Chapter 8
Methods to Prove Koszulity of an Operad

Nous voulons, tant ce feu nous brûle le cerveau, Plonger au fond du gouffre, Enfer ou Ciel, qu'importe? Au fond de l'Inconnu pour trouver du nouveau!

Charles Baudelaire

This chapter extends to the operadic level the various methods, obtained in Chap. 4, to prove that algebras are Koszul. In Chap. 7, we have already given one method, based on the vanishing of the homology of the free $\mathscr{P}$-algebra.

We begin by generalizing the rewriting method for associative algebras given in Sect. 4.1 to nonsymmetric operads. To extend it much further to symmetric operads, we need to introduce the notion of shuffle operad, which sits in between the notion of operad and nonsymmetric operad. It consists of the same kind of compositions as in an operad but without the symmetric groups action. For instance, the free symmetric operad is isomorphic to the free shuffle operad as $\mathbb{K}$-modules, thereby providing a $\mathbb{K}$-linear basis in terms of shuffle trees for the first one.

With this notion of shuffle operad at hand, we adapt the rewriting method, the reduction by filtration method, the Diamond Lemma, the PBW bases and the Gröbner bases of Chap. 4 from associative algebras to operads.

We give then yet another method. Starting from two operads $\mathscr{A}$ and $\mathscr{B}$, one can sometimes cook up a third one on the underlying $\mathbb{S}$-module $\mathscr{A} \circ \mathscr{B}$, by means of a distributive law $\mathscr{B} \circ \mathscr{A} \to \mathscr{A} \circ \mathscr{B}$. One can interpret this data as a rewriting rule, which pulls the elements of $\mathscr{B}$ above those of $\mathscr{A}$. This interpretation allows us to show the same kind of results as the ones obtained by the aforementioned methods. For instance, we give a Diamond Lemma for distributive laws which proves that an operad obtained from two Koszul operads by means of a distributive law is again Koszul. Notice that, retrospectively, such a method applies to associative algebras as well.

Instead of Backelin's lattice criterion used in the algebra case, we introduce a partition poset method in the operad case. The idea is to associate a family of partition type posets to a set operad. The main theorem asserts that the linear operad

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generated by the set operad is Koszul if and only if the homology groups of the operadic partition posets are concentrated in top dimension (Cohen–Macaulay posets). On the one hand, the many combinatorial criteria to show that a poset is Cohen–Macaulay provide ways to prove that an operad is Koszul. On the other hand, it gives a method to compute explicitly the homology groups of some partition type posets as $\mathbb{S}$-modules since the top homology groups are isomorphic to the Koszul dual cooperad.

In a last section, we extend the definition and the properties of Manin products to operads.

The material of this chapter mainly comes from Hoffbeck [Hof10c], Dotsenko and Khoroshkin [DK10], Markl [Mar96a], [Val07b], Ginzburg and Kapranov [GK94, GK95b], and [Val08].

In the first five sections of this chapter, we work with *reduced* $\mathbb{S}$-modules M: $M(0)=0$, respectively $M(0)=\emptyset$ in the set theoretic case.

8.1 Rewriting Method for Binary Quadratic NS Operads

In this section, we explain how the rewriting method for associative algebras of Sect. 4.1 extends to binary quadratic ns operads. It provides a short algorithmic method, based on the rewriting rules given by the relations, to prove that an operad is Koszul. The general theory for operads (not necessarily binary) requires new definitions that will be given in the following sections.

Let $\mathscr{P}(E, R)$ be a binary quadratic ns operad.

Step 1. We consider an ordered basis $\{\mu_1, \mu_2, \dots, \mu_k\}$ for the generating space E of binary operations. The ordering $\mu_1 < \mu_2 < \cdots < \mu_k$ will play a key role in the sequel.

Step 2. The ternary operations, which span the weight 2 part of the free ns operad, are of the form $\mu_i \circ_a \mu_j$, where $a = 1, 2$. We put a total order on this set as follows:

$$\begin{cases} \mu_i \circ_2 \mu_j < \mu_i \circ_1 \mu_j, & \text{for any } i, j, \\ \mu_i \circ_a \mu_j < \mu_k \circ_a \mu_l, & \text{whenever } i<k, \quad a=1 \text{ or } 2, \text{ and for any } j,l, \\ \mu_i \circ_a \mu_j < \mu_i \circ_a \mu_l, & \text{whenever } j<l, \quad a=1 \text{ or } 2. \end{cases}$$

The operad $\mathscr{P}$ is determined by the space of relations R, which is spanned by a set of relators written in this basis as

$$r = \lambda \mu_i \circ_a \mu_j - \sum \lambda_{k,b,l}^{i,a,j} \mu_k \circ_b \mu_l, \quad \lambda, \lambda_{k,b,l}^{i,a,j} \in \mathbb{K} \quad \text{and} \quad \lambda \neq 0,$$

so that the sum runs over the indices satisfying $\mu_i \circ_a \mu_j > \mu_k \circ_b \mu_l$. The operation $\mu_i \circ_a \mu_j$ is called the *leading term* of the relator (r). One can always suppose that λ is equal to 1, that the leading terms of the set of relators are all distinct and that there is no leading term of any other relator in the sum in the right-hand side. This is called a *normalized form* of the presentation.

Step 3. Observe that such a relator gives rise to a *rewriting rule* in the operad $\mathscr{P}$:

$$\mu_i \circ_a \mu_j \mapsto \sum \lambda_{k,b,l}^{i,a,j} \mu_k \circ_b \mu_l.$$

Given three generating binary operations μ_i, μ_j, μ_k, one can compose them in 5 different ways: they correspond to the 5 planar binary trees with 3 vertices. Such a monomial, i.e. decorated planar tree, is called *critical* if the two sub-trees with 2 vertices are leading terms.

Step 4. There are at least two ways of rewriting a critical monomial ad libitum, that is, until no rewriting rule is applicable any more. If all these ways lead to the same element, then the critical monomial is said to be *confluent*.

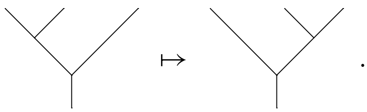
Conclusion. If each critical monomial is confluent, then the ns operad $\mathscr{P}$ is Koszul.

This assertion is a consequence of the following result.

Theorem 8.1.1 (Rewriting method for ns operads). *Let $\mathscr{P}(E, R)$ be a reduced quadratic ns operad. If its generating space E admits an ordered basis for which there exists a suitable order on planar trees such that every critical monomial is confluent, then the ns operad $\mathscr{P}$ is Koszul.*

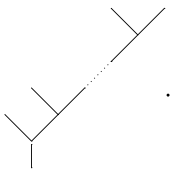
In this case, the ns operad $\mathscr{P}$ admits a $\mathbb{K}$-linear basis made up of some planar trees called a PBW-basis, see Sect. 8.5. The proof is analogous to the proof of Theorem 4.1.1; it follows from Sect. 8.5.4.

EXAMPLE. Consider the ns operad *As* encoding associative algebras, see Chap. 9. It is generated by one operation of arity 2:  . The rewriting rule expressing associativity reads:



There is only one critical monomial (the left comb), which gives the following confluent graph (Fig. 8.1).

Therefore, the nonsymmetric operad *As* is Koszul. It admits a PBW basis made up of the right combs:



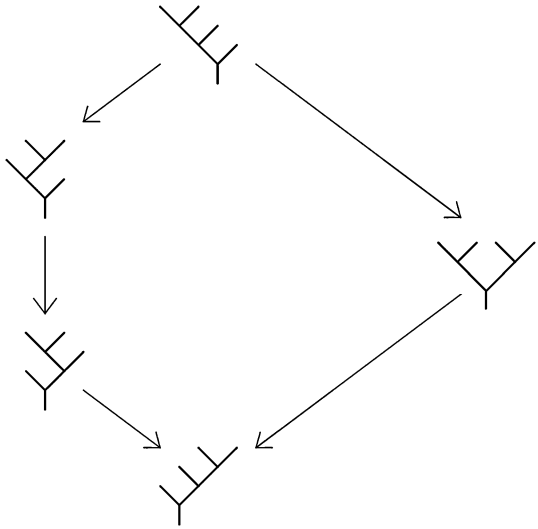
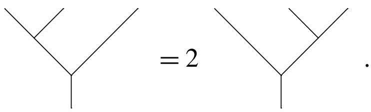


Fig. 8.1 The diamond for the nonsymmetric operad *As*

REMARK. Notice first that one recovers the associativity pentagon of monoidal categories, see Appendix B.3. Moreover, the Diamond Lemma applied to the ns operad *As* is exactly Mac Lane's coherence theorem for (nonunital) monoidal categories. The first one states that any graph built out of the left combs, under the associativity relation, is confluent. The second one states that any graph built with the associativity relation is commutative. With this remark in mind, the reading of [ML95, Sect. VII-2] enjoys another savor.

COUNTER-EXAMPLE. We consider the same example but with the modified associativity relation



In this case, the above graph is not confluent because $2^3 \neq 2^2$. Whatever the suitable order is, the graph will never be confluent since it can be proved that this ns operad is not Koszul, see Exercise 8.10.9.

8.2 Shuffle Operad

The forgetful functor $\mathsf{Op} \to \mathsf{ns\,Op}$, from symmetric operads to nonsymmetric operads, forgets the action of the symmetric groups, see Sect. 5.9.11. It factors through the category of shuffle operads $\mathsf{Op}_{\sqcup\sqcup}$. Shuffle operads have the advantage of being based on arity-graded vector spaces like nonsymmetric operads, while retaining the whole structure of a symmetric operad.

The notion of shuffle operad is due to Dotsenko–Khoroshkin [DK10] after ideas of Hoffbeck [Hof10c].

8.2.1 Shuffle Composite Product

For any subset X of $\underline{n} := \{1, \dots, n\}$ we denote by $\min(X)$ the smallest element of X. Any partition P of $\underline{n}$ into k subsets can be written uniquely

$$\mathrm{P} = (\mathrm{P}_1, \dots, \mathrm{P}_k)$$

under the requirement

$$\min(\mathrm{P}_1) < \min(\mathrm{P}_2) < \cdots < \min(\mathrm{P}_k).$$

Writing the elements of P_i in order and concatenating them for all i, it defines the preimages of $\{1, \dots, n\}$ under a permutation σ_P of $\mathbb{S}_n$.

$$\{1, 3, 4\} < \{2, 7\} < \{5, 6, 8\} \quad \mapsto \quad [1\ 4\ 2\ 3\ 6\ 7\ 5\ 8].$$

The associated permutation σ_P is a $(i_1, \dots, i_k)$-unshuffle, cf. Sect. 1.3.2, where $i_j := |\mathrm{P}_j|$,

$$\begin{aligned}
&\sigma^{-1}(1) < \cdots < \sigma^{-1}(i_1),\\
&\sigma^{-1}(i_1 + 1) < \cdots < \sigma^{-1}(i_1 + i_2),\\
&\vdots\\
&\sigma^{-1}(i_1 + \cdots + i_{k-1} + 1) < \cdots < \sigma^{-1}(n),
\end{aligned}$$

satisfying the extra property

$$\sigma^{-1}(1) < \sigma^{-1}(i_1 + 1) < \cdots < \sigma^{-1}(i_1 + \cdots + i_{k-1} + 1).$$

Such unshuffles are called *pointed* $(i_1, \dots, i_k)$*-unshuffles*, or simply *pointed unshuffles* when the underlying type is understood. We denote the associated set by $Ш(i_1, \dots, i_k)$. For instance, the set $Ш(2, 1)$ has two elements, namely [1 2 3] and [1 3 2]; the set $Ш(1, 2)$ has only one element, namely [1 2 3].

For any arity-graded spaces M and N, we define the *shuffle composite product* $M \circ_Ш N$ as follows:

$$(M \circ_Ш N)_n := \bigoplus_{\substack{k \geq 1, (i_1, \dots, i_k) \\ i_1 + \cdots + i_k = n}} M_k \otimes N_{i_1} \otimes \cdots \otimes N_{i_k} \otimes \mathbb{K}\big[Ш(i_1, \dots, i_k)\big].$$

Proposition 8.2.1. *The shuffle composition product makes the category of arity-graded spaces* $(\mathbb{N}\text{-Mod}, \circ_Ш, \mathrm{I})$ *into a monoidal category.*

Proof. The associativity of $\circ_{\sqcup}$ is proved by direct inspection, see [DK10, Sect. 2] for more details. □

8.2.2 Shuffle Trees

A *shuffle tree* is a reduced planar rooted tree equipped with a labeling of the leaves by integers $\{1, 2, \ldots, n\}$ satisfying some condition stated below. First, we label each edge of the tree as follows. The leaves are already labeled. Any other edge is the output of some vertex v. We label this edge by $\min(v)$ which is the minimum of the labels of the inputs of v. Second, the condition for a labeled tree to be called a shuffle tree is that, for each vertex, the labels of the inputs, read from left to right, are increasing.

EXAMPLE. See Fig. 8.2.

The relationship between shuffle trees and pointed unshuffles is the following: Any shuffle tree with two levels corresponds to a partition written in order and vice versa.

EXAMPLE. See Fig. 8.3.

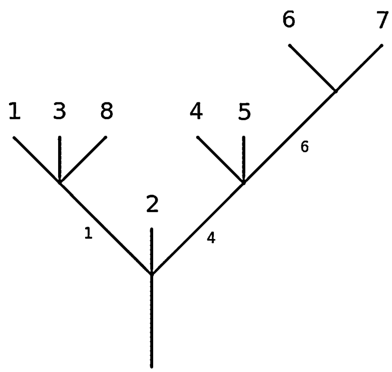


Fig. 8.2 Example of a shuffle tree

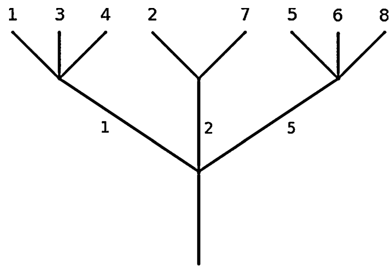


Fig. 8.3 Example of a 2-leveled shuffle tree

8.2.3 Monoidal Definition of Shuffle Operad

By definition a *shuffle operad* is a monoid $(\mathscr{P}, \gamma_{\sqcup\sqcup}, \eta)$ in the monoidal category $(\mathbb{N}\text{-Mod}, \circ_{\sqcup\sqcup}, \mathrm{I})$. Explicitly, it is an arity-graded vector space $\mathscr{P}$ equipped with an associative composition map $\gamma_{\sqcup\sqcup} : \mathscr{P} \circ_{\sqcup\sqcup} \mathscr{P} \to \mathscr{P}$ and a unit map $\eta : \mathrm{I} \to \mathscr{P}$.

Equivalently a shuffle operad can be defined by maps

$$\gamma_\sigma : \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \to \mathscr{P}_n$$

for any pointed unshuffle $\sigma \in \sqcup\sqcup(i_1, \ldots, i_k)$. Assembling these maps, we get $\gamma : \mathscr{P} \circ_{\sqcup\sqcup} \mathscr{P} \to \mathscr{P}$. Associativity of γ can be written explicitly in terms of the individual maps γ_σ as in Proposition 5.3.1.

8.2.4 Partial Definition of Shuffle Operad

Let m and i, $1 \le i \le m$, be positive integers. Any monotonic injection

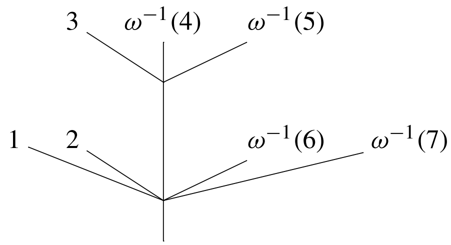
$$\{i+1, i+2, \ldots, i+n-1\} \to \{i+1, i+2, \ldots, m+n-1\}$$

is completely determined by a $(n-1, m-i)$-unshuffle ω that we let act on the set $\{i+1, \ldots, m+n-1\}$. This data is equivalent to a partition of type

$$\begin{aligned} P = \big(&\{1\}, \ldots, \{i-1\}, \big\{i, \omega^{-1}(i+1), \ldots, \omega^{-1}(i+n-1)\big\}, \\ &\big\{\omega^{-1}(i+n)\big\}, \ldots, \big\{\omega^{-1}(m+n-1)\big\}\big), \end{aligned} \tag{8.1}$$

where all the subsets but one are made up of one element.

Such a partition is equivalent to a shuffle tree with two vertices:



The associated pointed unshuffle σ_{P} is of type $(1, \ldots, 1, n, 1, \ldots, 1)$ and it determines a map

$$\gamma_{\sigma_{\mathrm{P}}} : \mathscr{P}(m) \otimes \mathscr{P}(1) \otimes \cdots \otimes \mathscr{P}(1) \otimes \mathscr{P}(n) \otimes \mathscr{P}(1) \otimes \cdots \otimes \mathscr{P}(1) \to \mathscr{P}(m+n-1).$$

By evaluating $\gamma_{\sigma_{\mathrm{P}}}$ on the elements $\mathrm{id} \in \mathscr{P}(1)$ we get a map:

$$\circ_{i,\omega} : \mathscr{P}(m) \otimes \mathscr{P}(n) \to \mathscr{P}(m+n-1).$$

They are called the *partial shuffle products* in the shuffle operad framework.

The partial operations $\circ_{i,\omega}$ generate, under composition, all the shuffle compositions. They satisfy some relations. For instance if the permutation shuffles

$$\sigma \in Sh(m,n), \quad \lambda \in Sh(m+n,r), \quad \delta \in Sh(n,r), \quad \gamma \in Sh(m,n+r)$$

satisfy the relation

$$(\sigma \times 1_r)\lambda = (1_m \times \delta)\gamma \quad \text{in } \mathbb{S}_{m+n+r},$$

then the partial operations satisfy the relation

$$(x \circ_{1,\sigma} y) \circ_{1,\lambda} z = x \circ_{1,\gamma} (y \circ_{1,\delta} z)$$

for any $x \in \mathscr{P}_{1+m}$, $y \in \mathscr{P}_{1+n}$, $z \in \mathscr{P}_{1+r}$.

We leave it to the reader to find the complete set of relations, which presents a shuffle operad out of the partial shuffle products.

8.2.5 Combinatorial Definition of Shuffle Operad

The *combinatorial definition of a shuffle operad* is the same as the combinatorial definition of a ns operad, cf. Sect. 5.9.5, except that we have to replace the planar rooted trees by the shuffle trees. The only subtle point is the substitution of shuffle trees, which is obtained as follows.

Let t be a shuffle tree and v be a vertex of t whose inputs are labeled by $(i_1, \ldots, i_k)$. So we have $i_1 < i_2 < \cdots < i_k$. Let s be a shuffle tree with k leaves and let $(j_1, \ldots, j_k)$ be the sequence of labels of the leaves. So $(j_1, \ldots, j_k)$ is a permutation of $\underline{k}$. Then, the substitution of s at v gives a new planar rooted tree, cf. Sect. 5.9.5, whose labeling is obtained as follows: each label j_l is changed into i_{j_l} for $l = 1, \ldots, k$ and the other labels are unchanged.

EXAMPLE. See Fig. 8.4.

Proposition 8.2.2. *An algebra over the monad of shuffle trees is a shuffle operad.*

Proof. The proof is left to the reader as a good exercise. □

As a result we get a description of the free shuffle operad over a reduced arity graded module M as follows. The underlying reduced arity-graded module $\mathscr{T}_{\sqcup}M$ is spanned by the shuffle trees with vertices indexed by elements of M, respecting the number of inputs. Its shuffle composition $\gamma_{\sqcup}$ is defined by the grafting of shuffle trees, as the example of Fig. 8.5 shows.

Theorem 8.2.3. *The shuffle operad $(\mathscr{T}_{\sqcup}M, \gamma_{\sqcup})$ is free over M among the shuffle operads.*

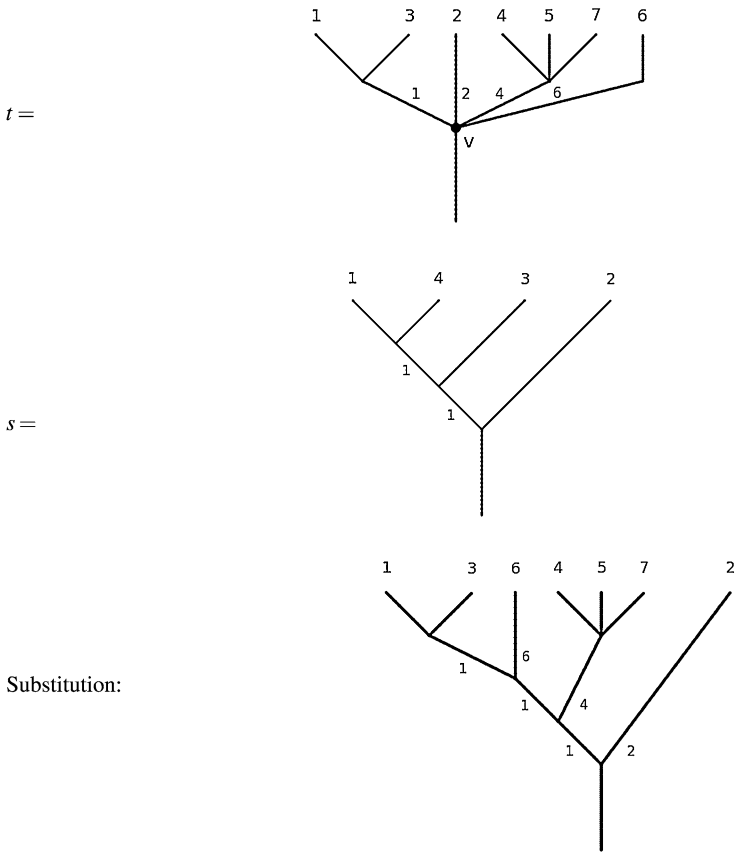


Fig. 8.4 Example of substitution of shuffle trees

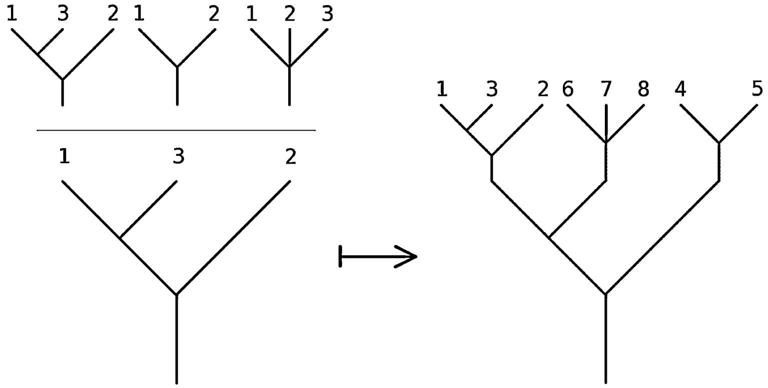


Fig. 8.5 Example of shuffle composition in the free shuffle operad

Proof. The shuffle tree space is equal to the same colimit as in Sect. 5.5.1 or equivalently as in Sect. 5.5.5 but applied to the shuffle composite product $\circ_{\sqcup\sqcup}$ instead of the composite product $\circ$. Thus, the proof of the present case follows from the same arguments. □

8.2.6 Group and Pre-Lie Algebra Associated to a Shuffle Operad

In Sect. 5.4.2, resp. Sect. 5.4.3, we associate to any symmetric operad a group, resp. a pre-Lie algebra. These constructions consist in summing over operations which are determined by two-level shuffle trees. Hence they make sense for shuffle operads as well.

8.2.7 From Symmetric Operads to Shuffles Operads

For any $\mathbb{S}$-module $\mathscr{P} = \{\mathscr{P}(n)\}_{n\geq 1}$, we denote by $\mathscr{P}^f$ the underlying arity-graded module:

$$\left(\mathscr{P}^f\right)_n := \mathscr{P}(n).$$

This is the forgetful functor from $\mathbb{S}$-Mod to $\mathbb{N}$-Mod. The crucial property of the shuffle composite product states that for any $\mathbb{S}$-modules $\mathscr{P}$ and $\mathscr{Q}$ there is an isomorphism

$$(\mathscr{P} \circ \mathscr{Q})^f \cong \mathscr{P}^f \circ_{\sqcup\sqcup} \mathscr{Q}^f.$$

Recall that the set of unshuffles $Sh^{-1}_{i_1,\ldots,i_k}$ provides representatives for the quotient of $\mathbb{S}_n$ under the left action by $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}$. The set of pointed unshuffles $\sqcup\sqcup(i_1, \ldots, i_k)$ provides representatives for the quotient of $Sh^{-1}_{i_1,\ldots,i_k}$ by the action of $\mathbb{S}_k$.

