obtain some new operations on $H(A)$ which generalize the higher Massey products.

9.4.1 Transferring the Algebra Structure

Let (A, d_A) be a dga algebra. We suppose that (V, d_V) is a *homotopy retract* of the chain complex (A, d_A):

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V),$$

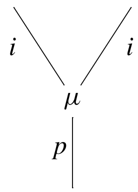
$$\mathrm{Id}_A - ip = d_A h + h d_A,$$

the map $i : V \to A$ being a quasi-isomorphism. From this hypothesis it follows immediately that the homology of (V, d_V) is the same as the homology of (A, d_A) and so $H_\bullet(V, d_V)$ is a graded associative algebra. It is natural to ask oneself what kind of algebraic structure there exists on V which implies that its homology is a graded associative algebra. One can define a binary operation $m_2 : V \otimes V \to V$ by the formula

$$m_2(u, v) := p\mu\big(i(u), i(v)\big)$$

where μ stands for the product in A:

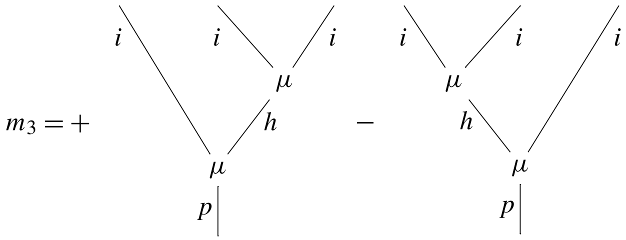
$m_2 =$



It is straightforward to see that there is no reason for m_2 to be associative. However the obstruction to associativity is measured as follows.

Lemma 9.4.1. *The ternary operation m_3 on V defined by the formula*

$$m_3(u, v, w) := -p\mu\big(i(u), h\mu\big(i(v), i(w)\big)\big) + p\mu\big(h\mu\big(i(u), i(v)\big), i(w)\big),$$



satisfies the relation

$$\partial(m_3) = -m_2 \circ (m_2, \text{id}) + m_2 \circ (\text{id}, m_2)$$

(*as already mentioned* $\partial(m_3) := d_V m_3 + m_3 d_{V^{\otimes 3}}$).

Proof. In order to ease the computation, we write the proof in the case of a deformation retract. So we think of V as a subspace of A (whence suppressing the notation i), so p becomes an idempotent of A satisfying $dh = \text{id} - p - hd$. Let us compute $\partial(p\mu(h\mu(i, i), i))$, that is $\partial(p\mu(h\mu, \text{id}))$ under our convention:

$$\begin{aligned}
&\partial\big(p\mu(h\mu, \text{id})\big)\\
&\quad = dp\mu(h\mu, \text{id}) + \big[\big(p\mu\big(h\mu(d, \text{id}), \text{id}\big)\big) + \big(p\mu\big(h\mu(\text{id}, d)\big), \text{id}\big) + \big(p\mu(h\mu, d)\big)\big]\\
&\quad = p\mu(dh\mu, \text{id}) - p\mu(h\mu, d) + [\cdots]\\
&\quad = p\mu(\mu, \text{id}) - p\mu(p\mu, \text{id})\\
&\qquad - \big(p\mu\big(h\mu(d, \text{id}), \text{id}\big)\big) - \big(p\mu\big(h\mu(\text{id}, d)\big), \text{id}\big) - p\mu(h\mu, d) + [\cdots]\\
&\quad = p\mu(\mu, \text{id}) - m_2(m_2, \text{id}).
\end{aligned}$$

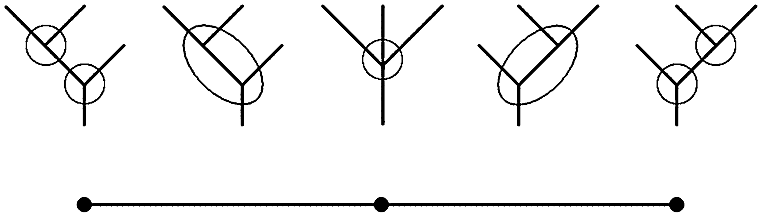
Similarly one gets

$$\partial\big(p\mu(\text{id}, h\mu)\big) = p\big(\mu(\text{id}, \mu)\big) - m_2(\text{id}, m_2).$$

Since μ is associative the terms $p\mu(\mu, \text{id})$ and $p\mu(\text{id}, \mu)$ are equal, and we get the expected formula. □

9.4.2 Geometric Interpretation of m_3

Let us consider the cubical decomposition of the interval, whose cells are labeled by the "circled trees", cf. Appendix C.2.2.



