Chapter 3
Koszul Duality for Associative Algebras

> *In the process of its internal development and prompted by its inner logic, mathematics, too, creates virtual worlds of great complexity and internal beauty which defy any attempt to describe them in natural language but challenge the imagination of a handful of professionals in many successive generations.*
>
> Yuri I. Manin in "Mathematics as metaphor"

A minimal model for the associative algebra A is a quasi-free resolution $(T(W), d)$ such that the differential map d maps W into $\oplus_{n\geq 2} W^{\otimes n}$. We would like to find a method to construct this minimal model when A is quadratic, that is $A = T(V)/(R)$ where the ideal (R) is generated by $R \subset V^{\otimes 2}$ (this is the quadratic hypothesis). We will see that the quadratic data (V, R) permits us to construct explicitly a coalgebra $A^{¡}$ and a twisting morphism $\kappa : A^{¡} \to A$. Then, applying the theory of Koszul morphisms given in the previous chapter, we obtain a simple condition which ensures that the cobar construction on the Koszul dual coalgebra, that is $\Omega A^{¡}$, is the minimal model of A.

If one tries to construct by hand the space W, then one is led to take $W = V \oplus R \oplus (R \otimes V \cap V \otimes R) \oplus \cdots$. In fact, $\mathbb{K} \oplus V \oplus R \oplus (R \otimes V \cap V \otimes R)$ is the beginning of a certain sub-coalgebra of the cofree coalgebra over V, which is uniquely determined by V and R. This is precisely the expected coalgebra $A^{¡}$, up to suspension. The twisting morphism κ is simply the composite $A^{¡} \twoheadrightarrow V \rightarrowtail A$. The expected condition is the acyclicity of the Koszul complex $A^{¡} \otimes_\kappa A$. This is the Koszul duality theory for homogeneous quadratic algebras as introduced by Stewart Priddy in [Pri70]. In practice it is easier to work with algebras instead of coalgebras. When V is finite dimensional we consider the "graded linear dual" of $A^{¡}$ which is, up to suspension, a quadratic algebra $A^{!}$, usually called the *Koszul dual algebra* of A.

The quadratic hypothesis $R \subset V^{\otimes 2}$ can be weakened by only requiring $R \subset V^{\otimes 2} \oplus V$. In this case, we say that the algebra is *inhomogeneous quadratic*. We show how to modify the preceding method to handle the inhomogeneous quadratic case, also done in [Pri70]. Two examples are: the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (original example due to J.-L. Koszul) and the Steenrod algebra.

J.-L. Loday, B. Vallette, *Algebraic Operads*,
Grundlehren der mathematischen Wissenschaften 346,
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Inhomogeneous Koszul duality theory gives a proof of a general *Poincaré–Birkhoff–Witt theorem*, which, applied to $U(\mathfrak{g})$, gives the classical one.

In our treatment of Koszul duality of associative algebras, we keep algebras and coalgebras on the same footing. Working with coalgebras allows us to avoid the finite dimensional hypothesis. Moreover we give conceptual proofs so that they can be generalized to other monoidal categories. Our interest for Koszul duality of associative algebras is to serve as a paradigm for Koszul duality of algebraic operads.

Koszul algebras have applications in many fields of mathematics, which will not be discussed at all here (see the introduction of [PP05]). Classical references on Koszul duality of associative algebras include: S. Priddy [Pri70], Yu.I. Manin [Man87, Man88], R. Fröberg [Frö99], A. Polishchuk and L. Positselski [PP05].

3.1 Quadratic Data, Quadratic Algebra, Quadratic Coalgebra

We start with a quadratic data (V, R) to which we associate an algebra and a coalgebra

$$\begin{tikzcd} & (V, R) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ A(V, R) & & C(V, R). \end{tikzcd}$$

In this chapter we suppose that $\mathbb{K}$ is a field, though most of the definitions and constructions are valid over a commutative ring.

3.1.1 Quadratic Data

By definition a *quadratic data* (V, R) is a graded vector space V and a graded subspace $R \subseteq V \otimes V$. A *morphism* of quadratic data $f : (V, R) \to (W, S)$ is a graded linear map $f : V \to W$ such that $(f \otimes f)(R) \subseteq S$.

3.1.2 Quadratic Algebra

The *quadratic algebra* $A(V, R) := T(V)/(R)$ is, by definition, the quotient of the free associative algebra over V by the two-sided ideal (R) generated by $R \subseteq V^{\otimes 2}$. In other words, $A(V, R)$ is the quotient of $T(V)$ which is *universal* among the quotient algebras A of $T(V)$ such that the composite

$$R \rightarrowtail T(V) \twoheadrightarrow A$$

is 0. It means that, for any such algebra A, there is a unique algebra morphism $A(V, R) \to A$ which makes the following diagram commutative:

$$\begin{tikzcd}
T(V) \arrow[rr, two heads] \arrow[dr] & & A \\
& A(V, R). \arrow[ur] &
\end{tikzcd}$$

Since (R) is a homogeneous ideal, it follows that $A(V, R)$ is graded and augmented. This degree is called the *weight* and denoted as a superscript in parentheses. Explicitly it is given by:

$$A = \bigoplus_{n \in \mathbb{N}} A^{(n)} = \mathbb{K}1 \oplus V \oplus \big(V^{\otimes 2}/R\big) \oplus \cdots \oplus \left(V^{\otimes n} \Big/ \sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\right) \oplus \cdots .$$

Any basis of V is called a set of *generators* of A. Any basis $\{r_i\}$ of R determines a set of *relations* $r_i = 0$ in A. By abuse of terminology r_i, which should be called a *relator*, is often called a relation.

A morphism of quadratic data, $f : (V, R) \to (W, S)$ induces a natural morphism of weight graded algebras $A(V, R) \to A(W, S)$. Any morphism of algebras which respects the weight grading is of this form. But it is not the case for every morphism of algebras.

3.1.3 Quadratic Coalgebra

The *quadratic coalgebra* $C(V, R)$ is, by definition, the sub-coalgebra of the cofree coassociative coalgebra $T^c(V)$ which is *universal* among the sub-coalgebras C of $T^c(V)$ such that the composite

$$C \rightarrowtail T^c(V) \twoheadrightarrow V^{\otimes 2}/R$$

is 0. It means that, for any such coalgebra C, there is a unique coalgebra morphism $C \to C(V, R)$ which makes the following diagram commutative:

$$\begin{tikzcd}
& C(V, R) \arrow[dr] & \\
C \arrow[ur] \arrow[rr, tail] & & T^c(V).
\end{tikzcd}$$

The coalgebra $C(V, R)$ is weight graded. Explicitly it is given by:

$$C = \bigoplus_{n \in \mathbb{N}} C^{(n)} = \mathbb{K}1 \oplus V \oplus R \oplus \cdots \oplus \left(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\right) \oplus \cdots .$$

Observe that the restriction of the coproduct of C (that is the deconcatenation) to the weight 2 component $C^{(2)} = R$ is given by

$$r = r_1 \otimes r_2 \mapsto r \otimes 1 + r_1 \otimes r_2 + 1 \otimes r \in \big(V^{\otimes 2} \otimes \mathbb{K}\big) \oplus (V \otimes V) \oplus \big(\mathbb{K} \otimes V^{\otimes 2}\big).$$

We will say that $C(V, R)$ is *cogenerated* by V with *corelations* R in $T^c(V)$. Observe that the coalgebra $C(V, R)$ is conilpotent, cf. Sect. 1.2.4.

A morphism of quadratic data, $f : (V, R) \to (W, S)$ induces a natural morphism of weight graded coalgebras $C(V, R) \to C(W, S)$. Any morphism of coalgebras which respects the weight grading is of this form. But it is not the case for every morphism of coalgebras.

3.1.4 The Graded Framework

Both constructions $A(V, R)$ and $C(V, R)$ can be extended to the category of graded vector spaces. In this framework, V is a graded module and R is a graded submodule of the graded module $V^{\otimes 2}$. Then the algebra $A(V, R)$, resp. the coalgebra $C(V, R)$, is bigraded by degree and weight (cf. Sect. 1.5.1). Both $A(V, R)$ and $C(V, R)$ are connected weight graded in the sense of Sect. 1.5.10, with trivial differential.

3.2 Koszul Dual of a Quadratic Algebra

We construct the Koszul dual coalgebra and the Koszul dual algebra of a quadratic algebra. We work here in the homogeneous framework. The inhomogeneous framework, where it is only supposed that $R \subset V \oplus V^{\otimes 2}$, is treated in Sect. 3.6.

3.2.1 Koszul Dual Coalgebra of a Quadratic Algebra

Let (V, R) be a graded quadratic data. By definition the *Koszul dual coalgebra* of the quadratic algebra $A(V, R)$ is the coalgebra

$$A^{¡} := C\big(sV, s^2R\big),$$

where s^2R is the image of R in $(sV)^{\otimes 2}$ under the map $V^{\otimes 2} \to (sV)^{\otimes 2}$, $vw \mapsto svsw$. The upside down exclamation point ¡ (left exclamation point in the Spanish language) is usually pronounced “anti-shriek”. If V is a graded space concentrated in degree 0, then sV is concentrated in degree 1. Observe that $C(sV, s^2R)$ is equal to $C(V, R)$ as a coalgebra. The decoration “s” is modifying the degree of the objects. It plays a role when we apply the Koszul sign rule to morphisms. We can omit it in the notation at the expense of changing the signs of the maps accordingly.