This equality enables us to define the composite

$$\gamma_{\sqcup\sqcup} : \mathscr{P}^f \circ_{\sqcup\sqcup} \mathscr{P}^f \cong (\mathscr{P} \circ \mathscr{P})^f \xrightarrow{\gamma} \mathscr{P}^f,$$

when $\mathscr{P} = (\mathscr{P}, \gamma, \eta)$ is an operad. Then, it is straightforward to see that $(\mathscr{P}^f, \gamma_{\sqcup\sqcup}, \eta)$ is a shuffle operad.

Proposition 8.2.4. *The forgetful functor*

$$(\mathbb{S}\text{-Mod}, \circ) \longrightarrow (\mathbb{N}\text{-Mod}, \circ_{\sqcup\sqcup})$$

is a strong monoidal functor, see Appendix B.3.3. *Therefore it induces the following functor*

$$\mathsf{Op} \longrightarrow \mathsf{Op}_{\sqcup\sqcup}, \qquad (\mathscr{P}, \gamma, \eta) \mapsto \left(\mathscr{P}^f, \gamma_{\sqcup\sqcup}, \eta\right).$$

Proof. It is straightforward to check the axioms of strong monoidal functors with the previous discussion. □

8.2.8 From Shuffle Operads to NS Operads

For any partition $n = i_1 + \cdots + i_k$, the identity permutation is an element of $\sqcup\!\sqcup(i_1, \ldots, i_k)$. Hence, for any arity graded modules M and N we have $M \circ N \subset M \circ_{\sqcup\!\sqcup} N$ and

$$\bigoplus M_k \otimes N_{i_1} \otimes \cdots \otimes N_{i_k} \subset \bigoplus M_k \otimes N_{i_1} \otimes \cdots \otimes N_{i_k} \otimes \mathbb{K}\big[\sqcup\!\sqcup(i_1, \ldots, i_k)\big].$$

Proposition 8.2.5. *The functor*

$$(\mathbb{N}\text{-}\mathsf{Mod}, \circ_{\sqcup\!\sqcup}) \longrightarrow (\mathbb{N}\text{-}\mathsf{Mod}, \circ)$$

is a monoidal functor, see Appendix B.3.3. *Therefore it induces the following functor*

$$\mathsf{Op}_{\sqcup\!\sqcup} \longrightarrow \mathsf{nsOp}.$$

Proof. It is a straightforward to check the axioms of monoidal functors with the previous discussion. □

Finally, we have two functors

$$\mathsf{Op} \longrightarrow \mathsf{Op}_{\sqcup\!\sqcup} \longrightarrow \mathsf{nsOp},$$

whose composite is the forgetful functor $\mathscr{P} \to \tilde{\mathscr{P}}$ mentioned in Sect. 5.9.11.

8.2.9 From Shuffle Operads to Permutads

Some families of shuffle trees are closed under substitution. The first example, mentioned above, is the set of planar trees which is in bijection with the shuffle trees with numbering $\{1, 2, 3, \ldots\}$. The associated monad defines the nonsymmetric operads.

Another example of a family of shuffle trees closed under substitution is made up of the "shuffle left combs", that is shuffle trees whose underlying planar tree is a left comb, see Fig. 8.6.

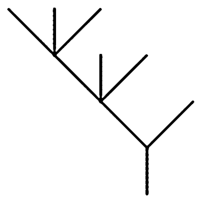


Fig. 8.6 Example of a left comb

An algebra over this monad is a permutad, see Sect. 13.14.7. It can be shown to coincide with the notion of shuffle algebra introduced by M. Ronco in [Ron11].

8.2.10 Koszul Duality Theory for Shuffle Operads

In the preceding chapters, we have developed the Koszul duality theory for symmetric and nonsymmetric operads following a certain pattern (twisting morphisms, bar and cobar constructions, twisting composite products, Koszul morphisms, comparison lemma). The same scheme applies to shuffle operads mutatis mutandis. In this way, one can develop the Koszul duality theory for shuffle operads as well. We leave the details to the reader as a very good exercise.

8.3 Rewriting Method for Operads

With the help of shuffle trees, we settle the rewriting method of Sects. 4.1 and 8.1 for (symmetric) operads. It works in the same way as in the algebra case except that one has to use a suitable order on shuffle trees in this case.

Let $\mathscr{P}(E, R)$ be a quadratic operad (not necessarily binary), for instance, the operad *Lie* encoding Lie algebras

$$Lie = \mathscr{P}\big(\mathbb{K}_{\text{sgn}}c, c \circ (c \otimes \text{id}) + c \circ (c \otimes \text{id})^{(123)} + c \circ (c \otimes \text{id})^{(321)}\big),$$

where $\mathbb{K}_{\text{sgn}}$ stands for the signature representation of $\mathbb{S}_2$.

Step 1. We choose an ordered $\mathbb{K}$-linear basis $\{e_i\}_{i=1,\dots,m}$ for the $\mathbb{S}$-module of generators E.

Step 2. We consider, for instance, the induced path-lexicographic ordered basis on the shuffle trees $\mathscr{T}_{\sqcup}^{(2)}$ with 2 vertices, see Fig. 8.7. (For the complete definition of the path-lexicographic order, we refer the reader to Sect. 8.4.) Notice that one can use any other suitable order on shuffle trees, see loc. cit.

We consider the induced $\mathbb{K}$-linear basis of the $\mathbb{S}$-module R. Any element is of the form

$$r = \lambda t(e_i, e_j) - \sum_{t'(k,l) < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} t'(e_k, e_l); \quad \lambda \neq 0,$$

where the notation $t(i, j)$ (resp. $t(e_i, e_j)$) represents a shuffle tree with 2 vertices labeled by i and j (resp. by e_i and e_j). The monomial tree $t(e_i, e_j)$ is called the *leading term* of r. As usual, we can always change this basis for one with *normalized form*.

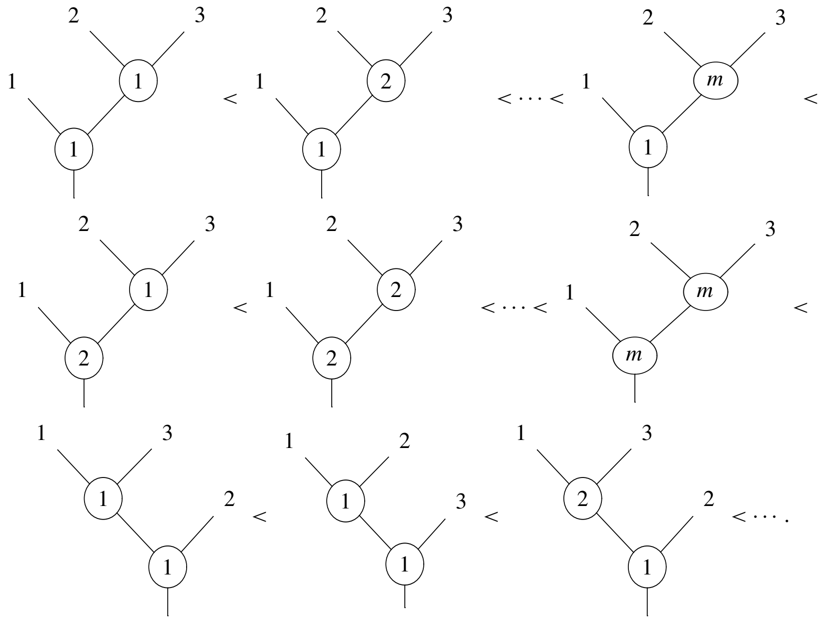
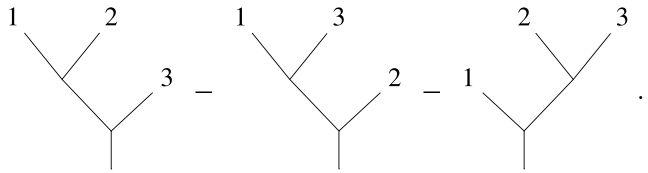


Fig. 8.7 Path-lexicographic order on 2-vertices binary shuffle trees

In the example of the operad *Lie*, the space of relations R admits the following normalized basis:



Since there is only one label $e_1 = c$ for the vertices, we do not mention it here.

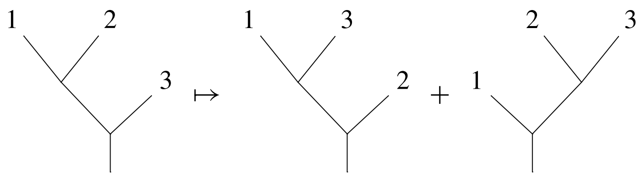
Step 3. These choices provide rewriting rules of the form

$$t(e_i, e_j) \mapsto \sum_{t'(k,l) < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} \, t'(e_k, e_l),$$

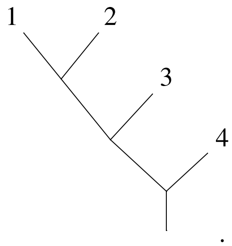
$$\text{leading term} \mapsto \text{sum of lower and non-leading terms},$$

for any relator r in the normalized basis of R. A tree monomial $t(e_i, e_j, e_k)$ with 3 vertices is called *critical* if its two shuffle subtrees with 2 vertices are leading terms.

In the case of the operad *Lie*, there is one rewriting rule



and only one critical tree monomial



Step 4. Any critical tree monomial gives rise to a graph made up of the successive applications of the rewriting rules aforementioned. Any critical monomial is called *confluent* if the associated graph has only one terminal vertex (confluent graph).

In the guiding example of the operad *Lie*, the graph associated to the only critical monomial is confluent, see Fig. 8.8.

REMARK. Notice that the graph given here is a compact version of the full rewriting graph of the operad *Lie*: when it is possible to apply the rewriting rule to two different trees of a sum, we have only drawn one arrow, applying the two rewriting rules at once. There is yet another way to draw this rewriting diagram, which gives the Zamolodchikov tetrahedron equation. It leads to the categorical notion of *Lie 2-algebra*, see Baez–Crans [BC04, Sect. 4].

Conclusion. If each critical monomial is confluent, then the operad $\mathscr{P}$ is Koszul.

It is a consequence of the following result.

Theorem 8.3.1 (Rewriting method for operads). *Let $\mathscr{P}(E, R)$ be a quadratic operad. If its generating space E admits a $\mathbb{K}$-linear ordered basis, for which there exists a suitable order on shuffle trees, such that every critical monomial is confluent, then the operad $\mathscr{P}$ is Koszul.*

In this case, the operad $\mathscr{P}$ admits an induced shuffle tree basis sharing nice properties, called a PBW basis, see Sect. 8.5.

Therefore, the operad *Lie* is Koszul and admits a PBW basis, see Sect. 13.2.4 for more details.

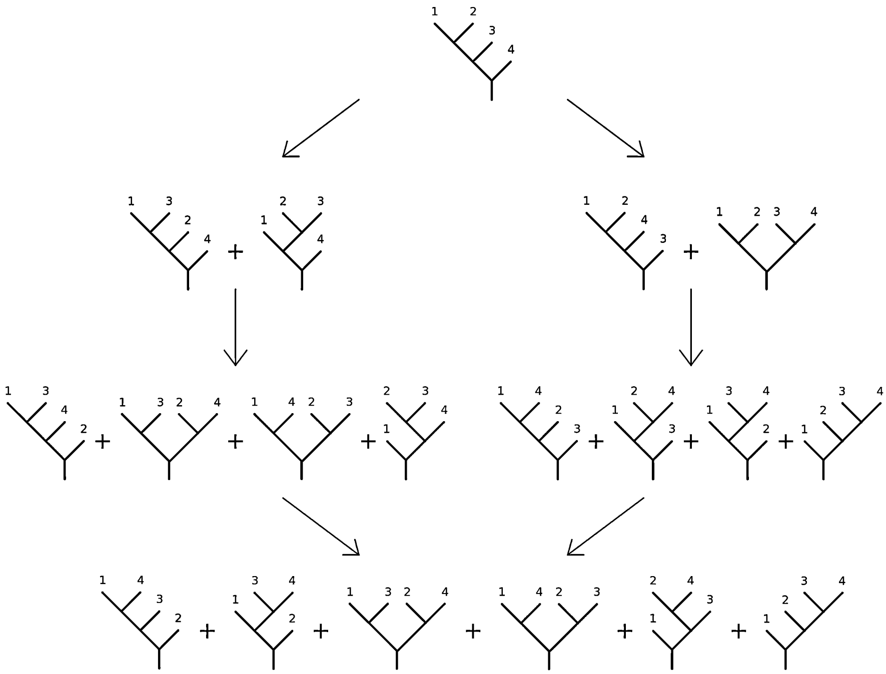


Fig. 8.8 The diamond for the operad *Lie*

8.4 Reduction by Filtration

In this section, we extend to operads the "reduction by filtration" method for algebras as described in Sect. 4.2. The only real new points lie in the use of the notion of shuffle operad and in suitable orders for the free (shuffle) operad. The proofs follow the same pattern as in Chap. 4 for associative algebras, so we skip most of them.

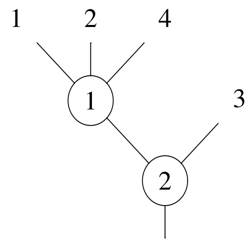
8.4.1 Suitable Order on Shuffle Trees

We consider the set of shuffle trees with vertices labeled by $\{1, \ldots, m\}$, where m can be infinite. We denote it simply by $\mathscr{T}_{\sqcup\sqcup}$. Notice that it labels a basis of the shuffle operad $\mathscr{T}_{\sqcup\sqcup}(E)$, on a reduced arity-graded module such that $\dim E_n = m$, for any $n \geq 1$. We choose a bijection between this set and the set of nonnegative integers $\mathbb{N}$, with its total order. (We require the identity tree | being sent to 0.) It endows the set of labeled shuffle trees with a total order denoted by ${\mathscr{T}_{\sqcup\sqcup}}^p$. We consider the partial shuffle products

$$\circ_{i,\omega} : {\mathscr{T}_{\sqcup\sqcup}}^p \times {\mathscr{T}_{\sqcup\sqcup}}^q \to {\mathscr{T}_{\sqcup\sqcup}}^{\chi(\sigma;p,q)}.$$

We ask that all these maps are strictly increasing, with respect to the lexicographic order on the left-hand side. In this case, we say that the order on labeled shuffle trees is a *suitable order*.

EXAMPLE (Path-lexicographic order). For simplicity, we restrict ourselves to the set of labeled shuffle trees whose vertices are at least trivalent, and such that m is finite. To any tree of arity n, we associate a sequence of $n+1$ words as follows. The n first words are obtained by reading the tree from the root to each leaf and by recording the labels indexing the vertices. The last word is given by the ordered labeling of the leaves, or equivalently by the image of the inverse of the associated pointed unshuffle. For example, one associates to the following tree



the sequence

$$(21, 21, 2, 21; 1324).$$

We leave it to the reader to verify that such a sequence characterizes the reduced labeled shuffle tree. We consider the following total order on this type of sequences.

1. We order them according to the number of elements of the sequence, that is the arity.
2. We consider the lexicographic order, with reversed order for the last word.

The example $m = 2$ is given in Fig. 8.9. This ordering endows the set of reduced labeled shuffle trees with a suitable order, see [Hof10c, Proposition 3.5] and [DK10, Proposition 5]. These two references provide other examples of suitable orders on reduced trees. We refer the reader to Exercise 8.10.4 for an example of a suitable order on labeled shuffle trees.

8.4.2 Associated Graded Shuffle Operad

Let $\mathscr{P}(E, R) = \mathscr{T}(E)/(R)$ be a homogeneous quadratic operad, i.e. the $\mathbb{S}$-module of relations satisfies $R \subset \mathscr{T}(E)^{(2)}$. Suppose that the generating space E comes equipped with an extra datum: a decomposition into $\mathbb{S}$-modules $E \cong E_1 \oplus \cdots \oplus E_m$. Notice that E and the E_i can be degree graded $\mathbb{S}$-modules.

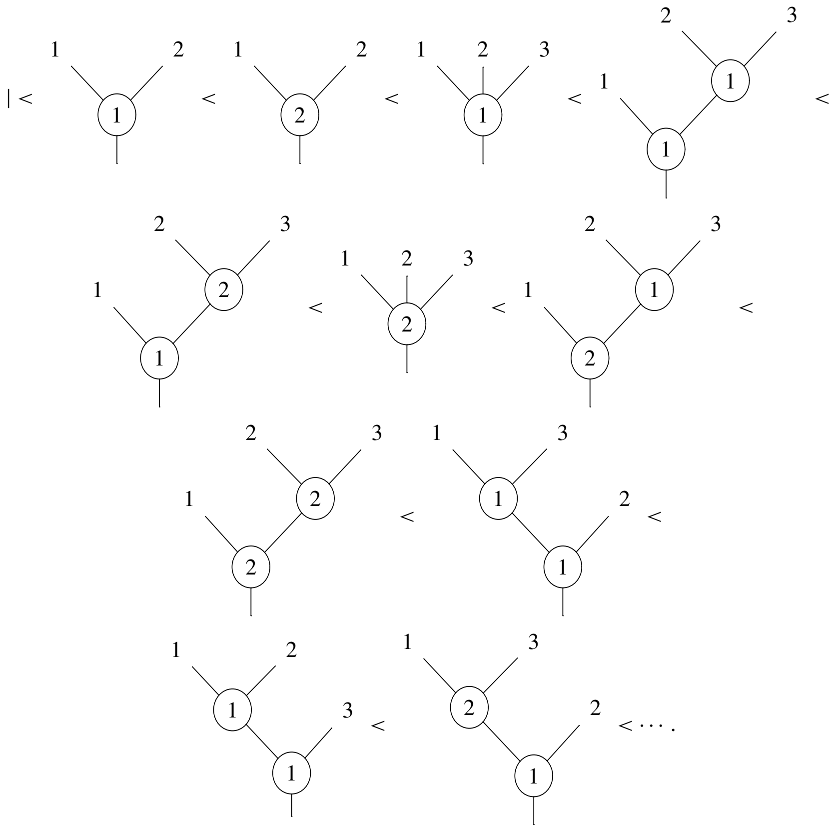


Fig. 8.9 Path-lexicographic order in the case $m = 2$

Since the forgetful functor $(\mathbb{S}\text{-Mod}, \circ) \to (\mathbb{N}\text{-Mod}, \circ_{\sqcup\!\sqcup})$, $\mathscr{M} \mapsto \mathscr{M}^f$, is a strong monoidal functor by Proposition 8.2.4, we automatically get the isomorphisms of arity-graded modules

$$\big(\mathscr{T}(E)\big)^f \cong \mathscr{T}_{\sqcup\!\sqcup}\big(E^f\big) \quad \text{and} \quad \big(\mathscr{P}(E,R)\big)^f \cong \mathscr{P}_{\sqcup\!\sqcup}\big(E^f, R^f\big).$$

In other words, the underlying space of the free operad is equal to the free shuffle operad and the underlying space of the quadratic operad $\mathscr{P}$ is equal to the quadratic shuffle operad, which we denote by $\mathscr{P}_{\sqcup\!\sqcup}$.

We consider a suitable order on the set of labeled shuffle trees. In this case, we say that the operad $\mathscr{P}$ is equipped with an *extra ordered grading*. Notice that if the generating space E is concentrated in some arities, it is enough to consider a suitable order only on the induced trees. For example, if $E(1) = 0$, then the shuffle trees are reduced and the aforementioned path-lexicographic order applies.

The total order on shuffle trees induces the following filtration of the free shuffle operad $F_p\mathcal{T}_{\sqcup\sqcup}(E) := \bigoplus_{q=0}^{p} \mathcal{T}_{\sqcup\sqcup}(E)^q$. This image under the canonical projection $\mathcal{T}_{\sqcup\sqcup}(E) \twoheadrightarrow \mathcal{P}_{\sqcup\sqcup} = \mathcal{P}_{\sqcup\sqcup}(E, R)$ defines a filtration on the quadratic shuffle operad denoted by $\mathrm{F}_p\mathcal{P}_{\sqcup\sqcup}$. The associated χ-graded arity-graded module is denoted by $\mathrm{gr}_\chi\, \mathcal{P}_{\sqcup\sqcup}$ or simply by $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$.

Proposition 8.4.1. *Any quadratic operad $\mathcal{P}(E, R)$ equipped with an extra order grading induces a shuffle operad structure on the χ-graded arity-graded module* $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$.

Proof. By the definition of a suitable order, any partial shuffle product of the shuffle operad $\mathcal{P}_{\sqcup\sqcup}$ induces a well-defined partial shuffle product on the graded module $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$:

$$\bar{\circ}_{i,\omega} : \mathrm{gr}_p\, \mathcal{P}_{\sqcup\sqcup} \otimes \mathrm{gr}_q\, \mathcal{P}_{\sqcup\sqcup} \to \mathrm{gr}_{\chi(\sigma;p,q)}\, \mathcal{P}_{\sqcup\sqcup},$$

for any i and ω. These data endow the graded module $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$ with a shuffle operad structure. □

Notice that, since the extra grading on $\mathcal{T}_{\sqcup\sqcup}(E)$ refines the weight grading, the shuffle operad $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$ is also weight graded.

8.4.3 The Koszul Property

Theorem 8.4.2. *Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad equipped with an extra ordered grading. If the shuffle operad* $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$ *is Koszul, then the operad $\mathcal{P}$ is also Koszul.*

Proof. The proof is essentially the same as the proof of Theorem 4.2.1, once the following point is understood. The isomorphism

$$\mathrm{B}\mathcal{P} \cong \mathrm{B}_{\sqcup\sqcup}\mathcal{P}_{\sqcup\sqcup}$$

from the bar construction of the operad $\mathcal{P}$ to the shuffle bar construction of the associated shuffle operad $\mathcal{P}_{\sqcup\sqcup}$, as differential graded arity-graded module, follows from Proposition 8.2.4 again. So if the homology of $\mathrm{B}_{\sqcup\sqcup}\mathcal{P}_{\sqcup\sqcup}$ is concentrated in syzygy degree 0, then so is the homology of $\mathrm{B}\mathcal{P}$. To conclude, it is enough to apply the methods of Theorem 4.2.1 to the shuffle operad $\mathcal{P}_{\sqcup\sqcup}$ to prove that $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$ Koszul implies $\mathcal{P}_{\sqcup\sqcup}$ Koszul. □

From now on, we concentrate on trying to prove that the shuffle operad $\mathrm{gr}\,\mathcal{P}_{\sqcup\sqcup}$ is Koszul in order to show that the operad $\mathcal{P}$ is Koszul.

8.4.4 Quadratic Analog

We consider the kernel R_{lead} of the restriction to R of the projection $\mathscr{T}_{\sqcup\sqcup}(E) \twoheadrightarrow \operatorname{gr}\mathscr{P}_{\sqcup\sqcup}$. Any relator in R can be written $r = T_1 + \cdots + T_p$, where any T_i is a tree monomial in $\mathscr{T}_{\sqcup\sqcup}(E)^{(2)}$, and such that $T_i < T_{i+1}$ and $T_p \neq 0$. This latter term T_p is called the *leading term* or r. The space R_{lead} is linearly spanned by the leading terms of the elements of R.

We consider the following quadratic shuffle algebra

$$\mathring{\mathscr{P}}_{\sqcup\sqcup} := \mathscr{T}_{\sqcup\sqcup}(E)/(R_{\text{lead}}),$$

which is the best candidate for being a quadratic presentation of the shuffle operad $\operatorname{gr}\mathscr{P}_{\sqcup\sqcup}$.

Proposition 8.4.3. *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad equipped with an extra ordered grading. There is a commutative diagram of epimorphisms of χ-graded, thus weight graded, shuffle operads*

$$\begin{array}{ccc} \mathscr{T}_{\sqcup\sqcup}(E) & & \\ \downarrow & \searrow & \\ \psi : \mathring{\mathscr{P}}_{\sqcup\sqcup} = \mathscr{T}_{\sqcup\sqcup}(E)/(R_{\text{lead}}) & \longrightarrow & \operatorname{gr}\mathscr{P}_{\sqcup\sqcup}. \end{array}$$

If the shuffle operad $\mathring{\mathscr{P}}_{\sqcup\sqcup}$ is Koszul and if the canonical projection $\mathring{\mathscr{P}}_{\sqcup\sqcup} \cong \operatorname{gr}\mathscr{P}_{\sqcup\sqcup}$ is an isomorphism, then the operad $\mathscr{P}$ is Koszul.