The two summands of m_3 correspond to the two 1-cells of this cubical decomposition and the formula $\partial(m_3) = m_2 \circ (\mathrm{id}, m_2) - m_2 \circ (m_2, \mathrm{id})$ is simply the computation of this boundary.

9.4.3 Higher Structure on the Homotopy Retract

Lemma 9.4.1 suggests that (V, d) inherits an A_∞-structure from the associative structure of (A, d). The geometric interpretation shows us the route to construct m_n explicitly: use the cubical decomposition of the Stasheff polytope:

$$m_n := \sum_{t \in PBT_{n+2}} \pm m_t,$$

where, for any planar binary tree (pb tree) t, the n-ary operation m_t is obtained by putting i on the leaves, μ on the vertices, h on the internal edges and p on the root, as in the case

$m_3 = -m$  $+\, m$  .

Theorem 9.4.2 (T. Kadeishvili [Kad80]). *Let*

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V),$$

$$\mathrm{Id}_A - ip = d_A h + h d_A,$$

be a retract. If (A, d_A) is a dga algebra, then (V, d_V) inherits an A_∞-algebra structure $\{m_n\}_{n \geq 2}$, which extends functorially the binary operation of A.

Proof. The proof is analogous to the proof of Lemma 9.4.1. It is done by induction on the size of the trees. □

This statement is a particular case of Theorem 9.4.7 which gives an even more precise result.

Corollary 9.4.3. *Let*

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V),$$

$$i = \textit{quasi-isomorphism}, \qquad \mathrm{Id}_A - ip = d_A h + h d_A,$$

be a homotopy retract (e.g. deformation retract). The homotopy class of the dga algebra (A, d) (considered as an A_∞-algebra) is equal to the homotopy class of the A_∞-algebra (V, d).

Lemma 9.4.4. *Let $\mathbb{K}$ be a field. Under a choice of sections any chain complex admits its homology as a deformation retract.*

Proof. Since we are working over a field, we can choose sections in the chain complex (A, d) so that $A_n \cong B_n \oplus H_n \oplus B_{n-1}$ where H_n is the homology and B_n the space of boundaries in degree n. The boundary map is 0 on $B_n \oplus H_n$ and identifies B_{n-1} with its copy in A_{n-1}. The homotopy h is 0 on $H_n \oplus B_{n-1}$ and identifies B_n with its copy in A_{n+1}. These choices make $(H_\bullet(A), 0)$ a deformation retract of (A, d):

	Id	$-ip$	dh	hd
B_n	Id	0	Id	0
H_n	Id	$-$Id	0	0
B_{n-1}	Id	0	0	Id

□

Corollary 9.4.5 (T. Kadeishvili [Kad80]). *For any dga algebra (A, d), there is an A_∞-algebra structure on $H_\bullet(A, d)$, with 0 differential, such that these two A_∞-algebras are homotopy equivalent.*

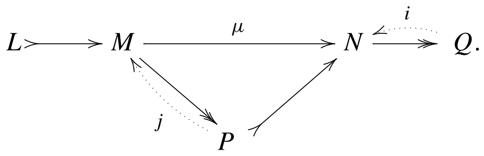
Proof. We apply Theorem 9.4.2 to the deformation retract constructed in Lemma 9.4.4. The homotopy equivalence of these two A_∞-algebras follows from the existence of an ∞-quasi-isomorphism. □