3.2.2 Koszul Dual Algebra of a Quadratic Algebra

The algebra obtained as the linear dual of the coalgebra $A^{¡}$ carries a desuspension sign. In the literature, one sometimes finds its unsuspended analog, denoted by $A^!$ and called the *Koszul dual algebra of the quadratic algebra* A. Explicitly it is defined by

$$\left(A^!\right)^{(n)} := s^n \left(A^{¡*}\right)^{(n)}$$

and carries the obvious associative algebra structure.

Dualizing linearly the exact sequence

$$0 \to R \rightarrowtail V^{\otimes 2} \twoheadrightarrow V^{\otimes 2}/R \to 0,$$

provides the exact sequence

$$0 \leftarrow R^* \twoheadleftarrow \left(V^*\right)^{\otimes 2} \leftarrowtail R^\perp \leftarrow 0.$$

In other words the *orthogonal space* $R^\perp$ is defined as the image of $(V^{\otimes 2}/R)^*$ in $(V^*)^{\otimes 2}$ under the isomorphism $(V^{\otimes 2})^* \cong V^* \otimes V^*$, cf. Sect. 1.2.2.

Proposition 3.2.1. *The Koszul dual algebra* $A^!$ *admits the following quadratic presentation*

$$A^! = A\left(V^*, R^\perp\right).$$

Proof. First notice that the linear dual of the quadratic coalgebra $A^{¡} = C(sV, s^2R)$ is the quadratic algebra $A^{¡*} = A(s^{-1}V^*, s^{-2}R^\perp)$. The last step can be proved either directly or by using the notion of Manin products of Sect. 4.5.1: the Koszul dual algebra is equal to $A^! = (A^{¡*}) \bigcirc T(s\mathbb{K}) = A(V^*, R^\perp)$. □

3.2.3 Koszul Dual Algebra of a Coalgebra

It is also useful to introduce the *Koszul dual algebra* of a quadratic coalgebra

$$C^{¡} := A\left(s^{-1}V, s^{-2}R\right) \quad \text{for } C = C(V, R).$$

It follows immediately that

$$\left(A^{¡}\right)^{¡} = A \quad \text{and} \quad \left(C^{¡}\right)^{¡} = C.$$

As an immediate consequence we have, under finite dimensionality assumption:

$$\left(A^!\right)^! = A.$$

Observe that the coalgebra $A^{¡}$ is well-defined even in the graded framework and without any finiteness hypothesis.

3.2.4 Examples

1. Let V be a finite dimensional vector space and let $R = 0$. Then we have $A = T(V)$. Its Koszul dual algebra is the *algebra of dual numbers* $A^! = D(V^*) := \mathbb{K}1 \oplus V^*$, with trivial multiplication.
2. The symmetric algebra $S(V)$ is the quadratic algebra $T(V)/(R)$, where the space of relations R is the subvector space of $V^{\otimes 2}$ spanned by the elements $x \otimes y - y \otimes x$ for $x, y \in V$. The coalgebra $\Lambda^c(sV)$ is the subcoalgebra of $T^c(sV)$ satisfying the universal property of Sect. 3.1.3 with the subspace $s^2R = \langle sx \otimes sy - sy \otimes sx \mid x, y \in V\rangle$. Therefore, its component of weight n is equal to

$$\Lambda^c(sV)^{(n)} = \left\langle \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) s^n x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \;\middle|\; x_1, \ldots, x_n \in V \right\rangle.$$

   The coalgebra structure is given by the deconcatenation coproduct and is cocommutative. When V is an n-dimensional vector space with basis $\{x_1, \ldots, x_n\}$ in degree 0, one gets the polynomial algebra $S(V) = \mathbb{K}[x_1, \ldots, x_n]$. In this case, its Koszul dual algebra is the exterior algebra $S(V)^! = \Lambda(V^*)$, since $R^\perp$ is spanned by the elements $x_i^* x_j^* + x_j^* x_i^*$, where $\{x_1^*, \ldots, x_n^*\}$ is the dual basis.
3. We refer to [Pri70, Man87, Man88, Frö99, PP05] for many more examples.

3.3 Bar and Cobar Construction on a Quadratic Data

We make explicit the dga coalgebra $\mathrm{B}A$ and the dga algebra ΩC in the quadratic case. The Koszul dual objects are shown to be equal to the syzygy degree 0 homology group in both cases.

3.3.1 Bar Construction on a Quadratic Algebra

The bar construction $\mathrm{B}A := T^c(s\bar{A})$ over the quadratic dga algebra $A = A(V, R)$ (whose differential is trivial) is equipped with a homological degree and a weight grading. We now introduce the *syzygy degree*.

The *weight grading* on $\mathrm{B}A$ is defined by the sum of the weight of each element: $\omega(sa_1, \ldots, sa_k) := \omega(a_1) + \cdots + \omega(a_k)$. Since A is a connected wgda algebra, the augmentation ideal $\bar{A}$ is concentrated in weight grading ≥ 1. We define another degree on $\bar{A}$ by the weight grading of A minus 1. It induces a new nonnegative degree on the bar construction, called the *syzygy degree* which is equal to $\omega(a_1) + \cdots + \omega(a_k) - k$. The component of syzygy degree d of $\mathrm{B}A$ is denoted by $\mathrm{B}^d A$, whereas the homological degree r component is denoted by $(\mathrm{B}A)_r$.

Since A has trivial internal differential, the differential on $\mathrm{B}A$ reduces to d_2, which raises the syzygy degree by 1 and preserves the weight grading. So it forms a cochain complex with respect to the syzygy degree, which splits with respect to the weight grading. Hence the associated cohomology groups will be bigraded, by the syzygy degree and by the weight grading.

The following diagram depicts this decomposition. The syzygy degree is indicated on the last row, so we delete the notation s for simplicity. We write $\bar{A} = V \oplus V^2/R \oplus V^3/(RV + VR) \oplus \cdots$ for more clarity, the tensor product notation being reserved for the one in $\mathrm{B}A$ only.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & (4) \\
0 & V^3/(VR+RV) \arrow[l] & \left(V^2/R \otimes V\right) \oplus \left(V \otimes V^2/R\right) \arrow[l] & V \otimes V \otimes V \arrow[l] & (3) \\
 & 0 & V^2/R \arrow[l] & V \otimes V \arrow[l] & (2) \\
 & & 0 & V \arrow[l] & (1) \\
 & & & \mathbb{K} & (0) \\
3 & 2 & 1 & 0 &
\end{tikzcd}$$

On the weight (3) row the map from $V \otimes V \otimes V$ is

$$u \otimes v \otimes w \mapsto [uv] \otimes w - u \otimes [vw]$$

where $[-]$ denotes the class in the quotient. The other map on this row is

$$\left([uv] \otimes w, u' \otimes \left[v'w'\right]\right) \mapsto [uvw] + \left[u'v'w'\right].$$

From this description we see immediately that the syzygy degree 0 column forms the cofree coalgebra $T^c(sV)$. Hence the Koszul dual coalgebra $A^{¡} = \mathbb{K} \oplus sV \oplus s^2R \oplus \cdots$ is a subspace of this column.

Let $x \in A^{¡}$ and $\bar{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$. The boundary map of $\Omega A^{¡}$ is given explicitly by the formula

$$d_2\left(s^{-1}x\right) = \sum (-1)^{|x_{(1)}|} s^{-1}x_{(1)} \otimes s^{-1}x_{(2)}.$$

The next proposition shows that $A^{¡}$ is equal to the kernel of the boundary map.

Proposition 3.3.1. *Let* (V, R) *be a quadratic data,* $A = A(V, R)$ *the quadratic algebra and* $A^{¡} = C(sV, s^2R)$ *its Koszul dual coalgebra. The natural coalgebra inclusion* $i : A^{¡} \rightarrowtail \mathrm{B}A$ *induces an isomorphism of graded coalgebras*:

$$i : A^{¡} \xrightarrow{\cong} H^0\left(\mathrm{B}^\bullet A\right), \quad \textit{i.e.} \quad A^{¡(n)} \cong H^0\left(\mathrm{B}^\bullet A\right)^{(n)} \quad \textit{for any } n.$$

Proof. We claim that, for each n, the inclusion $A^{¡(n)} \to (sV)^{\otimes n}$ is exactly the kernel of the horizontal differential, that is $H^0(\mathrm{B}^\bullet A)^{(n)}$. It is obvious for $n = 0$ and $n = 1$. For $n = 2$ the boundary map in $(\mathrm{B}A)^{(2)}$ is the quotient map $V^{\otimes 2} \to V^{\otimes 2}/R$, hence its kernel is R. More generally, since the boundary map is a derivation, it is given in degree 0 by the sum of the maps $(sV)^{\otimes n} \to (sV)^{\otimes i} \otimes sV^{\otimes 2}/R \otimes (sV)^{\otimes j}$. So the kernel is

$$\bigcap_{i+2+j=n} (sV)^{\otimes i} \otimes s^2R \otimes (sV)^{\otimes j} = A^{¡\,(n)}. \qquad \square$$

3.3.2 Cobar Construction on a Quadratic Coalgebra

Like the bar construction, the cobar construction $\Omega C = T(s^{-1}\overline{C})$ over the quadratic dga coalgebra $C = C(V, R)$ (whose differential is trivial) has several gradings.

We introduce the same definitions as for the bar construction. We consider the *weight grading* $(\Omega C)^{(n)}$, which is the sum of the weights of the elements of $\overline{C}$. The *syzygy degree* of ΩC is induced by the weight of elements of $\overline{C}$ minus 1 in the same way. We denote it by $\Omega_d C$.

Since the internal differential of the coalgebra C is trivial, the differential of the cobar construction ΩC reduces to d_2, which lowers the syzygy degree by 1. Hence, $(\Omega_\bullet C, d_2)$ becomes a chain complex. Since the differential d_2 preserves the weight of the elements of C, this chain complex splits with respect to the weight: it is isomorphic to the following direct sum of sub-chain complexes $\Omega C \cong \bigoplus_{n\geq 0}(\Omega C)^{(n)}$.

