Chapter 2
Twisting Morphisms

... remember young fellow, Ω is left adjoint ...
Dale Husemöller, MPIM (Bonn),
private communication

In this chapter, we introduce the bar construction and the cobar construction as follows. A *twisting morphism* is a linear map $f : C \to A$, from a dga coalgebra C to a dga algebra A, which satisfies the Maurer–Cartan equation:

$$\partial(f) + f \star f = 0.$$

The set of twisting morphisms $\mathrm{Tw}(C, A)$ is shown to be representable both in C and in A. More precisely, the cobar construction is a functor Ω from dga coalgebras to dga algebras and the bar construction is a functor B from dga algebras to dga coalgebras which satisfy the following properties: there are natural isomorphisms

$$\mathrm{Hom}_{\mathsf{dga\ alg}}(\Omega C, A) \cong \mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathsf{dga\ coalg}}(C, \mathrm{B}A).$$

As an immediate consequence the functors cobar and bar are adjoint to each other. Then we investigate the twisting morphisms which give rise to quasi-isomorphisms under the aforementioned identifications. We call them *Koszul morphisms*.

The main point is the following characterization of the Koszul morphisms. Any linear map $\alpha : C \to A$ gives rise to a map $d_\alpha : C \otimes A \to C \otimes A$, which is a differential if and only if α is a twisting morphism. Moreover, α is a Koszul morphism if and only if the chain complex $(C \otimes A, d_\alpha)$ is acyclic. This is the first step of Koszul duality theory, which will be treated in the next chapter.

As a corollary, it is shown that the unit and the counit of the bar–cobar adjunction

$$C \to \mathrm{B}\Omega C \quad \text{and} \quad \Omega\mathrm{B}A \to A,$$

are quasi-isomorphisms. Hence, the latter provides a canonical free resolution of A.

This chapter is inspired by H. Cartan [Car55], E. Brown [Bro59], J.C. Moore [Moo71], Husemoller–Moore–Stasheff [HMS74], A. Prouté [Pro86] and K. Lefèvre-Hasegawa [LH03].

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2.1 Twisting Morphisms

We introduce the Maurer–Cartan equation in the convolution algebra. Its solutions are called twisting morphisms (sometimes called twisting cochains in the literature). To such a twisting morphism we associate a twisted structure on the convolution algebra and on the tensor product, thereby introducing the notion of twisted tensor product of chain complexes.

In this section (C, d_C) is a differential graded coaugmented coalgebra and (A, d_A) is a differential graded augmented algebra, where the differentials are both of degree -1.

2.1.1 Convolution in the DG Framework

We extend the result of Sect. 1.6.1 to graded vector spaces, that is $\mathrm{Hom}(C, A)$ is a graded associative algebra under the convolution product $\star$ (called *cup-product* in [HMS74]). The derivative ∂ of graded linear maps defined in p. 25 makes $\mathrm{Hom}(C, A)$ into a dg vector space.

Proposition 2.1.1. *The convolution algebra* $(\mathrm{Hom}(C, A), \star, \partial)$ *is a dga algebra.*

Proof. It suffices to prove that the derivative ∂ is a derivation for the convolution product $\star$. Let f and g be two maps of degree p and q respectively. We have

$$\begin{aligned}
\partial(f \star g) &= d_A \circ (f \star g) - (-1)^{p+q}(f \star g) \circ d_C \\
&= d_A \circ \mu \circ (f \otimes g) \circ \Delta - (-1)^{p+q}\mu \circ (f \otimes g) \circ \Delta \circ d_C \\
&= \mu \circ (d_A \otimes \mathrm{id} + \mathrm{id} \otimes d_A) \circ (f \otimes g) \circ \Delta \\
&\quad -(-1)^{p+q}\mu \circ (f \otimes g) \circ (d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_C) \circ \Delta \\
&= \mu \circ \big((d_A \circ f) \otimes g + (-1)^p f \otimes (d_A \circ g) \\
&\quad -(-1)^p (f \circ d_C) \otimes g - (-1)^{p+q} f \otimes (g \circ d_C)\big) \circ \Delta \\
&= \mu \circ \big(\partial(f) \otimes g + (-1)^p f \otimes \partial(g)\big) \circ \Delta \\
&= \partial(f) \star g + (-1)^p f \star \partial(g).
\end{aligned}$$

□

2.1.2 Maurer–Cartan Equation, Twisting Morphism

In the dga algebra $\mathrm{Hom}(C, A)$ we consider the *Maurer–Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = 0.$$

By definition a *twisting morphism* (terminology of John Moore [Moo71], "fonctions tordantes" in H. Cartan [Car58]) is a solution $\alpha : C \to A$ of degree -1 of the

Maurer–Cartan equation, which is null when composed with the augmentation of A and also when composed with the coaugmentation of C.