9.4.4 MacLane Invariant of a Crossed Module

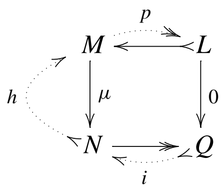
Though it was constructed decades before m_3, the MacLane invariant of a crossed module can be interpreted as a nonlinear variation of m_3. Let us recall the framework. A *crossed module* is a group homomorphism $\mu : M \to N$ together with an action of N on M, denoted ${}^n m$, such that the following relations hold

$$\text{a)} \quad \mu({}^n m) = n\mu(m)n^{-1},$$
$$\text{b)} \quad {}^{\mu(m)}m' = mm'm^{-1}.$$

Let $Q := \operatorname{Coker}\mu$ and $L := \operatorname{Ker}\mu$. From the axioms it is easily seen that L is abelian and equipped with a Q-module structure. In [ML95, Chap. IV, Sect. 8], MacLane constructed an element $\alpha \in H^3(Q, L)$ as follows. Let P be the image of μ. We choose set-theoretic sections i and j such that $i(1) = 1$, $j(1) = 1$:



They permit us to construct set bijections: $N \cong P \times Q$ and $M \cong L \times P$. Hence, viewing the crossed module as a nonabelian chain complex (so μ plays the role of the differential) its homology $L \xrightarrow{0} Q$ can be seen as a (nonlinear) deformation retract:



where p and h are the following composites:

$$p : M \cong L \times P \twoheadrightarrow L, \qquad h : N \cong P \times Q \twoheadrightarrow P \xrightarrow{j} M.$$

For $u, v \in Q$ we define

$$\varphi(u, v) := h\big(i(u)i(v)\big) = j\big(i(u)i(v)i(uv)^{-1}\big) \in M,$$

so that $i(u)i(v) = \mu(\varphi(u, v))i(uv)$. We compute as in the linear case:

$$\begin{aligned}\big(i(u)i(v)\big)i(w) &= \mu\big(\varphi(u, v)\big)\mu\big(\varphi(uv, w)\big)i(uvw),\\ i(u)\big(i(v)i(w)\big) &= \mu\big({}^u\varphi(v, w)\big)\mu\big(\varphi(u, vw)\big)i(uvw).\end{aligned}$$

Comparing these two equalities, it follows that there exists a unique element $m_3(u, v, w) \in L = \operatorname{Ker}\mu$ such that

$${}^u\varphi(v, w)\varphi(u, vw) = m_3(u, v, w)\varphi(u, v)\varphi(uv, w) \in M.$$

MacLane showed that this element is a 3-cocycle and that its cohomology class in $H^3(Q, L)$ does not depend on the choice of the sections i and j. Moreover any morphism of crossed modules inducing an isomorphism on the kernel and the cokernel gives rise to the same invariant.

The topological interpretation is the following, cf. [Lod82]. The crossed module $\mu : M \to N$ defines a simplicial group whose classifying space is a topological space with only π_1, equal to Q, and π_2, equal to L. So its homotopy type is completely determined by the Postnikov invariant which is an element of $H^3(BQ, L) = H^3(Q, L)$. It is precisely the MacLane invariant. So, as in the linear case, the triple (Q, L, m_3) completely determines the homotopy type of the crossed module.

9.4.5 Massey Product

We know that the homology of a dga algebra (A, d) is a graded associative algebra $H_\bullet(A)$. Corollary 9.4.5 tells us that we have more structure: for any $n \geq 3$ there is an n-ary operation m_n which is nontrivial in general. They are called *Massey products* because they generalize the classical Massey products constructed in algebraic topology (cf. [Mas58, Kra66, May69]). Let X be a connected topological space, and let $C^\bullet_{\mathrm{sing}}(X)$ be the singular cochain complex with homology $H^\bullet_{\mathrm{sing}}(X)$. The product structure is given by the cup product of cochains. Then the *triple Massey product* $\langle x, y, z\rangle$ is classically defined for homology classes x, y, z such that $x \cup y = 0 = y \cup z$ as follows. Let us still denote by x, y, z the cycles representing the homology classes. Because of the hypothesis there exist chains a and b such that $(-1)^{|x|} x \cup y = da$, $(-1)^{|y|} y \cup z = db$. Then the chain

$$\langle x, y, z\rangle := (-1)^{|x|} x \cup b + (-1)^{|a|} a \cup z$$

is a cycle. Its homology class is well-defined in the quotient $H/(xH + Hz)$ and called the *triple Massey product* of x, y, z.

