Chapter 4
Methods to Prove Koszulity of an Algebra

"Là, tout n'est qu'ordre et beauté, Luxe, calme et volupté.
Charles Baudelaire

After having introduced the notion of Koszul algebra in the preceding chapter, we give here methods to prove that an algebra is Koszul together with constructions to produce new Koszul algebras.

We begin by describing a short algorithmic method, called *rewriting method*. It amounts to choosing first an ordered basis of the generating space. Then, we interpret the relations as rewriting rules, replacing each leading term by a sum of lower terms, with respect to a suitable ordering on monomials. If applying the rewriting rules to the critical monomials leads to the same element (confluence property), then the algebra is Koszul.

This method is the simplest case of a general one, which relies on an extra data: a decomposition of the generating space $V \cong V_1 \oplus \cdots \oplus V_k$ of a quadratic algebra $A = A(V, R)$ and a suitable order on the set of tuples in $\{1, \ldots, k\}$. Such a data induces a filtration on the algebra A. When the associated graded algebra $\operatorname{gr} A$ is Koszul, the algebra A itself is also Koszul. So the problem reduces to the graded algebra $\operatorname{gr} A$, whose product is simpler than the product of A. What about its underlying module? We have a tentative quadratic presentation $\mathring{A} := A(V, R_{\text{lead}}) \twoheadrightarrow \operatorname{gr} A$, where the module R_{lead} is made up of the leading terms of the relations. The *Diamond Lemma* asserts that it is enough to prove the injectivity of this map in weight 3 and that the algebra $\mathring{A}$ is Koszul, to get the isomorphism of algebras $\mathring{A} \cong \operatorname{gr} A$. It implies that A is Koszul. This method reduces the problem to proving the Koszulity of the simpler quadratic algebra $\mathring{A}$.

The particular case where each component V_i is one-dimensional gives rise to the notion of *Poincaré–Birkhoff–Witt (PBW) basis* of a quadratic algebra. Here the quadratic algebra $\mathring{A}$ is a *quadratic monomial algebra*, which is always a Koszul algebra, thereby simplifying the theory. For instance, any quadratic algebra admitting a PBW basis is a Koszul algebra. In this case, we refine even further the Diamond Lemma to give a simple way to check whether a quadratic algebra admits a PBW

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basis. This is the aforementioned rewriting method. We also introduce the notion of quadratic *Gröbner basis* for the ideal (R) and prove that it is equivalent to a PBW basis for the quotient algebra $T(V)/(R)$.

The last method uses a family of lattices associated to any quadratic data. The *Backelin criterion* states that these lattices are distributive if and only if the quadratic data is Koszul.

Finally, we introduce the two Manin products, white $\bigcirc$ and black $\bullet$, in the category of quadratic data. They are sent to one another under the Koszul dual functor and they preserve the Koszul property by Backelin's criterion. This allows us to construct a new chain complex, called the *Manin complex*, on the white product $A \bigcirc A^{!}$ of a quadratic algebra and its Koszul dual algebra (not coalgebra). Dually, the black product $A \bullet A^{!}$ is endowed with a Hopf algebra structure.

This chapter is essentially extracted from Priddy [Pri70], Bergman [Ber78], Backelin [Bac83], Manin [Man87, Man88] and Polishchuk–Positselski [PP05].

4.1 Rewriting Method

In this section, we give a short algorithmic method, based on the rewriting rules given by the relations, to prove that an algebra is Koszul. We give no proof here since this method is a particular case of a more general theory explained in detail in the next two sections.

Let $A(V, R)$ be a quadratic algebra, for instance

$$A\big(v_1, v_2, v_3; v_1^2 - v_1v_2, v_2v_3 + v_2v_2, v_1v_3 + 2v_1v_2 - v_1^2\big).$$

Step 1. We choose a basis $\{v_i\}_{i=1,\dots,k}$ for the space of generators V. We consider the ordering $v_1 < v_2 < \cdots < v_k$.

Step 2. We consider the induced basis of $V^{\otimes 2}$, which we order lexicographically:

$$v_1v_1 < v_1v_2 < \cdots < v_1v_k < v_2v_1 < \cdots .$$

(One can choose other suitable orders, like

$$v_1v_1 < v_1v_2 < v_2v_1 < v_1v_3 < v_2v_2 < v_3v_1 < v_1v_4 < \cdots$$

see the discussion at the end of Sect. 4.2.1.)

We choose a basis of R. Any one of its elements is of the form

$$r = \lambda v_iv_j - \sum_{(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_kv_l, \quad \lambda \neq 0.$$

The monomial v_iv_j is called the *leading term* of r. We can always change this basis for one with the following *normalized form*. First, the coefficient of the leading term

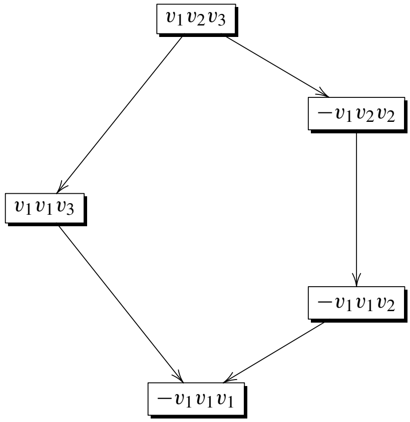


Fig. 4.1 Pentagonal diamond

can always be supposed to be 1 since $\mathbb{K}$ is a field. Then, we can always suppose that two different relators in the basis have different leading terms and that the sum in the right-hand side of any relator contains no leading term of any other relator.

In the example at hand, the space of relations R admits the following normalized basis:

$$\left\{v_1v_2 - \left(v_1^2\right), v_2v_3 - (-v_2v_2), v_1v_3 - \left(-v_1^2\right)\right\}.$$

The three leadings terms are v_1v_2, v_2v_3 and v_1v_3.

Step 3. These choices provide rewriting rules of the form

$$v_iv_j \mapsto \sum_{(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_kv_l,$$
$$\text{leading term} \mapsto \text{sum of lower terms},$$

for any relator r in the normalized basis of R. A monomial $v_iv_jv_k$ is called *critical* if both v_iv_j and v_jv_k are leading terms. Any critical monomial gives rise to a graph made up of the successive application of the aforementioned rewriting rule.

In the example at hand, we have the following rewriting rules

$$v_1v_2 \mapsto v_1^2, \qquad v_2v_3 \mapsto -v_2v_2, \qquad v_1v_3 \mapsto -v_1^2.$$

There is only one critical monomial: $v_1v_2v_3$.

Step 4. Any critical monomial $v_iv_jv_k$ gives a graph under the rewriting rules. It is *confluent*, if it has only one terminal vertex.

In the example at hand, the only critical monomial induces the confluent graph shown in Fig. 4.1.

Conclusion. If each critical monomial is confluent, then the algebra A is Koszul.

This assertion is a consequence of the following result.

Theorem 4.1.1 (Rewriting method). *Let $A = A(V, R)$ be a quadratic algebra. If its generating space V admits an ordered basis, for which there exists a suitable order on the set of tuples, such that every critical monomial is confluent, then the algebra A is Koszul.*

Proof. This result is Theorem 4.3.7. □

In this case, the algebra A is equipped with an induced basis sharing nice properties, called a PBW basis, see Sect. 4.3. For other examples, like the symmetric algebra, and for more details, we refer the reader to Sect. 4.3.5.

4.2 Reduction by Filtration

The idea of the "reduction by filtration" method can be shortened as follows: when a quadratic algebra $A = A(V, R)$ admits a filtration with nice properties, there exists a morphism of algebras

$$\mathring{A} := A(V, R_{\text{lead}}) \twoheadrightarrow \operatorname{gr} A := \operatorname{gr}\big(A(V, R)\big)$$

from the quadratic algebra defined by the associated graded presentation to the associated graded algebra. If the quadratic algebra $\mathring{A}$ is Koszul and if this map is an isomorphism (in weight 3), then the algebra A itself is Koszul. This reduces the problem of the Koszulity of the algebra A to the algebra $\mathring{A}$. Koszulity of $\mathring{A}$ is easier to check in general.

4.2.1 Extra Ordered Grading

The aim of this section is to endow the free algebra $T(V)$ with an extra grading, which refines the weight grading, such that the product is strictly increasing: $a < a'$, $b < b'$ implies $ab < a'b'$.

Let $A = A(V, R) = T(V)/(R)$ be a quadratic algebra, i.e. $R \subset V^{\otimes 2}$. We suppose here that V is equipped with an extra grading $V \cong V_1 \oplus \cdots \oplus V_k$, which is finite. This induces the following grading on $T(V)$:

$$V^{\otimes n} \cong \bigoplus_{(i_1,\ldots,i_n)\in\{1,\ldots,k\}^n} V_{i_1} \otimes \cdots \otimes V_{i_n},$$

under the lexicographical order

$$0 < 1 < \cdots < k < (1,1) < (1,2) < \cdots < (k,k) < (1,1,1) < (1,1,2) < \cdots,$$

where $\mathbb{K}1$ is in degree 0. This lexicographical order induces a bijection of totally ordered sets between the set of tuples in $\{1,\ldots,k\}$ and the set of integers $\mathbb{N}$. For any tuple $(i_1, \ldots, i_n)$ sent to $p \in \mathbb{N}$, we will denote the sub-space $V_{i_1} \otimes \cdots \otimes V_{i_n}$ of $V^{\otimes n}$ simply by $T(V)_p$. Under this bijection, there exists a map χ which corresponds to the concatenation of tuples:

$$\begin{array}{ccc} ((i_1, \ldots, i_n), (j_1, \ldots, j_m)) & \longmapsto & (i_1, \ldots, i_n, j_1, \ldots, j_m) \\ \downarrow \cong & & \downarrow \cong \\ (p, q) & \overset{\chi}{\longmapsto} & \chi(p, q). \end{array}$$

Under the lexicographical order on $\mathbb{N} \times \mathbb{N}$, the map $\chi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is strictly increasing. The concatenation product on the free associative algebra $T(V)$ satisfies

$$\mu : T(V)_p \otimes T(V)_q \to T(V)_{\chi(p,q)}.$$

Hence, this grading $(\mathbb{N}, \chi)$ refines the weight grading of $T(V)$. Notice that the map χ defines a monoid structure $(\mathbb{N}, \chi, 0)$.

Associated to this grading, we consider the increasing and exhaustive filtration $F_p T(V) := \bigoplus_{q=0}^{p} T(V)_q$ on $T(V)$. The image of this filtration under the canonical projection $T(V) \twoheadrightarrow A$ defines an increasing filtration

$$\mathrm{F}_0 A \subset \mathrm{F}_1 A \subset \mathrm{F}_2 A \subset \cdots \subset \mathrm{F}_p A \subset \mathrm{F}_{p+1} A \subset \cdots$$

of the underlying module of A. The strictly increasing map χ allows us to define a χ-graded product on the associated graded module $\mathrm{gr}_p A := \mathrm{F}_p A / \mathrm{F}_{p-1} A$:

$$\bar{\mu} : \mathrm{gr}_p A \otimes \mathrm{gr}_q A \to \mathrm{gr}_{\chi(p,q)} A.$$

This algebra is denoted by $\mathrm{gr}_\chi A$, or simply by $\mathrm{gr}\, A$, when there is no possible confusion. Since the extra grading refines the weight grading, the algebra $\mathrm{gr}\, A$ is also weight graded.