We denote by $\mathrm{Tw}(C, A)$ the set of twisting morphisms from C to A. Recall from Sect. 1.1.9 that a graded associative algebra is a graded Lie algebra, with the graded bracket defined by $[a, b] := a \star b - (-1)^{|a|\cdot|b|} b \star a$. When 2 is invertible in the ground ring $\mathbb{K}$, we have $\alpha \star \alpha = \frac{1}{2}[\alpha, \alpha]$, when α has degree -1. Therefore, the "associative" Maurer–Cartan equation, written above, is equivalent to the "classical" Maurer–Cartan equation $\partial(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0$ in the Lie convolution algebra $(\mathrm{Hom}(C, A), [-, -])$.

Until the end of next section, we assume that the characteristic of the ground field is not equal to 2.

2.1.3 Twisted Structure on the Hom Space

Let $\alpha \in \mathrm{Hom}(C, A)$ be a map of degree -1. We define a *twisted derivation* ∂_α on $\mathrm{Hom}(C, A)$ by the formula

$$\partial_\alpha(f) := \partial(f) + [\alpha, f].$$

Lemma 2.1.2. *Let* $(\mathrm{Hom}(C, A), [,], \partial)$ *be the dg Lie convolution algebra. For any map* $\alpha \in \mathrm{Hom}(C, A)$ *of degree* -1 *the twisted derivation* $\partial_\alpha(x) := \partial(x) + [\alpha, x]$ *satisfies*

$$\partial_\alpha^2(x) = \big[\partial(\alpha) + \alpha \star \alpha, x\big].$$

Proof. We have

$$\begin{aligned}
\partial_\alpha^2(x) &= \partial_\alpha\big(\partial(x) + [\alpha, x]\big) \\
&= \partial^2(x) + \partial\big([\alpha, x]\big) + \big[\alpha, \partial(x)\big] + \big[\alpha, [\alpha, x]\big] \\
&= \big[\partial(\alpha), x\big] + \big[\alpha, [\alpha, x]\big] \quad \big(\partial \text{ is a derivation for } [,]\big) \\
&= \big[\partial(\alpha), x\big] + [\alpha \star \alpha, x] \quad (\text{graded Jacobi relation}) \\
&= \big[\partial(\alpha) + \alpha \star \alpha, x\big].
\end{aligned}$$

□

As a consequence, when α is a twisting morphism in $\mathrm{Hom}(C, A)$, the map ∂_α is a differential. We denote by $\mathrm{Hom}^\alpha(C, A) := (\mathrm{Hom}(C, A), \partial_\alpha)$ this chain complex.

Proposition 2.1.3. *Let* α *be a twisting morphism. The convolution algebra* $(\mathrm{Hom}^\alpha(C, A), \star, \partial_\alpha)$ *is a dga algebra.*

Proof. The twisted derivation ∂_α is the sum of a derivation ∂ with $[\alpha, -]$. Therefore, it is enough to prove that the latter is a derivation with respect to the convolution product $\star$:

$$\begin{aligned}&[\alpha, f]\star g+(-1)^p f\star[\alpha, g]\\&\quad=\alpha\star f\star g-(-1)^p f\star\alpha\star g+(-1)^p f\star\alpha\star g-(-1)^{p+q} f\star g\star\alpha\\&\quad=[\alpha, f\star g],\end{aligned}$$

for f of degree p and g of degree q. □

The dga algebras of the form $(\mathrm{Hom}^\alpha(C,A),\star,\partial_\alpha)$ are called *twisted convolution algebras*. We leave it to the reader to prove that $(\mathrm{Hom}^\alpha(C,A),[,],\partial_\alpha)$ is a dg Lie algebra *twisted* by the twisting morphism α.

2.1.4 Twisted Tensor Product

We saw in Sect. 1.6.2 that the differential on the free A-module (resp. cofree C-comodule) $C\otimes A$ is a derivation (resp. coderivation). Any map $\alpha: C\to A$ induces a unique (co)derivation on $C\otimes A$, which we denote by d^r_α here. Since C and A are dga (co)algebras, we consider the total (co)derivation

$$d_\alpha := d_{C\otimes A}+d^r_\alpha = d_C\otimes \mathrm{Id}_A+\mathrm{Id}_C\otimes d_A+d^r_\alpha.$$

So d_α is a perturbation of the differential of the tensor product.

Lemma 2.1.4. *The (co)derivation d_α satisfies*

$$d_\alpha{}^2 = d^r_{\partial(\alpha)+\alpha\star\alpha}.$$

Therefore, α satisfies the Maurer–Cartan equation if and only if the (co)derivation d_α satisfies $d_\alpha{}^2=0$.