8.4.5 Diamond Lemma

When the quadratic shuffle operad $\mathring{\mathscr{P}}_{\sqcup\sqcup}$ is Koszul, it is enough to check that the canonical projection $\psi : \mathring{\mathscr{P}}_{\sqcup\sqcup} \twoheadrightarrow \operatorname{gr}\mathscr{P}_{\sqcup\sqcup}$ is injective in weight 3, to show that it is an isomorphism.

Theorem 8.4.4 (Diamond Lemma for quadratic operads). *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad equipped with an extra ordered grading. Suppose that the quadratic shuffle operad $\mathring{\mathscr{P}}_{\sqcup\sqcup} = \mathscr{T}_{\sqcup\sqcup}(E)/(R_{\text{lead}})$ is Koszul. If the canonical projection $\mathring{\mathscr{P}}_{\sqcup\sqcup} \twoheadrightarrow \operatorname{gr}\mathscr{P}_{\sqcup\sqcup}$ is injective in weight 3, then it is an isomorphism. In this case, the operad $\mathscr{P}$ is Koszul and its underlying arity-graded module is isomorphic to $\mathscr{P} \cong \mathring{\mathscr{P}}_{\sqcup\sqcup}$.*

8.4.6 The Inhomogeneous Case

For an inhomogeneous quadratic operad $\mathscr{P} = \mathscr{P}(E, R)$, that is $R \subset E \oplus \mathscr{T}(E)^{(2)}$, we require that the presentation satisfies the conditions (ql_1) and (ql_2) of Sect. 7.8.

We suppose that the associated homogeneous quadratic operad $q\mathscr{P} := \mathscr{P}(E, qR)$ admits an extra ordered grading. In this case, there exists a commutative diagram

$$
\begin{array}{ccccc}
q\mathring{\mathscr{P}}_{\sqcup\!\sqcup} & & & & \\
\Big\downarrow{\scriptstyle \mathsf{Op}_{\sqcup\!\sqcup}} & & & & \\
\mathrm{gr}_\chi\, q\mathscr{P}_{\sqcup\!\sqcup} & \xrightarrow[\mathbb{N}\text{-}\mathsf{Mod}]{\cong} & q\mathscr{P} & & \\
\Big\downarrow{\scriptstyle \mathsf{Op}_{\sqcup\!\sqcup}} & & \Big\downarrow{\scriptstyle \mathsf{Op}} & & \\
\mathrm{gr}_\chi\, \mathscr{P}_{\sqcup\!\sqcup} & \xrightarrow[\mathbb{N}\text{-}\mathsf{Mod}]{\cong} & \mathrm{gr}\,\mathscr{P} & \xrightarrow[\mathbb{S}\text{-}\mathsf{Mod}]{\cong} & \mathscr{P},
\end{array}
$$

where the type of the morphisms is indicated on the arrows. The morphisms of the first column preserve the χ-grading and the morphisms of the second column preserve the weight grading.

Theorem 8.4.5 (Diamond Lemma for inhomogeneous quadratic operads). *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic-linear operad with a presentation satisfying conditions (ql_1) and (ql_2). We suppose that $\mathscr{T}_{\sqcup\!\sqcup}(E)$ comes equipped with an extra ordered grading.*

If the quadratic operad $q\mathring{\mathscr{P}}_{\sqcup\!\sqcup}$ is Koszul and if the canonical projection $q\mathring{\mathscr{P}}_{\sqcup\!\sqcup} \twoheadrightarrow \mathrm{gr}_\chi\, q\mathscr{P}_{\sqcup\!\sqcup}$ is injective in weight 3, *then the operad $\mathscr{P}$ is Koszul and all the maps of the above diagram are isomorphisms*, *in particular*:

$$q\mathring{\mathscr{P}}_{\sqcup\!\sqcup} \cong \mathrm{gr}_\chi\, q\mathscr{P}_{\sqcup\!\sqcup} \cong q\mathscr{P} \cong \mathrm{gr}\,\mathscr{P} \cong \mathscr{P}.$$

8.4.7 Reduction by Filtration Method for Nonsymmetric Operads

The same theory holds for nonsymmetric operads. In this simpler case, there is no need to use the notions of shuffle operad and shuffle trees. One works with the set of planar trees PT from the very beginning and one remains in the context of nonsymmetric operads. The only point is to consider a suitable order on planar trees, that is a total order such that any partial composite product

$$\circ_i : PT_k \times PT_l \to PT_n$$

is an increasing map. The adaptation of the path-lexicographic order gives an example of such a suitable order. Notice that reduced planar trees are particular examples of shuffle trees and that partial composite products are particular examples of partial shuffle products. We leave the details to the reader as a good exercise.

8.5 PBW Bases and Gröbner Bases for Operads

In this section, we study the particular case of the preceding section when the generating space E is equipped with an extra grading $E \cong E_1 \oplus \cdots \oplus E_m$ such that each sub-space E_i is one-dimensional. This gives rise to the notion of Poincaré–Birkhoff–Witt basis for (shuffle) operads. Quadratic operads which admit such a basis share nice properties. For instance, they are Koszul operads.

We introduce the equivalent notion of (quadratic) Gröbner basis, which is to the (quadratic) ideal (R) what PBW basis is to the quotient operad $\mathscr{T}(E)/(R)$.

For operads, the notion of PBW basis is due to Hoffbeck [Hof10c] and the notion of Gröbner basis is due to Dotsenko and Khoroshkin [DK10].

8.5.1 Ordered Bases

Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad with a decomposition of the generating space $E \cong E_1 \oplus \cdots \oplus E_m$ into one-dimensional vector spaces. This data is equivalent to an ordered basis $\{e_1, \ldots, e_m\}$ of the $\mathbb{K}$-module E. Together with a suitable order on shuffle trees $\mathscr{T}_{\sqcup\sqcup}$, it induces a totally ordered basis of $\mathscr{T}_{\sqcup\sqcup}(E)$, made up of tree monomials. In this case, we say that $\mathscr{T}_{\sqcup\sqcup}(E)$ is equipped with a *suitable ordered basis*.

In this basis, the space of relations is equal to

$$R = \left\{ \lambda t(e_i, e_j) - \sum_{t'(k,l) < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} t'(e_k, e_l); \lambda \neq 0 \right\},$$

where the notation $t(i, j)$ (resp. $t(e_i, e_j)$) stands for a shuffle tree with two vertices indexed by i and j (resp. by e_i and e_j). We denote by $\overline{T}^{(2)}$ the subset of $\mathscr{T}_{\sqcup\sqcup}$ made up of labeled shuffle trees which appear as leading terms of some relators. We denote by $T^{(2)}$ the complement of $\overline{T}^{(2)}$ in $\mathscr{T}_{\sqcup\sqcup}^{(2)}$. Notice that the space of relations admits a normalized basis of the form

$$R = \left\langle t(e_i, e_j) - \sum_{t'(k,l) \in T^{(2)} < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} t'(e_k, e_l) \right\rangle.$$

Proposition 8.5.1. *Let $\mathscr{P}$ be a quadratic operad $\mathscr{P}(E, R)$, with $\mathscr{T}_{\sqcup\sqcup}(E)$ equipped with a suitable ordered basis. The associated quadratic shuffle operad $\mathring{\mathscr{P}}_{\sqcup\sqcup}$ is equal to the quadratic shuffle operad $\mathscr{P}_{\sqcup\sqcup}(E, R_{\text{lead}})$, with $R_{\text{lead}} \cong \langle t(e_i, e_j); t(i, j) \in \overline{T}^{(2)} \rangle$.*

8.5.2 Quadratic Monomial Shuffle Operads

A *quadratic monomial shuffle operad* is a quadratic shuffle operad $\mathring{\mathscr{P}}_{\sqcup\sqcup} = \mathscr{P}_{\sqcup\sqcup}(E, R)$ with a (non-necessarily ordered) basis $\{e_i\}_{1 \leq i \leq m}$ of E such that the space of relations R is linearly spanned by a set of trees $\{t(e_i, e_j)\}_{t(i,j)\in \overline{T}^{(2)}}$, where $\overline{T}^{(2)} \subset \mathcal{T}_{\sqcup}^{(2)}$. Hence the complement $T^{(2)}$ of $\overline{T}^{(2)}$ in $\mathcal{T}_{\sqcup}^{(2)}$ labels a basis of the quotient $\mathring{\mathcal{P}}_{\sqcup}^{(2)} = \mathcal{T}_{\sqcup}(E)^{(2)}/R$.

We set $T^{(0)} := \{|\}$ and $T^{(1)} := \mathcal{T}_{\sqcup}^{(1)}$. For any $n \geq 2$, we define the subset $T^{(n)} \subset \mathcal{T}_{\sqcup}^{(n)}$ as the set of labeled shuffle trees with n vertices such that for any internal edge, the associated two-vertices subtree is in $T^{(2)}$. Finally, we consider $T = \bigsqcup_{n\in\mathbb{N}} T^{(n)} \subset \mathcal{T}_{\sqcup}$. We define the subset $\overline{T} \subset \mathcal{T}_{\sqcup}$ in the same way, that is a tree lives in $\overline{T}^{(n)} \subset \mathcal{T}_{\sqcup}^{(n)}$ if for any internal edge, the associated two-vertices subtree is in $\overline{T}^{(2)}$.

Proposition 8.5.2. *Any quadratic monomial shuffle operad $\mathring{\mathcal{P}}_{\sqcup} = \mathcal{P}_{\sqcup}(E, R)$ admits a basis labeled by the subset $T \subset \mathcal{T}_{\sqcup}$. Its Koszul dual shuffle cooperad $(\mathring{\mathcal{P}}_{\sqcup})^{¡}$ admits a basis labeled by $\overline{T} \subset \mathcal{T}_{\sqcup}$.*

Theorem 8.5.3. *Any quadratic monomial shuffle operad is Koszul.*

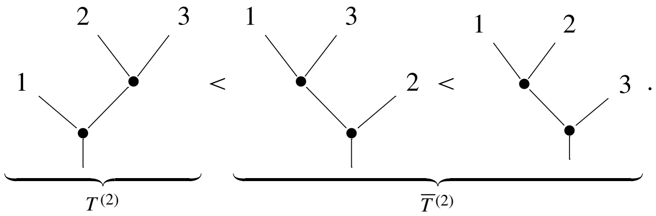
Proof. By the preceding proposition, the underlying space of the Koszul complex is equal to $(\mathring{\mathcal{P}}_{\sqcup})^{¡} \circ_{\sqcup} \mathring{\mathcal{P}}_{\sqcup}$, which admits a basis of the following form. We consider the set of labeled shuffle trees with a horizontal partition into two parts; any two-vertices subtrees in the upper part belong to $T^{(2)}$ and any two-vertices subtrees in the lower part belong to $\overline{T}^{(2)}$. Finally, we conclude with the same argument as in the proof of Theorem 4.3.4. The differential map amounts to pulling up one top vertex from the lower part to the upper part. When it is 0, it produces a cycle element, which is easily shown to be a boundary element. Hence, this chain complex is acyclic. □

This result is a key point in the PBW basis theory because it simplifies the statements of Sect. 8.4. When the decomposition $E \cong E_1 \oplus \cdots \oplus E_m$ is made up of one-dimensional sub-spaces, the quadratic analog $\mathring{\mathcal{P}}_{\sqcup}$ is always a Koszul shuffle operad.

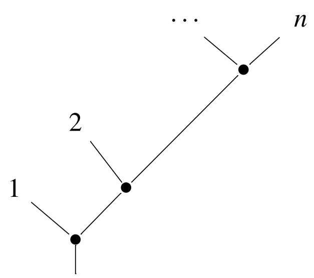
8.5.3 PBW Basis

The image of the monomial basis $\{t(e_{\bar{i}})\}_{t\in T}$ of the shuffle operad $\mathring{\mathcal{P}}_{\sqcup}$, given in Proposition 8.5.2, under the successive morphisms of graded (and also arity-graded) modules $\mathring{\mathcal{P}}_{\sqcup} \twoheadrightarrow \mathrm{gr}\, \mathcal{P}_{\sqcup} \cong \mathcal{P}_{\sqcup} \cong \mathcal{P}$ provides a family of elements $\{\bar{t}(e_{\bar{i}})\}_{t\in T}$, which span the operad $\mathcal{P}$. When these elements are linearly independent, they form a basis of the operad $\mathcal{P}$, called a *Poincaré–Birkhoff–Witt basis*, or *PBW basis* for short. This is equivalent to having an isomorphism $\psi : \mathring{\mathcal{P}}_{\sqcup} \cong \mathrm{gr}\, \mathcal{P}_{\sqcup}$. We say that an operad $\mathcal{P} = \mathcal{P}(E, R)$ admits a PBW basis if the free shuffle operad $\mathcal{T}_{\sqcup}(E)$ admits a suitable ordered basis such that the associated elements $\{\bar{t}(e_{\bar{i}})\}_{t\in T}$ form a basis of the operad $\mathcal{P}$.

EXAMPLE. The operad *Com* is generated by a one-dimensional space concentrated in arity 2: $E = \mathbb{K}\bullet$. We consider the suitable order obtained by restriction of the path-lexicographic order Sect. 8.4.1 on binary shuffle trees. On trees with 2 vertices, it is equal to



The space of relations is the linear span of the three terms made up of the difference of such two trees. Hence the set $T^{(2)}$ is made up of the first tree. It generates the set T made up of the left combs:



Since $\dim_{\mathbb{K}} Com(n) = 1$, this forms a PBW basis of the operad *Com*.

Theorem 8.5.4. *Any quadratic operad endowed with a PBW basis is Koszul.*

As for algebras, the existence of a PBW basis gives a purely algebraic condition to prove that an algebra is Koszul, without having to compute any homology group.

8.5.4 Diamond Lemma for PBW Bases

The following Diamond Lemma gives an easy way to prove that one has a PBW basis.

Theorem 8.5.5 (Diamond Lemma for PBW bases of operads). *Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad, with $\mathscr{T}_{\sqcup}(E)$ equipped with a suitable ordered basis $\{t(e_{\bar{i}})\}_{t \in T}$. If the associated elements $\{\bar{t}\}_{t \in T^{(3)}}$ are linearly independent in $\mathscr{P}^{(3)}$, then the elements $\{\bar{t}(e_{\bar{i}})\}_{t \in T}$ form a PBW basis of $\mathscr{P}$. In that case, the operad $\mathscr{P}$ is Koszul.*

To check the assumption of this theorem, one uses the rewriting method of Sect. 8.3: one shows that every critical monomial is confluent. Notice that this theorem and Theorem 4.3.7 give a proof of the rewriting method Theorem 8.3.1.

8.5.5 Partial Shuffle Product of Elements of a PBW Basis

The composite $\mathring{\mathscr{P}}_{\sqcup\sqcup} \twoheadrightarrow \mathrm{gr}\,\mathscr{P}_{\sqcup\sqcup} \cong \mathscr{P}_{\sqcup\sqcup} \cong \mathscr{P}$ is an epimorphism of arity-graded modules, but not of (shuffle) operads. Therefore, the partial shuffle product of two elements of the generating family $\{\bar{t}(e_{\bar{i}})\}_{t\in T}$ is not always equal to an element of this family, but sometimes to a sum of lower terms.

Proposition 8.5.6. *The elements $\{\bar{t}(e_{\bar{i}})\}_{t\in T}$ satisfy the following properties.*

(1) *Let σ be a partial pointed shuffle and let $t, s \in T$ be a pair of trees with matching arity. If the partial shuffle product of trees is not in T, $t \circ_\sigma s \notin T$, then the partial shuffle product $\bar{t} \circ_\sigma \bar{s}$ in the operad $\mathscr{P}$ is equal to a linear combination of strictly lower terms labeled by T:*

$$\bar{t} \circ_\sigma \bar{s} = \sum_{u\in T,\, u<t\circ_\sigma s} \lambda_u^{t,s}\, \bar{u},$$

with $\lambda_u^{t,s} \in \mathbb{K}$.

(2) *Any shuffle tree lies in T, if and only if any shuffle subtree $s \subset t$ lies in T.*

In the above example of the operad *Com*, only the partial shuffle composite $i = k$ and $\sigma = \mathrm{id}$ produces an element of the basis. The other composites are equal to the left comb in *Com*, which is a strictly lower but in the basis.

We leave it to the reader to prove that a tree basis T satisfying conditions (1) and (2) is a PBW basis. This equivalent definition is the definition originally given by Hoffbeck in [Hof10c].

8.5.6 Gröbner Bases for Operads

Following [DK10], we introduce the notion of Gröbner basis for an operadic ideal I of the free (shuffle) operad. In the quadratic case, when $I = (R)$, it is equivalent to a PBW basis for the quotient operad $\mathscr{P} = \mathscr{T}(E)/(R)$.

Any element $t \in \mathscr{T}_{\sqcup\sqcup}(E)$ of the free shuffle operad is a linear combination of tree monomials. When $\mathscr{T}_{\sqcup\sqcup}(E)$ is equipped with a suitable ordered basis, we denote by t_{lead} the leading term of t. For any subset $M \subset \mathscr{T}_{\sqcup\sqcup}(E)$, we consider the set made up of the leading terms of any element of M and we denote it by $\mathrm{Lead}(M)$. Under this notation, the space of relations R_{lead} of Sect. 8.5.1 is equal to the linear span of $\mathrm{Lead}(R)$: $R_{\mathrm{lead}} = \langle \mathrm{Lead}(R) \rangle$.

A *Gröbner basis of an ideal* I *in* $\mathcal{T}_{\sqcup}(E)$ is a set $G \subset I$ which generates the ideal I, i.e. $(G) = I$, such that the leading terms of G and the leading terms of the elements of I generate the same ideal: $(\mathrm{Lead}(G)) = (\mathrm{Lead}(I))$.

Proposition 8.5.7. *Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad such that $\mathcal{T}_{\sqcup}(E)$ is equipped with a suitable ordered basis. The elements $\{\bar{t}(e_{\bar{i}})\}_{t \in T}$ form a PBW basis of $\mathcal{P}$ if and only if the elements*

$$\left\{ t(e_i, e_j) - \sum_{t'(k,l) \in T^{(2)} < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} t'(e_k, e_l) \right\}_{t(i,j) \in \overline{T}^{(2)}},$$

spanning R, form a Gröbner basis of the ideal (R) in $\mathcal{T}_{\sqcup}(E)$.

In the quadratic case, the two notions of PBW basis and Gröbner basis are equivalent dual notions. The terminology "PBW basis" refers to the basis of the quotient operad while the terminology "Gröbner basis" refers to the ideal (R). We refer to [DK10] for more details on Gröbner bases for operads.

8.5.7 PBW Bases for Inhomogeneous Quadratic Operads

Following Sect. 8.4.6, we say that *an inhomogeneous quadratic operad $\mathcal{P}$ admits a PBW basis* if there exists a presentation $\mathcal{P} = \mathcal{P}(E, R)$, satisfying conditions (ql_1) and (ql_2), such that the associated quadratic algebra $q\mathcal{P} = \mathcal{P}(E, qR)$ admits a PBW basis. In this case, the image $\{\bar{t}(e_{\bar{i}})\}_{t \in T} \subset \mathcal{P}$ of the tree basis elements $\{t(e_{\bar{i}})\}_{\bar{i} \in T}$ of the quadratic monomial shuffle operad $q\mathring{\mathcal{P}}_{\sqcup}$ gives a basis of the inhomogeneous quadratic operad $\mathcal{P}$. Such a result is once again proved using the following version of the Diamond Lemma.

Theorem 8.5.8 (Diamond Lemma for PBW bases of inhomogeneous operads). *Let $\mathcal{P} = \mathcal{P}(E, R)$ be an inhomogeneous quadratic operad with a quadratic-linear presentation satisfying conditions (ql_1) and (ql_2) and such that $\mathcal{T}_{\sqcup}(E)$ is equipped with a suitable ordered basis.*

If the images of the tree elements $\{t(e_{\bar{i}})\}_{\bar{i} \in T^{(3)}}$ in $q\mathcal{P}$ are linearly independent, then the images $\{\bar{t}(e_{\bar{i}})\}_{t \in T}$ of the elements $\{t(e_{\bar{i}})\}_{t \in T}$ form a basis of $\mathcal{P}$, and the operad $\mathcal{P}$ is Koszul.

In the inhomogeneous case too, the notion of PBW basis is equivalent and dual to that of Gröbner basis.

Proposition 8.5.9. *Let $\mathcal{P}$ be an inhomogeneous quadratic operad with a quadratic-linear presentation $\mathcal{P} = \mathcal{P}(E, R)$ satisfying conditions (ql_1) and (ql_2) and such that $\mathcal{T}_{\sqcup}(E)$ is equipped with a suitable ordered basis. Let $\varphi : qR \to E$ be* *the linear map whose graph gives* R. *The elements* $\{\bar{t}(e_{\bar{i}})\}_{t\in T} \subset \mathscr{P}$ *form a PBW basis of* $\mathscr{P}$ *if and only if the elements*

$$\left\{(\mathrm{Id}-\varphi)\left(t(e_i,e_j) - \sum_{t'(k,l)\in T^{(2)} < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)}\, t'(e_k,e_l)\right)\right\}_{t(i,j)\in \overline{T}^{(2)}}$$

spanning R, *form a Gröbner basis of the ideal* (R) *in* $\mathscr{T}_{\sqcup}(E)$.

8.5.8 PBW/Gröbner Bases for Nonsymmetric Operads

Once again, the same results hold for nonsymmetric operads. One has just to replace the set of shuffle trees by the one of planar trees.

8.6 Distributive Laws

In this section, we show how to build a new operad out of two operads by means of an extra datum, called *distributive law*. When the two first operads are quadratic, we describe how to get a distributive law from these presentations. The main result states that, in this case, the resulting operad is Koszul if the two first ones are Koszul.

The notion of distributive law goes back to Jon Beck [Bec69]. Its application to Koszul operads was first introduced by Martin Markl in [Mar96a, FM97] and then refined in [Val07b, Dot07].

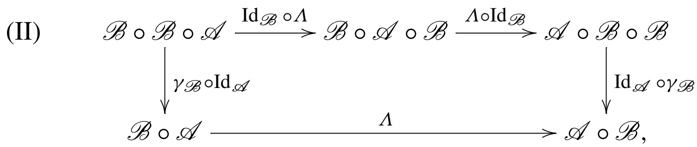
8.6.1 Definition of a Distributive Law

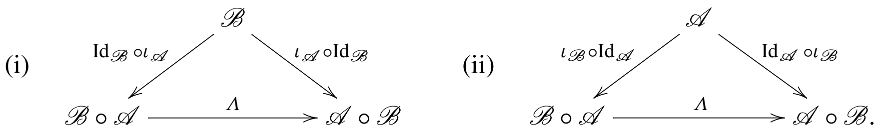
Let $\mathscr{A} = (\mathscr{A}, \gamma_{\mathscr{A}}, \iota_{\mathscr{A}})$ and $\mathscr{B} = (\mathscr{B}, \gamma_{\mathscr{B}}, \iota_{\mathscr{B}})$ be two operads. In order to put an operad structure on the composite $\mathscr{A} \circ \mathscr{B}$, one needs a morphism of $\mathbb{S}$-modules

$$\Lambda : \mathscr{B} \circ \mathscr{A} \to \mathscr{A} \circ \mathscr{B}.$$

It is called a *distributive law* if the following diagrams are commutative:

$$\text{(I)}\qquad \begin{array}{ccccc} \mathscr{B}\circ\mathscr{A}\circ\mathscr{A} & \xrightarrow{\Lambda\circ \mathrm{Id}_{\mathscr{A}}} & \mathscr{A}\circ\mathscr{B}\circ\mathscr{A} & \xrightarrow{\mathrm{Id}_{\mathscr{A}}\circ\Lambda} & \mathscr{A}\circ\mathscr{A}\circ\mathscr{B} \\ \downarrow{\scriptstyle \mathrm{Id}_{\mathscr{B}}\circ\gamma_{\mathscr{A}}} & & & & \downarrow{\scriptstyle \gamma_{\mathscr{A}}\circ\mathrm{Id}_{\mathscr{B}}} \\ \mathscr{B}\circ\mathscr{A} & & \xrightarrow{\quad\Lambda\quad} & & \mathscr{A}\circ\mathscr{B}, \end{array}$$





The associativity isomorphisms and the identifications like $\mathscr{B}\circ \mathrm{I}\cong\mathscr{B}\cong \mathrm{I}\circ\mathscr{B}$ are implicit in these diagrams.