The prototypical example of a nonzero triple Massey product is given by the Borromean rings. We consider the complement (in the 3-sphere) of the Borromean rings:

Each "ring" (i.e. solid torus) determines a 1-cocycle: take a loop from the base-point with linking number one with the circle. Since any two of the three circles are disjoint, the cup product of the associated cohomology classes is 0. The nontriviality of the triple Massey product of these three cocycles detects the entanglement of the three circles (cf. [Mas58, Sta97b]).

Lemma 9.4.6. *For any cohomology classes x, y, z such that $x \cup y = 0 = y \cup z$ the triple Massey product is given by the operation m_3:*

$$\langle x, y, z\rangle = -(-1)^{|x|+|y|} m_3(x, y, z).$$

Proof. Let (A, d) be a dga algebra. We denote by $\cup$ its product (formerly denoted by μ) as well as the product induced on homology (formerly denoted by m_2). We use the homotopy equivalence data described in the proof of Lemma 9.4.5. If x, y, z are cycles and a, b chains such that $(-1)^{|x|} x \cup y = d(a)$, $(-1)^{|y|} y \cup z = d(b)$, then we can choose a and b such that $hd(a) = a$, $hd(b) = b$. Therefore one has

$$\begin{aligned} -m_3(x, y, z) &= p\mu\big(i(x), h\mu\big(i(y), i(z)\big)\big) - p\mu\big(h\mu\big(i(x), i(y)\big), i(z)\big) \\ &= x \cup h(y \cup z) - h(x \cup y) \cup z \\ &= (-1)^{|y|} x \cup hd(b) - (-1)^{|x|} hd(a) \cup z \\ &= (-1)^{|y|} x \cup b - (-1)^{|x|} a \cup z \\ &= (-1)^{|x|+|y|} \langle x, y, z\rangle. \end{aligned}$$ □

9.4.6 A Quadruple Massey Product

Let x, y, z, t be cycles in the dga algebra (A, d) such that there exist chains a, b, c satisfying

$$(-1)^{|x|} xy = d(a), \qquad (-1)^{|y|} yz = d(b), \qquad (-1)^{|z|} zt = d(c),$$

and such that there exist chains $\alpha, \beta, \beta', \gamma$ satisfying:

$$\begin{aligned} (-1)^{|a|} az &= d(\alpha), & (-1)^{|x|} xb &= d(\beta), \\ (-1)^{|b|} bt &= d\big(\beta'\big), & (-1)^{|y|} yc &= d(\gamma). \end{aligned}$$

One can check that the element

$$\langle x, y, z, t\rangle := (-1)^{|x|} \alpha t + (-1)^{|x|} \beta t + (-1)^{|x|} ac + (-1)^{|x|} x\beta' + (-1)^{|x|} x\gamma$$

is a cycle and defines a homology class in $H_\bullet(A, d)$. The five elements of this sum correspond to the five 0-cells of the cellular decomposition of the pentagon, cf. Fig. 9.2. In fact one can check that

$$\langle x, y, z, t\rangle = \pm m_4(x, y, z, t).$$

9.4.7 Homotopy Invariance of A_∞-Algebras

The homotopy transfer theorem for dga algebras can be generalized into a homotopy transfer theorem for A_∞-algebras.