Proof. The first relation comes from $d_\alpha{}^2=(d_{C\otimes A}+d^r_\alpha)^2=d_{C\otimes A}\circ d^r_\alpha+d^r_\alpha\circ d_{C\otimes A}+d^r_\alpha{}^2$. We saw in Proposition 1.6.2 that $d^r_\alpha{}^2=d^r_{\alpha\star\alpha}$. And we have $d_{C\otimes A}\circ d^r_\alpha+d^r_\alpha\circ d_{C\otimes A}=d^r_{d_A\circ\alpha+\alpha\circ d_C}=d^r_{\partial(\alpha)}$.

Hence, if $\alpha\in\mathrm{Tw}(C,A)$, then $d_\alpha{}^2=d^r_0=0$. Conversely, we notice that the restriction of d^r_f on $C\otimes\mathbb{K}1_A\to\mathbb{K}1_C\otimes A$ is equal to f. So if $d_\alpha{}^2=0$, then $\partial(\alpha)+\alpha\star\alpha=0$. □

From the preceding lemma, it follows that, when $\alpha: C\to A$ is a twisting morphism, there exists a chain complex

$$C\otimes_\alpha A := (C\otimes A, d_\alpha)$$

which is called the *(right) twisted tensor product* (or twisted tensor complex). Since the tensor product is symmetric, this construction is also symmetric in A and C. So we can define a left twisted tensor product $A\otimes_\alpha C$. Warning: even if the two underlying modules $C \otimes A$ and $A \otimes C$ are isomorphic, the left and the right twisted tensor products are not isomorphic as chain complexes in general. The twisting term of the differential is not symmetric; it uses one particular side of the coproduct of the coalgebra and one particular side of the product of the algebra but not the same ones. If C were cocommutative and if A were commutative, then they would be isomorphic. Since the two constructions are symmetric, we will only state the related properties for the right twisted tensor product in the rest of this chapter.

This construction is functorial both in C and in A. Let $g : A \to A'$ be a morphism of dga algebras and $f : C \to C'$ be a morphism of dga coalgebras. Consider $C \otimes_\alpha A$ and $C' \otimes_{\alpha'} A'$ two twisted tensor products. We say that the morphisms f and g are *compatible* with the twisting morphisms α and α' if $\alpha' \circ f = g \circ \alpha$. One can show that $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ is then a morphism of chain complexes.

In the weight-graded context, we require that the twisting morphisms preserve the weight. In this case, the following lemma states that if two among these three morphisms are quasi-isomorphisms, then so is the third one. This result first appeared in the Cartan seminar [Car55].

Lemma 2.1.5 (Comparison Lemma for twisted tensor product). *Let $g : A \to A'$ be a morphism of wdga connected algebras and $f : C \to C'$ be a morphism of wdga connected coalgebras. Let $\alpha : C \to A$ and $\alpha' : C' \to A'$ be two twisting morphisms, such that f and g are compatible with α and α'.*

If two morphisms among f, g and $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ (or $g \otimes f : A \otimes_\alpha C \to A' \otimes_{\alpha'} C'$) are quasi-isomorphisms, then so is the third one.

Proof. We postpone the proof to the end of the chapter (see Sect. 2.5). □

2.2 Bar and Cobar Construction

We construct the cobar and bar functors and we prove that they give representing objects for the twisting morphisms bifunctor $\mathrm{Tw}(-,-)$. As a consequence the bar and cobar functors form a pair of adjoint functors. The bar construction goes back to Samuel Eilenberg and Saunders Mac Lane [EML53] and the cobar construction goes back to Franck Adams [Ada56].

2.2.1 Bar Construction

We are going to construct a functor from the category of augmented dga algebras to the category of conilpotent dga coalgebras:

$$\mathrm{B} : \{\text{aug. dga algebras}\} \longrightarrow \{\text{con. dga coalgebras}\}$$

called the *bar construction*.

Let A be an augmented algebra: $A = \mathbb{K}1 \oplus \bar{A}$ (concentrated in degree 0) with product μ. The bar construction of A is a differential graded coalgebra defined on the cofree coalgebra $T^c(s\bar{A})$ over the suspension $s\bar{A} = \mathbb{K}s \otimes \bar{A}$ as follows. We denote it by $\mathrm{B}A$, using a slight but usual abuse of notation.

Consider the map $\Pi_s : \mathbb{K}s \otimes \mathbb{K}s \to \mathbb{K}s$ of degree -1 induced by $\Pi_s(s \otimes s) := s$. The restriction $\mu_{\bar{A}}$ of the product of the algebra A to $\bar{A}$ induces the following map

$$f : T^c(s\bar{A}) \twoheadrightarrow \mathbb{K}s \otimes \bar{A} \otimes \mathbb{K}s \otimes \bar{A} \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} \mathbb{K}s \otimes \mathbb{K}s \otimes \bar{A} \otimes \bar{A} \xrightarrow{\Pi_s \otimes \mu_{\bar{A}}} \mathbb{K}s \otimes \bar{A}.$$