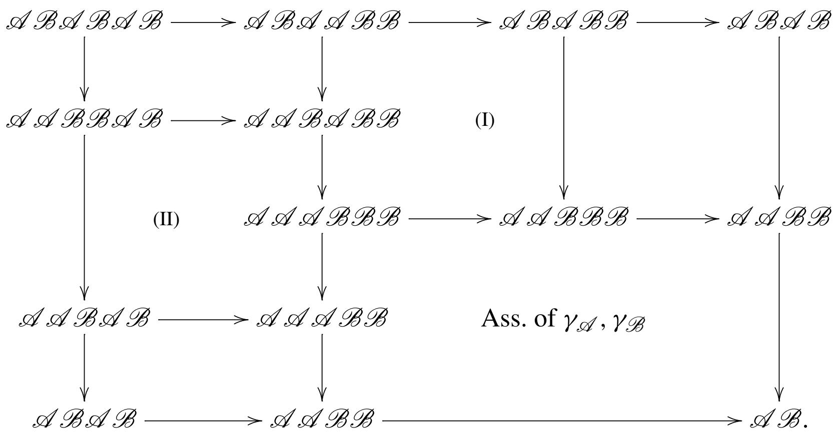
Proposition 8.6.1. *If $\Lambda:\mathscr{B}\circ\mathscr{A}\to\mathscr{A}\circ\mathscr{B}$ is a distributive law for the operad structures of $\mathscr{A}$ and $\mathscr{B}$, then $\mathscr{A}\circ\mathscr{B}$ is an operad for the composition*

$$\gamma_\Lambda := (\gamma_{\mathscr{A}}\circ\gamma_{\mathscr{B}})(\mathrm{Id}_{\mathscr{A}}\circ\Lambda\circ\mathrm{Id}_{\mathscr{B}}) : (\mathscr{A}\circ\mathscr{B})\circ(\mathscr{A}\circ\mathscr{B})\to\mathscr{A}\circ\mathscr{B},$$

and for the unit

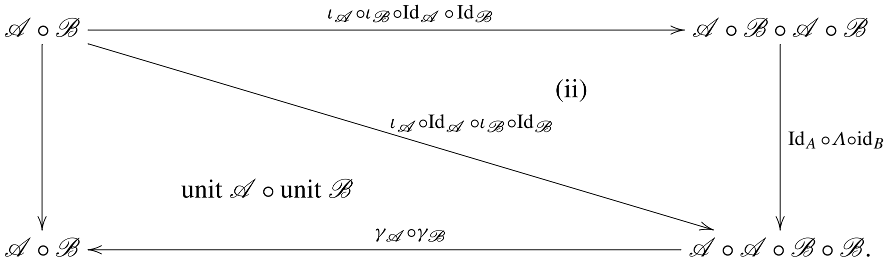
$$\iota_\Lambda := \iota_{\mathscr{A}}\circ\iota_{\mathscr{B}} : \mathrm{I}\to\mathscr{A}\circ\mathscr{B}.$$

Proof. In order to simplify the notation we write $\mathscr{A}\mathscr{B}$ in place of $\mathscr{A}\circ\mathscr{B}$. The associativity condition for γ_Λ is the commutativity of the outer square diagram. It follows from the commutativity of the inner diagrams, which are either straightforward, or follows from the associativity of $\gamma_{\mathscr{A}}$ and $\gamma_{\mathscr{B}}$, or from the hypothesis.



In this diagram the maps are composite products of Λ, $\gamma_{\mathscr{A}}$, $\gamma_{\mathscr{B}}$ and the identities. The exact combination is clear from the source and the target. For instance the leftmost arrow of the first row is $\mathrm{Id}_{\mathscr{A}}\circ\mathrm{Id}_{\mathscr{B}}\circ\mathrm{Id}_{\mathscr{A}}\circ\Lambda\circ\mathrm{Id}_{\mathscr{B}}$.

The left unit property is proved in the same way by the following commutative diagram.



The proof of the right unit property uses (i). □

This notion of distributive law is the exact application to Schur functors of the general one for monads due to Beck [Bec69]. For a converse result, we refer the reader to Exercise 8.10.10.

8.6.2 Distributive Laws for Quadratic Data

Let $\mathscr{A} := \mathscr{P}(V, R) = \mathscr{T}(V)/(R)$ and $\mathscr{B} := \mathscr{P}(W, S) = \mathscr{T}(W)/(S)$ be two quadratic operads. Their coproduct $\mathscr{A} \vee \mathscr{B}$ in the category of operads is again a quadratic operad with presentation $\mathscr{A} \vee \mathscr{B} = \mathscr{P}(V \oplus W, R \oplus S)$. We suppose now that there exists a compatibility relation between the generating operations V of $\mathscr{A}$ and the generating operations W of $\mathscr{B}$ of the following form.

Under the notation $\circ_{(1)}$, introduced in Sect. 6.1.1, we have a natural isomorphism of $\mathbb{S}$-modules:

$$\mathscr{T}(V \oplus W)^{(2)} = \mathscr{T}(V)^{(2)} \oplus V \circ_{(1)} W \oplus W \circ_{(1)} V \oplus \mathscr{T}(W)^{(2)}.$$

Let $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ be a morphism of $\mathbb{S}$-modules that we call a *rewriting rule*. The graph of λ gives the weight 2 space of relation

$$D_\lambda := \big\langle T - \lambda(T), T \in W \circ_{(1)} V \big\rangle \subset \mathscr{T}(V \oplus W)^{(2)}.$$

Finally, we consider the operad $\mathscr{A} \vee_\lambda \mathscr{B}$ which is, by definition, the quotient of the coproduct $\mathscr{A} \vee \mathscr{B}$ by the ideal generated by D_λ. This operad admits the following quadratic presentation:

$$\mathscr{A} \vee_\lambda \mathscr{B} = \mathscr{P}(V \oplus W, R \oplus D_\lambda \oplus S).$$

So, in this operad, the map λ has to been seen as a rewriting rule, which allows us to move the operations of V under the operations of W, see the figures of Sect. 8.6.3 below. It remains to see whether this *local* rewriting rule induces or not a *global* distributive law $\mathscr{B} \circ \mathscr{A} \to \mathscr{A} \circ \mathscr{B}$.

The composite $\mathscr{T}(V) \circ \mathscr{T}(W) \to \mathscr{T}(V \oplus W) \twoheadrightarrow \mathscr{T}(V \oplus W)/(R \oplus D_\lambda \oplus S)$ induces the following epimorphism of $\mathbb{S}$-modules

$$p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}.$$

Similarly, there exists a morphism of $\mathbb{S}$-modules $\mathscr{B} \circ \mathscr{A} \to \mathscr{A} \vee_\lambda \mathscr{B}$.

Proposition 8.6.2. *Let $\mathscr{A} = \mathscr{P}(V, R)$ and $\mathscr{B} = \mathscr{P}(W, S)$ be two quadratic operads. For any morphism of $\mathbb{S}$-modules $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$, such that $p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ is an isomorphism, the composite*

$$\Lambda : \mathscr{B} \circ \mathscr{A} \to \mathscr{A} \vee_\lambda \mathscr{B} \xrightarrow{p^{-1}} \mathscr{A} \circ \mathscr{B}$$

is a distributive law.

In that case, the map $p : (\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda, \iota_\Lambda) \to \mathscr{A} \vee_\lambda \mathscr{B}$ is an isomorphism of operads.

Proof. Conditions (i) and (ii) of Sect. 8.6.1 are trivially satisfied. The inclusion $i : \mathscr{A} \rightarrowtail \mathscr{A} \vee \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ satisfies properties like the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{B} \circ \mathscr{A} \circ \mathscr{A} & \xrightarrow{p \circ \mathrm{Id}_{\mathscr{A}}} & (\mathscr{A} \vee_\lambda \mathscr{B}) \circ \mathscr{A} & \xrightarrow{\mathrm{Id} \circ i} & (\mathscr{A} \vee_\lambda \mathscr{B}) \circ (\mathscr{A} \vee_\lambda \mathscr{B}) \\
\downarrow{\scriptstyle \mathrm{Id}_{\mathscr{B}} \circ \gamma_{\mathscr{A}}} & & & & \downarrow{\scriptstyle \gamma} \\
\mathscr{B} \circ \mathscr{A} & & \xrightarrow{\qquad p \qquad} & & (\mathscr{A} \vee_\lambda \mathscr{B}),
\end{array}$$

where Id and γ refer to $\mathscr{A} \vee_\lambda \mathscr{B}$.

Therefore, "Condition (I) of a distributive law" is a consequence of the commutativity of the diagram:

$$\begin{array}{ccccccccc}
\mathscr{B}\mathscr{A}\mathscr{A} & \longrightarrow & (\mathscr{A} \vee_\lambda \mathscr{B})\mathscr{A} & \longrightarrow & \mathscr{A}\mathscr{B}\mathscr{A} & \longrightarrow & \mathscr{A}(\mathscr{A} \vee_\lambda \mathscr{B}) & \longrightarrow & \mathscr{A}\mathscr{A}\mathscr{B} \\
\downarrow & & \downarrow & \swarrow & & \searrow & \downarrow & & \downarrow \\
 & & (\mathscr{A} \vee_\lambda \mathscr{B})(\mathscr{A} \vee_\lambda \mathscr{B}) & & & & (\mathscr{A} \vee_\lambda \mathscr{B})(\mathscr{A} \vee_\lambda \mathscr{B}) & & \\
 & & & \searrow & & \swarrow & & & \\
\mathscr{B}\mathscr{A} & \longrightarrow & & & \mathscr{A} \vee_\lambda \mathscr{B} & \longrightarrow & & & \mathscr{A}\mathscr{B}.
\end{array}$$

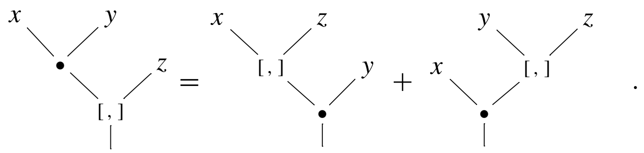
The commutativity of the middle square comes from the associativity of the composite of $\mathscr{A} \vee_\lambda \mathscr{B}$: $\mathscr{A}\mathscr{B}\mathscr{A} \rightarrowtail (\mathscr{A} \vee_\lambda \mathscr{B})^{\circ 3} \to \mathscr{A} \vee_\lambda \mathscr{B}$. □

Notice the similarity with the rewriting method: the elements of $\mathscr{A} \circ \mathscr{B}$ play the same role as the chosen monomials which can form a PBW basis. They always linearly span the final operad but it remains to prove that they actually form a basis of it. The rest of this section is written in the same way as the previous sections.

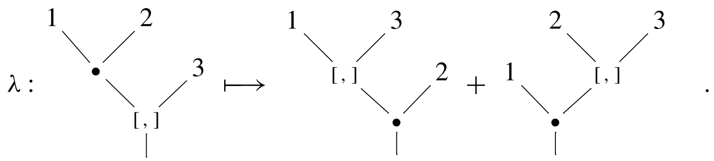
For a more general study of distributive laws, see P.-L. Curien [Cur12].

8.6.3 The Example of the Operad *Pois*

Let us consider the example of the operad *Pois* encoding Poisson algebras, see Sect. 13.3 for more details. A Poisson algebra is a vector space (or dg module) endowed with an associative and commutative product $\bullet$ and with a Lie bracket $[\,,\,]$, which satisfy the Leibniz relation $[x \bullet y, z] = [x, z] \bullet y + x \bullet [y, z]$. The graphical representation of this relation is



It is an example of the preceding definition. Here the operad $\mathscr{A} = Com$ is the operad of commutative algebras with $V = \mathbb{K}\bullet$, the operad $\mathscr{B} = Lie$ is the operad of Lie algebras with $W = \mathbb{K}[\,,\,]$ and the rewriting rule λ is equal to



Hence the $\mathbb{S}$-module $Com \circ Lie$ linearly spans the operad *Pois*. Actually, the rewriting rule λ induces a distributive law and the $\mathbb{S}$-module $Com \circ Lie$ is isomorphic to *Pois*. Since the proof is quite involved, we postpone it to Sect. 8.6.7 after Theorem 8.6.5.

8.6.4 Koszul Duality of Operads with Trivial Distributive Law

Let $\mathscr{A} = \mathscr{P}(V, R)$ and $\mathscr{B} = \mathscr{P}(W, S)$ be two quadratic operads and consider the trivial rewriting rule $\lambda \equiv 0$. On the level of $\mathbb{S}$-modules $\mathscr{A} \vee_0 \mathscr{B} \cong \mathscr{A} \circ \mathscr{B}$, so it induces a global distributive law, which is the trivial one $\Lambda \equiv 0$. This yields the isomorphism of operads $\mathscr{A} \vee_0 \mathscr{B} \cong (\mathscr{A} \circ \mathscr{B}, \gamma_0)$.

Proposition 8.6.3. *Let $\mathscr{A} = \mathscr{P}(V, R)$ and $\mathscr{B} = \mathscr{P}(W, S)$ be two quadratic operads. The underlying module of the Koszul dual cooperad of the operad defined by the trivial rewriting rule is equal to*

$$(\mathscr{A} \vee_0 \mathscr{B})^{¡} \cong \mathscr{B}^{¡} \circ \mathscr{A}^{¡}.$$

If moreover, the operads $\mathscr{A}$, $\mathscr{B}$ are Koszul, then the operad $\mathscr{A} \vee_0 \mathscr{B}$ is Koszul.

Proof. The isomorphism $(\mathscr{A} \vee_0 \mathscr{B})^{\text{i}} \cong \mathscr{B}^{\text{i}} \circ \mathscr{A}^{\text{i}}$ is a direct consequence of Proposition 7.3.1. The Koszul complex of the operad $\mathscr{A} \vee_0 \mathscr{B}$ is isomorphic to $(\mathscr{B}^{\text{i}} \circ \mathscr{A}^{\text{i}}) \circ (\mathscr{A} \circ \mathscr{B})$. We filter this chain complex by the weight of the elements of $\mathscr{B}^{\text{i}}$. Hence the first term of the associated spectral sequence is equal to

$$\left(E^0, d^0\right) \cong \mathscr{B}^{\text{i}} \circ \left(\mathscr{A}^{\text{i}} \circ_{\kappa_{\mathscr{A}}} \mathscr{A}\right) \circ \mathscr{B}.$$

So its homology is equal to $(E^1, d^1) \cong \mathscr{B}^{\text{i}} \circ_{\kappa_{\mathscr{B}}} \mathscr{B}$. It finally gives $E^2 \cong \text{I}$ and we conclude by the convergence theorem of spectral sequences (Theorem 1.5.1). □

8.6.5 Distributive Law Implies Koszul

The Koszul property is stable under the construction of operads via distributive laws.

Theorem 8.6.4. *Let $\mathscr{A} = \mathscr{P}(V, R)$ and $\mathscr{B} = \mathscr{P}(W, S)$ be two quadratic operads endowed with a rewriting rule $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ which induces a distributive law.*

The operads $\mathscr{A}$ and $\mathscr{B}$ are Koszul if and only if the operad $\mathscr{A} \vee_\lambda \mathscr{B}$ is Koszul.

Proof. (⇒) We follow the same ideas as in the proof of Theorem 4.2.1. We consider the bar construction $\text{B}^{-\bullet}(\mathscr{A} \vee_\lambda \mathscr{B})$ with the opposite of the syzygy degree as homological degree and we introduce the following filtration. Since λ defines a distributive law, the operad $\mathscr{A} \vee_\lambda \mathscr{B}$ is isomorphic to $(\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda)$. The bar construction $\text{B}(\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda)$ is made up of linear combinations of trees with vertices labeled by 2-leveled trees made up of elements of $\mathscr{A}$ in the bottom part and elements of $\mathscr{B}$ in the top part. We say that one internal edge carries an inversion if it links a nontrivial element of $\mathscr{B}$ in the lower vertex with a nontrivial element of $\mathscr{A}$ in the upper vertex. For such a tree, we define its number of inversions by the sum, over the internal edges, of inversions. The filtration F_p of $\text{B}(\mathscr{A} \vee_\lambda \mathscr{B})$ is equal to the sub-$\mathbb{S}$-module generated by trees with number of inversions less than or equal to p. This filtration is stable under the boundary map. The first term of the associated spectral sequence (E^0, d^0) is isomorphic to the bar construction $\text{B}(\mathscr{A} \circ \mathscr{B}, \gamma_0) \cong \text{B}(\mathscr{A} \vee_0 \mathscr{B})$ of the operad defined by the trivial rewriting rule. Since this latter operad is Koszul by Proposition 8.6.3, the homology of its bar construction is concentrated in syzygy degree 0. Therefore, the second page E^1 is concentrated on the diagonal $E^1_{p\,-p}$. So it collapses at rank 1 and the limit E^∞ is also concentrated on the diagonal $E^\infty_{p\,-p}$. The filtration F_p being exhaustive and bounded below, the classical convergence theorem of spectral sequences (Theorem 1.5.1) ensures that the homology of $\text{B}(\mathscr{A} \vee_\lambda \mathscr{B})$ is concentrated in syzygy degree 0. Hence the operad $\mathscr{A} \vee_\lambda \mathscr{B}$ is Koszul.

(⇐) The bar constructions $\text{B}\mathscr{A}$ and $\text{B}\mathscr{B}$ are sub-chain complexes of the bar construction $\text{B}(\mathscr{A} \vee_\lambda \mathscr{B})$. If the homology of the latter one is concentrated in syzygy degree 0 ($\mathscr{A} \vee_\lambda \mathscr{B}$ Koszul operad), it is also true for $\text{B}\mathscr{A}$ and $\text{B}\mathscr{B}$ ($\mathscr{A}$, $\mathscr{B}$ Koszul operads). □

8.6.6 The Diamond Lemma for Distributive Laws

We prove an analog of the Diamond Lemma for distributive laws. We denote by $(\mathscr{A} \circ \mathscr{B})^{(\omega)}$ the sub-$\mathbb{S}$-module of $\mathscr{A} \circ \mathscr{B}$ made up of elements of total weight ω. The map $p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ is always injective on the components of weight 0, 1 and 2. As in the case of the Diamond Lemma for PBW bases (Theorem 8.5.5), if we have an isomorphism in weight (3), then we have an isomorphism in any weight.

Theorem 8.6.5 (Diamond Lemma for distributive laws). *Let $\mathscr{A} = \mathscr{P}(V, R)$ and $\mathscr{B} = \mathscr{P}(W, S)$ be two Koszul operads endowed with a rewriting rule $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ such that the restriction of $p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ on $(\mathscr{A} \circ \mathscr{B})^{(3)}$ is injective. In this case, the morphism p is an isomorphism, the map λ defines a distributive law and the induced operad $(\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda)$ is Koszul.*

Proof. We use the same filtration as in the proof of Theorem 8.6.4, together with the ideas of the proof of Theorem 4.2.4.

Step 1. Since the map $p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ is an isomorphism in weight 1, 2 and 3, the underlying $\mathbb{S}$-modules of the components of syzygy degree 0, 1 and 2 of the bar constructions $\mathrm{B}^\bullet(\mathscr{A} \vee_\lambda \mathscr{B})$ and $\mathrm{B}^\bullet(\mathscr{A} \vee_0 \mathscr{B})$ are isomorphic. Therefore the three first lines of the first term of the spectral sequence are equal to $\bigoplus_{p\in\mathbb{N}} E^0_{p\,p+q} \cong \mathrm{B}^{-\bullet}(\mathscr{A} \vee_0 \mathscr{B})$, for $p+q = -\bullet = 0, 1, 2$. It implies the isomorphisms of $\mathbb{S}$-modules $\bigoplus_{p\in\mathbb{N}} E^1_{p\,-p} \cong \mathscr{B}^{¡} \circ \mathscr{A}^{¡}$ and $\bigoplus_{p\in\mathbb{N}} E^1_{p\,-p} \cong 0$. Finally, the convergence of the spectral sequence, Theorem 1.5.1, gives the isomorphisms

$$H^0\big(\mathrm{B}^\bullet(\mathscr{A} \vee_\lambda \mathscr{B})\big) \cong (\mathscr{A} \vee_\lambda \mathscr{B})^{¡} \cong \bigoplus_{p\in\mathbb{N}} E^\infty_{p\,-p} \cong \bigoplus_{p\in\mathbb{N}} E^1_{p\,-p} \cong \mathscr{B}^{¡} \circ \mathscr{A}^{¡}.$$

Step 2. Let us first construct the induced decomposition coproduct on $\mathscr{B}^{¡} \circ \mathscr{A}^{¡}$ from the one of the cooperad $(\mathscr{A} \vee_0 \mathscr{B})^{¡}$. It is given by the following composite

$$\mathscr{B}^{¡} \circ \mathscr{A}^{¡} \xrightarrow{\Delta\circ\mathrm{Id}} \mathscr{B}^{¡} \circ \mathscr{B}^{¡} \circ \mathscr{A}^{¡} \twoheadrightarrow \mathscr{B}^{¡} \circ \big(\mathrm{I} \circ \mathscr{A}^{¡} \oplus \overline{\mathscr{B}}^{¡} \circ \mathscr{A}^{¡}\big) \xrightarrow{\mathrm{Id}\circ(\mathrm{Id}\circ\Delta+\mathrm{Id})}$$
$$\mathscr{B}^{¡} \circ \big(\mathrm{I} \circ \mathscr{A}^{¡} \circ \mathscr{A}^{¡} \oplus \overline{\mathscr{B}}^{¡} \circ \mathscr{A}^{¡}\big) \cong \mathscr{B}^{¡} \circ \big(\mathscr{A}^{¡} \circ \mathrm{I} \circ \mathscr{A}^{¡} \oplus \mathrm{I} \circ \overline{\mathscr{B}}^{¡} \circ \mathscr{A}^{¡}\big)$$
$$\to \big(\mathscr{B}^{¡} \circ \mathscr{A}^{¡}\big) \circ \big(\mathscr{B}^{¡} \circ \mathscr{A}^{¡}\big).$$

This is proved using the following commutative diagram of cooperads

$$\begin{array}{ccc} (\mathscr{A} \vee_0 \mathscr{B})^{¡} & \rightarrowtail & \mathscr{T}^c(sV \oplus sW) \\ \downarrow \cong & & \uparrow \\ \mathscr{B}^{¡} \circ \mathscr{A}^{¡} & \rightarrowtail & \mathscr{T}^c(sW) \circ \mathscr{T}^c(sV). \end{array}$$

We consider the same kind of number of inversions, changing this time B by Ω, $\mathscr{A}$ by $\mathscr{B}^{¡}$ and $\mathscr{B}$ by $\mathscr{A}^{¡}$. We consider the decreasing filtration induced by the number of inversions on the components of fixed total weight of the cobar construction $\Omega((\mathscr{A} \vee_\lambda \mathscr{B})^{¡}) \cong \Omega(\mathscr{B}^{¡} \circ \mathscr{A}^{¡})$, so that the induced spectral sequence converges. One concludes by the same arguments as before. The first term of the spectral sequence is equal to $(E^0, d^0) \cong \Omega((\mathscr{A} \vee_0 \mathscr{B})^{¡})$, since d^0 corresponds to the aforementioned cooperad structure on $\mathscr{B}^{¡} \circ \mathscr{A}^{¡}$. Since the operad $\mathscr{A} \vee_0 \mathscr{B}$ is Koszul, the spectral sequence collapses at rank 1 and

$$\begin{aligned}\mathscr{A} \vee_\lambda \mathscr{B} \cong H_0\big(\Omega\big((\mathscr{A} \vee_\lambda \mathscr{B})^{¡}\big)\big) &\cong \bigoplus_{p\in\mathbb{N}} E^{\infty}_{p\,-p} \\ &\cong \bigoplus_{p\in\mathbb{N}} E^{1}_{p\,-p} \cong H_0\big(\Omega(\mathscr{A} \vee_0 \mathscr{B})\big) \cong \mathscr{A} \circ \mathscr{B}. \qquad \square\end{aligned}$$

This methods gives an easy way to prove that the operad $\mathscr{A} \vee_\lambda \mathscr{B}$ is isomorphic to $(\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda, \iota_\Lambda)$.