Theorem 9.4.7. [Kad82] *Let*

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V),$$

$$i = \textit{quasi-isomorphism}, \qquad \mathrm{Id}_A - ip = d_A h + h d_A,$$

be a homotopy retract. If (A, d_A) *is an* A_∞*-algebra, then* (V, d_V) *inherits a* A_∞*-algebra structure* $\{m_n\}_{n\geq 2}$ *such that the quasi-isomorphism* i *extends to an* ∞*-quasi-isomorphism.*

Proof. In the proof of Lemma 9.4.1 we used the fact that the binary product $\mu = \mu_2$ on A is associative. Suppose now that it is only associative up to homotopy, that is, there exists a ternary operation μ_3 on A such that

$$\partial(\mu_3) = -\mu_2 \circ (\mathrm{id}, \mu_2) + \mu_2 \circ (\mu_2, \mathrm{id}).$$

Then one needs to modify the ternary operation m_3 on V by adding the extra term $p\mu_3 i$:

$$m_3 := -p\mu_2\big(i, h\mu_2(i,i)\big) + p\mu_2\big(h\mu_2(i,i), i\big) + p\mu_3(i,i,i).$$

After this modification we get the formula

$$\partial(m_3) = -m_2 \circ (m_2, \mathrm{id}) + m_2 \circ (\mathrm{id}, m_2)$$

as in Lemma 9.4.1. Observe that the term which has been added corresponds to the corolla of the figure in Sect. 9.4.2.

Similarly the higher order operations m_n are defined by using not only the binary trees, but all the planar trees, with vertices indexed by the operations μ_n given by the A_∞-algebra structure of A. The proof is done by induction on the size of the trees, see for instance [KS00] by Kontsevich and Soibelman or [Mer99] by Merkulov. It uses the fact that the homotopy retract determines a morphism of dg cooperads $\mathrm{B}\,\mathrm{End}_A \to \mathrm{B}\,\mathrm{End}_V$. The A_∞-structure of A is encoded by a morphism of dg cooperads $As^{¡} \to \mathrm{B}\,\mathrm{End}_A$. By composition we get the expected A_∞-structure on V. □

This result is a particular case of a more general statement valid for any Koszul operad. Its complete proof is given in Theorem 10.3.1.

9.4.8 Variations on the Homotopy Transfer Theorem

There are various proofs and several generalizations of the Homotopy Transfer Theorem. The proof given here follows the method of Kontsevich and Soibelman [KS00], see also [Mer99] by Sergei Merkulov. Another method, closer to the original proof of Kadeishvili, consists in applying the Perturbation Lemma, see [HK91].

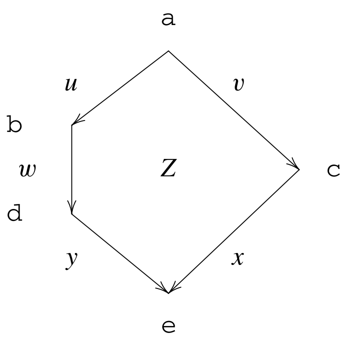
In [Mar06], Martin Markl showed that p can also be extended to an ∞-quasi-isomorphism, and h to an ∞-homotopy.

Table 9.1 The binary operation m_2 on (α, β)

$\alpha\backslash\beta$	a*	b*	c*	d*	e*	u^*	v^*	w^*	x^*	y^*	Z^*
a*	a*					u^*	v^*				Z^*
b*		b*						w^*			
c*			c*						x^*		
d*				d*						y^*	
e*					e*						
u^*		u^*						Z^*		Z^*	
v^*			v^*						$-Z^*$		
w^*				w^*						Z^*	
x^*					x^*						
y^*					y^*						
Z^*					Z^*						

9.5 An Example of an A_∞-Algebra with Nonvanishing m_3

Let us consider the cochain complex of the Stasheff polytope $\mathscr{K}^2$ (pentagon). We denote by $\mathtt{a}^*, \ldots, u^*, \ldots, Z^*$ the cochains which are linear dual of the cells $\mathtt{a}, \ldots, u, \ldots, Z$ of $\mathscr{K}^2$:



We make it into an A_∞-algebra as follows. First, we put $m_n = 0$ for any $n \geq 4$. Second, m_3 is zero except on the triple of 1-cochains (u^*, w^*, y^*) where

$$m_3\bigl(u^*, w^*, y^*\bigr) = Z^*.$$

Third, the binary operation m_2 on (α, β) is given by Table 9.1.