Since $T^c(s\bar{A})$ is cofree, by Proposition 1.2.2 there is a unique coderivation $d_2 : T^c(s\bar{A}) \to T^c(s\bar{A})$ which extends the map $f : T^c(s\bar{A}) \to s\bar{A}$:

$$\begin{array}{ccccccccc}
 & (0) & & (1) & & (2) & & (3) & \\
T^c(s\bar{A}) = & \mathbb{K} & & \bar{A} & & \bar{A}^{\otimes 2} & & \bar{A}^{\otimes 3} & \cdots \\
\downarrow d_2 & & \swarrow^{0} & & \swarrow & & \swarrow & & \swarrow \\
T^c(s\bar{A}) = & \mathbb{K} & & \bar{A} & & \bar{A}^{\otimes 2} & & \bar{A}^{\otimes 3} & \cdots
\end{array}$$

Proposition 2.2.1. *The associativity of* μ *implies that* $(d_2)^2 = 0$, *hence* $(T^c(s\bar{A}), d_2)$ *is a chain complex.*

Proof. We will give the proof in the dual case in Proposition 2.2.4. It is also a direct consequence of the next lemma. □

The complex $\mathrm{B}A := (T^c(s\bar{A}), d_2)$ is a conilpotent differential graded coalgebra, called the *bar construction* of the augmented graded algebra A. It is obviously a functor from the category of augmented graded algebras to the category of conilpotent differential graded coalgebras.

Lemma 2.2.2. *For any augmented associative algebra* A, *concentrated in degree* 0, *the bar complex of* A *can be identified with the nonunital Hochschild complex of* $\bar{A}$:

$$\cdots \to \bar{A}^{\otimes n} \xrightarrow{b'} \bar{A}^{\otimes n-1} \to \cdots \to \bar{A} \to \mathbb{K},$$

where $b'[a_1 \mid \ldots \mid a_n] = \sum_{i=1}^{n-1} (-1)^{i-1} [a_1 \mid \ldots \mid \mu(a_i, a_{i+1}) \mid \ldots \mid a_n]$.

Proof. Here we have adopted Mac Lane's notation $[a_1 \mid \ldots \mid a_n] \in \bar{A}^{\otimes n}$. Since $\bar{A}$ is in degree 0, the space $s\bar{A}$ is in degree 1 and $(s\bar{A})^{\otimes n}$ is in degree n. So the module of n-chains can be identified with $\bar{A}^{\otimes n}$. Let us identify the boundary map. Since d_2 is induced by the product and is a derivation, it has the form indicated in the statement. The signs come from the presence of the shift s. For instance:

$$\begin{aligned}
[a_1 \mid a_2 \mid a_3] = (sa_1, sa_2, sa_3) \mapsto & \big(d_2(sa_1, sa_2), sa_3\big) - \big(sa_1, d_2(sa_2, sa_3)\big) \\
& = \big[\mu(a_1, a_2) \mid a_3\big] - \big[a_1 \mid \mu(a_2, a_3)\big].
\end{aligned}$$

The minus sign appears because d_2 "jumps" over sa_1 which is of degree one. □

In general, the formula is the same with only the following change of sign:

$$d_2(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} (-1)^{i-1+|a_1|+\cdots+|a_i|} sa_1 \otimes \cdots \otimes s\mu(a_i, a_{i+1}) \otimes \cdots \otimes sa_n.$$

One can extend this functor to the case where (A, d_A) is an augmented differential graded algebra. Indeed, the differential $d_A : A \to A$ induces a differential on $A^{\otimes n}$ by

$$d_1 := \sum_{i=1}^{n} (\mathrm{id}, \ldots, \mathrm{id}, d_A, \mathrm{id}, \ldots, \mathrm{id}).$$

We denote by d_1 the differential on $T^c(s\bar{A})$. Since μ_A is a morphism of differential graded vector spaces, one can check that d_1 and d_2 anticommute: $d_1 \circ d_2 + d_2 \circ d_1 = 0$. The chain complex associated to the total differential $d_1 + d_2$ is called the *bar construction* of the augmented differential graded algebra

$$\mathrm{B}A := \big(T^c(s\bar{A}), d_{\mathrm{B}A} = d_1 + d_2\big).$$

The analogous construction in algebraic topology (classifying space of a topological group) is also called bar construction and denoted by B.