8.6.7 Example: The Poisson Operad

In the case of the operad *Pois* introduced in Sect. 8.6.3, we leave it to the reader to check that $Com \circ Lie(4) \cong Pois(4)$. Hence the Diamond Lemma implies that the operad *Pois* is isomorphic to the operad defined on $Com \circ Lie$ by the distributive law.

8.6.8 Counter-Example

The following operad, introduced in [Mer04], gives an example of an operad defined by a rewriting rule λ but which does not induce a distributive law Λ.

We consider *noncommutative Poisson algebras*, whose definition is the same as Poisson algebras except that we do not require the associative product $*$ to be commutative. So this related operad is equal to $NCPois \cong Ass \vee_\lambda Lie$.

There are two ways to rewrite the critical monomial $[x * y, z * t]$ depending on which side we apply the relation first:

$$\begin{array}{ccc} & [x * y, z * t] & \\ \overset{=}{\swarrow} & & \overset{=}{\searrow} \\ [x, z * t] * y + x * [y, z * t] & & [x * y, z] * t + z * [x * y, t] \\ \downarrow & & \downarrow \\ \begin{array}{l}[x, z] * t * y + z * [x, t] * y \\ +x * [y, z] * t + x * z * [y, t]\end{array} & = & \begin{array}{l}[x, z] * y * t + x * [y, z] * t \\ +z * x * [y, t] + z * [x, t] * y.\end{array} \end{array}$$

This yields the relation

$$[x,z]*t*y+x*z*[y,t]=[x,z]*y*t+z*x*[y,t],$$

which produces a nontrivial element in the kernel of the map $(Ass\circ Lie)^{(3)} \to Ass \vee_\lambda Lie \cong NCPois$. Therefore the map λ does not induce a distributive law in this case and the $\mathbb{S}$-module $NCPois \cong Ass \vee_\lambda Lie$ is not isomorphic to $Ass \circ Lie$.

We refer the reader to Exercise 8.10.12 for another counterexample.

8.6.9 Distributive Laws and the Koszul Duals

When an operad is given by a distributive law, it is also the case for its Koszul dual operad.

Proposition 8.6.6. *Let $\mathscr{A} = \mathscr{P}(V,R)$ and $\mathscr{B} = \mathscr{P}(W,S)$ be two quadratic operads endowed with a rewriting rule $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ which induces a distributive law.*

The underlying $\mathbb{S}$-module of the Koszul dual cooperad of $\mathscr{A} \vee_\lambda \mathscr{B}$ is isomorphic to

$$(\mathscr{A} \vee_\lambda \mathscr{B})^{¡} \cong \mathscr{B}^{¡} \circ \mathscr{A}^{¡}.$$

Proof. It is a direct corollary of the preceding proof and Proposition 8.6.3:

$$\bigoplus_{p\in\mathbb{N}} E^{\infty}_{p,-p} \cong (\mathscr{A} \vee_\lambda \mathscr{B})^{¡} \cong \bigoplus_{p\in\mathbb{N}} E^{0}_{p,-p} \cong (\mathscr{A} \vee_0 \mathscr{B})^{¡} \cong \mathscr{B}^{¡} \circ \mathscr{A}^{¡}.$$

□

Proposition 8.6.7. *Let $\mathscr{A} := \mathscr{P}(V,R)$ and $\mathscr{B} := \mathscr{P}(W,S)$ be two finitely generated binary quadratic operads and let $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ be a rewriting rule. The Koszul dual operad of $\mathscr{A} \vee_\lambda \mathscr{B}$ has the following presentation:*

$$(\mathscr{A} \vee_\lambda \mathscr{B})^! \cong \mathscr{B}^! \vee_{{}^t\lambda} \mathscr{A}^!,$$

where ${}^t\lambda$ is the transpose of λ.

Proof. We apply the general formula of Theorem 7.6.2 for the Koszul dual operad of a finitely generated binary quadratic operad: $(\mathscr{A} \vee_\lambda \mathscr{B})^! \cong \mathscr{P}(V^\vee \oplus W^\vee, R^\perp \oplus D_\lambda^\perp \oplus S^\perp)$. The space $D_\lambda^\perp$ is isomorphic to the graph $D_{{}^t\lambda}$ of the transpose map ${}^t\lambda$, which concludes the proof. □

Theorem 8.6.8. *Under the hypotheses of the previous proposition, if, moreover, the rewriting rule λ induces a distributive law, then the transpose rewriting rule ${}^t\lambda$ induces a distributive law on the Koszul dual operad*

$$\mathscr{B}^! \circ \mathscr{A}^! \cong \mathscr{B}^! \vee_{{}^t\lambda} \mathscr{A}^!.$$

Proof. By the definition of the Koszul dual operad of Sect. 7.2.3 (linear dual of the Koszul dual cooperad up to suspension and signature representations), it is a direct corollary of the two previous propositions. □

8.7 Partition Poset Method

In this section, we construct a family of posets associated to a set operad. Since they are a generalization of the poset of partitions of a set, they are called *operadic partition posets*. The main theorem asserts that the induced linear operad is Koszul if and only if every poset of the family is Cohen–Macaulay, i.e. its homology is concentrated in top dimension. When it is the case, these top homology groups are isomorphic to the Koszul dual cooperad. Notice that the proof relies on the properties of the simplicial bar construction of operads.

On the one hand, the various combinatorial ways to prove that a poset is Cohen–Macaulay give simple ways of proving that an operad is Koszul. On the other hand, this result provides a means to compute the homology of partition type posets.

The construction of the operadic partition posets was first described by M. Méndez and J. Yang in [MY91]. The case of the operad *Com* and the properties of the simplicial bar construction of operads were studied by B. Fresse in [Fre04]. The Koszul–Cohen–Macaulay criterion was proved in [Val07a].

8.7.1 Partition Poset

We denote by $\underline{n}$ the set $\{1, \ldots, n\}$. Recall that a *partition* of the set $\underline{n}$ is a non-ordered collection of subsets $I_1, \ldots, I_k$, called *blocks*, which are nonempty, pairwise disjoint and whose union is equal to $\underline{n}$.

For any integer n, there is a partial order $\leqslant$ on the set of partitions of $\underline{n}$ defined by the *refinement of partitions*. Let π and ρ be two partitions of $\underline{n}$. We have $\pi \leqslant \rho$ when π is *finer* than ρ, that is when the blocks of π are contained in the blocks of ρ, for instance, $\{\{1\}, \{3\}, \{2, 4\}\} \leqslant \{\{1, 3\}, \{2, 4\}\}$. This partially ordered set is called the *partition poset* (or partition lattice) and denoted by $\Pi(n)$. See Fig. 8.10 for the example of $\Pi(3)$.

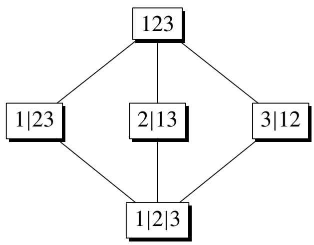


Fig. 8.10 Hasse diagram of $\Pi(3)$

The single block partition $\{\{1, \dots, n\}\}$ is the maximal partition of $\underline{n}$, and the collection $\{\{1\}, \dots, \{n\}\}$ is the minimal partition.

The set of partitions of $\underline{n}$ is equipped with a left action of the symmetric group $\mathbb{S}_n$. Let $\sigma : \underline{n} \to \underline{n}$ be a permutation, the image of the partition $\{\{i_1^1, \dots, i_{j_1}^1\}, \dots, \{i_1^k, \dots, i_{j_k}^k\}\}$ under σ is the partition

$$\big\{\{\sigma(i_1^1), \dots, \sigma(i_{j_1}^1)\}, \dots, \{\sigma(i_1^k), \dots, \sigma(i_{j_k}^k)\}\big\}.$$

8.7.2 Operadic Partitions

We introduce the notion of partitions enriched with operadic elements.

Let P be a set operad. Recall from Sects. 5.1.13, 5.3.5 and 5.6.1 that it can be either considered as a functor $\mathsf{Bij} \to \mathsf{Set}$ or as an $\mathbb{S}$-set, together with an associative and unital composition product. In one way, we use $\mathrm{P}(n) := \mathrm{P}(\underline{n})$ and, in the other way round, we use

$$\mathrm{P}(X) := \mathrm{P}(n) \times_{\mathbb{S}_n} \mathsf{Bij}(\underline{n}, X) = \left(\coprod_{\substack{f:\text{bijection} \\ \underline{n} \to X}} \mathrm{P}(n) \right)_{\mathbb{S}_n},$$

where the action of $\sigma \in \mathbb{S}_n$ on $(f; \mu)$ for $\mu \in M(n)$ is given by $(f\sigma; \mu^\sigma)$.

An element of $\mathsf{Bij}(\underline{n}, X)$ can be seen as an ordered sequence of elements of X, each element appearing only once. Therefore, an element in $\mathrm{P}(X)$ can be thought of as a sequence of elements of X indexed by an operation of $\mathrm{P}(n)$ with respect to the symmetry of this operation.

A P-*partition* of $\underline{n}$ is a set of elements called *blocks* $\{B_1, \dots, B_k\}$ such that each B_j belongs to $\mathrm{P}(I_j)$, for $\{I_1, \dots, I_k\}$ a partition of $\underline{n}$. A P-partition of $\underline{n}$ corresponds to a classical partition of $\underline{n}$ enriched by the operations of P.

EXAMPLES.

1. The operad *Com* for commutative algebras comes from a set operad which has only one element in arity $n \geq 1$, with trivial action of the symmetric group. Therefore, $\mathit{Com}(X)$ has only one element, which corresponds to the set X itself. As a consequence, a *Com*-partition of $\underline{n}$ is a classical partition of $\underline{n}$.
2. The set operad *Perm* is defined by the set $\mathit{Perm}(n) := \{e_1^n, \dots, e_n^n\}$ of n elements in arity n, where the action of the symmetric group is $e_k^n \cdot \sigma := e_{\sigma^{-1}(k)}^n$, see Sect. 13.4.6. The operadic composition is given by

$$e_k^n \circ_i e_l^m := \begin{cases} e_{k+m-1}^{n+m-1} & \text{for } i < k, \\ e_{k+l-1}^{n+m-1} & \text{for } i = k, \\ e_k^{n+m-1} & \text{for } i > k. \end{cases}$$

We leave it to the reader as a good exercise to check that *Perm* is an operad. As a species, *Perm*(X) can represented by the set of *pointed sets* associated to $X = \{x_1, \dots, x_n\}$:

$$\big\{\{\bar{x}_1, \dots, x_n\}, \{x_1, \bar{x}_2, \dots, x_n\}, \dots, \{x_1, \dots, \bar{x}_n\}\big\}.$$

The element e_k^n "singles out the kth element of X". Finally a *Perm*-partition is a *pointed partition* like $1\bar{3}4|\bar{2}6|57\bar{8}$.

The natural action of the symmetric group on $\underline{n}$ induces a right action of $\mathbb{S}_n$ on the set of P-partitions of $\underline{n}$. For instance,

$$\{1\bar{3}4|\bar{2}6|57\bar{8}\}^{(123)} = \bar{2}34|\bar{1}6|57\bar{8}.$$

8.7.3 Operadic Partition Poset

We now define a partial order on the set of operadic partitions as follows. We consider the following natural map on P-partitions. Let $\{B_1, \dots, B_t\}$ be a P-partition of a set X associated to a partition $\{I_1, \dots, I_t\}$. Each element B_j in $\mathrm{P}(I_j)$ can be represented as the class of an element $[\nu_j \times (x_1^j, \dots, x_{i_j}^j)]$, where $\nu_j \in \mathrm{P}(i_j)$ and $I_j = \{x_1^j, \dots, x_{i_j}^j\}$.

Lemma 8.7.1. *The map $\tilde{\gamma}$ given by the formula*

$$\tilde{\gamma} : \mathrm{P}(t) \times \big(\mathrm{P}(I_1) \times \cdots \times \mathrm{P}(I_t)\big) \to \mathrm{P}(I),$$
$$\nu \times (B_1, \dots, B_t) \mapsto \big[\gamma(\nu; \nu_1, \dots, \nu_t) \times \quad \big(x_1^1, \dots, x_{i_t}^t\big)\big],$$

is well defined and equivariant under the action of $\mathbb{S}_t$.

Proof. It is a direct consequence of the equivariance, under the action of the symmetric groups, in the definition of the composition of a set operad, see Sect. 5.3.5. □

Let $\pi = \{B_1, \dots, B_r\}$ and $\rho = \{C_1, \dots, C_s\}$ be two P-partitions of $\underline{n}$ associated to two partitions $\{I_1, \dots, I_r\}$ and $\{J_1, \dots, J_s\}$ of $\underline{n}$. The P-partition π is a *refinement* of ρ if, for any $k \in \{1, \dots, s\}$, there exist $\{p_1, \dots, p_t\} \subset \{1, \dots, r\}$ such that $\{I_{p_1}, \dots, I_{p_t}\}$ is a partition of J_k and if there exists an element ν in $\mathrm{P}(t)$ such that $C_k = \tilde{\gamma}(\nu \times (B_{p_1}, \dots, B_{p_t}))$. We denote this relation by $\pi \leqslant \rho$.

Proposition 8.7.2. *When* P *is a reduced set operad,* $\mathrm{P}(0) = \emptyset$, *such that* $\mathrm{P}(1) = \{\mathrm{id}\}$, *the relation $\leqslant$ defines a partial order on the set of* P*-partitions.*

Proof. The symmetry of $\leqslant$ comes from the unit of the operad, the reflexivity comes from $\mathrm{P}(1) = \{\mathrm{id}\}$ and the transitivity comes from the associativity of the operad. □

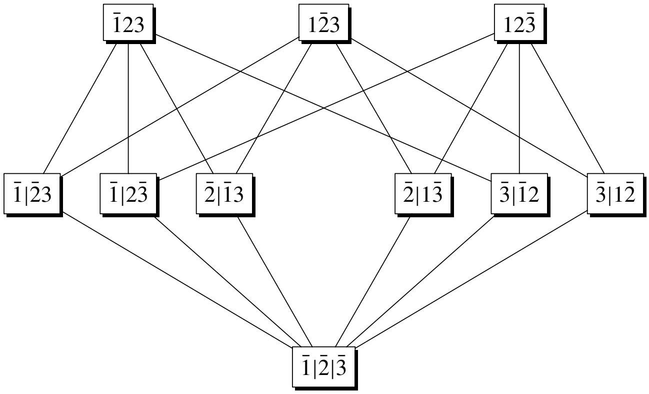


Fig. 8.11 Hasse diagram of $\Pi_{Perm}(3)$

We call this poset the P-*partition poset* associated to the operad P and we denote it by $\Pi_{\mathrm{P}}(n)$, for any integer $n \geq 1$.

EXAMPLES.

1. In the case of the operad *Com*, the *Com*-partition poset is exactly the classical partition poset.
2. In the case of the operad *Perm*, a pointed partition π is less than a pointed partition ρ if the underlying partition of π refines that of ρ and if the pointed elements of ρ belong to the pointed elements of π. See Fig. 8.11 for the example of $\Pi_{Perm}(3)$.

Since the set P(1) is reduced to the identity, the poset $\Pi_{\mathrm{P}}(n)$ has only one minimal element corresponding to the partition $\{\{1\}, \ldots, \{n\}\}$, where $\{i\}$ represents the unique element of $\mathrm{P}(\{i\})$. Following the classical notations, we denote this element by $\hat{0}$. The set of maximal elements is $\mathrm{P}(\underline{n}) \cong \mathrm{P}(n)$. Hence, the number of maximal elements of $\Pi_{\mathrm{P}}(n)$ is equal to the number of elements of $\mathrm{P}(n)$.

8.7.4 Graded Posets

We denote by $\mathrm{Min}(\Pi)$ and $\mathrm{Max}(\Pi)$ the sets of minimal and maximal elements of Π. When $\mathrm{Min}(\Pi)$ and $\mathrm{Max}(\Pi)$ have only one element, the poset is said to be *bounded*. In this case, we denote the unique element of $\mathrm{Min}(\Pi)$ by $\hat{0}$ and the unique element of $\mathrm{Max}(\Pi)$ by $\hat{1}$. For a pair $x \leqslant y$ in Π, we consider the *closed interval* $\{z \in \Pi \mid x \leqslant z \leqslant y\}$, denoted by $[x, y]$, and the *open interval* $\{z \in \Pi \mid x < z < y\}$, denoted by (x, y). For any $\alpha \in \mathrm{Min}(\Pi)$ and any $\omega \in \mathrm{Max}(\Pi)$, the closed interval $[\alpha, \omega]$ is a bounded poset. If Π is a bounded poset, the *proper part* $\overline{\Pi}$ of Π is the open interval $(\hat{0}, \hat{1})$.

For two elements $x < y$, we say that y *covers* x if there is no z such that $x < z < y$. The *covering relation* is denoted by $x \prec y$. A *chain* $\lambda_0 < \lambda_1 < \cdots < \lambda_l$ is a totally ordered sequence of elements of Π. Its *length* is equal to l. A *maximal chain* between x and y, is a chain $x = \lambda_0 \prec \lambda_1 \prec \cdots \prec \lambda_l = y$ which cannot be lengthened. A *maximal chain of* Π is a maximal chain between a minimal element of Π and a maximal element of Π. A poset is *pure* if, for any $x \leqslant y$, all maximal chains between x and y have the same length. If a poset is both bounded and pure, it is called a *graded* poset. For example, the partition poset of Sect. 8.7.1 is graded.

For more details on posets, we refer the reader to Chap. 3 of [Sta97a].

If the operad P is quadratic and generated by a homogeneous $\mathbb{S}$-set E concentrated in arity k, that is $E_n = \emptyset$ for $n \neq k$, then we have $\mathrm{P}(n) = \emptyset$ for $n \neq i(k-1)+1$ with $i \in \mathbb{N}$. Therefore, the P-partitions have restricted block size. The possible lengths for the blocks are $i(k-1)+1$ with $i \in \mathbb{N}$.

Proposition 8.7.3. *Let* P *be a set-theoretic quadratic operad generated by a homogeneous* $\mathbb{S}$*-set* E *concentrated in arity* k, *with* $k \geq 2$, *then all the maximal chains of* $\Pi_{\mathcal{P}}$ *have the same length.*

For any $\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(n)) = \mathrm{P}(\underline{n})$, *the subposets* $[\hat{0}, \omega]$ *are graded posets.*

Proof. If the operad P is generated by operations of arity k with $k \geq 2$, the set P(1) is reduced to the identity operations and the P-partition poset is well defined. Since P is generated only by operations of arity k, every block of size $i(k-1)+1$ can be refined if and only if $i > 1$.

The length of maximal chains between $\hat{0}$ and ω is equal to $i+1$ if $n = i(k-1)+1$. Hence, each closed interval of the form $[\hat{0}, \omega]$, for $\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(n)) = \mathrm{P}(\underline{n})$ is bounded and pure. It is also graded by definition. □

8.7.5 Order Complex

We consider the set of chains $\lambda_0 < \lambda_1 < \cdots < \lambda_l$ of a poset $(\Pi, \leqslant)$ such that $\lambda_0 \in \mathrm{Min}(\Pi)$ and $\lambda_l \in \mathrm{Max}(\Pi)$. This set is denoted by $\Delta_\bullet(\Pi)$, or simply by $\Delta(\Pi)$. More precisely, a chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l$ of length l belongs to $\Delta_l(\Pi)$.

The set $\Delta(\Pi)$ is equipped with the following face maps. For $0 < i < l$, the face map $d_i : \Delta_l(\Pi) \to \Delta_{l-1}(\Pi)$ is given by the omission of λ_i

$$d_i(\lambda_0 < \lambda_1 < \cdots < \lambda_l) := \lambda_0 < \lambda_1 < \cdots < \lambda_{i-1} < \lambda_{i+1} < \cdots < \lambda_l.$$

On the free module $\mathbb{K}[\Delta(\Pi)]$, we consider the induced linear maps still denoted d_i and we define the maps $d_0 = d_l = 0$ by convention. The module $\mathbb{K}[\Delta(\Pi)]$ (resp. the set $\Delta(\Pi)$) is a presimplicial module (resp. a presimplicial set), that is $d_i \circ d_j = d_{j-1} \circ d_i$, for $i < j$. These relations ensure that $d := \sum_{0 \leq i \leq l} (-1)^i d_i$ satisfies $d^2 = 0$. The chain complex $(\mathbb{K}[\Delta_\bullet(\Pi)], d)$ is called the *order complex* of Π. By definition, the *homology of a poset* $(\Pi, \leqslant)$ is the homology of its order complex, denoted by $H(\Pi)$.

The *reduced* homology of a poset is defined as follows. We denote by $\tilde{\Delta}_l(\Pi)$ the set of chains $\lambda_0 < \lambda_1 < \cdots < \lambda_l$, with no restriction on λ_0 and λ_l. The face maps d_i are defined by the omission of λ_i, for $0 \le i \le l$. By convention, this complex is augmented by $\tilde{\Delta}_{-1}(\Pi) = \{\emptyset\}$, that is $\mathbb{K}[\tilde{\Delta}_{-1}(\Pi)] = \mathbb{K}$. The associated homology groups are denoted by $\widetilde{H}(\Pi)$.

The relation between the two definitions is given by the following formula

$$\Delta_l(\Pi) = \bigsqcup_{(\alpha,\omega)\in \mathrm{Min}(\Pi)\times\mathrm{Max}(\Pi)} \tilde{\Delta}_{l-2}\big((\alpha,\omega)\big),$$

which induces a canonical isomorphism of presimplicial complexes. Therefore, we have

$$H_l(\Pi) = \bigoplus_{(\alpha,\omega)\in \mathrm{Min}(\Pi)\times\mathrm{Max}(\Pi)} \widetilde{H}_{l-2}\big((\alpha,\omega)\big).$$

If Π is bounded, its homology is equal to the reduced homology of its proper part, up to a degree shift.

The action of a group G on a poset $(\Pi, \leqslant)$ is *compatible* with the partial order $\leqslant$ if for every $g \in G$ and for every pair $x \leqslant y$, we still have $g \cdot x \leqslant g \cdot y$. In this case, the modules $\mathbb{K}[\Delta_l(\Pi)]$ are G-modules. Since, the chain map commutes with the action of G, the homology groups $H(\Pi)$ are also G-modules.

For example, the action of the symmetric group on the operadic partition posets is compatible with the partial order. So the module $\mathbb{K}[\Delta_\bullet(\Pi_{\mathrm{P}}(n))]$ is an $\mathbb{S}_n$-presimplicial module and $H_\bullet(\Pi_{\mathrm{P}}(n))$ is an $\mathbb{S}_n$-module.

8.7.6 Cohen–Macaulay Poset

Let Π be a graded poset. It is said to be *Cohen–Macaulay* over $\mathbb{K}$ if the homology of each interval is concentrated in top dimension, i.e. for every $x \leqslant y$, if m is the length of maximal chains between x and y, we have

$$H_l\big([x,y]\big) = \widetilde{H}_{l-2}\big((x,y)\big) = 0, \quad \text{for } l \neq m.$$

There are many sufficient combinatorial conditions for a poset to be Cohen–Macaulay, e.g. modular, distributive, shellable. For a comprehensive survey on these notions, we refer the reader to the article by A. Björner, A.M. Garsia and R.P. Stanley [BGS82]. We recall the one used in the sequel.

A pure poset is *semi-modular* if for every triple x, y and t such that x and y cover t, there exists z covering both x and y. A *totally semi-modular* poset is a pure poset such that each interval is semi-modular.

Proposition 8.7.4. [Bac76, Far79] *A totally semi-modular poset is Cohen–Macaulay.*

For instance, the partition posets are totally semi-modular (it is actually much more). We also leave it to the reader to prove that the pointed partition posets, coming from the operad *Perm*, are semi-modular.