There are two ways of generalizing the aforementioned arguments. First, one can allow k to be infinite, that is V can admit an extra grading labeled by $\mathbb{N}$: $V \cong \bigoplus_{i \in \mathbb{N}} V_i$. Then, one need not work only with the lexicographical order. Let $I := \{1, \ldots, k\}$ denote the labeling set of the extra grading on V. We consider any bijection $\bigsqcup_{n \geq 0} I^n \cong \mathbb{N}$. This endows the set of tuples $\bigsqcup_{n \geq 0} I^n$ with a total order isomorphic to $\mathbb{N}$. To define the graded algebra $\mathrm{gr}\, A$, it is enough to require that the map χ, or equivalently the concatenation product, be strictly increasing. In this case, we call the total order on the set of tuples a *suitable order*. For instance, when $k = \infty$, we can consider the following suitable total order

$$0 < 1 < 2 < (1,1) < 3 < (1,2) < (2,1) < (1,1,1) < 4 < (1,3) < (2,2) < \cdots,$$

isomorphic to $\mathbb{N}$.

Such a data, the decomposition of V and the suitable order on tuples, is called an *extra ordered grading*.

4.2.2 The Koszul Property

Recall from Sect. 3.4.3 that a connected weight graded algebra is called Koszul if the cohomology of its bar construction is concentrated in syzygy degree 0.

Proposition 4.2.1. *Let $A = A(V, R)$ be a quadratic algebra equipped with an extra ordered grading. If the algebra* $\mathrm{gr}\, A$ *is Koszul, then the algebra* A *is also Koszul.*

Proof. We consider the bar construction $\mathrm{B}^{-\bullet}A$ as a chain complex with the opposite of the syzygy degree (see Sect. 3.3.1). We extend the filtration on the free algebra $T(V)$ to its bar construction $\mathrm{B}T(V)$ as follows:

$$F_p\mathrm{B}T(V) := \left\{sx_1 \otimes \cdots \otimes sx_m \mid x_1x_2\ldots x_m \in F_pT(V)\right\}.$$

This filtration is stable under the differential map. The canonical projection $T(V) \twoheadrightarrow A$ induces an epimorphism of dg coalgebras $\mathrm{B}T(V) \twoheadrightarrow \mathrm{B}A$ between the bar constructions. The image under this map of the preceding filtration defines a filtration $\mathrm{F}_p\mathrm{B}A$ of the bar construction of A. The first page of the associated spectral sequence $\mathrm{E}^0_{pq} \cong \mathrm{F}_p\mathrm{B}^{-p-q}A/\mathrm{F}_{p-1}\mathrm{B}^{-p-q}A$ is isomorphic to the bar construction of the associated graded algebra $\mathrm{gr}\, A$:

$$\left(\mathrm{E}^0_{pq}, d^0\right) \cong \mathrm{B}_p^{-p-q}\, \mathrm{gr}\, A,$$

where the index p refers to the total grading induced by the finer grading on the bar construction, see Fig. 4.2.

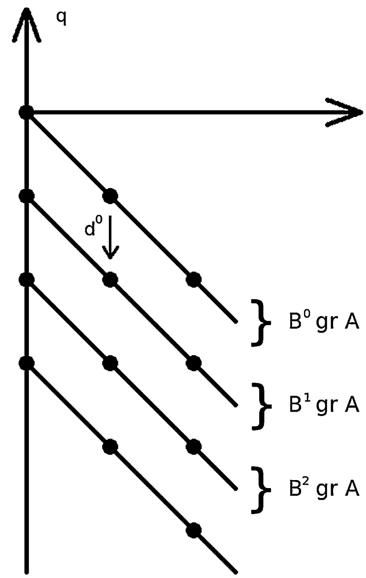


Fig. 4.2 The page E^0 of the spectral sequence

This latter algebra being Koszul, the homology of its bar construction is concentrated in syzygy degree $p+q=0$. This implies the collapsing of the spectral sequence at rank 1. The filtration being bounded below and exhaustive, it converges by Theorem 1.5.1:

$$\mathrm{E}^{\infty}_{pq} \cong \mathrm{E}^{1}_{pq} \cong \mathrm{gr}_p\, H^{-p-q}\big(\mathrm{B}^{\bullet} A\big) = 0, \quad \text{for } p+q \neq 0.$$

Since the homology of the bar construction of A is concentrated in syzygy degree 0, the algebra A is a Koszul algebra. □

Thanks to the finer filtration, we have reduced the Koszul problem for the algebra A to the algebra $\mathrm{gr}\, A$, whose product is simpler. But the underlying module of $\mathrm{gr}\, A$ might be difficult to describe. However, Exercise 3.8.1 implies that if the algebra $\mathrm{gr}\, A$ is Koszul, then it admits a quadratic presentation.

4.2.3 Quadratic Analog and Leading Space of Relations

By the universal property of the free algebra, there exists a morphism of algebras $T(V) \twoheadrightarrow \mathrm{gr}\, A$, which is an epimorphism of χ-graded algebras. Hence, it also preserves the weight grading. It is obviously an isomorphism in weights 0 and 1. Let us denote by R_{lead} the kernel of its restriction to $V^{\otimes 2}$. We consider the quadratic algebra defined by

$$\mathring{A} := T(V)/(R_{\mathrm{lead}}).$$

Let us now make R_{lead} explicit. Any element $r \in R$ decomposes according to the finer grading as $r = X_1 + \cdots + X_p$, with $X_i \in V^{\otimes 2}$ and where $X_p \neq 0$ is the term of greatest grading. We call X_p the *leading term* of r. The space R_{lead} is spanned by the leading terms of all the elements of R. Hence, we call it the *leading space of relations*.

Proposition 4.2.2. *Let $A = A(V, R)$ be a quadratic algebra equipped with an extra ordered grading. We have the following commutative diagram of epimorphisms of χ-graded, thus weight graded, algebras*

$$\begin{array}{ccc} T(V) & & \\ \downarrow & \searrow & \\ \psi : \mathring{A} = T(V)/(R_{\mathrm{lead}}) & \twoheadrightarrow & \mathrm{gr}\, A, \end{array}$$

where the space of relations R_{lead} is equal to

$$R_{\mathrm{lead}} = \langle X_p, r = X_1 + \cdots + \underbrace{X_p}_{\neq 0} \in R \rangle = \big\langle \textit{Leading Term } (r), r \in R \big\rangle.$$

By definition, the canonical projection of weight graded algebras $\psi : \mathring{A} \twoheadrightarrow \operatorname{gr} A$ is bijective in weights 0, 1 and 2. Therefore the algebra $\mathring{A}$ is the best candidate for a quadratic presentation of the weight graded algebra $\operatorname{gr} A$. Before studying this situation, let us give the following straightforward but useful result.

Proposition 4.2.3. *Let $A = A(V, R)$ be a quadratic algebra equipped with an extra ordered grading. If the algebra $\mathring{A} := T(V)/(R_{\text{lead}})$ is Koszul and if the canonical projection $\mathring{A} \cong \operatorname{gr} A$ is an isomorphism of algebras, then the algebra A is Koszul.*

Proof. In this case, the algebra $\operatorname{gr} A$ is Koszul and we conclude with Theorem 4.2.1. □

4.2.4 The Diamond Lemma

Under the assumption that the quadratic algebra $\mathring{A}$ is Koszul, the Diamond Lemma asserts that it is enough for the canonical projection $\psi : \mathring{A} \twoheadrightarrow \operatorname{gr} A$ to be injective in weight 3, to ensure that it is an isomorphism.

Theorem 4.2.4 (Diamond Lemma for quadratic algebras). *Let $A = A(V, R)$ be a quadratic algebra equipped with an extra ordered grading. Suppose that the quadratic algebra $\mathring{A} := T(V)/(R_{\text{lead}})$ is Koszul.*

If the canonical projection $\mathring{A} \twoheadrightarrow \operatorname{gr} A$ is injective in weight 3, then it is an isomorphism. In this case, the algebra A is Koszul.

Proof. We prove this theorem in two steps. We first filter the bar construction to get an isomorphism on the level of the Koszul dual coalgebras. Then we filter the cobar construction, in the same way, to get the final isomorphism.

Step 1. We consider the same filtration of the bar construction $\mathrm{B}^{-\bullet} A$ as in the proof of Theorem 4.2.1. Since the canonical projection ψ is an isomorphism in weight less than 3, the first page of the associated spectral sequence is equal to

$$\left(\mathrm{E}^0_{pq}, d^0\right) \cong \mathrm{B}_p^{-p-q} \operatorname{gr} A \cong \mathrm{B}_p^{-p-q} \mathring{A},$$

for $p + q \geq -2$ (the syzygy degree being defined by the weight grading minus 1). The algebra $\mathring{A}$ being Koszul, we get $\mathrm{E}^1_{pq} = 0$ for $p + q = -1$ and $\mathrm{E}^1_{p-p} = \mathring{A}^{¡}_p$, see Fig. 4.3.

We conclude by the same argument as in the proof of Theorem 4.2.1: the convergence of the spectral sequence shows that there is an isomorphism $\operatorname{gr}_p A^{¡} \cong \mathring{A}^{¡}_p$.