Proposition 2.2.3. *For any quasi-isomorphism $f : A \to A'$ of augmented dga algebras, the induced morphism $\mathrm{B}f : \mathrm{B}A \to \mathrm{B}A'$ is a quasi-isomorphism.*

Proof. We consider the filtration on $\mathrm{B}A$ defined by

$$F_p\mathrm{B}A := \Big\{\sum sa_1 \otimes \cdots \otimes sa_n \mid n \le p\Big\}.$$

It is stable under $d_{\mathrm{B}A}$, $d_1 : F_p \to F_p$ and $d_2 : F_p \to F_{p-1}$. This filtration is increasing, bounded below and exhaustive. Hence, the classical convergence theorem of spectral sequences (Theorem 1.5.1) applies. The first page is equal to

$$E^0_{pq}\mathrm{B}A = (F_p\mathrm{B}A)_{p+q}/(F_{p-1}\mathrm{B}A)_{p+q} \cong \big\{sa_1 \otimes \cdots \otimes sa_p \mid |a_1| + \cdots + |a_p| = q\big\}.$$

Finally $E^0_{p\bullet}(f) = (sf)^{\otimes p}$ is a quasi-isomorphism by Künneth formula. □

2.2.2 Cobar Construction

Analogously one can construct a functor from the category of coaugmented dga coalgebras to the category of augmented dga algebras:

$$\Omega : \{\text{coaug. dga coalgebras}\} \longrightarrow \{\text{aug. dga algebras}\}$$

called the *cobar construction*, as follows.

Let C be a coaugmented graded coalgebra: $C = \overline{C} \oplus \mathbb{K}1$ with coproduct Δ. The reduced coproduct $\bar{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$ is defined by the equality $\Delta(x) = x \otimes 1 + 1 \otimes x + \bar{\Delta}(x)$ for any $x \in \overline{C}$. It is obviously coassociative and of degree 0.

Consider now $\mathbb{K}s^{-1}$ equipped with the diagonal map $\Delta_s(s^{-1}) := -s^{-1} \otimes s^{-1}$ of degree -1, see Exercise 2.7.3. Then, one defines a map f on $s^{-1}\overline{C} = \mathbb{K}s^{-1} \otimes \overline{C}$ as the composite

$$f : \mathbb{K}s^{-1} \otimes \overline{C} \xrightarrow{\Delta_s \otimes \bar{\Delta}} \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1} \otimes \overline{C} \otimes \overline{C} \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} \mathbb{K}s^{-1} \otimes \overline{C} \otimes \mathbb{K}s^{-1} \otimes \overline{C}.$$

Consider the free algebra $T(s^{-1}\overline{C})$ over the desuspension $s^{-1}\overline{C}$. Since it is free, the degree -1 map $f : s^{-1}\overline{C} \to s^{-1}\overline{C} \otimes s^{-1}\overline{C}$ has a unique extension to $T(s^{-1}\overline{C})$ as a derivation by Proposition 1.1.2. We denote it by

$$d_2 : T\big(s^{-1}\overline{C}\big) \to T\big(s^{-1}\overline{C}\big),$$

$$\begin{array}{cccccc}
 & (0) & (1) & (2) & (3) & \\
T(s^{-1}\overline{C}) = & \mathbb{K} & \overline{C} & \overline{C}^{\otimes 2} & \overline{C}^{\otimes 3} & \cdots \\
\Big\downarrow d_2 & & \searrow^{0} & \searrow & \searrow & \searrow \\
T(s^{-1}\overline{C}) = & \mathbb{K} & \overline{C} & \overline{C}^{\otimes 2} & \overline{C}^{\otimes 3} & \cdots
\end{array}$$

Proposition 2.2.4. *The coassociativity of $\bar{\Delta}$ implies that $d_2 \circ d_2 = 0$ on $s^{-1}\overline{C}$. Therefore d_2 is a differential and $(T(s^{-1}\overline{C}), d_2)$ is a chain complex.*

Proof. For any $x \in \overline{C}$, let us write $\bar{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$. We also adopt the notation

$$(\bar{\Delta} \otimes \mathrm{id})\bar{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = (\mathrm{id} \otimes \bar{\Delta})\bar{\Delta}(x).$$

We have defined

$$d_2\big(s^{-1}x\big) := -\sum (-1)^{|x_{(1)}|} s^{-1}x_{(1)} \otimes s^{-1}x_{(2)} \in \overline{C}^{\otimes 2}.$$

Let us prove that $d_2 \circ d_2 = 0$. Let $p := |x_{(1)}|$, $q := |x_{(2)}|$, $r := |x_{(3)}|$. The term $s^{-1}x_{(1)} \otimes s^{-1}x_{(2)} \otimes s^{-1}x_{(3)}$ coming from $(\bar{\Delta} \otimes \mathrm{id})\bar{\Delta}$ under $d_2 \circ d_2$ comes with the sign $(-1)^{p+q}(-1)^{p}$. Indeed, the first one comes from the application of the first copy of d_2, the second one comes from the application of the second copy of d_2. The term $s^{-1}x_{(1)} \otimes s^{-1}x_{(2)} \otimes s^{-1}x_{(3)}$ coming from $(\mathrm{id} \otimes \bar{\Delta})\bar{\Delta}$ under $d_2 \circ d_2$ comes with the sign $(-1)^{p+q}(-1)^{1+p+q}(-1)^{q}$. Indeed, the first one comes from the application of the first copy of d_2, the second one comes from the fact that d_2, which is of degree -1, jumps over a variable of degree $p-1$, the third one comes from the application of the second copy of d_2.