8.7.7 Koszul–Cohen–Macaulay Criterion

The main theorem of this section requires the following assumption. For any element $(\nu_1, \ldots, \nu_t)$ of $\mathrm{P}(i_1) \times \cdots \times \mathrm{P}(i_t)$, we denote by $\gamma_{\nu_1,\ldots,\nu_t}$ the following map defined by the composition of the set operad P:

$$\gamma_{\nu_1,\ldots,\nu_t} : \mathrm{P}(t) \to \mathrm{P}(i_1 + \cdots + i_t),$$
$$\nu \mapsto \gamma(\nu; \nu_1, \ldots, \nu_t).$$

A set operad P is called a *basic-set operad* if the maps $\gamma_{\nu_1,\ldots,\nu_t}$ are injective, for any $(\nu_1, \ldots, \nu_t)$ in $\mathrm{P}(i_1) \times \cdots \times \mathrm{P}(i_t)$. The operads *Com* and *Perm* are examples of basic-set operads.

Theorem 8.7.5 (Koszul–Cohen–Macaulay criterion). *Let* P *be a quadratic basic-set operad generated by a homogeneous* $\mathbb{S}$*-set concentrated in arity* k, *with* $k \geq 2$.

1. *The linear operad* $\mathscr{P}(n) = \mathbb{K}[\mathrm{P}(n)]$ *is a Koszul operad if and only if, for every* $n \geq 1$ *and every* $\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(n))$, *the interval* $[\hat{0}, \omega]$ *is Cohen–Macaulay.*
2. *The top homology groups are isomorphic to the Koszul dual cooperad*

$$H_{\mathrm{top}}\big(\Pi_{\mathrm{P}}(n)\big) \cong \mathscr{P}^{¡}(n).$$

We postpone the proof until after the next section about the simplicial bar construction of operads, on which it relies.

Example. The aforementioned theorem provides a proof that the operads *Com* and *Perm* are Koszul. On the level of poset homology, it shows that the homology groups of the partition posets and of the pointed partition posets are concentrated in top dimension. The last point of Theorem 8.7.5 shows that they are respectively isomorphic to

$$H_{n-1}\big(\Pi_{Com}(n)\big) \cong Lie(n)^* \otimes \mathrm{sgn}_{\mathbb{S}_n} \quad \text{and} \quad H_{n-1}\big(\Pi_{Perm}(n)\big) \cong preLie(n)^* \otimes \mathrm{sgn}_{\mathbb{S}_n}$$

as $\mathbb{S}_n$-modules. The first result has already been proved with more classical methods. (We refer the reader to the prolog of [Fre04] for complete reference.) The second result was proved in [Val07a] using this operadic method. We refer the reader to Sect. 13.4 for more details about the operad *preLie* encoding pre-Lie algebras. For instance, this operad admits a basis labeled by rooted trees, which, in turn, induces a basis for the homology groups of the pointed partition poset.

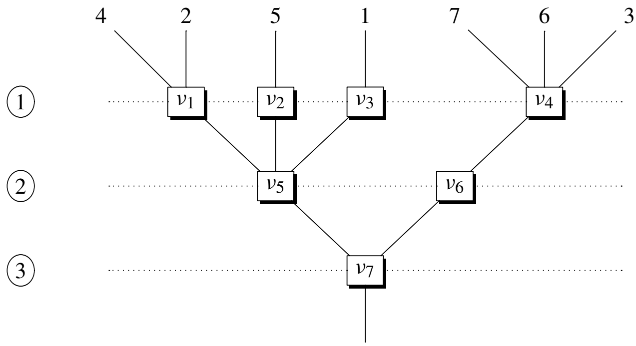


Fig. 8.12 An example of a 3-leveled tree with vertices indexed by elements of P

8.7.8 Simplicial and Normalized Bar Construction of Operads

In order to prove the Koszul–Cohen–Macaulay criterion, we introduce the simplicial and normalized bar construction of an operad.

Let $(\mathscr{P}, \gamma, \iota)$ be an operad. Its *simplicial bar construction* is the simplicial $\mathbb{S}$-module defined by $(\mathrm{C}\mathscr{P})_l := \mathscr{P}^{\circ l}$, equipped with the face and degeneracy maps

$$d_i = \mathrm{Id}^{\circ(i-1)} \circ \gamma \circ \mathrm{Id}^{\circ(l-i-1)} : \mathscr{P}^{\circ l} \to \mathscr{P}^{\circ(l-1)},$$

for $1 \leq l \leq l-1$, $d_0 = d_l = 0$, and by the face maps

$$s_j = \mathrm{Id}^{\circ j} \circ \iota \circ \mathrm{Id}^{\circ(l-j)} : \mathscr{P}^{\circ l} \to \mathscr{P}^{\circ(l+1)},$$

for $0 \leq j \leq l$.

Such a definition CP also holds for a set operad P, except for the face maps d_0 and d_l. The set $(\mathrm{CP})_l(n)$, which provides a basis for $(\mathrm{C}\mathscr{P})_l(n)$, is made up of l-leveled trees where the vertices are indexed by operations of P and where the leaves are labeled by $1, \ldots, n$. See Fig. 8.12 for an example.

We consider the *normalized bar construction* $\mathrm{N}\mathscr{P}$, given as usual by the quotient of the simplicial bar construction under the images of the degeneracy maps. Recall that the two chains complexes associated to the simplicial and to the normalized bar construction respectively are quasi-isomorphic.

Let $(\mathrm{NP})_l(n)$ be the subset of $(\mathrm{CP})_l(n)$ made up of l-leveled trees with at least one nontrivial operation on each level. When P is an augmented set operad, it means that there is at least one nontrivial operation on each level. This set is stable under the face maps and is a presimplicial set.

Lemma 8.7.6. *The presimplicial* $\mathbb{S}$*-set* NP *provides a basis for the normalized bar construction* $\mathrm{N}\mathscr{P}$, *where* $\mathscr{P}(n) = \mathbb{K}[\mathrm{P}(n)]$ *is the linear operad associated to* P.

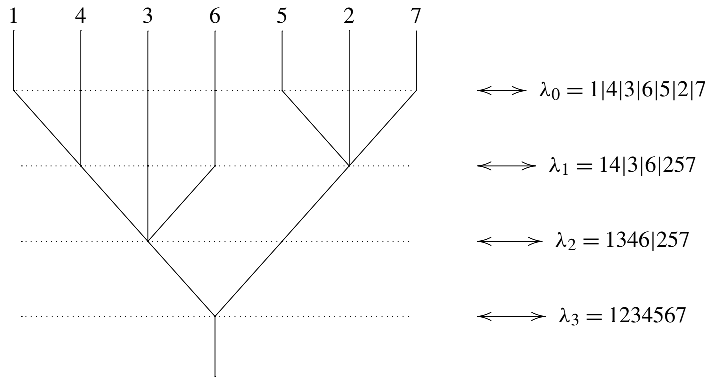


Fig. 8.13 An example of the image of Ψ in the case of the operad *Com*

8.7.9 Normalized Bar Construction and Order Complex

Theorem 8.7.7. *Let* P *be an augmented basic-set operad. For any* $n \geq 1$, *the presimplicial* $\mathbb{S}_n$*-set* $\Delta(\Pi_{\mathrm{P}}(n))$ (*resp. the presimplicial* $\mathbb{S}_n$*-module* $\mathbb{K}[\Delta(\Pi_{\mathrm{P}}(n))]$) *is in bijection with* (*resp. is isomorphic to*) *the presimplicial* $\mathbb{S}_n$*-set* $\mathrm{N}(\mathrm{P})(n)$ (*resp. the normalized bar construction* $\mathrm{N}(\mathscr{P})(n)$).

Proof. We define a bijection Ψ between $\mathrm{N}_l(\mathrm{P})(\underline{n})$ and $\Delta_l(\Pi_{\mathrm{P}}(n))$ as follows. Let $\mathbb{T}$ be a non-planar tree with l levels and n leaves whose vertices are indexed by elements of P. To such a tree, we associate a maximal chain of P-partitions of $\underline{n}$: we cut the tree $\mathbb{T}$ along the ith level and look upward. We get t indexed and labeled subtrees. By composing the operations indexing the vertices along the scheme given by the subtree, each of them induces an element of $\mathrm{P}(I_j)$, where $\{I_1, \ldots, I_t\}$ is a partition of $\underline{n}$. For every $0 \leq i \leq l$, the union of these blocks forms a P-partition λ_i of $\underline{n}$. Figure 8.13 shows an example in the case of the operad *Com*.

When $\mathbb{T}$ is a tree of $\mathrm{N}(\mathrm{P})(\underline{n})$, that is with at least one nontrivial operation on each level, λ_i is a strict refinement of λ_{i+1}. Since $\lambda_0 = \hat{0}$ and $\lambda_l \in \mathrm{P}(\underline{n})$, the chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l$ is maximal. The image of the tree $\mathbb{T}$ under Ψ is this maximal chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l$.

The surjectivity of the map Ψ comes from the definition of the partial order between the P-partitions. Since P is a basic-set operad, the injectivity of the maps $\gamma_{\nu_1,\ldots,\nu_t}$ induces the injectivity of Ψ. Therefore, Ψ is a bijection.

Composing the ith and the $(i+1)$th levels of the tree $\mathbb{T}$ corresponds, via Ψ, to removing the ith partition of the chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l$. Therefore, the map Ψ commutes with the face maps. Moreover Ψ preserves the action of the symmetric group $\mathbb{S}_n$. □

8.7.10 Bar Construction and Normalized Bar Construction

In [Fre04], Fresse defined the following morphism $\mathscr{L} : \mathrm{B}\mathscr{P} \to \mathrm{N}\mathscr{P}$ of dg $\mathbb{S}$-modules between the bar construction and the normalized bar construction of an operad. Recall that an element of $\mathrm{B}\mathscr{P}$ is a tree $\mathbb{T}$ with l vertices labeled by elements of $s\overline{\mathscr{P}}$. Its image under the map $\mathscr{L}$ is given by the sum over all l-leveled trees with one and only one nontrivial vertex on each level and which give $\mathbb{T}$ after forgetting the levels (together with proper desuspension and sign convention). We consider here the bar construction as a chain complex with the homological degree given by the number of vertices of the underlying tree.

Proposition 8.7.8. *For any reduced operad $\mathscr{P}$, such that $\mathscr{P}(1) = \mathbb{K}\,\mathrm{id}$, the levelization morphism $\mathscr{L} : \mathrm{B}\mathscr{P} \to \mathrm{N}\mathscr{P}$ is a quasi-isomorphism of dg $\mathbb{S}$-modules.*

Proof. The idea of the proof is to apply an adequate version of the Comparison Lemma 6.7.1 to the quasi-isomorphism $\mathscr{L} \circ \mathrm{Id} : \mathrm{B}\mathscr{P} \circ \mathscr{P} \to \mathrm{N}\mathscr{P} \circ \mathscr{P}$. The normalized bar construction is not a cooperad but a right $\mathscr{P}$-comodule. We refer the reader to Sect. 4.7 of [Fre04] for the details. □

8.7.11 Proof of the Koszul–Cohen–Macaulay Criterion

Lemma 8.7.9. *Let* P *be a quadratic basic-set operad generated by a homogeneous $\mathbb{S}$-set E concentrated in arity k with $k \geq 2$.*

1. *The operad $\mathscr{P}$ is Koszul if and only if $H_l(\Pi_{\mathrm{P}}(m(k-1)+1)) = 0$ for $l \neq m$.*
2. *We have $H_m(\Pi_{\mathrm{P}}(m(k-1)+1)) \cong (\mathscr{P}^{¡})^{(m)}$.*

Proof. Since the operad is quadratic, it is weight graded and both the bar and the simplicial bar constructions split with respect to this extra grading denoted by (m). In both cases, the part of weight (m) is equal to the part of arity $m(k-1)+1$. The levelization morphism $\mathscr{L}$ of Sect. 8.7.10 preserves this grading. Therefore $\mathscr{L}^{(m)} : (\mathrm{B}\mathscr{P}(m(k-1)+1))^{(m)} \to \mathrm{N}\mathscr{P}(m(k-1)+1)^{(m)}$ is a quasi-isomorphism, for every $m \geq 0$, by Proposition 8.7.8. Recall from the Koszul criterion (Theorem 7.4.2) that the operad $\mathscr{P}$ is Koszul if and only if $H_l((\mathrm{B}_\bullet\mathscr{P}(m(k-1)+1))^{(m)}) = 0$ for $l \neq m$. By the preceding quasi-isomorphism, it is equivalent to asking that $H_l((\mathrm{N}_\bullet\mathscr{P}(m(k-1)+1))^{(m)}) = 0$ for $l \neq m$. And by Theorem 8.7.7, it is equivalent to $H_l(\Pi_{\mathrm{P}}(m(k-1)+1)) = 0$ for $l \neq m$.

The levelization quasi-isomorphism $\mathscr{L}^{(m)}$ and the isomorphism of Theorem 8.7.7 show that $H_m(\Pi_{\mathrm{P}}(m(k-1)+1)) \cong H_m((\mathrm{B}_\bullet\mathscr{P}(m(k-1)+1))^{(m)})$. Recall from Proposition 7.3.1 that the top homology group of $(\mathrm{B}_\bullet\mathscr{P}(m(k-1)+1))^{(m)}$ is equal to the Koszul dual cooperad $H^0((\mathrm{B}^\bullet\mathscr{P})^{(m)}) = (\mathscr{P}^{¡})^{(m)}$. □

This lemma implies the equivalence between Koszul set operads and Cohen–Macaulay posets.

Proof of Theorem 8.7.5. ($\Rightarrow$) Proposition 8.7.3 shows that Π_{P} is pure. Hence each interval $[\hat{0}, \omega]$ is graded.

If the operad $\mathscr{P}$ is Koszul, then Lemma 8.7.9 implies that the homology of each poset $\Pi_{\mathrm{P}}(m(k-1)+1)$ is concentrated in top dimension m. Since

$$H_l\big(\Pi_{\mathrm{P}}\big(m(k-1)+1\big)\big) = \bigoplus_{\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(m(k-1)+1)} \widetilde{H}_{l-2}\big((\hat{0}, \omega)\big),$$

we have $H_l([\hat{0}, \omega]) = \widetilde{H}_{l-2}((\hat{0}, \omega)) = 0$ for every $\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(m(k-1)+1))$ and every $l \neq m$. Let $x \leqslant y$ be two elements of $\Pi_{\mathrm{P}}(m(t-1)+1)$. We denote the P-partition x by $\{B_1, \dots, B_r\}$ and the P-partition y by $\{C_1, \dots, C_s\}$. Each $C_t \in \mathrm{P}(I_t)$ is refined by some B_p. For $1 \le t \le s$, we consider the subposet $[x_t, y_t]$ of $\Pi_{\mathrm{P}}(I_t)$, where $y_t = C_t$ and y_t the corresponding set of B_p. There exists one $\omega_t \in \mathrm{Max}(\Pi_{\mathrm{P}}(|x_t|))$ such that the poset $[x_t, y_t]$ is isomorphic to $[\hat{0}, \omega]$, which is a subposet of $\Pi_{\mathrm{P}}(|x_t|)$. (The notation $|x_t|$ stands for the number of B_p in x_t.) This decomposition gives, with Künneth Theorem, the following formula

$$\widetilde{H}_{l-1}\big((x, y)\big) \cong \bigoplus_{l_1+\cdots+l_s=l} \bigotimes_{t=1}^{s} \widetilde{H}_{l_t-1}\big((x_t, y_t)\big) \cong \bigoplus_{l_1+\cdots+l_s=l} \bigotimes_{t=1}^{s} \widetilde{H}_{l_t-1}\big((\hat{0}, \omega)\big).$$

(We can apply Künneth formula since we are working with chain complexes of free modules over an hereditary ring $\mathbb{K}$. The extra Tor terms in Künneth formula come from homology groups of lower dimension which are null.) If we define m_t by $|x_t| = m_t(k-1)+1$, the homology groups $\widetilde{H}_{l_t-1}((\hat{0}, \omega))$ vanish for $l_t \neq m_t - 1$. Therefore, if l is different from $\sum_{t=1}^{s}(m_t - 1)$, we have $\widetilde{H}_{l-1}((x, y)) = 0$. Since the length of maximal chains between x and y is equal to $m = \sum_{t=1}^{s}(m_t - 1) + 1$, see Proposition 8.7.3, the homology of the interval $[x, y]$ is concentrated in top dimension.

($\Leftarrow$) Conversely, if the poset Π_{P} is Cohen–Macaulay over the ring $\mathbb{K}$, we have $\widetilde{H}_{l-2}((\hat{0}, \omega)) = 0$, for every $m \geqslant 1$, $l \neq m$, and every $\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(m(k-1)+1))$. Therefore, we get

$$H_l\big(\Pi_{\mathrm{P}}\big(m(k-1)+1\big)\big) = \bigoplus_{\omega \in \mathrm{Max}(\Pi_{\mathrm{P}}(m(k-1)+1))} \widetilde{H}_{l-2}\big((\hat{0}, \omega)\big) = 0,$$

for $l \neq m$. Finally we conclude by (1) of Lemma 8.7.9. □

Recall that one can generalize the Koszul duality of operads over a Dedekind ring, see [Fre04] for more details. The proof of the Koszul–Cohen–Macaulay criterion also holds in that case. So it provides a method for proving that an operad is Koszul over Dedekind rings, not only fields.

8.7.12 Applications

- ⋄ The triptych made up of the operad *Com*, the partition poset and the operad *Lie* plays a fundamental role in Goodwillie calculus in homotopy theory; we refer the reader to G. Arone and M. Mahowald [AM99] and to M. Ching [Chi05].
- ⋄ M. Mendez in his thesis [Mén89] and F. Chapoton and M. Livernet in [CL07] explained (independently) how to associate incidence Hopf algebras to operadic partition posets. In the particular case of the operad *NAP* of [Liv06], these two last authors recover Connes–Kreimer Hopf algebra involved in renormalization theory [CK98].
- ⋄ In [DK07], V. Dotsenko and A. Khoroshkin introduced the operad encoding a pair of compatible Lie brackets, which is an algebraic structure related to integrable Hamiltonian equations, like the KdV-equations. The Koszul dual operad is a basic-set operad, which was shown to be Koszul by H. Strohmayer in [Str08] using the present poset method. Notice that the associated posets are not totally semimodular this time, but only CL-shellable, that is yet another sufficient condition for being Cohen–Macaulay. Furthermore, he applied this result in the context of bi-Hamiltonian geometry in [Str10].

8.8 Manin Products

In this section, we extend the definition of Manin black and white products for quadratic algebras (Sect. 4.5) to operads. The conceptual approach followed here allows us to define the Manin products of pairs of operads given by any presentation, not necessarily quadratic. We explain how to compute some black products of operads. Finally, we study the behavior of Manin products of operads under Koszul duality; for instance, we state in the operadic context, the adjunction property between the black product and the white product.

Manin products for operads were first defined in the binary quadratic case by V. Ginzburg and M.M. Kapranov in [GK94, GK95b]. The more conceptual and general definition given here comes from [Val08].

8.8.1 White Product for Operads

Let V and W be two $\mathbb{S}$-modules. Let us denote by $i_V : V \rightarrowtail \mathscr{T}(V)$ the canonical inclusion of V into the free operad on V. There is a natural map $i_V \underset{\mathrm{H}}{\otimes} i_W : V \underset{\mathrm{H}}{\otimes} W \to \mathscr{T}(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}(W)$. Recall from Sect. 5.3.2 that the Hadamard product of two operads is again an operad. So the Hadamard product $\mathscr{T}(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}(W)$ is an operad. By the universal property of the free operad, there exists a (unique) morphism of operads Φ completing the following commutative diagram

$$\begin{array}{ccc} V \underset{H}{\otimes} W & \xrightarrow{i_{V \underset{H}{\otimes} W}} & \mathscr{T}(V \underset{H}{\otimes} W) \\ & \searrow^{i_V \underset{H}{\otimes} i_W} & \Big\downarrow{\exists!\,\Phi} \\ & & \mathscr{T}(V) \underset{H}{\otimes} \mathscr{T}(W). \end{array}$$

Let $\mathscr{P} = \mathscr{T}(V)/(R)$ and $\mathscr{Q} = \mathscr{T}(W)/(S)$ be two operads defined by generators and relations (not necessarily quadratic). We denote by $\pi_{\mathscr{P}} : \mathscr{T}(V) \twoheadrightarrow \mathscr{P}$ and by $\pi_{\mathscr{Q}} : \mathscr{T}(W) \twoheadrightarrow \mathscr{Q}$ the respective projections. The composite $(\pi_{\mathscr{P}} \underset{H}{\otimes} \pi_{\mathscr{Q}}) \circ \Phi$ is a morphism of operads. Hence its kernel is an operadic ideal and it admits the following factorization

$$\begin{array}{ccccc} \mathscr{T}(V \underset{H}{\otimes} W) & \xrightarrow{\Phi} & \mathscr{T}(V) \underset{H}{\otimes} \mathscr{T}(W) & \xrightarrow{\pi_{\mathscr{P}} \underset{H}{\otimes} \pi_{\mathscr{Q}}} & \mathscr{P} \underset{H}{\otimes} \mathscr{Q} \\ & \searrow & & \nearrow^{\overline{\Phi}} & \\ & & \mathscr{T}(V \underset{H}{\otimes} W)/\operatorname{Ker}((\pi_{\mathscr{P}} \underset{H}{\otimes} \pi_{\mathscr{Q}}) \circ \Phi). & & \end{array}$$

A direct inspection shows that the kernel of $(\pi_{\mathscr{P}} \underset{H}{\otimes} \pi_{\mathscr{Q}}) \circ \Phi$ is the ideal generated by $\Phi^{-1}(R \underset{H}{\otimes} \mathscr{T}(W) + \mathscr{T}(V) \underset{H}{\otimes} S)$, that is

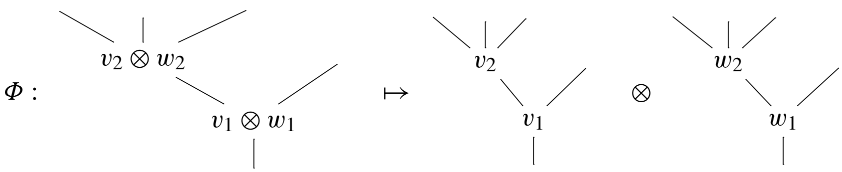
$$\operatorname{Ker}\Big(\Big(\pi_{\mathscr{P}} \underset{H}{\otimes} \pi_{\mathscr{Q}}\Big) \circ \Phi\Big) = \Big(\Phi^{-1}\Big(R \underset{H}{\otimes} \mathscr{T}(W) + \mathscr{T}(V) \underset{H}{\otimes} S\Big)\Big).$$

We define the *Manin white product of $\mathscr{P}$ and $\mathscr{Q}$* as being the quotient operad

$$\mathscr{P} \bigcirc \mathscr{Q} := \mathscr{T}\Big(V \underset{H}{\otimes} W\Big)/\Big(\Phi^{-1}\Big(R \underset{H}{\otimes} \mathscr{T}(W) + \mathscr{T}(V) \underset{H}{\otimes} S\Big)\Big).$$

By definition, the white product comes equipped with a natural monomorphism of operads $\overline{\Phi} : \mathscr{P} \bigcirc \mathscr{Q} \rightarrowtail \mathscr{P} \underset{H}{\otimes} \mathscr{Q}$.

The map Φ has the following form. The image of a tree, with vertices labeled by elements of $V \underset{H}{\otimes} W$, under the map Φ is the tensor product of two copies of the same tree, with vertices labeled by the corresponding elements of V, resp. W.



This description allows us to show that, in some cases, the white product of two operads is equal to their Hadamard product.

We suppose now that the generating space V is concentrated in arity 2, i.e. the operad $\mathscr{P}$ is binary. To any shuffle binary tree $\mathbb{T}$ of arity n, we associate the $\mathbb{K}$-module $\mathbb{T}(V)$, made up of copies of $\mathbb{T}$ with vertices labeled by elements of V. We consider the following map

$$\mathscr{C}_{\mathbb{T}}^{\mathscr{P}} : \mathbb{T}(V) \to \mathscr{T}(V)(n) \twoheadrightarrow \mathscr{P}(n),$$

which consists in composing in $\mathscr{P}$ the operations of V along the composition scheme given by the tree $\mathbb{T}$. (Recall that the set of binary shuffle trees provides a basis of the free operad over an $\mathbb{S}$-module concentrated in arity 2, see Sect. 8.2.5.)