Step 2. Dually, we apply the same method and consider the same kind of filtration on $T^c(sV)$ as on $T(V)$ and on $\Omega T^c(sV)$ as on $\mathrm{B}T(V)$:

$$F_p \Omega T^c(sV) := \left\{s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_m \mid x_1 x_2 \ldots x_m \in F_p T(V)\right\}.$$

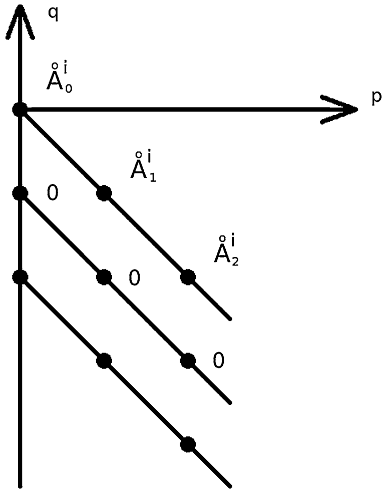


Fig. 4.3 The page E^1 of the spectral sequence

Since $A^{¡}$ is a sub-coalgebra of $T^c(sV)$, this filtration restricts to a filtration $\mathsf{F}_p\Omega A^{¡}$ on the cobar construction of $A^{¡}$. We consider the cobar construction $\Omega_\bullet A^{¡}$ with its syzygy degree, see Sect. 3.3.2. The first page of the associated spectral sequence $\mathsf{E}^0_{pq} \cong \mathsf{F}_p\Omega_{p+q}A^{¡}/\mathsf{F}_{p-1}\Omega_{p+q}A^{¡}$ is isomorphic to the cobar construction of the Koszul dual coalgebra $\mathring{A}^{¡}$:

$$\left(\mathsf{E}^0_{pq}, d^0\right) \cong \left(\Omega_{p+q}\,\mathrm{gr}\,A^{¡}\right)_p \cong \left(\Omega_{p+q}\mathring{A}^{¡}\right)_p.$$

The isomorphism between the underlying modules is induced by the aforementioned isomorphism $\mathrm{gr}_p A^{¡} \cong \mathring{A}^{¡}_p$. The part d^0 of the boundary map of $\Omega A^{¡}$ is the part which preserves the extra grading. Hence it is given by the deconcatenation of the leading terms of the elements of $A^{¡}$ or equivalently by the coproduct of the coalgebra $\mathring{A}^{¡}$ under the above isomorphism. This proves that d^0 is in one-to-one correspondence with the differential of $\Omega\mathring{A}^{¡}$. Since the quadratic algebra $\mathring{A}$ is Koszul, the homology of $\Omega\mathring{A}^{¡}$ is concentrated in syzygy degree $p+q=0$. Therefore, $\mathsf{E}^1_{pq}=0$ for $p+q\neq 0$ and $\mathsf{E}^1_{p\,-p}=\mathring{A}_p$. The convergence theorem for spectral sequences (Theorem 1.5.1) finally gives the desired isomorphism:

$$\psi : \mathring{A}_p \cong \mathsf{E}^1_{p\,-p} \cong \mathsf{E}^\infty_{p\,-p} \cong \mathrm{gr}_p\, H_0\left(\Omega_\bullet A^{¡}\right) = \mathrm{gr}_p\, A. \qquad \square$$

Notice that in the proof, we have proved the same isomorphism of χ-graded modules: $\mathrm{gr}_p A^{¡} \cong \mathring{A}^{¡}_p$, but on the level of the Koszul dual coalgebras.

4.2.5 The Inhomogeneous Case

When the associative algebra A is inhomogeneous quadratic, we choose, if possible, a presentation $A = A(V,R)$ with $R \subset V \oplus V^{\otimes 2}$ satisfying conditions (ql_1) and

(ql_2) of Sect. 3.6. Applying the previous propositions to the quadratic algebra $qA := A(V, qR)$, we get the same kind of results.

We start with the same refining data. Let the generating module V be endowed with an extra grading $V \cong V_1 \oplus \cdots \oplus V_k$ together with a suitable order on tuples. As above, this defines a filtration of the algebra A. It induces a χ-graded algebra, denoted $\mathrm{gr}_\chi A$, whose underlying module refines that of the weight graded algebra $\mathrm{gr}\, A$ of Sect. 3.6. This extra grading also induces a filtration on the quadratic algebra qA and the canonical projection $qA \twoheadrightarrow \mathrm{gr}\, A$ refines as follows

$$\begin{array}{ccccc} \mathring{q}A & & & & \\ \downarrow & & & & \\ \mathrm{gr}_\chi\, qA & \xrightarrow{\cong} & qA & & \\ \downarrow & & \downarrow & & \\ \mathrm{gr}_\chi\, A & \xrightarrow{\cong} & \mathrm{gr}\, A & \xrightarrow{\cong} & A, \end{array}$$

where the vertical maps are epimorphisms of algebras and where the horizontal maps are linear isomorphisms. The first column is made up of χ-graded algebras and the second column is made up of weight graded algebras. We only state the last theorem in the inhomogeneous case, leaving the details of the other ones to the reader.

Theorem 4.2.5 (Diamond Lemma for inhomogeneous quadratic algebras). *Let $A = A(V, R)$ be a quadratic-linear algebra with a presentation satisfying conditions (ql_1) and (ql_2). We suppose that $T(V)$ comes equipped with an extra ordered grading.*

If the quadratic algebra $\mathring{q}A$ is Koszul and if the canonical projection $\mathring{q}A \twoheadrightarrow \mathrm{gr}_\chi\, qA$ is injective in weight 3, *then the algebra A is Koszul and all the maps of the above diagram are isomorphisms*, *in particular*:

$$\mathring{q}A \cong \mathrm{gr}_\chi\, qA \cong qA \cong \mathrm{gr}\, A \cong A.$$

Proof. The Diamond Lemma 4.2.4, applied to the quadratic algebra qA, gives that the algebra qA is Koszul and the isomorphism $\mathring{q}A \cong \mathrm{gr}_\chi\, qA$. It implies that the inhomogeneous algebra A is Koszul. The last isomorphism is given by the PBW Theorem 3.6.4. □

In this case, we can compute the Koszul dual dg coalgebra of A from the isomorphism of weight graded modules $(\mathring{q}A)^{¡} \cong qA^{¡} \cong A^{¡}$.

4.2.6 Koszul Dual Algebra

In this section, we suppose that the generating space V is finite dimensional to be able to consider the Koszul dual algebra $A^! = T(V^*)/(R^\perp)$, see Sect. 3.2.2. The extra grading on $V \cong V_1 \oplus \cdots \oplus V_k$ induces a dual grading $V^* \cong V_k^* \oplus \cdots \oplus V_1^*$, that we choose to order as indicated. To any suitable order on the tuples indexing $T(V)$, we consider the locally reversed order for $T(V^*)$: we keep the global ordering but completely reverse the total order on the subset of n-tuples, for any n. For example, with the lexicographic order, it gives the following grading on $T(V^*)$:

$$T(V^*) = \mathbb{K} \bigoplus V_k^* \oplus \cdots \oplus V_1^* \bigoplus V_k^* \otimes V_k^* \oplus \cdots \oplus V_1^* \otimes V_1^* \bigoplus V_k^* \otimes V_k^* \otimes V_k^* \oplus \cdots .$$

This order allows us to consider the graded algebra $\mathrm{gr}(A^!)$, as in Sect. 4.2.1, which comes with its quadratic analog $(\mathring{A}^!) := A(V^*, (R^\perp)_{\mathrm{lead}}) \twoheadrightarrow \mathrm{gr}(A^!)$.

Lemma 4.2.6. *Let $A = A(V, R)$ be a finitely generated quadratic algebra equipped with an extra ordered grading. The isomorphism $(R^\perp)_{\mathrm{lead}} \cong (R_{\mathrm{lead}})^\perp$ of submodules of $(V^*)^{\otimes 2}$ induces the isomorphism of quadratic algebras $(\mathring{A}^!) \cong (\mathring{A})^!$.*

Proof. We write the proof for the lexicographic order, the general case being similar. Let us denote $\mathsf{R}_i := R_{\mathrm{lead}} \cap T(V)_{k+i}$, for $1 \le i \le k^2$. Therefore, $R_{\mathrm{lead}} = \mathsf{R}_1 \oplus \cdots \oplus \mathsf{R}_{k^2}$. Since the vector space V is finite dimensional, each $T(V)_{k+i}$ is finite dimensional and we can consider a direct summand $\breve{\mathsf{R}}_i$ such that $T(V)_{k+i} \cong \mathsf{R}_i \oplus \breve{\mathsf{R}}_i$. Hence the linear dual is equal to $(T(V)_{k+i})^* \cong \mathsf{R}_i^\perp \oplus \breve{\mathsf{R}}_i^\perp$.

By definition of R_{lead}, the space of relations R is linearly spanned by elements of the form $r = X_1 + \cdots + X_p$ with $X_i \in \breve{\mathsf{R}}_i$, for $1 \le i < p \le k^2$ and $X_p \ne 0$, $X_p \in \mathsf{R}_p$. Dually, any element $\rho \in R^\perp$ decomposes as $\rho = Y_q - Y_{q+1} - \cdots - Y_{k^2}$ with $Y_q \ne 0$, $Y_q \in \mathsf{R}_q^\perp$ and $Y_i \in \breve{\mathsf{R}}_i^\perp$, for $q < i \le k^2$. This implies finally the isomorphism $(R^\perp)_{\mathrm{lead}} \cong \mathsf{R}_1^\perp \oplus \cdots \oplus \mathsf{R}_{k^2}^\perp \cong (R_{\mathrm{lead}})^\perp$. □

Proposition 4.2.7. *Let $A = A(V, R)$ be a finitely generated quadratic algebra equipped with an extra ordered grading. Suppose that the quadratic algebra $\mathring{A} := T(V)/(R_{\mathrm{lead}})$ is Koszul. If the canonical projection $\mathring{A} \twoheadrightarrow \mathrm{gr}\, A$ is injective in weight 3, then the dual canonical projection $(\mathring{A}^!) \cong \mathrm{gr}(A^!)$ is an isomorphism.*

Proof. We pursue the proof of Theorem 4.2.4. The proper desuspension of the linear dual of the isomorphism $\mathrm{gr}_p\, A^{¡} \cong \mathring{A}^{¡}_p$ gives the isomorphism $\mathrm{gr}_p\, A^! \cong (\mathring{A})^!_p$. We conclude with the isomorphism of Lemma 4.2.6. □

4.3 Poincaré–Birkhoff–Witt Bases and Gröbner Bases

In this section, we study the particular case of the preceding section when the generating space V is equipped with an extra grading $V \cong V_1 \oplus \cdots \oplus V_k$ such that each sub-space V_i is one-dimensional. This gives rise to the notion of Poincaré–Birkhoff–Witt basis, or PBW basis for short. Quadratic algebras which admit such a basis share nice properties. For instance, they are Koszul algebras.

We introduce the equivalent notion of quadratic Gröbner basis (also called Gröbner–Shirshov basis), which is to the ideal (R) what PBW basis is to the quotient algebra $T(V)/(R)$.

The notion of PBW basis comes from Sect. 5 of the original paper of S. Priddy [Pri70]. We refer the reader to Chap. 4 of the book [PP05] for more details on the subject.

4.3.1 Ordered Bases

We now restrict ourself to the case where the generating space V of a quadratic algebra $A(V, R)$ is equipped with an extra grading $V \cong V_1 \oplus \cdots \oplus V_k$ such that each V_i is one-dimensional. This datum is equivalent to a *totally ordered basis* $\{v_i\}_{i\in\{1,\ldots,k\}}$ of V, which is a basis labeled by a totally ordered set, by definition. Let us denote $I := \{1, \ldots, k\}$ and let us use the convention $I^0 := \{0\}$. As in Sect. 4.2.1, we consider the set $J := \bigsqcup_{n\geq 0} I^n$ of tuples $\bar{i} = (i_1, \ldots, i_n)$ in $\{1, \ldots, k\}$ equipped with a suitable order, for instance the lexicographic order. With this definition, the set $\{v_{\bar{i}} = v_{i_1} v_{i_2} \cdots v_{i_n}\}_{\bar{i}\in J}$ becomes a totally ordered basis of $T(V)$. In this case, we say that $T(V)$ is equipped with a *suitable ordered basis*.