In Table 9.1, empty space means 0. This defines a cohomologically graded A_∞-algebra: $|\partial| = +1$ and $|m_3| = +1$. We leave it to the reader to verify the relations, that is

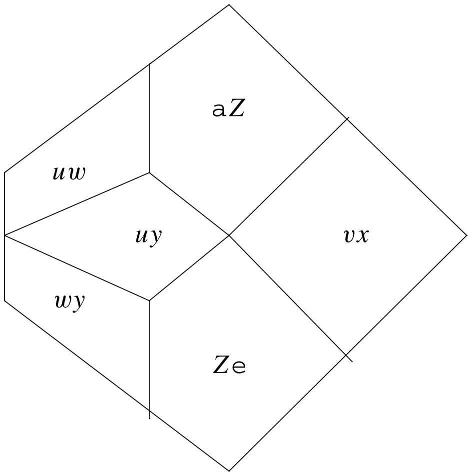
$$\partial(m_3) = m_2 \circ (m_2, \mathrm{id}) - m_2 \circ (\mathrm{id}, m_2)$$

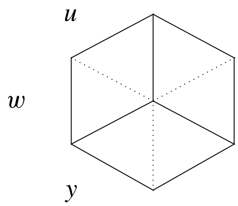
and, since $\partial(m_4) = 0$,

$$m_2(m_3, \mathrm{id}) - m_3(m_2, \mathrm{id}, \mathrm{id}) + m_3(\mathrm{id}, m_2, \mathrm{id}) - m_3(\mathrm{id}, \mathrm{id}, m_2) + m_2(\mathrm{id}, m_3) = 0.$$

For the first relation, it suffices to evaluate the two sides on (u^*, w^*, y^*). For the second relation there are six cases to consider (one per face of the cube).

The topological interpretation of these formulas is better seen on the dual statement, that is $C_\bullet(\mathscr{K}^2)$ is an A_∞-coalgebra. The coproducts Δ^2 and Δ^3 applied to the 2-cell Z are shown in the following pictures:





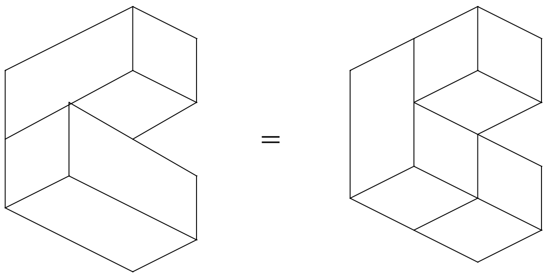
The identity

$$\big(\Delta^3, \mathrm{id}\big)\Delta^2 - \big(\Delta^2, \mathrm{id}, \mathrm{id}\big)\Delta^3 + \big(\mathrm{id}, \Delta^2, \mathrm{id}\big)\Delta^3 - \big(\mathrm{id}, \mathrm{id}, \Delta^2\big)\Delta^3 + \big(\mathrm{id}, \Delta^3\big)\Delta^2 = 0$$

which is equivalent to

$$\big(\Delta^2, \mathrm{id}, \mathrm{id}\big)\Delta^3 + \big(\mathrm{id}, \mathrm{id}, \Delta^2\big)\Delta^3 = \big(\Delta^3, \mathrm{id}\big)\Delta^2 + \big(\mathrm{id}, \Delta^2, \mathrm{id}\big)\Delta^3 + \big(\mathrm{id}, \Delta^3\big)\Delta^2$$

amounts to the identification of the following unions of cells:



More generally, the cochain complex $C^\bullet(\mathscr{K}^n)$ can be shown to be an A_∞-algebra for any n, cf. [Lod12].

9.6 Résumé

9.6.1 The Operad *Ass*

Symmetric operad Ass encoding associative algebras: $Ass(n) = \mathbb{K}[\mathbb{S}_n]$.
Nonsymmetric operad As encoding associative algebras: $As_n = \mathbb{K}\mu_n$.
Composition rule: $\mu_m \circ_i \mu_n = \mu_{m-1+n}$.