Proposition 8.8.1. *Let $\mathscr{P} = \mathscr{P}(V, R)$ be a binary operad. If the maps $\mathscr{C}_{\mathbb{T}}$ are surjective for any shuffle binary tree $\mathbb{T}$, then, for any binary operad $\mathscr{Q} = \mathscr{P}(W, S)$, the white product with $\mathscr{P}$ is isomorphic to the Hadamard product*:

$$\mathscr{P} \bigcirc \mathscr{Q} \cong \mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}.$$

Proof. From the definition of the white product, we only have to show that the composite $(\pi_{\mathscr{P}} \underset{\mathrm{H}}{\otimes} \pi_{\mathscr{Q}}) \circ \Phi$ is surjective in this case. Let $\mu \otimes \nu$ be an elementary tensor of $\mathscr{P}(n) \underset{\mathrm{H}}{\otimes} \mathscr{Q}(n)$. There exists a shuffle binary tree $\mathbb{T}$ of arity n such that ν lives in the image of $\mathscr{C}_{\mathbb{T}}^{\mathscr{Q}}$. By the assumption, the element μ lives in the image of $\mathscr{C}_{\mathbb{T}}^{\mathscr{P}}$. Therefore, the element $\mu \otimes \nu$ is the image of the tree $\mathbb{T}$ with vertices labeled by elements of $V \underset{\mathrm{H}}{\otimes} W$ under the map $(\pi_{\mathscr{P}} \underset{\mathrm{H}}{\otimes} \pi_{\mathscr{Q}}) \circ \Phi$. □

The operads *Com* and *Perm* satisfy the assumption of this proposition. Therefore, the operad *Com* is the unit object with respect to the white product $\mathscr{P} \bigcirc Com = \mathscr{P} \underset{\mathrm{H}}{\otimes} Com = \mathscr{P}$ in the category of binary operads.

The category of binary quadratic operads is stable under Manin white product. Since it is associative and symmetric, the category of binary quadratic operads, endowed with the white product and the operad *Com* as a unit, forms a symmetric monoidal category.

Let $A = A(V, R)$ and $B = A(W, S)$ be two quadratic algebras, which we consider as operads concentrated in arity 1. We leave it to the reader to check that the white product of A and B as operads is equal to their white product as algebras, defined in Sect. 4.5.1.

8.8.2 Black Product for Cooperads

Dualizing the previous arguments and working in the opposite category, we get the notion of *black product for cooperads* as follows.

Let $(\mathscr{C}, \Delta_{\mathscr{C}})$ and $(\mathscr{D}, \Delta_{\mathscr{D}})$ be two cooperads. Their Hadamard product $\mathscr{C} \underset{\mathrm{H}}{\otimes} \mathscr{D}$ is again a cooperad. The coproduct is given by the composite

$$\mathscr{C} \underset{\mathrm{H}}{\otimes} \mathscr{D} \xrightarrow{\Delta_{\mathscr{C}} \underset{\mathrm{H}}{\otimes} \Delta_{\mathscr{D}}} (\mathscr{C} \circ \mathscr{C}) \underset{\mathrm{H}}{\otimes} (\mathscr{D} \circ \mathscr{D}) \twoheadrightarrow (\mathscr{C} \underset{\mathrm{H}}{\otimes} \mathscr{D}) \circ (\mathscr{D} \underset{\mathrm{H}}{\otimes} \mathscr{C}),$$

where the second map is a treewise projection. We denote by $p_V : \mathscr{T}^c(V) \twoheadrightarrow V$ the canonical projection from the conilpotent cofree cooperad to V. Since $\mathscr{T}^c(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}^c(W)$ is a conilpotent cooperad, by the universal property of the conilpotent cofree cooperad, there exists a (unique) morphism of cooperads $\Psi : \mathscr{T}^c(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}^c(W) \to \mathscr{T}^c(V \underset{\mathrm{H}}{\otimes} W)$ which factors the map $p_V \underset{\mathrm{H}}{\otimes} p_W$.

$$\begin{array}{ccc} V \underset{\mathrm{H}}{\otimes} W & \xleftarrow{\;p_{V \underset{\mathrm{H}}{\otimes} W}\;} & \mathscr{T}^c(V \underset{\mathrm{H}}{\otimes} W) \\ & \nwarrow^{\,p_V \underset{\mathrm{H}}{\otimes} p_W} & \uparrow \exists! \Psi \\ & & \mathscr{T}^c(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}^c(W). \end{array}$$

Let $\mathscr{C} = \mathscr{C}(V, R)$ and $\mathscr{D} = \mathscr{C}(W, S)$ be two cooperads defined by cogenerators and corelators, not necessarily quadratic. Let us denote by $\iota_{\mathscr{C}} : \mathscr{C} \rightarrowtail \mathscr{T}^c(V)$ and by $\iota_{\mathscr{D}} : \mathscr{D} \rightarrowtail \mathscr{T}^c(W)$ the canonical inclusions. The composite of morphisms of cooperads $\Psi \circ (\iota_{\mathscr{C}} \underset{\mathrm{H}}{\otimes} \iota_{\mathscr{D}})$ factors through its image

$$\begin{array}{ccccc} \mathscr{T}^c(V \underset{\mathrm{H}}{\otimes} W) & \xleftarrow{\;\Psi\;} & \mathscr{T}^c(V) \underset{\mathrm{H}}{\otimes} \mathscr{T}^c(W) & \xleftarrow{\;\iota_{\mathscr{C}} \underset{\mathrm{H}}{\otimes} \iota_{\mathscr{D}}\;} & \mathscr{C} \underset{\mathrm{H}}{\otimes} \mathscr{D} \\ & \nwarrow & & \swarrow_{\bar{\Psi}} & \\ & & \mathscr{C} \bullet \mathscr{D} & & \end{array}$$

which we define to be *Manin black product of the cooperads* $\mathscr{C}$ *and* $\mathscr{D}$. It is the cooperad cogenerated by $V \underset{\mathrm{H}}{\otimes} W$ with corelations $\Psi(R \underset{\mathrm{H}}{\otimes} S)$:

$$\mathscr{C} \bullet \mathscr{D} := \mathscr{C}\Big(V \underset{\mathrm{H}}{\otimes} W, \Psi\Big(R \underset{\mathrm{H}}{\otimes} S\Big)\Big).$$

8.8.3 Black Products for Operads

Since it is easier to work with operads than cooperads, we will consider the linear dual of the previous definition. This provides a notion of black product for operads.

From now on, we will work in the category of finitely generated binary quadratic operads (concentrated in degree 0).

Recall from Sect. 7.6.3 the notation $V^\vee := V^* \otimes \mathrm{sgn}_2$, where the $\mathbb{S}$-module V is concentrated in arity 2. In that section, we introduced a basis of $\mathscr{T}(V)(3)$ and a scalar product $\langle -, - \rangle$, which induces an isomorphism $\theta_V : \mathscr{T}(V)(3) \xrightarrow{\cong} \mathscr{T}(V^\vee)(3)^*$. We define the morphism $\widetilde{\Psi}$ by the following composite

$$\begin{array}{ccc}
\mathscr{T}(V)(3) \underset{\mathrm{H}}{\otimes} \mathscr{T}(W)(3) & \xrightarrow{\quad \widetilde{\Psi} \quad} & \mathscr{T}(V \underset{\mathrm{H}}{\otimes} W \underset{\mathrm{H}}{\otimes} \mathrm{sgn}_{\mathbb{S}_2})(3) \\
\Big\downarrow {\scriptstyle \theta_V \underset{\mathrm{H}}{\otimes} \theta_W} & & \Big\uparrow {\scriptstyle \theta^{-1}_{V \underset{\mathrm{H}}{\otimes} W}} \\
\mathscr{T}(V^\vee)(3)^* \underset{\mathrm{H}}{\otimes} \mathscr{T}(W^\vee)(3)^* & & \mathscr{T}((V \underset{\mathrm{H}}{\otimes} W \underset{\mathrm{H}}{\otimes} \mathrm{sgn}_{\mathbb{S}_2})^\vee)(3)^* \\
\Big\downarrow {\scriptstyle \simeq} & & \Big\uparrow {\scriptstyle \simeq} \\
(\mathscr{T}(V^\vee)(3) \underset{\mathrm{H}}{\otimes} \mathscr{T}(W^\vee)(3))^* & \xrightarrow{\quad {}^t\Phi \quad} & \mathscr{T}(V^\vee \underset{\mathrm{H}}{\otimes} W^\vee)(3)^*,
\end{array}$$

where $\simeq$ stands for the natural isomorphism for the linear dual of a tensor product, since the modules are finite dimensional, and where Φ is the map defined in Sect. 8.8.1 applied to $V^\vee$ and $W^\vee$. The morphism $\widetilde{\Psi}$ defined here is a twisted version of Ψ.

The black product of two finitely generated binary quadratic operads $\mathscr{P} = \mathscr{P}(V, R)$ and $\mathscr{Q} = \mathscr{P}(W, S)$ is equal to

$$\mathscr{P} \bullet \mathscr{Q} := \mathscr{T}\left(V \underset{\mathrm{H}}{\otimes} W \underset{\mathrm{H}}{\otimes} \mathrm{sgn}_{\mathbb{S}_2}\right) / \left(\widetilde{\Psi}\left(R \underset{\mathrm{H}}{\otimes} S\right)\right).$$

8.8.4 Examples of Computations of Black Products

In order to compute black products for operads (and white products by Koszul duality), we use the basis and notations given in Sect. 7.6.4.

Proposition 8.8.2. *The following isomorphism holds*

$$\mathit{PreLie} \bullet \mathit{Com} \cong \mathit{Zinb}.$$

Proof. We denote by μ the generating operation of the operad *PreLie*. In a pre-Lie algebra the binary operation is such that its associator is right symmetric, that is $\mu(\mu(a, b), c) - \mu(a, \mu(b, c)) = \mu(\mu(a, c), b) - \mu(a, \mu(c, b))$. This relation corresponds to $v_i - v_{i+1} + v_{i+2} - v_{i+3}$ for $i = 1, 5, 9$ with our conventions. We denote by ν the commutative generating operation of *Com* and by w_1, w_5, w_9 the associated generators of $\mathscr{T}(\mathbb{K}\nu)(3)$. The associativity relation of ν reads in this basis:

$w_1 - w_5 = 0$ and $w_5 - w_9 = 0$. We have

(1) $\widetilde{\Psi}((v_1 - v_2 + v_3 - v_4) \otimes (w_1 - w_5)) = \widetilde{\Psi}(v_1 \otimes w_1 + v_4 \otimes w_5)$,
(2) $\widetilde{\Psi}((v_1 - v_2 + v_3 - v_4) \otimes (w_5 - w_9)) = \widetilde{\Psi}((v_2 - v_3) \otimes w_9 - v_4 \otimes w_5)$,
(3) $\widetilde{\Psi}((v_5 - v_6 + v_7 - v_8) \otimes (w_1 - w_5)) = \widetilde{\Psi}((v_7 - v_6) \otimes w_1 - v_5 \otimes w_5)$,
(4) $\widetilde{\Psi}((v_5 - v_6 + v_7 - v_8) \otimes (w_5 - w_9)) = \widetilde{\Psi}(v_5 \otimes w_5 + v_8 \otimes w_9)$,
(5) $\widetilde{\Psi}((v_9 - v_{10} + v_{11} - v_{12}) \otimes (w_1 - w_5)) = \widetilde{\Psi}(-v_{12} \otimes w_1 + (v_{10} - v_{11}) \otimes w_5)$,
(6) $\widetilde{\Psi}((v_9 - v_{10} + v_{11} - v_{12}) \otimes (w_5 - w_9)) = \widetilde{\Psi}((v_{11} - v_{10}) \otimes w_5 - v_9 \otimes w_9)$.

Using for instance Fig. 7.1, one can see that the action of (132) sends (1) to (4), (3) to (6) and (5) to (2). The image of (1) under (13) is (3). Therefore, we only need to make (1) and (2) explicit. If we identify $(\mathbb{K}\mu \oplus \mathbb{K}\mu') \otimes \mathbb{K}\nu \otimes \mathbb{K}\,\mathrm{sgn}_{\mathbb{S}_2}$ with $\mathbb{K}\gamma \oplus \mathbb{K}\gamma'$ via the isomorphism of $\mathbb{S}_2$-modules

$$\mu \otimes \nu \otimes 1 \mapsto \gamma,$$
$$\mu' \otimes \nu \otimes 1 \mapsto -\gamma',$$

the morphism $\widetilde{\Psi}$ becomes

$$\widetilde{\Psi}\big((\mu \circ_{\mathrm{I}} \mu) \otimes (\nu \circ_{\mathrm{I}} \nu)\big) = \widetilde{\Psi}(v_1 \otimes w_1) = \gamma \circ_{\mathrm{I}} \gamma = z_1 \quad \text{and}$$
$$\widetilde{\Psi}\big((\mu' \circ_{\mathrm{II}} \mu) \otimes (\nu \circ_{\mathrm{II}} \nu)\big) = \widetilde{\Psi}(v_2 \otimes w_1) = -\gamma' \circ_{\mathrm{I}} \gamma = -z_2.$$

The images of the other elements are obtained from these two by the action of $\mathbb{S}_3$. For instance, we have $\widetilde{\Psi}(v_3 \otimes w_1) = -z_3$, $\widetilde{\Psi}(v_4 \otimes w_1) = z_4$ and $\widetilde{\Psi}(v_5 \otimes w_5) = z_5$.

So, we get

$$\widetilde{\Psi}(v_1 \otimes w_1 + v_4 \otimes w_5) = \gamma \circ_{\mathrm{I}} \gamma - \gamma \circ_{\mathrm{III}} \gamma',$$
$$\widetilde{\Psi}\big((v_2 - v_3) \otimes w_9 - v_4 \otimes w_5\big) = -\gamma' \circ_{\mathrm{II}} \gamma - \gamma' \circ_{\mathrm{II}} \gamma' + \gamma \circ_{\mathrm{III}} \gamma'.$$

Finally, if we represent the operation $\gamma(x, y)$ by $x \prec y$, then we have

$$(x \prec y) \prec z = (x \prec z) \prec y,$$
$$(x \prec z) \prec y = x \prec (z \prec y) + x \prec (y \prec z),$$

where we recognize the axioms of a Zinbiel algebra, see Sect. 13.5.2. □

8.8.5 Manin Products and Koszul Duality

Theorem 8.8.3. *For any pair of finitely generated binary quadratic operads $\mathscr{P} = \mathscr{P}(V, R)$ and $\mathscr{Q} = \mathscr{P}(W, S)$, their black and white products are sent to one another under Koszul duality*

$$(\mathscr{P} \bullet \mathscr{Q})^! = \mathscr{P}^! \circ \mathscr{Q}^!.$$

Proof. Let us denote by $\langle -,-\rangle_E$ the scalar product on $\mathscr{T}(E^\vee)(3)\otimes\mathscr{T}(E)(3)$. The orthogonal space of $\widetilde{\Psi}(R\underset{\mathrm{H}}{\otimes}S)$ for $\langle -,-\rangle_{V\underset{\mathrm{H}}{\otimes}W\underset{\mathrm{H}}{\otimes}\mathrm{sgn}}$ is $\Phi^{-1}(R^\perp\underset{\mathrm{H}}{\otimes}\mathscr{T}(W^\vee)+\mathscr{T}(V^\vee)\underset{\mathrm{H}}{\otimes}S^\perp)$. By definition of the transpose of Φ, we have

$$\begin{aligned}\Big\langle X,\widetilde{\Psi}\Big(r\underset{\mathrm{H}}{\otimes}s\Big)\Big\rangle_{V\underset{\mathrm{H}}{\otimes}W\underset{\mathrm{H}}{\otimes}\mathrm{sgn}} &= \Big\langle\Phi(X),r\underset{\mathrm{H}}{\otimes}s\Big\rangle_{(\mathscr{T}(V)\underset{\mathrm{H}}{\otimes}\mathscr{T}(V^\vee))\times(\mathscr{T}(W)\underset{\mathrm{H}}{\otimes}\mathscr{T}(W^\vee))}\\ &= \big(\langle -,r\rangle_V.\langle -,s\rangle_W\big)\circ\Phi(X),\end{aligned}$$

for every $(r,s)\in R\times S$ and every $X\in\mathscr{T}(V^\vee\underset{\mathrm{H}}{\otimes}W^\vee)$.

Therefore, we have

$$\begin{aligned}&\widetilde{\Psi}\Big(R\underset{\mathrm{H}}{\otimes}S\Big)^\perp\\ &\quad=\Big\{X\in\mathscr{T}\Big(V^\vee\underset{\mathrm{H}}{\otimes}W^\vee\Big)(3)\mid\forall(r,s)\in R\times S\big(\langle -,r\rangle_V.\langle -,s\rangle_W\big)\circ\Phi(X)=0\Big\}\\ &\quad=\Big\{X\in\mathscr{T}\Big(V^\vee\underset{\mathrm{H}}{\otimes}W^\vee\Big)(3)\mid\Phi(X)\in R^\perp\underset{\mathrm{H}}{\otimes}\mathscr{T}\big(W^\vee\big)+\mathscr{T}\big(V^\vee\big)\underset{\mathrm{H}}{\otimes}S^\perp\Big\}\\ &\quad=\Phi^{-1}\Big(R^\perp\underset{\mathrm{H}}{\otimes}\mathscr{T}\big(W^\vee\big)+\mathscr{T}\big(V^\vee\big)\underset{\mathrm{H}}{\otimes}S^\perp\Big).\end{aligned}$$

□

Since the operad *Com* is the unit object for the white product, the operad *Lie* is the unit object for the black product, that is

$$\mathit{Lie}\bullet\mathscr{P}=\mathscr{P},$$

for any finitely generated binary quadratic operad $\mathscr{P}$. The black product is also an associative product. Therefore the category of finitely generated binary quadratic operads, equipped with the black product and the operad *Lie* as unit, is a symmetric monoidal category.

Corollary 8.8.4. *The following isomorphism holds*

$$\mathit{Perm}\bigcirc\mathit{Lie}\cong\mathit{Leib}.$$

Proof. It is a direct corollary of Proposition 8.8.2 and Theorem 8.8.3. □

Contrarily to associative algebras, Manin black or white product of two Koszul operads is not necessarily a Koszul operad. A counterexample is given in Exercise 8.10.16.

8.8.6 Remark

In Sect. 7.2.3, we defined the Koszul dual operad as $\mathscr{P}^! := (\mathscr{S}^c\underset{\mathrm{H}}{\otimes}\mathscr{P}^{¡})^*$. Let Com_{-1} be the operad which is spanned by sgn_n (signature representation) of degree $n-1$

in arity n. Then we have $\mathscr{P}^! = (\mathscr{P}^{¡})^* \bigcirc Com_{-1}$. This formula explains the presentation of $\mathscr{P}^!$ in terms of the presentation of $\mathscr{P}$ given in Proposition 7.2.1.

8.8.7 Adjunction and Internal (Co)Homomorphism

Theorem 8.8.5. *There is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Quad\text{-}Op}}(\mathscr{P} \bullet \mathscr{Q}, \mathscr{R}) \cong \mathrm{Hom}_{\mathsf{Quad\text{-}}Op}\big(\mathscr{P}, \mathscr{Q}^! \bigcirc \mathscr{R}\big),$$

where Quad-Op *stands for the category of finitely generated binary quadratic operads.*

In other words, the functors

$$- \bullet \mathscr{Q} : \mathsf{Quad\text{-}Op} \rightleftarrows \mathsf{Quad\ Op} : \mathscr{Q}^! \bigcirc -$$

form a pair of adjoint functors, for any finitely generated binary quadratic operad $\mathscr{Q}$.

Proof. Let $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ be three operads presented by $\mathscr{P} = \mathscr{P}(V, R)$, $\mathscr{Q} = \mathscr{P}(W, S)$ and $\mathscr{R} = \mathscr{P}(X, T)$. There is a bijection between maps $f : V \underset{\mathrm{H}}{\otimes} W \underset{\mathrm{H}}{\otimes} \mathrm{sgn}_{\mathbb{S}_2} \to X$ and maps $g : V \to W^\vee \underset{\mathrm{H}}{\otimes} X$. Using the same arguments as in the proof of Theorem 8.8.3, we can see that

$$\begin{aligned}
&\Big\langle \Big(\Phi^{-1}\Big(S^\perp \underset{\mathrm{H}}{\otimes} \mathscr{T}(X) + \mathscr{T}\big(W^\vee\big) \underset{\mathrm{H}}{\otimes} T\Big)\Big)^\perp, \mathscr{T}(g)(R)\Big\rangle_{W^\vee \underset{\mathrm{H}}{\otimes} X} \\
&\quad = \Big\langle \widetilde{\Psi}\Big(S \underset{\mathrm{H}}{\otimes} T^\perp\Big), \mathscr{T}(g)(R)\Big\rangle_{W^\vee \underset{\mathrm{H}}{\otimes} X} \\
&\quad = \Big\langle T^\perp, \mathscr{T}(f)\Big(\widetilde{\Psi}\Big(R \underset{\mathrm{H}}{\otimes} S\Big)\Big)\Big\rangle_X .
\end{aligned}$$

Therefore $\mathscr{T}(f)(\widetilde{\Psi}(R \underset{\mathrm{H}}{\otimes} S)) \subset T$ is equivalent to $\mathscr{T}(g)(R) \subset \Phi^{-1}(S^\perp \underset{\mathrm{H}}{\otimes} \mathscr{T}(X) + \mathscr{T}(W^\vee) \underset{\mathrm{H}}{\otimes} T)$, which concludes the proof. □

In other words, $\mathsf{Hom}(B, C) := B^! \bigcirc C$ is the internal "Hom" functor in the monoidal category of finitely generated binary quadratic operads with the black product. Dually, $\mathsf{CoHom}(A, B) := A \bullet B^!$ is the internal "coHom" (or inner) functor in the monoidal category of finitely generated binary quadratic operads with the white product.

For another point of view on this type of adjunction and coHom objects in the general operadic setting, we refer the reader to the paper [BM08] by D. Borisov and Yu.I. Manin.

Let $\mathscr{P} = \mathscr{P}(V, R)$ be any finitely generated binary quadratic operad. We apply Proposition 8.8.5 to the three operads *Lie*, $\mathscr{P}$ and $\mathscr{P}$. Since *Lie* is the unit object for the black product, we have a natural bijection $\mathrm{Hom}_{\mathsf{Quad\ Op}}(\mathscr{P}, \mathscr{P}) \cong \mathrm{Hom}_{\mathsf{Quad\ Op}}(\mathit{Lie}, \mathscr{P}^! \bigcirc \mathscr{P})$. The image of the identity of $\mathscr{P}$ under this bijection provides a canonical morphism of operads $\mathit{Lie} \to \mathscr{P}^! \bigcirc \mathscr{P}$. The composite with $\overline{\Phi} : \mathscr{P}^! \bigcirc \mathscr{P} \to \mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P}^!$ gives a morphism of operads from $\mathit{Lie} \to \mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P}^!$. We leave it to the reader to verify that this morphism of operads is equal to the one given in Theorem 7.6.5, thereby providing a more conceptual proof of this property.

8.8.8 Hopf Operads

Proposition 8.8.6. *Let $\mathscr{P} = \mathscr{P}(V, R)$ be a finitely generated binary quadratic operad. The black product $\mathscr{P}^! \bullet \mathscr{P}$ is a Hopf operad.*

Proof. The proof is similar to the proof of Proposition 4.5.4. □

8.8.9 Manin Products for Nonsymmetric Operads

We can perform the same constructions in the category of arity-graded vector spaces. Some constructions of this type have been devised in [EFG05]. This defines *Manin black and white products for nonsymmetric operads*, which we denote by ■ and by □. (The details are left to the reader.) Notice that the black or white product of two ns operads does not give in general the "same" result as the black or white product of the associated symmetric operads. One can also introduce the notion of Manin products for shuffle operads. In this case, the forgetful functor from operads to shuffle operads commutes with Manin white product, see Exercise 8.10.17.

The black and white products for ns operads are associative products. For any finitely generated binary quadratic ns operad $\mathscr{P}$, the ns operad As of associative algebras is the unit object for both products, that is $\mathit{As} \,\square\, \mathscr{P} = \mathit{As} \,\blacksquare\, \mathscr{P} = \mathscr{P}$. Therefore, the category of finitely generated binary quadratic ns operads carries two symmetric monoidal category structures provided by the black and white product and the operad As.

The following result is proved with the same argument as in the symmetric operad framework.