The diagram below represents this weight decomposition. The syzygy degree is indicated on the last row, so we delete the notation s^{-1} for simplicity.

$$\begin{array}{cccccccc}
\cdots & & \cdots & & \cdots & & \cdots & (4)\\
0 & \to & VR\cap RV & \to & (V\otimes R)\oplus(R\otimes V) & \to & V\otimes V\otimes V & (3)\\
 & & 0 & \to & R & \to & V\otimes V & (2)\\
 & & & & 0 & \to & V & (1)\\
 & & & & & & \mathbb{K} & (0)\\
3 & & 2 & & 1 & & 0 &
\end{array}$$

In degrees 0 and 1, the maps $R \to V^{\otimes 2}$ and $(V \otimes R) \oplus (R \otimes V) \to V^{\otimes 3}$ are simply the inclusions. The map $VR \cap RV \to (V \otimes R) \oplus (R \otimes V)$ is $\mathrm{inc}_1 - \mathrm{inc}_2$ where inc_1, resp. inc_2 is the inclusion of the first, resp. second, summand. From this description we see immediately that the syzygy degree 0 column forms the free algebra $T(s^{-1}V)$ and that the algebra

$$C^{¡} = \mathbb{K} \oplus s^{-1}V \oplus \big(s^{-1}V\big)^{\otimes 2}/s^{-2}R \oplus \cdots$$

is a quotient of it.

Proposition 3.3.2. *Let $C = C(V, R)$ be the quadratic coalgebra associated to the quadratic data (V, R), and let $C^{¡} := A(s^{-1}V, s^{-2}R)$ be its Koszul dual algebra.*

The natural algebra projection $p : \Omega C \twoheadrightarrow C^{¡}$ *induces an isomorphism of graded algebras*:

$$p : H_0(\Omega_\bullet C) \xrightarrow{\cong} C^{¡}, \quad \textit{i.e.} \quad H_0(\Omega_\bullet C)^{(n)} \cong C^{¡\,(n)} \quad \textit{for any } n.$$

Proof. The proof is analogous to the proof of Proposition 3.3.1. ☐

3.4 Koszul Algebras

For any quadratic data, we define a twisting morphism from the Koszul dual coalgebra to the quadratic algebra. This gives a twisted tensor product, called the *Koszul complex*, which plays a central role in the Koszul duality theory. We state and prove the main theorem of this chapter which says that the Koszul complex is acyclic if and only if the cobar construction over the Koszul dual coalgebra gives the minimal model of the algebra. The other definitions of a Koszul algebra appearing in the literature are given and we conclude with examples.

3.4.1 The Koszul Complex of a Quadratic Data

Starting with a quadratic data (V, R) we define $\kappa : C(sV, s^2R) \to A(V, R)$ as the linear map of degree -1 which is 0 everywhere except on V where it identifies sV to V:

$$\kappa : C\big(sV, s^2R\big) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V, R).$$

Observe that the shift s in the definition of $A^{¡}$ makes κ a degree -1 map. The following result shows that κ is a twisting morphism.

Lemma 3.4.1. *We have* $\kappa \star \kappa = 0$, *and therefore* $\kappa \in \mathrm{Tw}(A^{¡}, A)$.

Proof. Since κ is 0 almost everywhere, the convolution product $\kappa \star \kappa$ is 0 except maybe on $V^{\otimes 2}$. Computing $\kappa \star \kappa$ explicitly on $V^{\otimes 2}$ we find that it is equal to the composite

$$C^{(2)} = R \to V \otimes V \to V^{\otimes 2}/R = A^{(2)},$$

hence it is 0 as expected.

So the map κ is a twisting morphism by Sect. 2.1.2. ☐

Proposition 3.4.2. *The twisting morphism* $\kappa : C(sV, s^2R) \twoheadrightarrow V \rightarrowtail A(V, R)$ *induces a map* d_κ *which makes*

$$A^{¡}c \otimes_\kappa A := \big(C\big(sV, s^2R\big) \otimes A(V, R), d_\kappa\big)$$

(*respectively* $A \otimes_\kappa A^{¡}$) *into a weight graded chain complex.*

Proof. The differential d_κ was constructed out of κ in Sect. 1.6.1. It is a differential by Lemmas 3.4.1 and 1.6.2. Since κ has degree -1 and weight 0, it is the same for the differential d_κ. Hence this chain complex splits with respects to the total weight. ☐

The chain complex $A^{¡} \otimes_\kappa A$ (resp. $A \otimes_\kappa A^{¡}$) is called the *Koszul complex*, or left Koszul complex (resp. right Koszul complex) of the quadratic algebra $A(V, R)$. Its summand $(A^{¡} \otimes_\kappa A)^{(n)}$ of weight (n) is equal to:

$$0 \to A^{¡(n)} \to A^{¡(n-1)} \otimes A^{(1)} \to \cdots \to A^{¡(1)} \otimes A^{(n-1)} \to A^{¡(n)} \to 0.$$

3.4.2 Koszul Criterion

In this section, we derive the main theorem of Koszul duality theory for associative algebras from the preceding chapter.

Proposition 3.4.3. *The maps corresponding to the twisting morphism* $\kappa : A^{i} \to A$ *under the isomorphisms of Theorem* 2.2.6 *are exactly* $i = f_\kappa : A^{i} \rightarrowtail \mathrm{B}A$ *and* $p = g_\kappa : \Omega A^{i} \twoheadrightarrow A$.

Proof. By direct inspection. ☐

Theorem 3.4.4 (Koszul criterion). *Let* (V, R) *be a quadratic data. Let* $A := A(V, R)$ *be the associated quadratic algebra and let* $A^{i} := C(sV, s^2R)$ *be the associated quadratic coalgebra. Then the following assertions are equivalent*:

1. *the right Koszul complex* $A^{i} \otimes_\kappa A$ *is acyclic*,
2. *the left Koszul complex* $A \otimes_\kappa A^{i}$ *is acyclic*,
3. *the inclusion* $i : A^{i} \rightarrowtail \mathrm{B}A$ *is a quasi-isomorphism*,
4. *the projection* $p : \Omega A^{i} \twoheadrightarrow A$ *is a quasi-isomorphism.*

When these assertions hold, the cobar construction on A^{i} *gives a minimal resolution of* A.

Proof. Theorem 2.3.1 can be applied to $A := A(V, R)$, $C := A^{¡} = C(sV, s^2R)$ and to $\alpha = \kappa$ since by Lemma 3.4.1 κ is a twisting morphism and since the connectivity and weight grading assumptions are satisfied.

Let us verify that $\Omega A^{¡}$ is the minimal model of A when the Koszul complex is acyclic. First, the dga algebra $\Omega A^{¡}$ is free as a graded algebra by construction (but not as a dga algebra). Second, its differential $d_{\Omega A^{¡}} = d_2$ satisfies the minimal hypothesis $d(W) \subset \bigoplus_{n\geq 2} W^{\otimes n}$ also by construction. Third, by Proposition 3.3.2 we have $H_0(\Omega_\bullet A^{¡}) = A$ and by (4) the resulting map $p : \Omega A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism. ☐

Observe that starting with $C(V, R)$ instead of $A(V, R)$ with the following twisting morphism

$$\begin{tikzcd} C = C(V, R) \arrow[r, two heads] & V \arrow[r, "s^{-1}"] & s^{-1}V \arrow[r, tail] & C^{¡} = A\big(s^{-1}V, s^{-2}R\big) \end{tikzcd}$$

gives the same result up to a shift of grading. So we get Koszul duality theory for coalgebras.

3.4.3 Definition of a Koszul Algebra

A quadratic data (resp. quadratic algebra, resp. quadratic coalgebra) is said to be *Koszul* if its Koszul complex is acyclic.

By Theorem 3.4.4 we see that A is Koszul if and only if there is an isomorphism $A^{¡} \cong H^{\bullet}(\mathrm{B}A)$ (resp. $H_{\bullet}(\Omega A^{!}) \cong A$). By Propositions 3.3.1 and 3.3.2, this is equivalent to the vanishing of the (co)homology groups: $H^d(\mathrm{B}^{\bullet}A)$ and $H_d(\Omega_{\bullet}A^{¡}) = 0$ for $d > 0$. More generally, a connected weight graded algebra A is said to be Koszul if the cohomology $H^d(\mathrm{B}^{\bullet}A) = 0$ of its bar construction is concentrated in syzygy degree $d = 0$. In this case, Exercise 3.8.1 shows that A admits a quadratic presentation. Therefore, there is no restriction to treat only the quadratic case.

The bar–cobar construction $\Omega\mathrm{B}A$ is always a resolution of A. To simplify it, one idea is to apply the cobar construction to the homology $H^{\bullet}(\mathrm{B}A)$ rather than to $\mathrm{B}A$. When A is Koszul, the homology of $\mathrm{B}A$ is exactly $A^{¡}$ and one gets the resolution $\Omega A^{¡}$ of A. For any quadratic algebra we have the following commutative diagrams:

$$\begin{tikzcd} \Omega A^{¡} \arrow[r, tail] \arrow[rr, bend left, two heads] & \Omega\mathrm{B}A \arrow[r, two heads, "\sim"] & A \end{tikzcd}$$

$$\begin{tikzcd} A^{¡} \arrow[r, tail, "\sim"] \arrow[rr, bend right, tail] & \mathrm{B}\Omega A^{¡} \arrow[r, two heads] & \mathrm{B}A \end{tikzcd}$$

The algebra A is Koszul if and only if all these maps are quasi-isomorphisms by Corollary 2.3.2 and Theorem 3.4.4. Both $\Omega A^{¡}$ and $\Omega\mathrm{B}A$ are models of A and $\Omega A^{¡}$ is the minimal model.