Adding up these two elements we get 0 as expected. □

By definition the *cobar construction* of the coaugmented graded coalgebra C is the augmented dga algebra

$$\Omega C := \left(T\left(s^{-1}\overline{C}\right), d_2\right).$$

It obviously gives a functor Ω from the category of coaugmented graded coalgebras to the category of augmented differential graded algebras.

One easily extends this functor to coaugmented differential graded coalgebras (C, Δ, d_C) by adding to d_2 the differential d_1 induced by the differential d_C. Since Δ is a morphism of chain complexes, d_1 and d_2 anticommute and one has a well-defined bicomplex. The chain complex associated to this total differential is called the *cobar construction* of the coaugmented coalgebra

$$\Omega C := \left(T\left(s^{-1}\overline{C}\right), d_{\Omega C} = d_1 + d_2\right).$$

The notation Ω is by analogy with the loop space construction in algebraic topology.

A nonnegatively graded dga coalgebra C is called 2-*connected* if $C_0 = \mathbb{K}1$ and $C_1 = 0$.

Proposition 2.2.5. *Let $f : C \to C'$ be a quasi-isomorphism between two 2-connected dga coalgebras. The induced morphism $\Omega f : \Omega C \to \Omega C'$ between the cobar constructions is a quasi-isomorphism.*

Proof. We consider the following filtration on the cobar construction

$$F_p \Omega C := \left\{ \sum s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_n \mid n \geq -p \right\}.$$

This increasing filtration is preserved by the differential of the cobar construction, $d_1 : F_p \to F_p$ and $d_2 : F_p \to F_{p-1}$. So the first term of the associated spectral sequence is equal to

$$\begin{aligned} E^0_{pq} &= (F_p \Omega C)_{p+q} / (F_{p-1} \Omega C)_{p+q} \\ &\cong \left\{ \sum s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_p \| c_1| + \cdots + |c_p| = 2p + q \right\}, \end{aligned}$$

with $d^0 = d_1$. Since $E^0_{p\bullet}(\Omega f) = (s^{-1}f)^{\otimes p}$, it is a quasi-isomorphism by Künneth formula. Since C (respectively C') is 2-connected, the degree of an element $s^{-1}c \in s^{-1}\overline{C}$ is at least 1 and $(F_p \Omega C)_n = 0$ for $p < -n$. The filtration being exhaustive and bounded below, this spectral sequence converges to the homology of the cobar construction by the classical convergence theorem of spectral sequences (Theorem 1.5.1), which concludes the proof. □

This result does not hold when the dga coalgebras are not 2-connected. We give a counterexample in Proposition 2.4.3. Beyond the 2-connected case, the relationship between the cobar construction and quasi-isomorphisms is more subtle. This question is fully studied in Sect. 2.4.

2.2.3 Bar–Cobar Adjunction

We show that the bar and cobar constructions form a pair of adjoint functors

$$\Omega : \{\text{con. dga coalgebras}\} \rightleftharpoons \{\text{aug. dga algebras}\} : \mathrm{B}.$$

More precisely, this adjunction is given by the space of twisting morphisms. When A is augmented and C coaugmented, a twisting morphism between C and A is supposed to send $\mathbb{K}$ to 0 and $\overline{C}$ to $\bar{A}$.

Theorem 2.2.6. *For every augmented dga algebra A and every conilpotent dga coalgebra C there exist natural bijections*

$$\mathrm{Hom}_{\text{dga alg}}(\Omega C, A) \cong \mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\text{dga coalg}}(C, \mathrm{B}A).$$

Proof. Let us make the first bijection explicit. Since $\Omega C = T(s^{-1}\overline{C})$ is a free algebra, any morphism of algebras from ΩC to A is characterized by its restriction to $\overline{C}$ (cf. Proposition 1.1.1). Let φ be a map from $\overline{C}$ to A of degree -1. Define the map $\bar{\varphi} : s^{-1}\overline{C} \to A$ of degree 0 by the formula $\bar{\varphi}(s^{-1}c) := \varphi(c)$. Similarly, $\bar{\varphi}$ induces a unique morphism Φ of algebras from ΩC to A. The map Φ commutes with the differentials, meaning $d_A \circ \Phi = \Phi \circ (d_1 + d_2)$, or equivalently to $d_A \circ \varphi = -\varphi \circ d_C - \varphi \star \varphi$. Finally, we get $\partial(\varphi) + \varphi \star \varphi = 0$. Notice that the map φ lands in $\bar{A}$ since the map Φ is a morphism of augmented algebras.