Theorem 8.8.7. *There is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Quad\text{-}nsOp}}(\mathscr{P} \blacksquare \mathscr{Q}, \mathscr{R}) \cong \mathrm{Hom}_{\mathsf{Quad\text{-}nsOp}}\big(\mathscr{P}, \mathscr{Q}^! \,\square\, \mathscr{R}\big),$$

where Quad-nsOp *stands for the category of finitely generated binary quadratic ns operads.*

As a direct corollary, there exists a morphism of ns operads $\mathit{As} \to \mathscr{P}^! \,\square\, \mathscr{P}$ for any finitely generated binary quadratic ns operad $\mathscr{P}$. This yields a canonical morphism of ns operads $As \to \mathscr{P}^! \underset{\mathrm{H}}{\otimes} \mathscr{P}$, which is the one given in Theorem 7.7.2. This result will be crucial in the study of operations on the cohomology of algebras over a ns operad, see Sect. 13.3.11.

8.9 Résumé

8.9.1 Shuffle Operads

Shuffle monoidal product $\circ_{\sqcup\!\sqcup}$ on $\mathbb{N}$-Mod: Monoidal functors

$$(\mathbb{S}\text{-Mod}, \circ) \longrightarrow (\mathbb{N}\text{-Mod}, \circ_{\sqcup\!\sqcup}) \longrightarrow (\mathbb{N}\text{-Mod}, \circ),$$

the first one being strong. Induced functors

$$\mathsf{Op} \longrightarrow \mathsf{Op}_{\sqcup\!\sqcup} \longrightarrow \mathsf{ns\ Op}.$$

Shuffle trees $\mathscr{T}_{\sqcup\!\sqcup}$: $\mathbb{K}$-linear basis of the free shuffle operad and free operad.
Partial shuffle product:

$$\begin{aligned} \circ_\sigma : P_k \otimes P_l &\to P_n, \\ \mu \otimes \nu &\mapsto \mu \circ_\sigma \nu. \end{aligned}$$

8.9.2 Rewriting Method

Let $\mathscr{P}(E, R)$ be a quadratic operad such that $E = \oplus_{i=1}^m \mathbb{K} e_i$ is a vector space equipped with a finite ordered basis. We consider a suitable order on shuffle trees with 2 vertices $\mathscr{T}_{\sqcup\!\sqcup}^{(2)}$ indexed by the $\{1, \ldots, m\}$.

Typical relation:

$$t(e_i, e_j) = \sum_{t'(k,l) < t(i,j)} \lambda_{t'(k,l)}^{t(i,j)} t'(e_k, e_l).$$

The element $t(e_i, e_j)$ is called a *leading term.* A tree monomial with 3 vertices $t(e_i, e_j, e_k)$ is called *critical* if both 2-vertices subtrees $t'(e_i, e_j)$ and $t''(e_j, e_k)$ are leading terms.

Theorem. *Confluence for all the critical tree monomials $\Rightarrow$ Koszulity of the operad.*

8.9.3 Reduction by Filtration and Diamond Lemma

Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad. Any grading on $E \cong E_1 \oplus \cdots \oplus E_m$ together with a suitable order on shuffle trees induce a filtration on the shuffle operad $\mathscr{P}_{\sqcup\sqcup}$ and

$$\psi : \mathring{\mathscr{P}}_{\sqcup\sqcup} = \mathscr{T}_{\sqcup\sqcup}(E)/(R_{\text{lead}}) \twoheadrightarrow \text{gr}\,\mathscr{P}_{\sqcup\sqcup},$$

with $R_{\text{lead}} = \langle \text{leading term}(r), r \in R \rangle$.

DIAMOND LEMMA FOR QUADRATIC OPERADS.

$$\mathring{\mathscr{P}}_{\sqcup\sqcup}\ \textit{Koszul and}\ (\mathring{\mathscr{P}}_{\sqcup\sqcup})^{(3)} \rightarrowtail (\text{gr}\,\mathscr{P}_{\sqcup\sqcup})^{(3)} \implies \mathscr{P}\ \textit{Koszul and}\ \mathscr{P} \cong \text{gr}\,\mathscr{P}_{\sqcup\sqcup} \cong \mathring{\mathscr{P}}_{\sqcup\sqcup}.$$

INHOMOGENEOUS CASE.

$$\begin{matrix} q\,\mathring{\mathscr{P}}_{\sqcup\sqcup}\ \textit{Koszul and} \\ (q\,\mathring{\mathscr{P}}_{\sqcup\sqcup})^{(3)} \rightarrowtail (\text{gr}_\chi\, q\,\mathscr{P}_{\sqcup\sqcup})^{(3)} \end{matrix} \implies \begin{matrix} \mathscr{P}\ \textit{Koszul and} \\ q\,\mathring{\mathscr{P}}_{\sqcup\sqcup} \cong \text{gr}_\chi\, q\,\mathscr{P}_{\sqcup\sqcup} \cong q\,\mathscr{P} \cong \text{gr}\,\mathscr{P} \cong \mathscr{P}. \end{matrix}$$

8.9.4 PBW Basis, Gröbner Basis and Diamond Lemma

Particular case:

$$\forall i \in I = \{1, \ldots, m\}, \dim(E_i) = 1 \quad \Leftrightarrow \quad \{e_i\}_{i \in I}\ \mathbb{K}\text{-linear basis of } E,$$
$$\mathring{\mathscr{P}}_{\sqcup\sqcup}\ \text{monomial shuffle operad} \quad \Rightarrow \quad \mathring{\mathscr{P}}_{\sqcup\sqcup}\ \text{Koszul and basis } \{t(e_{\bar{i}})\}_{t \in T}.$$

PBW basis of $\mathscr{P}(E, R)$:

$$\text{basis } \{\bar{t}(e_{\bar{i}})\}_{t \in T} := \text{image of } \{t(e_{\bar{i}})\}_{t \in T} \text{ under } \mathring{\mathscr{P}}_{\sqcup\sqcup} \twoheadrightarrow \text{gr}\,\mathscr{P}_{\sqcup\sqcup} \cong \mathscr{P}_{\sqcup\sqcup} \cong \mathscr{P}.$$

MAIN PROPERTIES OF PBW BASES.

$$\mathscr{P}(E, R)\ \text{PBW basis} \quad \Rightarrow \quad \mathscr{P}(E, R)\ \textit{Koszul algebra}.$$

DIAMOND LEMMA.

$$\{\bar{t}(e_{\bar{i}})\}_{t \in T^{(3)}}\ \textit{linearly independent} \implies \{\bar{t}(e_{\bar{i}})\}_{t \in T}\ \textit{PBW basis}.$$

GRÖBNER BASIS.

$$\textit{Gröbner basis of}\ (R) \subset \mathscr{T}(E) \iff \textit{PBW basis of}\ \mathscr{T}(E)/(R).$$

PBW bases for inhomogeneous quadratic algebras:

$$q\,\mathscr{P} = \mathscr{P}(E, qR)\ \text{PBW basis} \quad \Rightarrow \quad \mathscr{P}(E, R)\ \text{PBW basis}.$$

PBW/Gröbner bases for nonsymmetric operads: We consider planar trees instead of shuffle trees.

8.9.5 Distributive Laws

A *distributive law* $\Lambda : \mathscr{B} \circ \mathscr{A} \to \mathscr{A} \circ \mathscr{B}$ induces an operad structure on $\mathscr{A} \circ \mathscr{B}$ by

$$\gamma_\Lambda := (\gamma_{\mathscr{A}} \circ \gamma_{\mathscr{B}})(\mathrm{Id}_{\mathscr{A}} \circ \Lambda \circ \mathrm{Id}_{\mathscr{B}}).$$

LOCAL TO GLOBAL. For $\mathscr{A} := \mathscr{P}(V, R)$ and $\mathscr{B} := \mathscr{P}(W, S)$, any (local) rewriting rule $\lambda : W \circ_{(1)} V \to V \circ_{(1)} W$ induces (global) distributive laws if $p : \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{A} \vee_\lambda \mathscr{B}$ is an isomorphism.

DIAMOND LEMMA FOR DISTRIBUTIVE LAW.

$$\begin{matrix} \mathscr{A}, \mathscr{B} \text{ Koszul and} \\ (\mathscr{A} \circ \mathscr{B})^{(3)} \rightarrowtail (\mathscr{A} \vee_\lambda \mathscr{B})^{(3)} \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \mathscr{A} \vee_\lambda \mathscr{B} \text{ Koszul and} \\ \mathscr{A} \vee_\lambda \mathscr{B} \cong (\mathscr{A} \circ \mathscr{B}, \gamma_\Lambda). \end{matrix}$$

8.9.6 Partition Poset Method

Set operad P $\longrightarrow$ family of *operadic partition posets* $\{\Pi_{\mathrm{P}}(n)\}_n$.

Theorem.

◇ $\mathscr{P} := \mathbb{K}[\mathrm{P}]$ *Koszul iff* $\Pi_{\mathrm{P}}(n)$ *Cohen–Macaulay, for all* n,

◇ $H_{\mathrm{top}}(\Pi_{\mathrm{P}}) \cong \mathscr{P}^{¡}$.

8.9.7 Manin Black and White Products for Operads

$$\mathscr{P}(V, R) \bigcirc \mathscr{P}(W, S) := \mathscr{P}\Big(V \underset{\mathrm{H}}{\otimes} W, \Phi^{-1}\Big(R \underset{\mathrm{H}}{\otimes} \mathscr{T}(W) + \mathscr{T}(V) \underset{\mathrm{H}}{\otimes} S\Big)\Big),$$

$$\mathscr{C}(V, R) \bullet \mathscr{C}(W, S) := \mathscr{C}\Big(V \underset{\mathrm{H}}{\otimes} W, \Psi\Big(R \underset{\mathrm{H}}{\otimes} S\Big)\Big),$$

$$\mathscr{P}(V, R) \bullet \mathscr{P}(W, S) = \mathscr{P}\Big(V \underset{\mathrm{H}}{\otimes} W \underset{\mathrm{H}}{\otimes} \mathrm{sgn}_{\mathbb{S}_2}, \widetilde{\Psi}\Big(R \underset{\mathrm{H}}{\otimes} S\Big)\Big).$$

For finitely generated quadratic operads:

$$(\mathscr{P} \bullet \mathscr{Q})^! = \mathscr{P}^! \bigcirc \mathscr{Q}^!.$$

Among finitely generated binary quadratic operads:
Unit for the white product: *Com*. Unit for the black product: *Lie*.

Theorem. $\mathrm{Hom}_{\textsf{Quad-Op}}(\mathscr{P} \bullet \mathscr{Q}, \mathscr{R}) \cong \mathrm{Hom}_{\textsf{Quad-Op}}(\mathscr{P}, \mathscr{Q}^! \bigcirc \mathscr{R})$.

$$Lie \to \mathscr{P}^! \bigcirc \mathscr{P}; \qquad \mathscr{P}^! \bullet \mathscr{P} \quad \text{is a Hopf operad.}$$

Manin black and white product for nonsymmetric operads: $\blacksquare$ and $\square$.

8.10 Exercises

Exercise 8.10.1 (Dendriform operad). Using the method described in Sect. 8.1, show that the dendriform operad, see Sect. 13.6.1, is a Koszul operad.

Exercise 8.10.2 (Unshuffles and pointed unshuffles). For a given partition $n = i_1 + \cdots + i_k$, show that the set of unshuffles

$$\left\{\sigma \in Sh^{-1}_{i_{\tau(1)},\ldots,i_{\tau(k)}};\ \tau \in \mathbb{S}_k\right\}$$

admits a left action of $\mathbb{S}_k$ and that the set of pointed unshuffles $\sqcup\!\!\sqcup(i_1,\ldots,i_k)$ gives one representative in every orbit.

Exercise 8.10.3 (From shuffle operads to symmetric operads). We denote by $M \mapsto M^{tr}$ the functor from arity-graded modules to $\mathbb{S}$-modules, which associates to M, the same underlying module, with the trivial action of the symmetric groups.

There is a natural isomorphism of arity-graded modules $M^{tr} \circ N^{tr} \cong (M \circ_{\sqcup\!\sqcup} N)^{tr}$ given by

$$\begin{aligned}
&\bigoplus M^{tr}(k) \otimes_{\mathbb{S}_k} \big(N^{tr}(i_1) \otimes \cdots \otimes N^{tr}(i_k) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \mathbb{K}[\mathbb{S}_n]\big)\\
&\quad \xrightarrow{\cong} \bigoplus M^{tr}(k) \otimes_{\mathbb{S}_k} \big(N^{tr}(i_1) \otimes \cdots \otimes N^{tr}(i_k) \otimes \mathbb{K}[\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k} \backslash \mathbb{S}_n]\big)\\
&\quad \xrightarrow{\cong} \bigoplus M^{tr}(k)\big(N^{tr}(i_1) \otimes \cdots \otimes N^{tr}(i_k) \otimes \mathbb{K}[\sqcup\!\!\sqcup_{i_1,\ldots,i_k}]\big).
\end{aligned}$$

Show that this isomorphism does not commute with the action of the symmetric groups. (Therefore, it does not induce a monoidal functor and does not send a shuffle operad to a symmetric operad.)

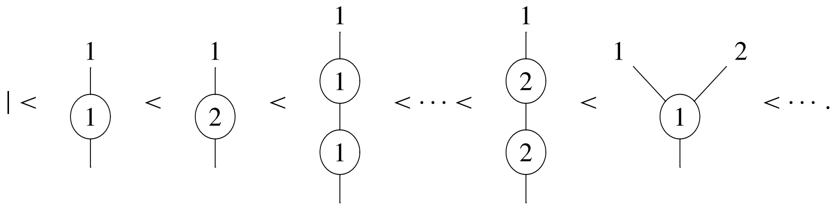
Exercise 8.10.4 (Suitable order on trees ★). We consider the following generalization of the path-lexicographic ordering of Sect. 8.4.1. It applies to the set of shuffle trees with bivalent vertices and whose vertices are labeled by $\{1,\ldots,m\}$, m being finite. To any such tree of arity n, we associate a sequence of $n+1$ words as follows. The n first words are obtained by reading the tree from the root to each leaf and by recording the labels indexing the vertices. If one encounters an arity 4 vertex labeled by 2, for instance, then the letter will be 2_4. The last word is given by the ordered labeling of the leaves, or equivalently the image of the inverse of the associate pointed unshuffle.

Prove that a such a sequence characterizes the labeled shuffle tree.

We say that $i_j < k_l$ when $(i, j) < (k, l)$ with the lexicographic order. We consider the following total order on this type of sequences.

1. We order them according to the total number of "letters" composing the words of the sequence.
2. We use the length of the last word (arity).
3. We consider the lexicographic order.

For $m = 2$, it gives



Show that this provides a suitable order.

Exercise 8.10.5 (Koszul dual operad with extra ordered grading). Let $\mathscr{P} = \mathscr{P}(E, R)$ be a finitely generated binary quadratic operad endowed with an extra ordered grading. Give a presentation of the Koszul dual operad $\mathscr{P}^!$ of $\mathscr{P}$ following the methods of Sect. 4.2.6.

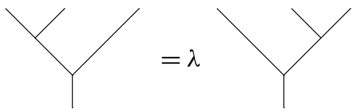
Refine this result as in Sect. 4.3.9: when $\mathscr{P} = \mathscr{P}(E, R)$ is a finitely generated binary quadratic operad endowed with a PBW basis, give a presentation of the Koszul dual operad $\mathscr{P}^!$ of $\mathscr{P}$.

Exercise 8.10.6 (Reduction by filtration method for nonsymmetric operads). Write the entire Sect. 8.4 for nonsymmetric operads as proposed in Sect. 8.4.7.

Exercise 8.10.7 (Computations of PBW bases). Describe a PBW basis for the quadratic operad *Pois* of Poisson algebras (Sect. 13.3), for the quadratic operad *Perm* of permutative algebras and for the quadratic operad *preLie* of pre-Lie algebras (Sect. 13.4).

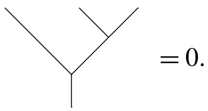
Exercise 8.10.8 (Computations of PBW bases in the inhomogeneous framework ★). Describe a PBW basis for the inhomogeneous operad BV encoding Batalin–Vilkovisky algebras, see Sects. 7.8.7 or 13.7 for the definition.

Exercise 8.10.9 (Parametrized operad *As*). We consider the nonsymmetric operad $\hbar As$ generated a binary product  , which satisfies the associativity relation



with parameter λ.

Show that the ns operad λAs is Koszul if and only if $\lambda = 0, 1$ or ∞, where the latter case means



Exercise 8.10.10 (From operad structure to distributive law). Let $\mathscr{A} = (\mathscr{A}, \gamma_{\mathscr{A}}, \iota_{\mathscr{A}})$ and $\mathscr{B} = (\mathscr{B}, \gamma_{\mathscr{B}}, \iota_{\mathscr{B}})$ be two operads. Let $\mathscr{A} \circ \mathscr{B}$ be endowed with an operad structure $(\gamma, \iota_{\mathscr{A} \circ \mathscr{B}})$ such that $\iota_{\mathscr{A} \circ \mathscr{B}} = \iota_{\mathscr{A}} \circ \iota_{\mathscr{A}}$.

Suppose that

$$\mathscr{A} \xrightarrow{\mathrm{Id}_{\mathscr{A}} \circ \iota_{\mathscr{B}}} \mathscr{A} \circ \mathscr{B} \xleftarrow{\iota_{\mathscr{A}} \circ \mathrm{Id}_{\mathscr{B}}} \mathscr{B}$$

are morphisms of operads and that the following diagram commutes

$$\begin{array}{ccc} & \mathscr{A} \circ \mathscr{B} & \\ {\scriptstyle \mathrm{Id}_{\mathscr{A}} \circ \iota_{\mathscr{B}} \circ \iota_{\mathscr{A}} \circ \mathrm{Id}_{\mathscr{B}}} \swarrow & & \searrow {\scriptstyle =} \\ \mathscr{A} \circ \mathscr{B} \circ \mathscr{A} \circ \mathscr{B} & \xrightarrow{\quad \gamma \quad} & \mathscr{A} \circ \mathscr{B}. \end{array}$$

Prove that

$$\Lambda := \mathscr{B} \circ \mathscr{A} \xrightarrow{\iota_{\mathscr{A}} \circ \mathrm{Id}_{\mathscr{B}} \circ \mathrm{Id}_{\mathscr{A}} \circ \iota_{\mathscr{B}}} \mathscr{A} \circ \mathscr{B} \circ \mathscr{A} \circ \mathscr{B} \xrightarrow{\gamma} \mathscr{A} \circ \mathscr{B}$$

is a distributive law (cf. [Bec69]).

Give another proof of Proposition 8.6.2 using this result. (Define the operadic composition on $\mathscr{A} \circ \mathscr{B}$ by transporting the operadic composition of $\mathscr{A} \vee_{\lambda} \mathscr{B}$ under the isomorphism p.)

Exercise 8.10.11 (Distributive law for the operad *Pois*). Prove the isomorphism $Com \circ Lie(4) \cong Pois(4)$.

Exercise 8.10.12 (Counter-Example, [Vladimir Dotsenko]). Let $\mathscr{A} = Com$ and $\mathscr{B} = Nil$, where a *Nil*-algebra is a vector space equipped with an antisymmetric nilpotent operation $[x, y]$, i.e. satisfying the quadratic relation $[[x, y], z] = 0$ for any x, y, z. We denote by $x \cdot y$ the commutative binary operation of *Com*. We consider the rewriting law λ given by the Leibniz relation:

$$[x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y$$

like in the Poisson case.

Show that this rewriting rule does not induce a distributive law.

HINT. Compute $[[x \cdot y, z], t]$ in two different ways.

Exercise 8.10.13 (Multi-pointed partition posets ★). A *multi-pointed partition* $\{B_1, \ldots, B_k\}$ of $\underline{n}$ is a partition of $\underline{n}$ on which at least one element of each block B_i is pointed, like $1\bar{3}\bar{4}|\bar{2}6|\bar{5}7\bar{8}$ for instance. Let π and ρ be two multi-pointed partitions. We say the ρ is larger than π, $\pi \leqslant \rho$, when the underlying partition of π is a refinement of the underlying partition of ρ and when, for each block of π, its pointed elements are either all pointed or all unpointed in ρ. For instance

$$1\bar{3}\bar{4}|\bar{2}6|\bar{5}7\bar{8} \leqslant 12\bar{3}\bar{4}6|\bar{5}7\bar{8}.$$

Prove that it is a graded poset and compute its top homology groups.

HINT. Introduce the operad *ComTrias* encoding *commutative trialgebras*, see [Val07a]. It is an algebraic structure defined by two binary operations $*$ and $\bullet$ such that the product $*$ is permutative (encoded by the operad *Perm*)

$$(x*y)*z = x*(y*z) = x*(z*y),$$

the product $\bullet$ is associative and commutative

$$\begin{cases} x\bullet y = y\bullet x, \\ (x\bullet y)\bullet z = x\bullet(y\bullet z), \end{cases}$$

and such that the two operations $*$ and $\bullet$ satisfy the following compatibility relations

$$\begin{cases} x*(y\bullet z) = x*(y*z), \\ (x\bullet y)*z = x\bullet(y*z). \end{cases}$$

Equivalently it is a triassociative algebra, cf. [LR04], satisfying some commutativity property.

Exercise 8.10.14 (Nijenhuis operad ★). Consider the following quadratic operad

$$Nij := \mathscr{P}\big(\mathbb{K}[\mathbb{S}_2]m \oplus \mathbb{K}c,\, R_{preLie} \oplus R_{Jacobi} \oplus R_{comp}\big),$$

where $\mathbb{K}c$ stands for the trivial representation of $\mathbb{S}_2$ and where $|m| = 0$ and $|c| = 1$. The space of relations are given by

$$\begin{cases} R_{preLie}\colon m\circ_1 m - m\circ_2 m - (m\circ_1 m)^{(23)} - (m\circ_2 m)^{(23)}, \\ R_{Jacobi}\colon c\circ_1 c + (c\circ_1 c)^{(123)} + (c\circ_1 c)^{(321)}, \\ R_{comp}\colon\ m\circ_1 c + m\circ_2 c + (m\circ_1 c)^{(12)} - c\circ_2 m - (c\circ_2 m)^{(12)}. \end{cases}$$

In plain words, the binary product m is a pre-Lie product and the binary product c is a "degree 1 Lie bracket".

Show that this operad is Koszul. For all the methods proposed in this chapter, try to see whether they can be applied. Notice that this operad provides the first example which requires the use of PBW bases.

REMARK. The introduction and the study of this operad are prompted by Nijenhuis geometry. For more details on the application of its Koszul resolution in this direction, we refer the reader to [Mer05, Str09].

Exercise 8.10.15 (Computations of Manin products ★). Prove that $\mathit{Perm} \bigcirc \mathit{Ass} \cong \mathit{Dias}$. Then conclude that $\mathit{preLie} \bullet \mathit{Ass} \cong \mathit{Dend}$. Prove this last isomorphism by a direct computation. (We refer to Sect. 13.6 for the definitions of the operads *Dias* and *Dend*.)

Exercise 8.10.16 (Counter-example ★). Consider the operad

$$Nil := \mathscr{P}\big(\mathrm{sgn}_{\mathbb{S}_2}, \mathscr{T}(\mathrm{sgn}_{\mathbb{S}_2})(3)\big)$$

of skew-symmetric nilpotent algebras. It is generated by the signature representation in arity 2, with all possible relations.

Compute the black product $Nil \bullet preLie$. (We refer to Exercise 8.10.14 and to Sect. 13.4 for the definition of the operad *preLie*.)

Show that the operad $Nil \bullet preLie$ is not Koszul, though the two operads *Nil* and *preLie* are.

Exercise 8.10.17 (Manin products for shuffle operads ★). Following the same method as in Sect. 8.8, define the notion of Manin white product $\bigcirc_{\sqcup\!\sqcup}$ for shuffle operads with presentation.

Prove the existence of a canonical isomorphism of shuffle operads of the form

$$(\mathscr{P} \bigcirc \mathscr{Q})^f \cong \mathscr{P}^f \bigcirc_{\sqcup\!\sqcup} \mathscr{Q}^f,$$

for any pair of (symmetric) operads $\mathscr{P} = \mathscr{P}(V, R)$ and $\mathscr{Q} = \mathscr{P}(W, S)$.