With the aforementioned definitions, a quadratic algebra A is Koszul if and only if its Koszul dual coalgebra $A^{¡}$ is Koszul. The following proposition states the same property with the Koszul dual algebra.

Proposition 3.4.5. *Let* (V, R) *be a finite dimensional quadratic data. The quadratic algebra* $A = A(V, R)$ *is Koszul if and only if its Koszul dual algebra* $A^{!} = A(V^*, R^{\perp})$ *is Koszul.*

Proof. The left Koszul complex $A^{!} \otimes_{\kappa'} A^{!¡}$ associated to the twisting morphism $\kappa' : A^{!¡} \to A^{!}$ is made up of finite dimensional vector spaces in each degree and weight. Its linear dual is equal to the right Koszul complex $A^{¡} \otimes_{\kappa} A$, up to suspension.

Therefore one is acyclic if and only if the other one is acyclic and we conclude by Theorem 3.4.4. ☐

3.4.4 Other Equivalent Definitions

In the literature [Löf86, Frö99], one encounters the following equivalent definitions of a Koszul algebra.

Lemma 3.4.6. *Let* $A = A(V, R)$ *be a quadratic algebra. It is a Koszul algebra if and only if the homology of its bar construction* $H^\bullet(\mathrm{B}A)$ *is a sub-coalgebra of* $T^c(sV)$*.*

Proof. It is a direct consequence of Proposition 3.3.1 and Theorem 3.4.4. ☐

Let $A = A(V, R)$ be a finitely generated quadratic algebra. Recall that the derived Ext-functor $\mathrm{Ext}^\bullet_A(\mathbb{K}, \mathbb{K})$ is defined as the homology $H_\bullet(\mathrm{Hom}_A(R, \mathbb{K}))$, where $R \xrightarrow{\sim} \mathbb{K}$ is any projective resolution of $\mathbb{K}$ in the category of A-modules. It can be endowed with an associative algebra structure called the *Yoneda algebra*. Considering the quasi-free resolution $A \otimes_\iota \mathrm{B}A \xrightarrow{\sim} \mathbb{K}$, the Ext-functor can be computed by $\mathrm{Ext}^\bullet_A(\mathbb{K}, \mathbb{K}) = H_\bullet((\mathrm{B}A)^*)$, where $(\mathrm{B}A)^*$ is the degreewise and weightwise dual of $\mathrm{B}A$. Since it is the homology of the linear dual of a dga coalgebra, the Yoneda algebra structure is easily described.

Proposition 3.4.7. *A finitely generated quadratic algebra* $A(V, R)$ *is Koszul if and only if its Yoneda algebra* $\mathrm{Ext}^\bullet_A(\mathbb{K}, \mathbb{K})$ *is generated by its weight* 1 *elements.*

Proof. This proposition is linear dual to the previous lemma. ☐

Another equivalent definition of a Koszul algebra amounts to saying that the ground field $\mathbb{K}$ has a "linear minimal graded resolution of $\mathbb{K}$ with free A-modules". Such a resolution is provided by the Koszul complex $A \otimes_\kappa A^{¡}$. For the definitions of these terms and a proof of the equivalence, we refer the reader to [Frö99].

3.4.5 Examples

The symmetric algebra $S(V)$ and the exterior coalgebra $\Lambda^c(sV)$ are Koszul dual to each other. The tensor algebra and the dual numbers coalgebra are also Koszul dual to each other. Here are the proofs of the acyclicity of the associated Koszul complexes, which proves that they are Koszul.

Proposition 3.4.8. *The Koszul complex* $(\Lambda^c(sV) \otimes S(V), d_\kappa)$ *is acyclic.*

Proof. Though this statement is true over $\mathbb{Z}$, we will prove it only over a characteristic zero field. We represent any element $\sum_{\sigma\in\mathbb{S}_p} \operatorname{sgn}(\sigma)s^p x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}$ of $\Lambda^c(sV)^{(p)}$ simply by $x_1 \wedge \cdots \wedge x_p$, keeping in mind that $x_1 \wedge \cdots \wedge x_p = \operatorname{sgn}(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}$ holds for any $\sigma \in \mathbb{S}_p$, like in the Koszul dual algebra $\Lambda(V^*)$. (This identification is nothing but the isomorphism between $(\Lambda^c(sV))^*$ and $\Lambda(V^*)$, up to suspension.)

The boundary map

$$d = d_\kappa : \Lambda^c(sV)^{(p)} \otimes S(V)^{(q)} \longrightarrow \Lambda^c(sV)^{(p-1)} \otimes S(V)^{(q+1)}$$

is given by

$$d(x_1 \wedge \cdots \wedge x_p \otimes y_1 \cdots y_q) = \sum_{j=1}^{p} (-1)^{p-j} x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_p \otimes x_j y_1 \cdots y_q.$$

Define

$$h : \Lambda^c(sV)^{(p)} \otimes S(V)^{(q)} \longrightarrow \Lambda^c(sV)^{(p+1)} \otimes S(V)^{(q-1)}$$

by the formula

$$h(x_1 \wedge \cdots \wedge x_p \otimes y_1 \cdots y_q) := \sum_{i=1}^{q} x_1 \wedge \cdots \wedge x_p \wedge y_i \otimes y_1 \cdots \widehat{y_i} \cdots y_q.$$

One checks that $hd + dh = (p+q)\,\mathrm{id}$. Since we work in characteristic zero it shows that id is homotopic to 0 and therefore the complex is acyclic. ☐

Proposition 3.4.9. *For any graded vector space V, the Koszul complex $((\mathbb{K} \oplus V) \otimes T(V), d_\kappa)$ of the quadratic algebra $T(V)$ is acyclic.*

Proof. Since $T(V) = A(V, 0)$, we get $R = 0$ and therefore $C(V, R) \cong \mathbb{K} \oplus V$, where $\Delta(1) = 1 \otimes 1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$.

The boundary map $d = d_\kappa$ of the Koszul complex $(\mathbb{K} \oplus V) \otimes T(V)$ is zero on the component $\mathbb{K} \otimes T(V)$ and is the identification of $V \otimes T(V)$ with $\mathbb{K} \otimes T(V)^{\geq 1}$ on the other component. Indeed, it is a consequence of the formulas for Δ and of $\kappa(1) = 0, \kappa(v) = v$.

So, the homology of the Koszul complex is $\operatorname{Ker} d / \operatorname{Im} d = T(V)/T(V)^{\geq 1} = \mathbb{K}$ concentrated in bidegree $(0, 0)$. Hence the Koszul complex is acyclic. ☐

3.5 Generating Series

Let (V, R) be a quadratic data such that V is finite dimensional. The weight-graded algebra $A(V, R) = \oplus_{n\geq 0} A^{(n)}$ is such that $A_0 = \mathbb{K}1$ and $A^{(n)}$ is finite dimensional.

By definition the *generating series* or *Hilbert–Poincaré series* of A is

$$f^{A}(x) := \sum_{n\geq 0} \dim A^{(n)} x^{n}.$$

Theorem 3.5.1. *If (V, R) is a finite dimensional quadratic data which is Koszul, then the following identity holds between the generating series of A and $A^{!}$:*

$$f^{A^{!}}(x) f^{A}(-x) = 1.$$

Proof. The Euler–Poincaré characteristic of the sub-chain complex of weight (n) of the Koszul complex of A is equal to $\sum_{k=0}^{n}(-1)^{k} \dim A^{(k)} \dim A^{¡\,(n-k)}$. By definition, it is equal to the coefficient of x^{n} of $f^{A^{!}}(x) f^{A}(-x)$. When the quadratic data (V, R) is Koszul, the Koszul complex is acyclic. It implies that the Euler–Poincaré characteristic is equal to 0, for $n > 0$, and it is equal to 1, for $n = 0$, which concludes the proof. □

Notice that one can also define the generating series of a quadratic coalgebra. In that case, we have $f^{A^{¡}} = f^{A^{!}}$.

Let us apply this theorem to the examples of Sect. 3.4.5. When the dimension of V is equal to k, we have

$$f^{T(V)}(x) = \frac{1}{1-kx} \quad \text{and} \quad f^{D(V^{*})}(x) = 1 + kx,$$

which satisfy $f^{\mathbb{K}\oplus V}(x) f^{T(V)}(-x) = 1$. In the case of the symmetric algebra, we have

$$f^{S(V)}(x) = \frac{1}{(1-x)^{k}} \quad \text{and} \quad f^{\Lambda(V^{*})}(x) = (1+x)^{k},$$

which satisfy $f^{\Lambda(V^{*})}(x) f^{S(V)}(-x) = 1$.

Theorem 3.5.1 provides a method to prove that an algebra is not Koszul. One first computes the Hilbert–Poincaré series $f^{A}(x)$ of the quadratic algebra A and then its inverse series $f^{A}(-x)^{-1}$. If this last one has at least one strictly negative coefficient, then it cannot be the series associated to a quadratic algebra. Therefore, the algebra A is not a Koszul algebra. (See [PP05, Sect. 2.2] for an example.) For a more exhaustive treatment of generating series, we refer the reader to [PP05, Ufn95].

If a chain complex is acyclic, then its Euler–Poincaré characteristic is equal to zero; but the converse is not true. This motivates us to look for quadratic algebras satisfying the functional equation of Theorem 3.5.1 but which fail to be Koszul. Such examples are given in [Pos95, Roo95, Pio01]. In the next section, we give a necessary and sufficient combinatorial condition for an algebra to be Koszul and in Sect. 4.3 we give a sufficient algebraic condition for an algebra to be Koszul.