Written in this basis, the space of relations R is equal to

$$R = \left\{ \lambda v_i v_j - \sum_{(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l;\ \lambda \neq 0 \right\}.$$

Proposition 4.3.1. *Let A be a quadratic algebra $A(V, R)$, with $T(V)$ equipped with a suitable ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$. The associated quadratic algebra $\mathring{A}$ is equal to $A(V, R_{\text{lead}})$, with $R_{\text{lead}} \cong \langle v_i v_j, (i, j) \in \overline{L}^{(2)}\rangle$, where the set $\overline{L}^{(2)}$ is the set of labels of the leading terms of the relations of R.*

Proof. It is a direct corollary of Proposition 4.2.2. □

Notice that the space of relations admits a normalized basis of the form

$$R = \left\langle v_i v_j - \sum_{(k,l)\notin\overline{L}^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l;\ (i, j) \in \overline{L}^{(2)} \right\rangle.$$

The algebra $\mathring{A}$ depends only on the ordered basis of V and on the suitable order on tuples, but it is always a quadratic algebra whose ideal is generated by monomial elements.

4.3.2 Quadratic Monomial Algebras

We introduce the notion of quadratic monomial algebra, which is the structure carried by the quadratic algebra $\mathring{A}$ in the PBW bases theory.

A *quadratic monomial algebra* is a quadratic algebra $\mathring{A} = A(V, R) = T(V)/(R)$ with a (non-necessarily ordered) basis $\{v_i\}_{i\in I}$ of V such that the space of relations R is linearly spanned by a set $\{v_i v_j\}_{(i,j)\in\overline{L}^{(2)}}$, where $\overline{L}^{(2)} \subset I^2$. We denote by $L^{(2)}$ the complement of $\overline{L}^{(2)}$ in I^2, $L^{(2)} := I^2\backslash\overline{L}^{(2)}$, which labels a basis of the quotient $\mathring{A}^{(2)} = V^{\otimes 2}/R$. We set $L^{(0)} := \{0\}$ and $L^{(1)} := I$. We will prove that a quadratic monomial algebra is Koszul by making its Koszul complex explicit and by computing its homology.

Proposition 4.3.2. *For any quadratic monomial algebra $\mathring{A} = A(V, R)$, the subset $L = \bigsqcup_{n\in\mathbb{N}} L^{(n)} \subset J$ defined by*

$$\bar{i} = (i_1, \dots, i_n) \in L^{(n)} \iff (i_m, i_{m+1}) \in L^{(2)}, \quad \forall 1 \le m < n$$

labels a basis of the monomial algebra $\mathring{A}$.

Though this statement is obvious, it will play a key role in the definition of PBW bases. Dually, we make explicit a monomial basis for the Koszul dual coalgebra.

Proposition 4.3.3. *For any quadratic monomial algebra $\mathring{A} = A(V, R)$, the subset $\overline{L} = \bigsqcup_{n\in\mathbb{N}} \overline{L}^{(n)} \subset J$ defined by*

$$\bar{i} = (i_1, \dots, i_n) \in \overline{L}^{(n)} \iff (i_m, i_{m+1}) \in \overline{L}^{(2)}, \quad \forall 1 \le m < n$$

labels a basis of its Koszul dual coalgebra $\mathring{A}^{¡}$.

When the generating space V is finite dimensional, the Koszul dual algebra $\mathring{A}^!$ is also a monomial algebra, with presentation

$$\mathring{A}^! \cong A\big(V^*, R^\perp\big), \quad \textit{with } R^\perp = \big\langle v_i^* v_j^*, (i, j) \in L^{(2)}\big\rangle$$

and basis labeled by $\overline{L}$.

Proof. The elements $s^n v_{\bar{i}}$ for $\bar{i} \in \overline{L}^{(n)}$ form a basis of $(\mathring{A}^{¡})^{(n)}$ by the intersection formula $(\mathring{A}^{¡})^{(n)} = \bigcap_{i+2+j=n} (sV)^{\otimes i} \otimes s^2 R \otimes (sV)^{\otimes j}$ of Sect. 3.1.3. □

Theorem 4.3.4. *Any quadratic monomial algebra is a Koszul algebra.*

Proof. By the two preceding propositions, the Koszul complex $\mathring{A}^{¡} \otimes_\kappa \mathring{A}$ admits a basis of the form $s^k v_{\bar{i}} \otimes v_{\bar{j}}$ with $\bar{i} \in \overline{L}^{(k)}$ and with $\bar{j} \in L^{(l)}$. In this basis, its boundary map is equal to

$$d_\kappa\big(s^k v_{\bar{i}} \otimes v_{\bar{j}}\big) = \pm s^{k-1} v_{i_1} \dots v_{i_{k-1}} \otimes v_{i_k} v_{j_1} \dots v_{j_l}, \quad \text{when } (i_k, j_1) \in L^{(2)}, \quad \text{and}$$
$$d_\kappa\big(s^k v_{\bar{i}} \otimes v_{\bar{j}}\big) = 0, \quad \text{when } (i_k, j_1) \in \overline{L}^{(2)}.$$

In the latter case, such a cycle is a boundary since $(i_1, \dots, i_k, j_1) \in \overline{L}^{(k+1)}$ and $d_\kappa(s^{k+1} v_{i_1} \dots v_{i_k} v_{j_1} \otimes v_{j_2} \dots v_{j_l}) = \pm s^k v_{\bar{i}} \otimes v_{\bar{j}}$. Finally the Koszul complex is acyclic and the algebra $\mathring{A}$ is Koszul. □

This result is a key point in the PBW basis theory. It says that when the decomposition $V \cong V_1 \oplus \cdots \oplus V_k$ is made up of one-dimensional sub-spaces, the quadratic analog $\mathring{A}$ is always a Koszul algebra.

4.3.3 PBW Basis

The image of the monomial basis $\{v_{\bar{i}}\}_{\bar{i}\in L}$ of $\mathring{A}$, given in Proposition 4.3.2, under the successive morphisms of graded modules $\mathring{A} \twoheadrightarrow \mathrm{gr}\, A \cong A$ provides a family of elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$, which linearly span the algebra A. When these elements are linearly independent, they form a basis of the algebra A, called a *Poincaré–Birkhoff–Witt basis*, or *PBW basis* for short. This condition corresponds to the bijection of the canonical projection $\psi : \mathring{A} \twoheadrightarrow \mathrm{gr}\, A$. We say that an algebra $A = A(V, R)$ admits a PBW basis if there exists a totally ordered basis of V and a suitable order on tuples such that the associated elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ form a basis of the algebra A.

EXAMPLE. The symmetric algebra $S(v_1, \ldots, v_k)$ admits the following PBW basis, with the lexicographic order: $\{v_1^{\nu_1} \ldots v_k^{\nu_k}\}$ with $\nu_1, \ldots, \nu_k \in \mathbb{N}$.

The main property of PBW bases lies in the following result.

Theorem 4.3.5. *Any quadratic algebra endowed with a PBW basis is Koszul.*

Proof. Since the monomial algebra $\mathring{A}$ is always Koszul by Proposition 4.3.4, it is a direct corollary of Proposition 4.2.3. □

The existence of a PBW basis gives a purely algebraic condition to prove that an algebra is Koszul, without having to compute any homology group.

There are Koszul algebras which do not admit any PBW basis. The quadratic algebra $A(V, R)$ generated by $V := \mathbb{K}x \oplus \mathbb{K}y \oplus \mathbb{K}z$ with the two relations $x^2 - yz$ and $x^2 + 2zy$ is Koszul but does not admit a PBW basis. This example comes from Sect. 4.3 of [PP05] and is due to J. Backelin.

4.3.4 Diamond Lemma for PBW Bases

Since the canonical projection $\psi : \mathring{A} \twoheadrightarrow \mathrm{gr}\, A$ is bijective in weights 0, 1 and 2, the elements $\{a_{\bar{i}}\}_{\bar{i}\in L^{(n)}}$ form a basis of $A^{(n)}$, for $n \leq 2$. It is enough to check only the next case, $n = 3$, as the following theorem shows.

Theorem 4.3.6 (Diamond Lemma for PBW bases). *Let $A = A(V, R)$ be a quadratic algebra, with $T(V)$ equipped with a suitable ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$. If the elements $\{a_{\bar{i}}\}_{\bar{i}\in L^{(3)}}$ are linearly independent in $A^{(3)}$, then the elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ form a PBW basis of A. In that case, the algebra A is Koszul.*

Proof. It is a direct corollary of Theorems 4.3.4 and 4.2.4. □

EXAMPLE. Let us consider the quadratic algebra $A := A(V, R)$ generated by a two-dimensional vector space $V := \mathbb{K}v_1 \oplus \mathbb{K}v_2$ with relation $R := \mathbb{K}(v_1v_2 - v_1^2)$. With the lexicographic order $1 < 2 < (1, 1) < \cdots$, we have $R_{\text{lead}} = \mathbb{K}v_1v_2$, $\overline{L}^{(2)} = \{(1, 2)\}$ and $L^{(2)} = \{(1, 1), (2, 1), (2, 2)\}$. One easily verifies that the elements $a_2a_2a_2$, $a_2a_2a_1$, $a_2a_1a_1$ and $a_1a_1a_1$ are linearly independent in A. Therefore, the family of monomial elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ indexed by $L^{(n)} = \{(2, 2, \ldots, 2), (2, 2, \ldots, 2, 1), \ldots, (1, 1, \ldots, 1)\}$, for $n \in \mathbb{N}$, form a PBW basis of the algebra A.

COUNTER-EXAMPLE. Let us consider the same quadratic algebra with the extra relation $v_2^2 - v_1^2$. In this case, $R_{\text{lead}} = \mathbb{K}v_1v_2 \oplus \mathbb{K}v_2^2$, $\overline{L}^{(2)} = \{(1, 2), (2, 2)\}$ and $L^{(2)} = \{(1, 1), (2, 1)\}$. Therefore, the monomial basis of the quadratic algebra $\mathring{A}$ is indexed by $L^{(n)} = \{(2, 1, \ldots, 1), (1, 1, \ldots, 1)\}$. In weight 3, the relation $a_1a_1a_1 = a_2a_1a_1$, obtained by calculating $a_2a_2a_2$ by two different methods, shows that it does not form a PBW basis.

4.3.5 Recollection with the Classical Diamond Lemma

The aforementioned result can also be seen as a direct consequence of the classical Diamond Lemma of G.M. Bergman [Ber78], which comes from graph theory [New42].

Let us first recall the statement. Starting from a vertex in an oriented graph, one might have the choice of two outgoing edges. Such a configuration is called an *ambiguity* in rewriting systems. An ambiguity is called *solvable* or *confluent* if, starting from each of these two edges, there exists one path ending at a common vertex. In this case, we get a diamond shape graph like in Fig. 4.4. Under the condition that any path has an end (*termination hypothesis*), the classical Diamond Lemma asserts that, if every ambiguity is confluent, then any connected component of a graph has a unique terminal vertex.