The second bijection is given by the same method, so the rest of the proof is left to the reader as an exercise. Notice that we need the coalgebra C to be conilpotent in order to be able to extend a map $C \to sA$ into a morphism of coalgebras $C \to \mathrm{B}A = T^c(s\bar{A})$ (see Sect. 1.2.6). □

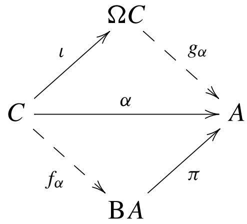
As a consequence of this proposition Ω and B form a pair of adjoint functors (Ω is left adjoint and B is right adjoint), which represent the bifunctor Tw.

2.2.4 Universal Twisting Morphisms

Several universal morphisms appear from this pair of adjoint functors. Applying Theorem 2.2.6 to $C = \mathrm{B}A$ we get the *counit* of the adjunction $\varepsilon : \Omega\mathrm{B}A \to A$ (see Appendix B.2.1) and the universal twisting morphism $\pi : \mathrm{B}A \to A$. Then applying Theorem 2.2.6 to $A = \Omega C$ we get the *unit* of the adjunction $\upsilon : C \to \mathrm{B}\Omega C$ (this is upsilon not v) and the universal twisting morphism $\iota : C \to \Omega C$.

By Theorem 2.2.6 the twisting morphisms π and ι have the following property.

Proposition 2.2.7. *Any twisting morphism $\alpha : C \to A$ factorizes uniquely through π and ι:*



where g_α is a dga algebra morphism and f_α is a dga coalgebra morphism.

2.2.5 Augmented Bar and Cobar Construction

The *universal twisting morphism* $\pi : \mathrm{B}A = T^c(s\bar{A}) \twoheadrightarrow s\bar{A} \cong \bar{A} \rightarrowtail A$ gives rise to the twisted tensor product $\mathrm{B}A \otimes_\pi A$ (cf. Sect. 2.1.4). It is called the *augmented bar construction of* A.

Dually, the universal twisting morphism $\iota : C \twoheadrightarrow \overline{C} \cong s^{-1}\overline{C} \rightarrowtail T(s^{-1}\overline{C}) = \Omega C$ gives rise to the *coaugmented cobar construction of* C denoted $C \otimes_\iota \Omega C = (C \otimes \Omega C, d_\iota)$.

Proposition 2.2.8. *The chain complexes* $\mathrm{B}A \otimes_\pi A$ (*resp.* $A \otimes_\pi \mathrm{B}A$) *and* $C \otimes_\iota \Omega C$ (*resp.* $\Omega C \otimes_\iota C$) *are acyclic.*

Proof. Once made explicit, the chain complex is the nonunital Hochschild complex with coefficients in A whose module of n-chains is $\bar{A}^{\otimes n} \otimes A$ and whose boundary map is b' given by

$$b'\big([a_1 \mid \dots \mid a_n]a_{n+1}\big) = \sum_{i=1}^{n-1} (-1)^{i-1}[a_1 \mid \dots \mid a_i a_{i+1} \mid \dots \mid a_n]a_{n+1} \\ + (-1)^{n-1}[a_1 \mid \dots \mid a_{n-1}]a_n a_{n+1}.$$

We consider the kernel K of the augmentation map

$$K \rightarrowtail \mathrm{B}A \otimes_\pi A \twoheadrightarrow \mathbb{K}.$$

It is immediate to check that the map $h : K \to K$ given by

$$[a_1 \mid \dots \mid a_n]a_{n+1} \mapsto (-1)^n \big[a_1 \mid \dots \mid a_n \mid a_{n+1} - \varepsilon(a_{n+1})\big]$$

is a homotopy from id_K to 0:

$$b'h + hb' = \mathrm{id}_K.$$

Hence the twisted tensor complex $\mathrm{B}A \otimes_\pi A$ is acyclic.

The proof for the other case is similar. □

2.3 Koszul Morphisms

We have just seen that the twisted tensor products associated to the two universal twisting morphisms π and ι are acyclic. When the twisted complex $C \otimes_\alpha A$, or equivalently $A \otimes_\alpha C$, happens to be acyclic, the twisting morphism α is called a *Koszul morphism*. We denote the set of Koszul morphisms by $\mathrm{Kos}(C, A)$.

In this section, we give the main theorem of this chapter which relates Koszul morphisms with bar and cobar resolutions. As a corollary, we prove that the unit and the counit of the bar–cobar adjunction are quasi-isomorphisms.

2.3.1 Koszul Criterion

Here we give the main result of this section, which is a criterion about Koszul morphisms. It comes from E. Brown's paper [Bro59].