3.6 Koszul Duality Theory for Inhomogeneous Quadratic Algebras

In the preceding sections, we dealt with Koszul duality of homogeneous quadratic algebras. In [Pri70] Priddy considered more general objects: inhomogeneous quadratic algebras with quadratic and linear relations. They are algebras whose relators contain not only quadratic terms but also possibly linear terms. The main example is the universal enveloping algebra of a Lie algebra: $U(\mathfrak{g}) = T(\mathfrak{g})/(R)$ where the relator is $[x, y] - x \otimes y + y \otimes x$. The purpose of this section is to adapt our treatment of Koszul duality theory to this more general framework. The modification consists in adding a suitable internal differential in the construction of the Koszul dual coalgebra.

There exists an even more general case allowing also constant terms in the space of relations, cf. [PP05].

3.6.1 Quadratic-Linear Algebra

A *quadratic-linear* data (V, R) is a graded vector space V together with a degree homogeneous subspace

$$R \subset V \oplus V^{\otimes 2}.$$

So, there may be linear terms in the space of relations. We still denote by $A = A(V, R) = T(V)/(R)$ the associated quotient. We consider $q : T(V) \twoheadrightarrow V^{\otimes 2}$ the projection onto the quadratic part of the tensor algebra. The image of R under q, denoted qR, is homogeneous quadratic, so (V, qR) is a quadratic data in the sense of Sect. 3.1. We denote by qA its associated algebra: $qA := A(V, qR)$. We assume that R satisfies the property

$$(ql_1) : R \cap V = \{0\}.$$

If it is not the case, by removing some elements of V one can choose another presentation of A which does satisfy (ql_1). This condition amounts to the minimality of the space of generators of A. Under this assumption, there exists a map $\varphi : qR \to V$ such that R is the graph of φ:

$$R = \left\{X - \varphi(X) \mid X \in qR\right\}.$$

For instance, if $A = U(\mathfrak{g})$, then $\varphi(x \otimes y - y \otimes x) = [x, y]$ and $qA = S\mathfrak{g}$. The weight grading on $T(V)$ induces a filtration which is compatible with the ideal (R). Hence the quotient algebra A is filtered by $\mathrm{F}_n A := \mathrm{Im}(\bigoplus_{k \le n} V^{\otimes k})$. The assumption $R \cap V = \{0\}$ implies $\mathrm{F}_1 A = \mathbb{K} \oplus V$. We denote by $\mathrm{gr}\, A$ the graded algebra associated to the filtration of A, $\mathrm{gr}_n A := \mathrm{F}_n A/\mathrm{F}_{n-1} A$. We denote by

$$p : qA \twoheadrightarrow \mathrm{gr}\, A$$

the resulting epimorphism. It is obviously an isomorphism in weight 0 and 1, but not necessarily in weight 2. A corollary of the present theory shows that p is an isomorphism provided that qA is Koszul, see Theorem 3.6.4. In the example $A = U(\mathfrak{g})$ the map $p : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$ is the PBW isomorphism.

3.6.2 Koszul Dual Coalgebra

The map φ permits us to construct the composite map

$$\tilde{\varphi} : (qA)^{¡} = C\big(sV, s^2qR\big) \twoheadrightarrow s^2qR \xrightarrow{s^{-1}\varphi} sV.$$

By Proposition 1.2.2 there exists a unique coderivation, $d_{\tilde{\varphi}} : (qA)^{¡} \to T^c(sV)$, which extends this composite.

Lemma 3.6.1.

(a) *If* $\{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset qR$, *then the image of the coderivation* $d_{\tilde{\varphi}}$ *lives in* $(qA)^i = C(sV, s^2qR) \subset T^c(sV)$, *thereby defining a coderivation* d_φ *of the coalgebra* $(qA)^i$.

(b) *If the condition*

$$(ql_2) : \{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset R \cap V^{\otimes 2}$$

is satisfied, *then the coderivation* d_φ *squares to* 0.

Proof. If $\{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset qR$, then we prove that $d_{\tilde{\varphi}}(C(sV, s^2qR)^{(3)}) \subset C(sV, s^2qR)^{(2)} = s^2qR$. The proof of the general case is done in the same way with the formula $(qA)^{¡\,(n)} = \bigcap_{i+2+j=n} (sV)^{\otimes i} \otimes s^2qR \otimes (sV)^{\otimes j}$. Since $C(sV, s^2qR)^{(3)}$ is equal to $s^2qR \otimes sV \cap sV \otimes s^2qR$, any of its elements can be written $Y = \sum s^2X \otimes sv = \sum sv' \otimes s^2X'$, with $v, v' \in V$ and $X, X' \in qR$. The formula for the unique coderivation on the cofree coalgebra $T^c(sV)$ of Proposition 1.2.2 gives

$$\begin{aligned} d_{\tilde{\varphi}}(Y) &= \sum \tilde{\varphi}\big(s^2X\big) \otimes sv - \sum (-1)^{|v'|} sv' \otimes \tilde{\varphi}\big(s^2X'\big) \\ &= \sum \big(s\varphi(X) - s^2X\big) \otimes sv + \sum sv' \otimes \big(s^2X' - (-1)^{|v'|} s\varphi\big(X'\big)\big). \end{aligned}$$

Hence, forgetting the suspension for simplicity, we have

$$\begin{aligned} d_{\tilde{\varphi}}(Y) &= \sum \big(\varphi(X) - X\big) \otimes v + \sum v' \otimes \big(X' - \varphi\big(X'\big)\big) \\ &\in \{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset qR = C(V, qR)^{(2)}. \end{aligned}$$

Since $\ker \varphi = R \cap V^{\otimes 2}$, we get ${d_\varphi}^2(C(V, qR)^{(3)}) = \{0\}$ if $\{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset R \cap V^{\otimes 2}$. Once again, the proof of the general case follows from the same pattern using the explicit formula of the coalgebra $(qA)^{¡}$. ☐

Since $R \cap V^{\otimes 2} \subset qR$, condition (ql_2) implies $\{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset qR$. Condition (ql_2) amounts to saying that one cannot create new quadratic relations in R by adding an element to the relations of the presentation.

Let (V, R) be a quadratic-linear data satisfying the conditions (ql_1) and (ql_2). By definition the *Koszul dual dga coalgebra* of $A = A(V, R)$ is the dga coalgebra

$$A^{¡} := \big((qA)^{¡}, d_\varphi\big) = \big(C\big(sV, s^2 qR\big), d_\varphi\big).$$

3.6.3 Koszulity in the Inhomogeneous Quadratic Framework

A quadratic-linear data (resp. a quadratic-linear algebra) is said to be *Koszul* if it satisfies conditions (ql_1), (ql_2) and if the quadratic data (V, qR), or equivalently the quadratic algebra qA, is Koszul in the sense of Sect. 3.4.

Notice that for a homogeneous quadratic data, Koszul in the classical sense is Koszul in this sense. In this case, the conditions (ql_1), (ql_2) are trivially satisfied and the inner coderivation d_φ vanishes.

3.6.4 Cobar Construction in the Inhomogeneous Quadratic Framework

Under the hypotheses (ql_1) and (ql_2), we have constructed a conilpotent dga coalgebra $A^{¡}$. Applying the cobar construction of Sect. 2.2.2, we get a dga algebra $\Omega A^{¡}$, whose differential is of the form $d_1 + d_2$. The internal derivation d_1 is the unique derivation which extends d_φ. The derivation d_2 is induced by the coalgebra structure of $A^{¡}$.

We consider the same map κ in this context

$$\kappa : A^{¡} = C\big(sV, s^2 qR\big) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A.$$

Lemma 3.6.2. *The map κ is a twisting morphism in* $\mathrm{Hom}(A^{i}, A)$, *that is* $\partial(\kappa) + \kappa \star \kappa = 0$.

Proof. We refine the proof of Lemma 3.4.1, taking care of the internal differential d_φ of $A^{¡}$. The Maurer–Cartan equation becomes $-\kappa \circ d_\varphi + \kappa \star \kappa = 0$. The map $-\kappa \circ d_\varphi + \kappa \star \kappa$ is equal to 0 everywhere except on $(A^{¡})^{(2)} = s^2 qR$ where its image is $\{-\varphi(X) + X \mid X \in qR\} = R$, which vanishes in A. ☐

The twisting morphism κ induces a morphism of dga algebras $g_\kappa : \Omega A^{¡} \to A$ by Theorem 2.2.6.

Theorem 3.6.3. *Let A be an inhomogeneous quadratic Koszul algebra satisfying the conditions (ql_1) and (ql_2). Let $A^{¡} = ((qA)^{¡}, d_\varphi)$ be its Koszul dual dga coalgebra. The morphism of dga coalgebras $g_\kappa : \Omega A^{¡} \xrightarrow{\sim} A$ is a quasi-isomorphism.*

Proof. In this proof, we consider the cobar construction as a chain complex graded by the syzygy degree as in Sect. 3.3.2: both the internal differential d_1 of $\Omega A^{¡}$ induced by d_φ and the differential d_2 induced by the coproduct of the coalgebra $(qA)^{¡}$ lower the syzygy degree by 1. So we have a well-defined nonnegatively graded chain complex.