The relationship with ring theory comes from the following graphical representation. We depict the elements of $T(V)$ by vertices of a graph with edges labeled by the relations of R oriented from the leading term to the rest. In the case of PBW bases, we restrict ourself to the generating relations given at the end of Sect. 4.3.1:

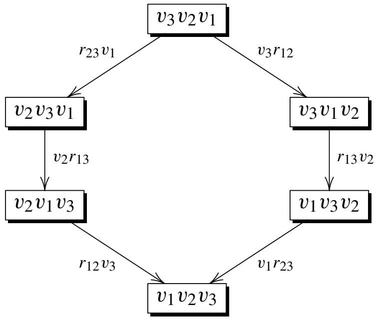
$$v_iv_j \longmapsto \sum_{(k,l)\in L^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_kv_l, \ (i, j) \in \overline{L}^{(2)}.$$

EXAMPLE. For instance, in the symmetric algebra

$$S(v_1, v_2, v_3) \\ := A\big(\{v_1, v_2, v_3\}, \{r_{12} = v_2v_1 - v_1v_2, r_{23} = v_3v_2 - v_2v_3, r_{13} = v_3v_1 - v_1v_3\}\big)$$

(see Fig. 4.4).

Fig. 4.4 Hexagonal diamond



In this example, there are two ways (ambiguity) of rewriting the monomial $v_3v_2v_1$, which finally end up on the same element (confluence). Notice that this diamond is exactly the Yang–Baxter equation.

Therefore, the connected graphs depict the successive relations applied to elements in $A^{(n)}$. The terminal vertices correspond to linear combinations of monomials labeled by elements of $L^{(n)}$; there is no more leading term of any relation inside them. Having two different terminal vertices would imply that the two associated elements are equal in A. Therefore, the set L labels a PBW basis of the algebra A if and only if every connected graph has a unique terminal vertex. The PBW basis is then made up of the labels of these terminal vertices, like the element $v_1v_2v_3$ in the above example.

To prove the Diamond Lemma for PBW bases from the classical Diamond Lemma, one has just to prove that any path is finite, which is given by the bounded below suitable order, and that any ambiguity is confluent. There are here only two types of ambiguities: the square type ambiguities where one applies two relations to two distinct sub-monomials, of the same monomial or of two different monomials of a sum, and the ones starting from a sub-monomial of length 3, where one either applies a relation to the two first elements or to the two last elements. The first type is obviously confluent. The confluence of the second type of ambiguities is precisely given by the assumption that the elements labeled by $L^{(3)}$ are linearly independent. These weight 3 monomials $v_iv_jv_k$ are called *critical*. They are such that v_iv_j and v_jv_k are both leading terms of some relator.

Bergman in [Ber78] extended the Diamond Lemma beyond the set theoretic case of graph theory and showed that it is enough to check the confluence condition on monomials as the next proposition shows.

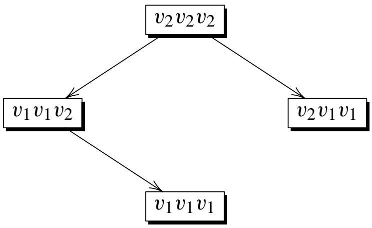
Theorem 4.3.7. *Let $A = A(V, R)$ be a quadratic algebra, with $T(V)$ equipped with an extra ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$. If the ambiguities coming from the critical monomials are confluent, then the elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ form a PBW basis of A and A is Koszul.*

Proof. The monomial elements with the second type of ambiguities are the elements labeled $\overline{L}^{(3)}$. Suppose that there exists a nontrivial linear combination between elements labeled by $L^{(3)}$. In terms of graph, it corresponds to a zig-zag like

$$X \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \cdots \leftarrow \bullet \rightarrow \bullet \rightarrow Y,$$

where $X = Y$ is the relation, with X and Y two sums of elements labeled by $L^{(3)}$. Since all the ambiguities are confluent by hypothesis, we can find another zig-zag, where the distance between X and the first $\leftarrow \bullet \rightarrow$ is strictly less than in the first zig-zag. By iteration, we prove the existence of a zig-zag of the shape $X \leftarrow \bullet \rightarrow \cdots$. Finally, by confluence, there exists an edge leaving X, which is impossible. We conclude with the Diamond Lemma 4.3.6. □

Finally, to prove that one has a PBW basis, it is enough to draw the graphs generated by elements of $\overline{L}^{(3)}$ only and to show that each of them has only one terminal vertex. We refer to the above figure for the example of the symmetric algebra. In the counterexample of Sect. 4.3.4, the element $v_2v_2v_2$ gives the following graph



We have finally proved here the rewriting method Theorem 4.1.1 given in Sect. 4.1.

4.3.6 Product of Elements of a PBW Basis

The canonical projection $\psi : \mathring{A} \twoheadrightarrow A$ is an epimorphism of graded modules, but not of algebras in general. Therefore, the product of two elements of the generating family $\{a_{\bar{i}}\}_{\bar{i}\in L}$ is not always equal to an element of this family, but to a sum of lower terms, as the following proposition shows.

Proposition 4.3.8. *The elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ satisfy the following property: for any pair $\bar{i}, \bar{j} \in L$, if $(\bar{i}, \bar{j}) \notin L$ the product $a_{\bar{i}}a_{\bar{j}}$ in A can be written as a linear combination of strictly lower terms labeled by L:*

$$a_{\bar{i}}a_{\bar{j}} = \sum_{\bar{l}\in L, \bar{l}<(\bar{i},\bar{j})} \lambda_{\bar{l}}^{\bar{i},\bar{j}}\, a_{\bar{l}},$$

with $\lambda_{\bar{l}}^{\bar{i},\bar{j}} \in \mathbb{K}$.

Proof. The proof is done by a simple induction argument with the suitable order on tuples. □

In the example of Sect. 4.3.4, we have $(a_2a_1)(a_2a_1) = a_2a_1a_1a_1$. The original definition of a PBW basis given by Priddy in [Pri70] is a basis made up of a family of monomial elements $\{a_{\bar{i}}\}_{\bar{i}\in L}$ of A labeled by a set $L \subset J$, which satisfies the property of Proposition 4.3.8 and: any $\bar{i} = (i_1, \ldots, i_n) \in L$ if and only if $(i_1, \ldots, i_m) \in L$ and $(i_{m+1}, \ldots, i_n) \in L$ for any $1 \le m < n$.

We leave it to the reader to check that this definition of a PBW basis is equivalent to the one given in Sect. 4.3.3.

4.3.7 Gröbner Bases

In this section, we introduce the notion of (noncommutative) Gröbner basis for an ideal I of the free algebra, see [Buc06]. In the quadratic case, when $I = (R)$, it is equivalent to a PBW basis for the quotient algebra $A = T(V)/(R)$.

Any element P in $T(V)$ is a linear combination of monomials. When $T(V)$ is equipped with a suitable ordered basis, we denote by P_{lead} the leading term of P. For any subset $M \subset T(V)$, we consider the set made up of the leading terms of any element of M and we denote it by $\mathrm{Lead}(M)$. Under this notation, the space of relations R_{lead} of Proposition 4.3.1 is equal to the linear span of $\mathrm{Lead}(R)$: $R_{\mathrm{lead}} = \langle \mathrm{Lead}(R)\rangle$.

A (*noncommutative*) *Gröbner basis of an ideal* I *in* $T(V)$ is a set $G \subset I$ which generates the ideal I, i.e. $(G) = I$, such that the leading terms of G and the leading terms of the elements of I generate the same ideal: $(\mathrm{Lead}(G)) = (\mathrm{Lead}(I))$.

Proposition 4.3.9. *Let $A = A(V, R)$ be a quadratic algebra such that $T(V)$ is equipped with a suitable ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$. The elements $\{a_{\bar{l}}\}_{\bar{l}\in L}$ form a PBW basis of A if and only if the elements*

$$\left\{ v_i v_j - \sum_{(k,l)\in L^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l \right\}_{(i,j)\in \overline{L}^{(2)}},$$

spanning R, form a Gröbner basis of the ideal (R) in $T(V)$.

Proof. (⇒) When L labels a PBW basis, the elements

$$\left\{ v_{\bar{i}} - \sum_{\bar{l}\in L, \bar{l}<\bar{i}} \lambda_{\bar{l}}^{\bar{i}} v_{\bar{l}} \right\}_{\bar{i}\in J\setminus L},$$

form a linear basis of (R). The leading terms of (R) are $\mathrm{Lead}(R) = \{\mathbb{K}v_{\bar{i}}\}_{\bar{i}\in J\setminus L}$ and $\mathrm{Lead}(v_i v_j - \sum_{(k,l)\in L^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l) = v_i v_j$. Condition (2) implies that $\bar{i} =$

$(i_1, \ldots, i_n) \in J\backslash L$ if and only if there exists $1 \le m < n$ such that $(i_m, i_{m+1}) \in \overline{L}^{(2)}$. Therefore, the two following ideals are equal $(v_i v_j, (i, j) \in \overline{L}^{(2)}) = (v_{\bar{i}}, \bar{i} \in J\backslash L)$.

($\Leftarrow$) We show that any $\bar{l} \in L$, $a_{\bar{l}}$ is not equal in A to a linear combination of strictly lower terms labeled by L. Suppose that there is an $\bar{l} \in L$ such that $a_{\bar{l}} = \sum_{\bar{k}\in L, \bar{k}<\bar{l}} \lambda_{\bar{k}}^{\bar{l}} a_{\bar{k}}$, with $\lambda_{\bar{l}}^{\bar{i}} \in \mathbb{K}$. This is equivalent to $v_{\bar{l}} - \sum_{\bar{k}\in L, \bar{k}<\bar{l}} \lambda_{\bar{k}}^{\bar{l}} v_{\bar{k}} \in (R)$, whose leading term is $v_{\bar{l}}$. By definition, this element belongs to the ideal generated by the elements $v_i v_j$ with $(i, j) \in \overline{L}^{(2)}$, which is impossible by condition (2). □

In the quadratic case, the two notions of PBW basis and Gröbner basis are equivalent dual notions. The terminology "PBW basis" refers to the basis of the quotient algebra while the terminology "noncommutative Gröbner basis" refers to the ideal (R).

We refer to Sect. 2.12 of [Ufn95] for the history of the Gröbner–Shirshov bases.

4.3.8 PBW Bases for Inhomogeneous Quadratic Algebras

Following Sect. 4.2.5, we say that *an inhomogeneous quadratic algebra A admits a PBW basis* if there exists a presentation $A = A(V, R)$, satisfying conditions (ql_1) and (ql_2), such that the associated quadratic algebra $qA = A(V, qR)$ admits a PBW basis. In this case, the image $\{a_{\bar{i}}\}_{\bar{i}\in L} \subset A$ of the basis elements $\{v_{\bar{i}}\}_{\bar{i}\in L}$ of the quadratic monomial algebra $q\mathring{A}$ gives a basis of the inhomogeneous quadratic algebra A. Such a result is once again proved using the following version of the Diamond Lemma.