9.6.2 Homotopy Associative Algebras

A an A_∞-algebra: $(T^c(sA), m^A) =$ dg coalgebra, equivalently, the chain complex (A, d) is equipped with an n-ary operation $m_n : A^{\otimes n} \to A$, $n \geq 2$, satisfying the relations:

$$\partial(m_n) = \sum (-1)^{p+qr} m_k \circ \left(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}\right).$$

Dgns operad A_∞: $A_\infty = \Omega As^{¡}$, $(A_\infty)_n = C_\bullet(\mathscr{K}^{n-2})$.

9.6.3 Cobar–Bar Construction

$$\left(\Omega \mathrm{B}\, As^{¡}\right)_n = C_\bullet\left(\mathscr{K}_{cub}^{n-2}\right).$$

9.6.4 Homotopy Transfer Theorem

If (V, d) is a homotopy retract (e.g. deformation retract) of (A, d), then any A_∞-algebra structure on (A, d) can be transferred through explicit formulas to an A_∞-algebra structure on (V, d), so that they represent the same homotopy class. Whence the slogan

"A_∞-algebras are stable under homotopy equivalence".

EXAMPLE. If (A, d) is a dga algebra, then $H_\bullet(A)$ is an A_∞-algebra with trivial differential. This structure induces the Massey products.

9.7 Exercises

Exercise 9.7.1 (Explicit homotopy). Construct the homotopy from Id to 0 for the Koszul complex $(As^{¡} \circ As, d_\kappa)$.

Exercise 9.7.2 (Acyclicity again). Consider the small chain complex of length one

$$C_\bullet : \qquad \cdots \to 0 \to C_1 = \mathbb{K} \xrightarrow{\text{id}} C_0 = \mathbb{K}.$$

Show that the arity n part of the Koszul complex of As is isomorphic to $(C_\bullet)^{\otimes n}$. Deduce another proof of the acyclicity of the Koszul complex of As. Compare with Exercise 3.8.7.

Exercise 9.7.3 (An explicit computation). Let As^* be the linear dual cooperad of As. Compute explicitly $\Delta(\mu_4^c)$, $\bar{\Delta}(\mu_4^c)$ and $\Delta_{(1)}(\mu_4^c)$.

Exercise 9.7.4 (Totally diassociative algebras [Zha12]). A *totally diassociative algebra* is defined by two binary operations $x * y$ and $x \cdot y$ such that any composition is associative:

$$(x * y) * z = x * (y * z),$$
$$(x \cdot y) * z = x \cdot (y * z),$$
$$(x * y) \cdot z = x * (y \cdot z),$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

We denote by $As^{(2)}$ the ns operad encoding these algebras. Show that $As_n^{(2)}$ is of dimension 2^{n-1}. Show that $As^{(2)}$ is self-dual for Koszul duality and is Koszul (apply the rewriting method). Describe explicitly the chain complex $(As_\infty^{(2)})_n$ in terms of the associahedron. What is an $As_\infty^{(2)}$-algebra?

Exercise 9.7.5 (Higher Massey products ★). In [Kra66], David Kraines defines higher Massey products for families of cochains $\underline{a} = \{a(i,j)\}_{1 \le i \le j \le k}$, in a cochain complex, satisfying

$$d\big(a(i,j)\big) = \sum_{r=i}^{j-1} (-1)^{|a(i,r)|} a(i,r)a(r+1,j)$$

as

$$c(\underline{a}) := \sum_{r=1}^{k-1} (-1)^{|a(1,r)|} a(i,r)a(r+1,k).$$

Interpret this construction and its properties in terms of A_∞-algebras. Compare with [May69, BM03b].

Exercise 9.7.6 (Classifying space of a crossed module ★). Let $G.$ be a simplicial group whose homotopy groups are trivial except $\pi_0 G = Q$ and $\pi_n G = L$ for some fixed n. Show that one can construct an analog $m_n : G^n \to L$ of MacLane invariant, which is a cocycle and gives a well-defined element in $H^n(Q, L)$ (cohomology of the discrete group Q with values in the Q-module L). Show that it is the Postnikov invariant of the classifying space $B|G.|$.