Since $(qA)^{¡}$ is a weight graded coalgebra, the underlying module $\Omega A^{¡} = T(s^{-1}\overline{(qA)^{¡}})$ of the bar construction is weight-graded. We consider the filtration F_r of $\Omega A^{¡}$ defined by its weight: the elements of F_r are the elements of weight less than r. The two components of the differential map $d = d_1 + d_2$ satisfy

$$d_2 : F_r \to F_r \quad \text{and} \quad d_1 : F_r \to F_{r-1}.$$

The filtration F_r is therefore stable under the boundary map d. Since it is bounded below and exhaustive, the associated spectral sequence $E^\bullet_{rs}$ converges to the homology of $\Omega A^{¡}$ by the classical convergence theorem of spectral sequences [ML95, Proposition 3.2, Chap. 11]. Hence, F_r induces a filtration F_r on the homology of $\Omega A^{¡}$ such that

$$E^\infty_{rs} \cong \mathrm{F}_r\big(H_{r+s}(\Omega A^{¡})\big)/\mathrm{F}_{r-1}\big(H_{r+s}(\Omega A^{¡})\big) =: \mathrm{gr}_r\big(H_{r+s}(\Omega A^{¡})\big).$$

The first term of this spectral sequence is equal to $E^0_{rs} = T(s^{-1}\overline{(qA)^{¡}})^{(r)}_{r+s}$, which is made up of the elements of syzygy degree equal to $r+s$ and grading equal to (r). The differential map d^0 is given by d_2. Since the algebra qA is Koszul, the spectral sequence is equal to $E^1_{rs} = qA^{(r)}$ at rank 1. More precisely E^1_{rs} is concentrated in the line $r+s=0$: $E^1_{rs} \cong qA^{(r)}$, for $r+s=0$ and $E^1_{rs}=0$, for $r+s \neq 0$. Therefore, the spectral sequence collapses at rank 1.

In conclusion, the convergence theorem gives

$$\begin{aligned} E^1_{r-r} &\cong qA^{(r)} \cong E^\infty_{r-r} \cong \mathrm{gr}_r\big(H_0(\Omega A^{¡})\big), \\ E^1_{rs} &\cong 0 \cong E^\infty_{rs} \cong \mathrm{gr}_r\big(H_{r+s}(\Omega A^{¡})\big), \quad \text{for } r+s \neq 0. \end{aligned}$$

The result of Proposition 3.3.2 still holds in the inhomogeneous case, that is $H_0(\Omega A^{¡}) \cong A$, with the syzygy degree. Hence the quotient $\mathrm{gr}_r(H_0(\Omega A^{¡}))$ is equal to $\mathrm{gr}_r A$ and the morphism $\Omega A^{¡} \xrightarrow{\sim} A$ is a quasi-isomorphism. ☐

Notice that, in the inhomogeneous case, this resolution is not minimal because of the internal differential d_1.

3.6.5 Poincaré–Birkhoff–Witt Theorem

Theorem 3.6.4 (Poincaré–Birkhoff–Witt Theorem). *When a quadratic-linear algebra A is Koszul, then the epimorphism $p : qA \twoheadrightarrow \mathrm{gr}\,A$ is an isomorphism of graded* *algebras*

$$qA \cong \operatorname{gr} A.$$

Proof. This theorem was already proved in the proof of the previous theorem, where the convergence of the spectral sequence gave

$$E^1_{r-r} \cong qA^{(r)} \cong E^\infty_{r-r} \cong \operatorname{gr}_r A. \qquad \square$$

Another proof of this theorem, based on deformation theory, can be found in [BG96]. Even if the Poincaré–Birkhoff–Witt theorem is a direct consequence of the proof of Proposition 3.6.3, it has the following two nontrivial consequences: Corollary 3.6.5 and Proposition 3.6.6.

Corollary 3.6.5. *Let $A(V, R)$ be an algebra with the quadratic-linear presentation (V, R). If the quadratic algebra $qA = A(V, qR)$ is Koszul, then conditions (ql_1) and (ql_2) are equivalent to conditions*

$$\bigl(ql_1{}'\bigr) : (R) \cap V = \{0\} \quad \textit{and} \quad \bigl(ql_2{}'\bigr) : R = (R) \cap \bigl\{V \oplus V^{\otimes 2}\bigr\}.$$

Proof. Condition $(ql_1{}')$ is the generalization of condition (ql_1) from R to (R). Condition $(ql_2{}')$ is the generalization of condition (ql_2) from $R \otimes V + V \otimes R$ to (R). In the other way round, if conditions (ql_1) and (ql_2) are satisfied and if the algebra $A(V, R)$ is Koszul, then we get the Poincaré–Birkhoff–Witt isomorphism $qA \cong \operatorname{gr} A$ of Theorem 3.6.4. In weight 1, it implies condition $(ql_1{}')$. In weight 2, it implies $qR = q((R) \cap \{V \oplus V^{\otimes 2}\})$, which is equivalent to condition $(ql_2{}')$ by condition $(ql_1{}')$. $\square$

Conditions $(ql_1{}')$ and $(ql_2{}')$ amount to say that the ideal generated by R does not create any new quadratic-linear relation. It is equivalent to the maximality of the space of relations in the presentation of the inhomogeneous quadratic algebra. Such conditions can be hard to check in practice because one would have to compute the full ideal generated by R. But this proposition shows that if one finds a quadratic-linear presentation of an algebra satisfying conditions (ql_1), (ql_2) and whose homogeneous quadratic data is Koszul, then the space of relations R is maximal.

REMARK. This result is "Koszul dual" to the Diamond Lemma 4.2.4, since we work with the cobar construction Ω instead of the bar construction B in Sect. 4.2.4. Here it gives, under condition (ql_1),

$$(qA)^{¡} \text{ Koszul } \& \ (ql_2) \quad \Rightarrow \quad A^{¡} \text{ Koszul } \& \ \bigl(ql_2{}'\bigr),$$

where condition (ql_2) has to be seen as the particular case of condition $(ql_2{}')$ in weight 3. These two conditions refer to the ideal generated by R, whereas the condition of the Diamond Lemma refers to the quotient by some ideal associated to R. Also, in a similar way, we get the following isomorphism between the Koszul dual algebras (of the aforementioned coalgebras): $qA \cong \operatorname{gr} A \cong A$ as a direct byproduct. This result is better seen as a Diamond Lemma for Gröbner bases, see Sect. 4.3.7.

3.6.6 Acyclicity of the Koszul Complex

As in the quadratic case, the Koszul complex associated to an inhomogeneous Koszul algebra is acyclic.

Proposition 3.6.6. *When $A(V,R)$ is a quadratic-linear Koszul algebra, its Koszul complexes $A^{¡} \otimes_\kappa A$ and $A \otimes_\kappa A^{¡}$ are acyclic.*

Proof. We consider the Koszul complex as a chain complex graded by the weight of the elements of $A^{¡}$. The two parts $d_\varphi \otimes \mathrm{id}_A$ and d_κ^r of the differential map lower this degree by -1, so it is a well-defined chain complex.

The natural filtration on A plus the weight grading on $(qA)^{¡}$ induce an exhaustive and bounded below filtration F_r on $A^{¡} \otimes_\kappa A$. The differential maps satisfy $d_\kappa^r : F_r \to F_r$ and $d_\varphi \otimes \mathrm{id}_A : F_r \to F_{r-1}$. Therefore, E^0 is equal to $A^{¡} \otimes_{\bar\kappa} \mathrm{gr} A$ where $\bar\kappa : A^{¡} \to \mathrm{gr} A$ is the associated twisting morphism and where $d^0 = d_{\bar\kappa}^r$.

By the Poincaré–Birkhoff–Witt Theorem, E^0 is equal to the twisted tensor product $(qA)^{¡} \otimes_{\tilde\kappa} qA$ of the Koszul quadratic algebra qA, with $\tilde\kappa : (qA)^{¡} \to qA$ being the Koszul twisting morphism. Therefore, it is acyclic and we conclude by the convergence theorem for spectral sequences [ML95, Proposition 3.2, Chap. 11]). □

In [Pri70], Priddy called *Koszul resolutions*, the resolution $A \otimes_\kappa A^{¡}$ (resp. $A^{¡} \otimes_\kappa A$) of $\mathbb{K}$ by free A-modules. They provide chain complexes, smaller than the augmented bar construction $A \otimes_\pi \mathrm{B}A$, which allow one to compute the Tor functors $\mathrm{Tor}_\bullet^A(\mathbb{K}, M)$ for any A-module M (see [CE56, ML95] for the definition of Tor functors). In the example of the universal enveloping algebra of a Lie algebra, Priddy recovers the original Koszul resolution [CE56], which computes the Chevalley–Eilenberg homology of Lie algebras, see Sect. 3.6.7. Applied to restricted Lie algebras, this gives May resolutions [May66]. For the Steenrod algebra, it provides resolutions based on the Λ (co)algebra of [BCK+66], see Sect. 3.6.8 for more details.

Dually, the twisted convolution algebra $\mathrm{Hom}^\kappa(A^{¡}, A)$ computes the homology functors $\mathrm{Ext}_A^\bullet(\mathbb{K}, A)$ as in [BCK+66] (see Exercise 3.8.11).

3.6.7 The Example of the Universal Enveloping Algebra

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is $U(\mathfrak{g}) := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y])$. So it is defined as a quadratic-linear algebra with $V = \mathfrak{g}$. Its associated quadratic algebra is the symmetric algebra on $\mathfrak{g}$: $q(U(\mathfrak{g})) \cong S(\mathfrak{g})$.