Theorem 4.3.10. *Let $A = A(V, R)$ be an inhomogeneous quadratic algebra with a quadratic-linear presentation satisfying conditions (ql_1) and (ql_2) and such that $T(V)$ is equipped with a suitable ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$.*

If the images of the elements $\{v_{\bar{i}}\}_{\bar{i}\in L^{(3)}}$ in qA are linearly independent, then the images $\{a_{\bar{i}}\}_{\bar{i}\in L}$ of the elements $\{v_{\bar{i}}\}_{\bar{i}\in L}$ form a basis of A and the algebra A is Koszul.

Proof. It is a particular case of Theorem 4.2.5. □

In the example of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (cf. Sect. 3.6.7), the symmetric monomials basis of $S(\mathfrak{g})$ induces a PBW basis of $U(\mathfrak{g})$. With the suitable order

$$0 < 1 < 2 < (1, 1) < 3 < (2, 1) < (1, 2) < (1, 1, 1) < 4 < (3, 1) < (2, 2) < \cdots,$$

the Cartan–Serre basis of Sect. 3.6.8 is a PBW basis of the Steenrod algebra.

In the inhomogeneous case too, the notion of PBW basis is equivalent and dual to that of Gröbner basis.

Proposition 4.3.11. *Let A be an inhomogeneous quadratic algebra with a quadratic-linear presentation $A = A(V, R)$ satisfying conditions (ql_1) and (ql_2) and such that $T(V)$ is equipped with a suitable ordered basis $\{v_{\bar{i}}\}_{\bar{i}\in J}$. Let $\varphi : qR \to V$ be the linear map whose graph gives R. The elements $\{a_{\bar{l}}\}_{\bar{l}\in L}$ form a PBW basis of A if and only if the elements*

$$\left\{(\mathrm{Id}-\varphi)\left(v_i v_j - \sum_{(k,l)\in L^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l\right)\right\}_{(i,j)\in \overline{L}^{(2)}}$$

spanning R, form a Gröbner basis of the ideal (R) in $T(V)$.

Proof. The proof of the inhomogeneous case is similar but uses the PBW Theorem 3.6.4. □

4.3.9 PBW Basis of the Koszul Dual Algebra

Proposition 4.2.7 shows that any PBW basis of a quadratic algebra induces a dual PBW basis on the Koszul dual algebra. In this section, we provide further details.

Let $\{v_i\}_{i\in I}$ be a finite ordered basis of the vector space V and consider a suitable order on tuples. The elements $\{a_k a_l\}_{(k,l)\in L^{(2)}}$ form a basis of $V^{\otimes 2}/R$. In this case, there are elements

$$\left\{v_i v_j - \sum_{(k,l)\in L^{(2)},(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l\right\}_{(i,j)\in \overline{L}^{(2)}},$$

which form a basis of R. So the complement set $\overline{L}^{(2)} = I^2 \setminus L^{(2)}$ labels a basis of R, which is not $\{v_i v_j\}_{(i,j)\in\overline{L}^{(2)}}$ itself in general. The dual elements $\{v_i^*\}_{i\in I}$ provide a dual basis of V^* and the elements

$$\left\{v_k^* v_l^* + \sum_{(i,j)\in \overline{L}^{(2)},(i,j)>(k,l)} \lambda_{k,l}^{i,j} v_i^* v_j^*\right\}_{(k,l)\in L^{(2)}}$$

provide a basis of $R^\perp$. Therefore the image of the elements $\{v_i^* v_j^*\}_{(i,j)\in\overline{L}^{(2)}}$ in $A^! = A(V^*, R^\perp)$, denoted $a_i^* a_j^*$, form a basis of $V^{*\otimes 2}/R^\perp$. We consider the opposite order $i \leqslant^{\mathrm{op}} j$, defined by $i \geqslant j$, on the labeling set I of the dual basis of V^*.

Theorem 4.3.12. *Let $A = A(V, R)$ be a quadratic algebra endowed with a PBW basis $\{a_{\bar{i}}\}_{\bar{i}\in L}$. Its Koszul dual algebra $A^!$ admits the PBW basis $\{a_{\bar{j}}^*\}_{\bar{j}\in\overline{L}}$, with opposite order.*

Proof. It is a direct corollary of Proposition 4.2.7. □

When the algebra $A = A(V, R)$ admits a PBW basis labeled by a set L, the Koszul dual coalgebra $A^{¡}$ admits a basis indexed by the set $\overline{L}$ by Proposition 4.2.3. More precisely, the Koszul dual coalgebra admits a basis of the form

$$\left\{ s^n v_{\bar{i}} + \sum_{\bar{j} < \bar{i},\ \bar{j} \in L^{(n)}} s^n \lambda_{\bar{j}}^{\bar{i}} v_{\bar{j}},\ \bar{i} \in \overline{L}^{(n)} \right\}_{n \in \mathbb{N}}.$$

4.4 Koszul Duality Theory and Lattices

We introduce a combinatorial criterion for Koszulity. It states that a certain family of lattices associated to a quadratic data is distributive if and only if the quadratic data is Koszul (Backelin criterion). It will allow us to prove that the Koszul property is stable under Manin products in the next section.

4.4.1 Poset and Lattice

This section recalls the basic properties of posets (partially ordered sets) and lattices. It mainly comes from R.P. Stanley's book [Sta97a].

In a poset with a partial order denoted $\leqslant$, a *least upper bound* z for two elements x and y, when it exists, is an upper bound, meaning $x \leqslant z$ and $y \leqslant z$, which is less than any other upper bound. When it exists, it is unique. It is denoted by $x \vee y$ and is called the *join*. Dually, there is the notion of *greatest lower bound* which is denoted by $x \wedge y$ and called the *meet*.

A *lattice* is a poset where the join and the meet exist for every pair of elements. These two operations are associative, commutative and idempotent, that is $x \vee x = x = x \wedge x$. They satisfy the *absorption law* $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and the partial order can be recovered by $x \leqslant y \iff x \wedge y = x \iff x \vee y = y$. A *sublattice generated by a subset* of a lattice L is the smallest sublattice of L stable for the operations join and meet. It is explicitly composed by the elements obtained by composing the generating elements with the operations join and meet.

A lattice is *distributive* if it satisfies the equivalent *distributivity relations*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \iff x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

The subsets of a set form a distributive lattice where the partial order is defined by the inclusion $\subset$ and where the join and meet are given by the union $\cup$ and the intersection $\cap$ respectively. Actually, any finite distributive lattice is of this form (fundamental theorem for finite distributive lattices). Notice also that a distributive sublattice generated by a finite number of elements is finite.

In the linear context, we will consider the lattice of sub-spaces of a vector space. The order is given by the inclusion $\subset$ and the join and meet are given by the sum

$+$ and the intersection $\cap$ respectively. In the sequel, we will study finitely generated sublattices of such a lattice. Our main tool will be the following result, which is the analog, in the linear setting, of the fundamental theorem for finite distributive lattices.

Lemma 4.4.1. *Let U be a vector space and let L be a finitely generated sublattice of the lattice of sub-spaces of U. The lattice L is distributive if and only if there exists a basis $\mathscr{B}$ of U such that, $\mathscr{B} \cap X$ is a basis of X, for any $X \in L$.*

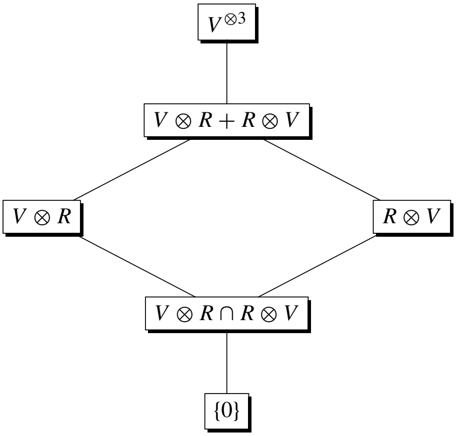
In this case, we say that the basis $\mathscr{B}$ *distributes* the sublattice L.

4.4.2 Lattice Associated to a Quadratic Data

Let (V, R) be a quadratic data. For every $n \in \mathbb{N}$, we consider the lattice of sub-spaces of $V^{\otimes n}$, where the order is given by the inclusion: $X \leqslant Y$ if $X \subset Y$. The join of two sub-spaces X and Y is their sum $X \vee Y := X + Y$ and their meet is the intersection $X \wedge Y := X \cap Y$.

For every $n \in \mathbb{N}$, we denote by $L(V, R)_{(n)}$ the sublattice of the lattice of sub-spaces of $V^{\otimes n}$ generated by the finite family $\{V^{\otimes i} \otimes R \otimes V^{\otimes n-2-i}\}_{i=0,\ldots,n-2}$.

The example of $L(V, R)_{(3)}$ is depicted below.



4.4.3 Backelin's Criterion

The following result belongs to the long list of properties between the Koszul duality theory and the poset theory.

Theorem 4.4.2 (Backelin [Bac83]). *A quadratic data (V, R) is Koszul if and only if the lattices $L(V, R)_{(n)}$ are distributive, for every $n \in \mathbb{N}$.*

In this case, the lattices $L(V,R)_{(n)}$, $n \geq 0$ are finite. We refer the reader to the Ph.D. thesis of J. Backelin [Bac83] and to the book of A. Polishchuk and L. Positselski [PP05] for the proof of this result and for more details about this subject.

4.5 Manin Products for Quadratic Algebras

In this section, extracted from Yu.I. Manin [Man87, Man88], we define two products $\bigcirc$ and $\bullet$ for quadratic data. They share nice properties with respect to Koszul duality theory. The black product is shown to produce Hopf algebras, some of which appear in quantum group theory.

4.5.1 Black and White Manin Products

Let (V,R) and (W,S) be two quadratic data. We denote by τ_{23} the isomorphism induced by the switching of the two middle terms:

$$\tau_{23} := \mathrm{Id}_V \otimes \tau \otimes \mathrm{Id}_W : V \otimes V \otimes W \otimes W \xrightarrow{\sim} V \otimes W \otimes V \otimes W.$$

By definition *Manin's white product of* (V,R) *and* (W,S) is the quadratic data given by

$$(V,R) \bigcirc (W,S) := \big(V \otimes W, \tau_{23}\big(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S\big)\big).$$

By definition *Manin's black product of* (V,R) *and* (W,S) is the quadratic data given by

$$(V,R) \bullet (W,S) := \big(V \otimes W, \tau_{23}(R \otimes S)\big).$$

The quadratic data $(\mathbb{K}x, 0)$ is the unit object for the white product $\bigcirc$, where $\mathbb{K}x$ stands for a one-dimensional vector space spanned by x. Dually, the quadratic data $(\mathbb{K}x, (\mathbb{K}x)^{\otimes 2})$ is the unit object for the black product $\bullet$. The associated algebras are the free associative algebra on one generator $\mathbb{K}[x]$ and the algebra of dual numbers $D(\mathbb{K}x) = \mathbb{K}[x]/(x^2)$ on one generator respectively.

We denote by $A(V,R) \bigcirc A(W,S)$ and by $A(V,R) \bullet A(W,S)$ the algebras associated to the quadratic data obtained by white and black products. Notice that there is a morphism of quadratic algebras

$$A(V,R) \bullet A(W,S) \to A(V,R) \bigcirc A(W,S),$$

for any pair of quadratic data. The algebra associated to the white product is isomorphic to the Hadamard (or Segre) product

$$A(V,R) \bigcirc A(W,S) \cong A(V,R) \underset{\mathrm{H}}{\otimes} A(W,S) := \bigoplus_{n \in \mathbb{N}} A(V,R)_{(n)} \otimes A(W,S)_{(n)},$$

which is the weight-wise tensor product.

4.5.2 Manin Products and Koszul Duality

Manin's black and white products behave well with respect to Koszul duality theory.

Proposition 4.5.1. *Let* (V, R) *and* (W, S) *be two quadratic data, where* V *and* W *are finite dimensional. Black and white products are sent one to the other under the Koszul duality functor*

$$\big(A(V, R) \bigcirc A(W, S)\big)^! = A(V, R)^! \bullet A(W, S)^!.$$

Proof. The quadratic algebra on the left-hand side is equal to

$$\begin{aligned} &A\big((V \otimes W)^*, \big(\tau_{23}\big(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S\big)\big)^{\perp}\big) \\ &\quad \cong A\big(V^* \otimes W^*, \tau_{23}\big(R^{\perp} \otimes W^{*\otimes 2} \cap V^{*\otimes 2} \otimes S^{\perp}\big)\big) \\ &\quad \cong A\big(V^* \otimes W^*, \tau_{23}\big(R^{\perp} \otimes S^{\perp}\big)\big). \end{aligned}$$ □

Theorem 4.5.2. [BF85] *If two quadratic data are Koszul, then their white product and their black product are Koszul.*

Proof. First we prove the white product property.

Let (V, R) and (W, S) denote two Koszul quadratic data. By Theorem 4.4.2, the sublattices $L(V, R)_{(n)}$ of $V^{\otimes n}$ and $L(W, S)_{(n)}$ of $W^{\otimes n}$ are distributive, for any $n \in \mathbb{N}$. By Lemma 4.4.1, there exist bases $\mathscr{B}'_{(n)}$ and $\mathscr{B}''_{(n)}$ of $V^{\otimes n}$ and $W^{\otimes n}$ respectively that distribute $L(V, R)_{(n)}$ and $L(W, S)_{(n)}$. For any $n \in \mathbb{N}$, the sublattice $L(V \otimes W, \tau_{23}(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S))_{(n)}$ of $(V \otimes W)^{\otimes n}$ is isomorphic to the sublattice of $V^{\otimes n} \otimes W^{\otimes n}$ generated by the finite family

$$\big\{V^{\otimes i} \otimes R \otimes V^{\otimes n-2-i} \otimes W^{\otimes n}, V^{\otimes n} \otimes W^{\otimes i} \otimes S \otimes W^{\otimes n-2-i}\big\}_{i=0,\ldots,n-2}.$$

Therefore, the basis $\mathscr{B}_{(n)} := \{\beta' \otimes \beta'' \mid \beta' \in \mathscr{B}'_{(n)}, \beta'' \in \mathscr{B}''_{(n)}\}$ distributes $L(V \otimes W, \tau_{23}(R \otimes W^{\otimes 2} + V^{\otimes 2} \otimes S))_{(n)}$. We conclude by using Theorem 4.4.2 in the other way round.

To prove the same result for the black product, we consider the Koszul dual algebras and apply Proposition 4.5.1 and Proposition 3.4.5. □

4.5.3 Adjunction and Internal (Co)Homomorphism

The white and black products satisfy the following adjunction formula.

Proposition 4.5.3. *There is a natural bijection in the category* Quad-alg *of quadratic algebras (or equivalently quadratic data):*

$$\mathrm{Hom}_{\mathsf{quad\ alg}}(A \bullet B, C) \cong \mathrm{Hom}_{\mathsf{quad\ alg}}\big(A, B^! \bigcirc C\big),$$

when B *is a finitely generated algebra.*

Proof. Let A, B and C be the three algebras associated to the three quadratic data (V,R), (W,S) and (X,T) respectively. There is a one-to-one correspondence between the maps $f : V \otimes W \to X$ and the maps $\tilde{f} : V \to W^* \otimes X$. Such a map satisfies $f^{\otimes 2} : \tau_{23}(R \otimes S) \to T$ if and only if $\tilde{f}^{\otimes 2} : R \to \tau_{23}(S^{\perp} \otimes X + (W^*)^{\otimes 2} \otimes T)$. □

In other words, $\mathsf{Hom}(B,C) := B^! \bigcirc C$ is the internal 'Hom' functor in the monoidal category of finitely generated quadratic algebras with the black product as tensor product. Dually, $\mathsf{CoHom}(A,B) := A \bullet B^!$ is the internal 'coHom' (or inner) functor in the monoidal category of finitely generated quadratic algebras with the white product as tensor product.

4.5.4 Manin Complexes

Let us apply this adjunction to the three quadratic algebras $(\mathbb{K}x, (\mathbb{K}x)^{\otimes 2})$, $A(V,R)$, $A(V,R)$, where V is finite dimensional. As usual we write $A = A(V,R)$. Since the first one is the unit object for the black product, we get the bijection

$$\mathrm{Hom}_{\mathsf{quad\ alg}}(A,A) \cong \mathrm{Hom}_{\mathsf{quad\ alg}}\big(\mathbb{K}[x]/\big(x^2\big), A^! \bigcirc A\big).$$

To the identity of A on the left-hand side corresponds a natural morphism of quadratic algebras $\mathbb{K}[x]/(x^2) \to A^! \bigcirc A = A^! \underset{\mathrm{H}}{\otimes} A$, which is equivalent to a square zero element ξ in $A^! \bigcirc A$. We define a differential d_ξ by multiplying elements of $A^! \underset{\mathrm{H}}{\otimes} A$ by ξ, that is $d_\xi(\alpha) := \alpha\xi$. The chain complex $(A^! \underset{\mathrm{H}}{\otimes} A, d_\xi)$ thereby obtained is called the *first Manin complex* denoted by $L(A)$ in [Man88, Chap. 9]. If we choose a basis $\{v_i\}_{i=1,\dots,n}$ for V and denote by $\{v_i^*\}_{i=1,\dots,n}$ the dual basis of V^*, the square zero element ξ is equal to $\sum_{i=1}^n v_i^* \otimes v_i$.

The *second Manin complex* $\widetilde{L}(A)$ is defined on the tensor product $A^! \otimes A$ by the differential $\tilde{d}_\xi(\alpha) := \alpha \mapsto \xi\alpha - (-1)^{|\alpha|}\alpha\xi$, where the cohomological degree is given by the weight of $A^!$. Under this degree convention, there is a isomorphism of graded modules

$$A^! \otimes A \cong \mathrm{Hom}\big(A^{\text{!`}}, A\big),$$

which sends the squarezero element ξ to the twisting morphism κ and the aforementioned differential to $\partial_\kappa(f) = [f,\kappa]$. This induces an isomorphism of cochain complexes

$$\widetilde{L}(A) = \big(A^! \otimes A, \tilde{d}_\xi\big) \cong \mathrm{Hom}^\kappa\big(A^{\text{!`}}, A\big).$$

So, when the algebra A is Koszul, the second Manin complex computes the homology functor $\mathrm{Ext}_A^\bullet(\mathbb{K}, A)$ and likewise the Hochschild cohomology of A with coefficients into itself, see Sect. 9.1.7.

4.5.5 Hopf Algebras

We show that the black product construction gives rise to Hopf algebras.

Proposition 4.5.4. *Let* (V, R) *be a quadratic data with* V *finite dimensional and let* $A = A(V, R)$*. The algebra* $A^! \bullet A$ *is a Hopf algebra.*

Proof. By general properties of adjunction (see Appendix B.2.1), $A^! \bullet A$ is a comonoid in the monoidal category $(\mathsf{quad\ alg}, \bigcirc = \underset{\mathrm{H}}{\otimes}, \mathbb{K}[\varepsilon])$. Since $A^! \underset{\mathrm{H}}{\otimes} A$ embeds into $A^! \otimes A$, the space $A^! \bullet A$ is a comonoid in the monoidal category of graded algebras with the classical tensor product, which makes it into a bialgebra. Since it is conilpotent, the antipode comes for free. □

This method was used by Yu.I. Manin to study quantum groups in [Man87, Man88].

4.6 Résumé

4.6.1 Rewriting Method

Let $A(V, R)$ be a quadratic algebra such that $V = \oplus_{i=1}^n \mathbb{K} v_i$ is a vector space equipped with a finite ordered basis. We order $V^{\otimes 2}$ by using, for instance, the lexicographical order:

$$v_1 v_1 < v_1 v_2 < \cdots < v_1 v_n < v_2 v_1 < \cdots < v_n v_n .$$

Typical relation:

$$v_i v_j = \sum_{(k,l)<(i,j)} \lambda_{k,l}^{i,j} v_k v_l, \quad \lambda_{k,l}^{i,j} \in \mathbb{K}.$$

The element $v_i v_j$ is called a *leading term*. The monomial $v_i v_j v_k$ is called *critical* if both $v_i v_j$ and $v_j v_k$ are leading terms.

Theorem. *Confluence for all the critical monomials* $\Rightarrow$ *Koszulity of the algebra.*

4.6.2 Reduction by Filtration and Diamond Lemma

Let $A = A(V, R)$ be a quadratic algebra. Any grading on $V \cong V_1 \oplus \cdots \oplus V_k$ together with a suitable order on tuples induce a filtration on the algebra A and

$$\psi : \mathring{A} := A(V, R_{\text{lead}}) \twoheadrightarrow \operatorname{gr} A,$$

with $R_{\text{lead}} = \langle \text{Leading Term}(r),\ r \in R \rangle$.

DIAMOND LEMMA FOR QUADRATIC ALGEBRAS.

$$\mathring{A} \quad \textit{Koszul} \quad \textit{and} \quad \mathring{A}^{(3)} \rightarrowtail (\operatorname{gr} A)^{(3)} \quad \Longrightarrow \quad A \quad \textit{Koszul} \quad \textit{and} \quad \mathring{A} \cong \operatorname{gr} A$$

INHOMOGENEOUS CASE.

$$\begin{array}{c} q\mathring{A} \quad \textit{Koszul} \quad \textit{and} \\ q\mathring{A}^{(3)} \rightarrowtail (\operatorname{gr}_\chi qA)^{(3)} \end{array} \quad \Longrightarrow \quad \begin{array}{c} A \quad \textit{Koszul} \quad \textit{and} \\ q\mathring{A} \cong \operatorname{gr}_\chi qA \cong qA \cong \operatorname{gr} A \cong A. \end{array}$$

4.6.3 PBW Basis, Gröbner Basis and Diamond Lemma

Particular case:

$$\forall i \in I = \{1, \dots, k\}, \quad \dim(V_i) = 1 \quad \Leftrightarrow \quad \{v_i\}_{i \in I} \text{ basis of } V,$$
$$\mathring{A} \text{ monomial algebra } \Rightarrow \mathring{A} \text{ Koszul and basis } \{v_{\bar{i}}\}_{\bar{i} \in L}.$$

PBW basis of $A(V, R)$*:* basis $\{a_{\bar{i}}\}_{\bar{i} \in L} :=$ image of $\{v_{\bar{i}}\}_{\bar{i} \in L}$ under $\mathring{A} \twoheadrightarrow \operatorname{gr} A$.