Proposition 3.6.7. *When the characteristic of the ground field is not* 2, *the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Koszul algebra.*

Proof. A direct inspection shows that condition (ql_1) is satisfied. Let us prove that condition (ql_2) is also satisfied. The subspace $R \cap V^{\otimes 2}$ of $V^{\otimes 2}$ is equal to $\{\sum x \otimes y \mid$

$\sum[x, y] = 0\}$. Let $\xi = \sum(x \otimes y \otimes z - y \otimes x \otimes z - [x, y] \otimes z) + \sum(t \otimes u \otimes v - t \otimes v \otimes u - t \otimes [u, v])$ be an element of $(R \otimes V + V \otimes R)$. It belongs to $V^{\otimes 2}$ if and only if $\sum(x \otimes y \otimes z - y \otimes x \otimes z) + \sum(t \otimes u \otimes v - t \otimes v \otimes u) = 0$. In this case, applying $[[-,-],-]$ to this element, we get $2\sum[[x, y], z] + 2\sum[[t, u], v] = 0$. This proves that $\xi \in R \cap V^{\otimes 2}$ and that (ql_2) holds, when the characteristic of $\mathbb{K}$ is not 2. Finally, Proposition 3.4.8 shows that $S(\mathfrak{g})$ is a Koszul algebra, therefore $U(\mathfrak{g})$ is a Koszul algebra. ☐

Among other consequences, Theorem 3.6.4 can be applied and gives the "classical" Poincaré–Birkhoff–Witt theorem: there is an isomorphism of graded algebras

$$S(\mathfrak{g}) \cong \operatorname{gr} U(\mathfrak{g}),$$

which is sometimes stated in terms of the monomial basis of the symmetric algebra.

Proposition 3.6.8. *The Koszul dual dga coalgebra of the universal enveloping algebra $U(\mathfrak{g})$ is the following dga coalgebra*

$$U(\mathfrak{g})^{i} \cong \big(\Lambda^c(s\mathfrak{g}), d_\varphi\big),$$

where d_φ is the Chevalley–Eilenberg boundary map defining the homology of the Lie algebra $\mathfrak{g}$.

Proof. First, we have $q(U(\mathfrak{g}))^{\mathrm{i}} = S(\mathfrak{g})^{\mathrm{i}} = \Lambda^c(s\mathfrak{g})$. Recall that $\Lambda^c(s\mathfrak{g})$ is linearly spanned by the elements $\sum_{\sigma\in\mathbb{S}_n} \operatorname{sgn}(\sigma) s^n x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, which we denote by $x_1 \wedge \cdots \wedge x_n$. The internal differential d_φ is the unique coderivation which extends $\varphi : x \otimes y - y \otimes x \mapsto [x, y]$. Therefore it is equal to

$$d_\varphi(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n,$$

which is the Chevalley–Eilenberg differential [CE48, Kos50], see Sect. 13.2.7. ☐

Corollary 3.6.9. *The twisted tensor product $U(\mathfrak{g}) \otimes_\kappa \Lambda^c(s\mathfrak{g})$ is a resolution of $\mathbb{K}$ by free $U(\mathfrak{g})$-modules.*

Proof. Direct corollary of Proposition 3.6.6 and Proposition 3.6.7. ☐

This is the original Koszul resolution which computes Chevalley–Eilenberg homology of Lie algebras [CE56].

3.6.8 The Example of the Steenrod Algebra

The *Steenrod algebra* $\mathscr{A}_2$ is the quadratic-linear algebra

$$\mathscr{A}_2 := A\big(\{Sq^i\}_{i\geq 1}, R_{Adem}\big)$$

over the characteristic 2 field $\mathbb{K} = \mathbb{F}_2$, where $|Sq^i| = i$ and where R_{Adem} stands for the *Adem relations*

$$\begin{aligned} Sq^i Sq^j &= \binom{j-1}{i} Sq^{i+j} \\ &\quad + \sum_{k=1}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k, \quad \forall i, j > 0 \text{ with } i < 2j. \end{aligned}$$

The quadratic analog $q\mathscr{A}_2$ is obtained by omitting the linear term $\binom{j-1}{i} Sq^{i+j}$. The images of the elements $\{Sq^{i_1} \cdots Sq^{i_k}; i_l \geq 2i_{l+1}\}$ form a basis of $q\mathscr{A}_2$ and $\mathscr{A}_2$, called the *Cartan–Serre basis* of *admissible monomials*.

The degree-wise linear dual of the Koszul dual dga coalgebra $\mathscr{A}_2^{\text{¡}}$ is a dga algebra, which is anti-isomorphic to the Λ *algebra* of [BCK+66]. Notice that its homology gives the second page of the Adams spectral sequence which computes homotopy groups of spheres. The dga algebra Λ is generated by the elements $\{\lambda_i\}_{i \geq 0}$ of degree $|\lambda_i| = i$ and satisfies the relations

$$\lambda_i \lambda_{2i+1+j} = \sum_{k \geq 0} \binom{j-k-1}{k} \lambda_{i+j-k} \lambda_{2i+1+k}.$$

Its differential is the unique derivation extending

$$\lambda_j \mapsto \sum_{k \geq 0} \binom{j-k-1}{k+1} \lambda_{j-k-1} \lambda_k.$$

The mod-p Steenrod algebra can be treated in the same way. For more details, we refer the reader to [Wan67, Pri70].

3.7 Résumé

3.7.1 Quadratic Data and Koszul Dual Constructions

$$\begin{tikzcd}
 & \begin{array}{c}(V, R)\\ \text{quadratic data}\end{array} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\begin{array}{c}A = A(V, R)\\ \text{quadratic algebra}\end{array} & & \begin{array}{c}C = C(V, R)\\ \text{quadratic coalgebra}\end{array}
\end{tikzcd}$$

The quadratic algebra:

$$\begin{aligned}A = A(V, R) = T(V)/(R) &\cong \bigoplus_{n\in\mathbb{N}} A^{(n)}\\ &= \mathbb{K}1 \oplus V \oplus \big(V^{\otimes 2}/R\big) \oplus \cdots \oplus \Bigg(V^{\otimes n}\Big/ \sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\Bigg) \oplus \cdots .\end{aligned}$$

The quadratic coalgebra:

$$\begin{aligned}C = C(V, R) \subset T^c(V) &\cong \bigoplus_{n\in\mathbb{N}} C^{(n)}\\ &= \mathbb{K}1 \oplus V \oplus R \oplus \cdots \oplus \Bigg(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\Bigg) \oplus \cdots .\end{aligned}$$

Koszul dual coalgebra of an algebra:

$$A(V, R)^{¡} := C\big(sV, s^2R\big).$$

Koszul dual algebra of a coalgebra:

$$C(V, R)^{¡} := A\big(s^{-1}V, s^{-2}R\big), \qquad \big(A^{¡}\big)^{¡} \cong A.$$

Koszul dual algebra of an algebra: When V is finite dimensional, the linear dual of the desuspension of $A^{¡}$ is the quadratic algebra $A^! \cong A(V^*, R^{\perp})$.

EXAMPLES: $T(V)^! \cong D(V^*)$ and $S(V)^! \cong \Lambda(V^*)$.

3.7.2 Koszul Duality Theory

Twisting morphism:

$$\kappa : A^{¡} = C\big(sV, s^2R\big) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V, R) = A.$$

Koszul complexes: $A \otimes_\kappa A^{¡}$ and $A^{¡} \otimes_\kappa A$,

$$A^{¡} \rightarrowtail \mathrm{B}A \quad \text{and} \quad \Omega A^{¡} \twoheadrightarrow A,$$

with the *syzygy degree*: $H^0(\mathrm{B}^\bullet A) \cong A^{¡}$ and $H_0(\Omega_\bullet A^{¡}) \cong A$.

The quadratic data (V, R) is *Koszul* when one of the following equivalent assertions is satisfied.

1. the right Koszul complex $A^{¡} \otimes_\kappa A$ is acyclic,
2. the left Koszul complex $A \otimes_\kappa A^{¡}$ is acyclic,
3. the inclusion $i : A^{¡} \rightarrowtail \mathrm{B}A$ is a quasi-isomorphism,

4. the projection $p : \Omega A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism,
5. $H^n(\mathrm{B}^\bullet A) = 0$ for $n \geq 1$,
6. $H_n(\Omega_\bullet A^{¡}) = 0$ for $n \geq 1$,
7. $H^\bullet(\mathrm{B}^\bullet A)$ is a sub-coalgebra of $T^c(sV)$,
8. the Yoneda algebra $\mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ is generated by its weight 1 elements [when V is finite dimensional].

EXAMPLES: $T(V)$, $D(V)$, $S(V)$, $\Lambda(V)$.

3.7.3 Generating Series or Hilbert–Poincaré Series

$$f^A(t) := \sum_{n\geq 0} \dim A^{(n)} t^n,$$

$$A \text{ Koszul} \quad \Longrightarrow \quad f^{A^!}(t) f^A(-t) = 1.$$

3.7.4 Inhomogeneous Koszul Duality Theory

Quadratic-linear data: (V, R), with $R \subset V \oplus V^{\otimes 2}$.

Quadratic analog: $qR := \mathrm{proj}_{V^{\otimes 2}}(R)$ and $qA := A(V, qR)$.

$$(ql_1) : R \cap V = \{0\} \quad \Rightarrow \quad R = \mathrm{Graph}(\varphi : qR \to V),$$
$$(ql_2) : \{R \otimes V + V \otimes R\} \cap V^{\otimes 2} \subset R \cap V^{\otimes 2}.$$

$(qA)^{¡} \twoheadrightarrow qR \xrightarrow{\varphi} V$ induces a coderivation $d_\varphi (qA)^{¡} \to T^c(V)$, (ql_1) and (ql_2) imply d_φ well-defined and $(d_\varphi)^2 = 0$.

Koszul dual dga coalgebra: $A^{¡} := ((qA)^{¡}, d_\varphi)$.

$A(V, R)$ *Koszul algebra* when (ql_1), (ql_2) and qA quadratic Koszul algebra. In this case:

- quasi-free resolution: $\Omega A^{¡} \overset{\sim}{\twoheadrightarrow} A$,
- Poincaré–Birkhoff–Witt theorem: $qA \cong \mathrm{gr}\, A$,
- Koszul complex: $A \otimes_\kappa A^{¡}$ acyclic.

EXAMPLE: $A = U(\mathfrak{g})$, universal enveloping algebra of a Lie algebra $\mathfrak{g}$,

- $U(\mathfrak{g})^{¡} = (\Lambda^c(s\mathfrak{g}), \text{Chevalley–Eilenberg differential})$,
- Original Poincaré–Birkhoff–Witt theorem: $S(V) \cong \mathrm{gr}\, U(\mathfrak{g})$,
- Original Koszul complex: $U(\mathfrak{g}) \otimes_\kappa \Lambda^c(s\mathfrak{g})$ acyclic.

EXAMPLE: $A = \mathscr{A}_2$, the mod-2 Steenrod algebra,

- Cartan–Serre basis,
- the dga algebra $(\mathscr{A}_2{}^{¡})^*$ is the Λ algebra.

3.8 Exercises

Exercise 3.8.1 (Koszul implies quadratic). Let A be a connected weight graded algebra (see Sect. 1.5.10). Its bar construction $\mathrm{B}A$ splits with respect to the weight and we consider the same syzygy degree as in Sect. 3.3.1. Show that if the homology of $\mathrm{B}A$ is concentrated in syzygy degree 0, then the algebra has a quadratic presentation.

Exercise 3.8.2 (Two-sided Koszul complex). Let (V, R) be a quadratic data. Under the notation of Sect. 2.1.4, we define the *two-sided Koszul complex* on $A \otimes A^{¡} \otimes A$ by the differential $d^l_\kappa \otimes \mathrm{Id}_A + \mathrm{Id}_A \otimes d^r_\kappa$ and we denote it by $A \otimes_\kappa A^{¡} \otimes_\kappa A$. Show that the quadratic data is Koszul if and only if the morphism of dg A-bimodules

$$A \otimes A^{¡} \otimes A \xrightarrow{\mathrm{Id}_A \otimes \varepsilon \otimes \mathrm{Id}_A} A \otimes \mathbb{K} \otimes A \cong A \otimes A \xrightarrow{\mu} A$$

is a resolution of A.

(★) When (V, R) is a quadratic-linear data satisfying conditions (ql_1) and (ql_2), we add the term $\mathrm{Id}_A \otimes d_\varphi \otimes \mathrm{Id}_A$ to the differential defining the two-sided Koszul complex. Prove the same result in this case.

Exercise 3.8.3 (Dual numbers algebra). Show that, for the quadratic algebra of dual numbers $A = \mathbb{K}[\varepsilon]/(\varepsilon^2 = 0)$, with ε of degree 0, the cobar construction of $A^{¡}$ is isomorphic to the dga algebra $\Omega A^{¡} = \mathbb{K}\langle t_1, t_2, \ldots, t_n, \ldots\rangle$, where $|t_n| = n - 1$ and $d(t_n) = -\sum_{i+j=n} (-1)^i t_i t_j$.

Exercise 3.8.4 (Inhomogeneous algebra ★). Let A be an inhomogeneous quadratic algebra. Show that if A is Koszul, then $f_\kappa : A^! \to \mathrm{B}A$ is a quasi-isomorphism of dga coalgebras.

Exercise 3.8.5 (Koszul complex of the symmetric algebra). Prove that $(\Lambda^c(sV) \otimes S(V), d_\kappa)$ is acyclic over $\mathbb{Z}$.

HINT. Use a suitable filtration.

Exercise 3.8.6 (Koszul complexes ★). Consider the three functors S, Λ and Γ (cf. Exercise 1.8.6). Show that there are acyclic complexes $\Lambda \otimes S$, $\Gamma \otimes \Lambda$. Show that $S \otimes \Lambda$ is not acyclic in characteristic p and defines the Cartier homomorphism, see for instance [Pir02b].

Exercise 3.8.7 (Koszul complex in local cohomology ★). Let A be a commutative algebra concentrated in degree 0. Let x be an element of A. We define the "Koszul complex" by

$$K^A_\bullet(x) : 0 \to A \to A \to 0,$$

concentrated in degrees 0 and 1, where the boundary map defined by $d(a) := ax$. More generally, for n elements $\{x_1, \ldots, x_n\}$ of A, the "Koszul complex" is defined by the tensor product

$$K_\bullet^A(x_1, \dots, x_n) := K_\bullet^A(x_1) \otimes \cdots \otimes K_\bullet^A(x_n)$$

of chain complexes.

Show that the degree -1 map

$$\tau : \Lambda^c(sx_1, \dots, sx_n) \twoheadrightarrow \mathbb{K}sx_1 \oplus \cdots \oplus \mathbb{K}sx_n \xrightarrow{s^{-1}} \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n \to A$$

is a twisting morphism from the symmetric cofree coalgebra on the suspension of the basis $\{x_1, \dots, x_n\}$ to the algebra A.

Prove that the "Koszul complex" $K_\bullet^A(x_1, \dots, x_n)$ is isomorphic to the twisted tensor product $\Lambda^c(sx_1, \dots, sx_n\mathbb{K}) \otimes_\tau A$.

Considering the canonical twisting morphism $\kappa : \Lambda^c(sx_1, \dots, sx_n) \to S(x_1, \dots, x_n)$, show that the "Koszul complex" $K_\bullet^A(x_1, \dots, x_n)$ is isomorphic to the relative tensor product

$$\big(\Lambda^c(sx_1, \dots, sx_n) \otimes_\kappa S(x_1, \dots, x_n)\big) \otimes_{S(x_1,\dots,x_n)} A,$$

where A is considered a left $S(x_1, \dots, x_n)$-module.

We say that $\{x_1, \dots, x_n\}$ is a *regular sequence* when the image of x_i in $A/(x_1, \dots, x_{i-1})A$ has no nonzero divisor, for $1 \leq i \leq n$. When it is the case, prove that the Koszul complex $K_\bullet^A(x_1, \dots, x_n)$ is a resolution of $A/(x_1, \dots, x_n)A$ by free A-modules.

This chain complex is used to compute local cohomology (see [Wei94, Sects. 4.5–4.6]).

Exercise 3.8.8 (Homological degree). Let (V, R) be a quadratic data such that V is concentrated in degree 0. We consider the bar construction $\mathrm{B}_\bullet A$ of the quadratic algebra $A = A(V, R)$ as a chain complex with the homological degree.

Show that this chain complex splits with respect to the weight grading: $\mathrm{B}_\bullet A = \bigoplus_{n \in \mathbb{N}} (\mathrm{B}_\bullet A)^{(n)}$. For $n \geq 1$, prove that the sub-chain complex $(\mathrm{B}_\bullet A)^{(n)}$ is finite, concentrated in degrees $1 \leq \bullet \leq n$ and that $H_n((\mathrm{B}_\bullet A)^{(n)}) \cong A^{¡\,(n)}$.

Show that the quadratic data (V, R) is Koszul if and only if the homology of the bar construction $\mathrm{B}_\bullet A$ is concentrated on the diagonal $\bigoplus_{n \in \mathbb{N}} H_n((\mathrm{B}_\bullet A)^{(n)})$.

Exercise 3.8.9 (Double Hilbert–Poincaré series). Pursuing the preceding exercise, we require here the vector space V to be finite dimensional. In this case, show that all the components $(\mathrm{B}_m A)^{(n)}$ of the bar construction of A are finite dimensional, for any $m, n \in \mathbb{N}$.

We define the *double Hilbert–Poincaré series* of A by

$$F^A(x, t) := \sum_{m,n \geq 0} \dim H_m\big((\mathrm{B}_\bullet A)^{(n)}\big) x^m t^n.$$

Show that the quadratic data is Koszul if and only if the double Hilbert–Poincaré series has only nontrivial coefficients in powers $x^m t^n$ for $m = n$. Prove that it is also equivalent to $F^A(x,t) = f^{A^!}(xt)$.

Prove the functional equation $f^A(t)F^A(-1,t) = 1$ and recover the equation of Theorem 3.5.1.

Exercise 3.8.10 (Every augmented algebra is inhomogeneous Koszul). Let A be an augmented associative algebra.

1. Show that $V := \bar{A}$ and $R := \{a \otimes b - ab \mid a, b \in \bar{A}^{\otimes 2}\}$ is a quadratic-linear presentation of A satisfying conditions (ql_1) and (ql_2).
2. Prove that qA is nilpotent and that the Koszul dual dg coalgebra $A^{¡} \cong \mathrm{B}A$ is isomorphic to the bar construction of the algebra A.
3. Finally, show that this quadratic-linear presentation is Koszul and that the Koszul resolution is nothing but the bar–cobar resolution.

Exercise 3.8.11 (BCKQRS spectral sequence as twisted convolution algebra ★). Show that the first page $E^1 X$ of the spectral sequence of [BCK+66] for any spectrum X is equal to the convolution algebra $\mathrm{Hom}^{\kappa}(\mathscr{A}_2{}^{¡}, \mathscr{A}_2)$ as follows: $H_\bullet(X)$ and $\mathrm{Hom}^{\kappa}(\mathscr{A}_2{}^{¡}, \mathscr{A}_2)$ are $\mathscr{A}_2$-modules and

$$E^1 X \cong \mathrm{Hom}^{\kappa}\left(\mathscr{A}_2{}^{¡}, \mathscr{A}_2\right) \otimes_{\mathscr{A}_2} H_\bullet(X).$$

Show that $E^2 X \cong \mathrm{Ext}^{\bullet}_{\mathscr{A}_2}(\mathbb{K}, H_\bullet(X))$.