MAIN PROPERTIES OF PBW BASES.

$$A(V, R) \textit{ PBW basis} \quad \Rightarrow \quad A(V, R) \textit{ Koszul algebra.}$$

DIAMOND LEMMA.

$$\{a_{\bar{i}}\}_{\bar{i} \in L^{(3)}} \textit{ linearly independent} \quad \Longrightarrow \quad \{a_{\bar{i}}\}_{\bar{i} \in L} \textit{ PBW basis.}$$

GRÖBNER BASIS.

$$\textit{Gröbner basis of } (R) \subset T(V) \quad \Longleftrightarrow \quad \textit{PBW basis of } T(V)/(R).$$

PBW bases for inhomogeneous quadratic algebras:

$$qA = A(V, qR) \text{ PBW basis} \quad \Rightarrow \quad A(V, R) \text{ PBW basis.}$$

PBW bases on Koszul dual algebra:

$$A = A(V, R) \text{ PBW basis} \quad \Longleftrightarrow \quad A^{!} = A(V^*, R^{\perp}) \text{ PBW basis.}$$

4.6.4 Backelin Criterion

$L(V, R)_{(n)}$: lattice of sub-spaces of $V^{\otimes n}$ generated by

$$\left\{V^{\otimes i} \otimes R \otimes V^{\otimes n-2-i}\right\}_{i=0,\dots,n-2},$$
$$(V, R) \text{ Koszul quadratic data} \quad \Longleftrightarrow \quad L(V, R)_{(n)} \text{ distributive lattice}, \quad \forall n \in \mathbb{N}.$$

4.6.5 Manin Black and White Products

$$(V,R)\bigcirc(W,S):=\big(V\otimes W,\tau_{23}\big(R\otimes W^{\otimes 2}+V^{\otimes 2}\otimes S\big)\big),$$
$$(V,R)\bullet(W,S):=\big(V\otimes W,\tau_{23}(R\otimes S)\big).$$

Unit for the white product: $\mathbb{K}[x]$. Unit for the black product: $\mathbb{K}[x]/(x^2)$.

$$(A\bigcirc B)^! = A^!\bullet B^!,$$

Theorem. *Manin products preserve the Koszul property.*

$$\mathrm{Hom}_{\text{quad alg}}(A\bullet B, C)\cong \mathrm{Hom}_{\text{quad alg}}\big(A, B^!\bigcirc C\big),$$
$$A^!\bigcirc A: \textit{Manin chain complex};\qquad A^!\bullet A: \textit{Hopf algebra}.$$

4.7 Exercises

Exercise 4.7.1 (An example). Show that the quadratic algebra presented by the generators x, y, z and the relators $xy-yz, zy-yx, xz-zx, y^2-zx$ is Koszul by the rewriting method.

Exercise 4.7.2 (Distributive law ★). Apply the method of Sect. 4.2 to the following case. Let $A(V\oplus W, R\oplus D_\lambda\oplus S)$ be a quadratic algebra, where $R\subset V^{\otimes 2}$, $S\subset W^{\otimes 2}$ and where $D_\lambda\subset V\otimes W\bigoplus W\otimes V$ is the graph of a linear morphism $\lambda: W\otimes V\to V\otimes W$. Let us use the following notations $A:=A(V,R)$, $B:=A(W,S)$ and $A\vee_\lambda B:=A(V\oplus W, R\oplus D_\lambda\oplus S)$.

We consider the following ordered grading $V_1:=V$ and $V_2:=W$ together with the lexicographic order. In this case, prove that $R_{\text{lead}}=R\bigoplus W\otimes V\bigoplus S$ and that $\mathring{A}=A\vee_0 B$. Show that the underlying module satisfies $\mathring{A}\cong A(V,R)\otimes A(W,S)$ and make the product explicit. Dually, show that the underlying module satisfies $\mathring{A}^{¡}\cong A(W,S)^{¡}\otimes A(V,R)^{¡}$ and make the coproduct explicit. We now suppose that the two quadratic data (V,R) and (W,S) are Koszul. Show that the quadratic data $(V\oplus W, R\bigoplus W\otimes V\bigoplus S)$ is also Koszul.

Finally, show that if the maps $V^{\otimes 2}/R\otimes W\to A$ and $V\otimes W^{\otimes 2}/S\to A$ are injective, then the algebra A is Koszul and its underlying graded module is isomorphic to $A\cong A(V,R)\otimes A(W,S)$.

Extra question: when the generating spaces V and W are finite dimensional, prove that the Koszul dual algebra has the same form:

$$A^! = A\big(V^*\oplus W^*, R^\perp\oplus D_{{}^t\lambda}\oplus S^\perp\big)=A^!\vee_{{}^t\lambda} B^!,$$

where ${}^t\lambda: V^*\otimes W^*\to W^*\otimes V^*$ is the transpose map.

Exercise 4.7.3 (Equivalent definitions of PBW bases ★). Let $A=(V,R)$ be a quadratic algebra endowed with a family of elements $\{a_{\bar{\imath}}\}_{\bar{\imath}\in L}$ of A labeled by a set $L\subset J$. Prove that, under condition (2) of Sect. 4.3.6, condition (1) is equivalent to

(1′) for any pair $(i, j) \in I^2$, if $(i, j) \notin L^{(2)}$, the product $a_i a_j$ in A can be written as a linear combination of strictly lower terms labeled by $L^{(2)}$:

$$a_i a_j = \sum_{(k,l) \in L^{(2)}, (k,l) < (i,j)} \lambda_{k,l}^{i,j} a_k a_l,$$

with $\lambda_{k,l}^{i,j} \in \mathbb{K}$.

In the same way, prove that, under condition (2), condition (1) and (1′) are equivalent to

(1″) for any $\bar{i} \in J$, if $\bar{i} \notin L$, then the element $a_{\bar{i}} \in A$ can be written as a linear combination of strictly lower terms labeled by L:

$$a_{\bar{i}} = \sum_{\bar{l} \in L, \bar{l} < \bar{i}} \lambda_{\bar{l}}^{\bar{i}} a_{\bar{l}},$$

with $\lambda_{\bar{l}}^{\bar{i}} \in \mathbb{K}$.

Exercise 4.7.4 (Hilbert–Poincaré series and PBW bases ★). Compute the Hilbert–Poincaré series of a quadratic algebra endowed with a PBW basis, see [PP05, Sect. 4.6].

Exercise 4.7.5 (From PBW to Koszul ★). A quadratic algebra $A(V, R)$ is called *n-PBW* if it admits an extra ordered grading $V \cong V_1 \oplus \cdots \oplus V_k$ such that $\dim V_i \leq n$, for any i, if the algebra $\mathring{A}$ is Koszul and if the isomorphism $\mathring{A} \cong \operatorname{gr} A$ holds. Notice that 1-PBW algebra are the algebras having a PBW basis.

Prove the following inclusions of categories

$$\mathsf{PBW} = \mathsf{1\text{-}PBW} \subset \mathsf{2\text{-}PBW} \subset \cdots \subset \mathsf{fg\ Koszul},$$

where the last category is the category of finitely generated Koszul algebras.

Show that the quadratic algebra $A(x, y, z; x^2 - yz, x^2 + 2zy)$ is 2-PBW but not 1-PBW.

Exercise 4.7.6 (Inhomogeneous Koszul duality theory with PBW-bases ★). Let $A(V, R)$ be an inhomogeneous Koszul algebra endowed with a PBW basis. Make the constructions of Sect. 3.6 explicit with this basis.

For instance, the degree-wise linear dual of the Koszul dual dg coalgebra $A^{¡} = ((qA)^{¡}, d_\varphi)$ is a dga algebra. Make it explicit with its differential. In the case of the Steenrod algebra, show that this gives the Λ algebra (see [BCK+66, Wan67, Pri70]).

Exercise 4.7.7 (PBW bases and Manin products ★). Let $A := A(V, R)$ and $B := A(W, S)$ be two quadratic algebras with ordered bases of V and W labeled by I_A and I_B respectively. Suppose that we have a PBW basis on A and on B labeled respectively by L_A and L_B. We denote by $\bar{i} = (i_1, \ldots, i_n)$ the elements of I_A^n and by $\bar{j} = (j_1, \ldots, j_n)$ the elements of I_B^n.

Show that the following set labels a PBW basis of the white product $A \bigcirc B$:

$$L_{A \bigcirc B} = \bigcup_{n \in \mathbb{N}} \left\{ (i_1, j_1, i_2, j_2, \ldots, i_n, j_n) \mid \bar{i} \in L_A^{(n)} \text{ and } \bar{j} \in L_B^{(n)} \right\}$$
$$\cong \bigcup_{n \in \mathbb{N}} L_A^{(n)} \times L_B^{(n)}.$$

Dually, show that the following set labels a PBW basis of the black product $A \bullet B$:

$$L_{A \bullet B} = \left\{ (i_1, j_1, i_2, j_2, \ldots, i_n, j_n) \mid \forall 1 \leq k < n, (i_k, i_{k+1}) \in L_A^{(2)} \right.$$
$$\left. \text{or } (j_k, j_{k+1}) \in L_B^{(2)} \right\}.$$

Exercise 4.7.8 (Coproduct ★). Let A and B be two nonunital associative algebras. Show that their coproduct $A \vee B$ in the category of nonunital associative algebras is given by a suitable product on

$$A \vee B = A \bigoplus B \bigoplus A \otimes B \bigoplus B \otimes A \bigoplus A \otimes B \otimes A \bigoplus B \otimes A \otimes B \bigoplus \cdots.$$

Let (V, R) and (W, S) be two quadratic data. Show that they admit a coproduct in the category of quadratic data, which is given by $(V \oplus W, R \oplus S)$. Prove that the quadratic algebra $A(V \oplus W, R \oplus S)$ is the coproduct of the quadratic algebras $A(V, R)$ and $A(W, S)$ in the category of unital associative algebras. When V and W are finite dimensional, compute its Koszul dual algebra.

Consider now the example of Exercise 4.7.2 and show that $A(V \oplus W, R \oplus D_\lambda \oplus S) \cong A \vee B/(D_\lambda)$, whence the notation $A \vee_\lambda B$.

Prove that if (V, R) and (W, S) are Koszul quadratic data, then so is their coproduct. [Give several different proofs, using the Koszul complex and the distributive lattices for instance.]

Prove that if $A := A(V, R)$ and $B := A(W, S)$ admit a PBW basis, then they can be used to construct a PBW basis on $A \vee B$.