Theorem 2.3.1 (Twisting morphism fundamental theorem). *Let A be a connected wdga algebra and let C be a connected wdga coalgebra. For any twisting morphism $\alpha : C \to A$ the following assertions are equivalent*:

1. *the right twisted tensor product $C \otimes_\alpha A$ is acyclic*,
2. *the left twisted tensor product $A \otimes_\alpha C$ is acyclic*,
3. *the dga coalgebra morphism $f_\alpha : C \xrightarrow{\sim} \mathrm{B}A$ is a quasi-isomorphism*,
4. *the dga algebra morphism $g_\alpha : \Omega C \xrightarrow{\sim} A$ is a quasi-isomorphism.*

Proof. Since we require A to be connected, we have $A = \bar{A} \oplus \mathbb{K}1$, where the elements of the augmentation ideal $\bar{A}$ have positive degree and positive weight. There is a similar statement for C. Recall that wdga (co)algebras were introduced in Sect. 1.5.10.

We first notice that the bar construction of a wgda connected algebra is a wgda connected coalgebra. And dually, the cobar construction of a wgda connected coalgebra is a wgda connected algebra. The weight of an element of $\mathrm{B}A$ is equal to the total weight $\omega(sa_1, \ldots, sa_k) = \omega(a_1) + \cdots + \omega(a_k)$.

We consider the commutative diagram of Sect. 2.2.4, where $f_\alpha : C \to \mathrm{B}A$, resp. $g_\alpha : \Omega C \to A$, is the morphism of wdga coalgebras, resp. algebras, associated to the twisting morphism α and respecting the weight grading. Notice that the universal twisting morphisms π and ι also preserve the weight.

(1) $\Leftrightarrow$ (3). Consider the tensor map $f_\alpha \otimes \mathrm{Id}_A : C \otimes A \to \mathrm{B}A \otimes A$. Since $\pi \circ f_\alpha = \alpha = \mathrm{Id}_A \circ \alpha$, the map $f_\alpha \otimes \mathrm{Id}_A$ is a morphism of chain complexes from $C \otimes_\alpha A$ to $\mathrm{B}A \otimes_\pi A$. We have seen in Proposition 2.2.8 that the augmented bar construction is always acyclic. Therefore, the twisted complex $C \otimes_\alpha A$ is acyclic if and only if $f_\alpha \otimes \mathrm{Id}_A$ is a quasi-isomorphism. The Comparison Lemma 2.1.5 implies that $C \otimes_\alpha A$ is acyclic if and only if f_α is a quasi-isomorphism.

(1) $\Leftrightarrow$ (4). We use the same method with the tensor map $\mathrm{Id}_C \otimes g_\alpha : C \otimes_\iota \Omega(C) \to C \otimes_\alpha A$. Since $g_\alpha \circ \iota = \alpha = \alpha \circ \mathrm{Id}_C$, the map $\mathrm{Id}_C \otimes g_\alpha$ is a morphism of chain complexes. The acyclicity of the coaugmented cobar construction (Proposition 2.2.8) and the Comparison Lemma 2.1.5 imply that the twisted chain complex $C \otimes_\alpha A$ is acyclic if and only if g_α is a quasi-isomorphism.

The proof of the equivalence (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is similar and uses the two other cases of Proposition 2.2.8 and Lemma 2.1.5. □

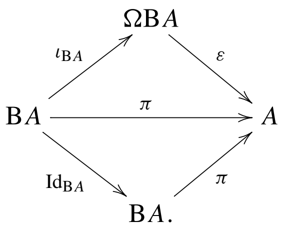
2.3.2 Bar–Cobar Resolution

We consider the counit $\varepsilon : \Omega \mathrm{B} A \to A$ and the unit $\upsilon : C \to \mathrm{B}\Omega C$ of the bar–cobar adjunction. The counit is a canonical resolution of A which is called the *bar–cobar resolution*. The following statement shows that it provides a quasi-free model for A, which is not minimal in general.

Corollary 2.3.2. *Let A be an augmented dga algebra and let C be a conilpotent dga coalgebra.*

The counit $\varepsilon : \Omega \mathrm{B} A \xrightarrow{\sim} A$ is a quasi-isomorphism of dga algebras. Dually, the unit $\upsilon : C \xrightarrow{\sim} \mathrm{B}\Omega C$ is a quasi-isomorphism of dga coalgebras.

Proof. We give a proof under the hypothesis that A (resp. C) is a connected wdga algebra (resp. connected wdga coalgebra). However the result holds in full generality (see [HMS74]). We apply Theorem 2.3.1 to the following diagram



Since $\mathrm{Id}_{\mathrm{B}A}$ is an isomorphism, it follows that the counit ε is a quasi-isomorphism.

Following the same method, since $\mathrm{Id}_{\Omega C}$ is an isomorphism, the unit υ is a quasi-isomorphism. □

2.4 Cobar Construction and Quasi-isomorphisms

Using the previous results, we study the relationship between the cobar construction and quasi-isomorphisms. The main source of inspiration for this section is Lefèvre-Hasegawa's thesis [LH03].

To any dga coalgebra C, we consider the graded modules associated to the coradical filtration: $\mathrm{gr}_r\, C := F_r C / F_{r-1} C$. Let $f : C \to C'$ be a morphism of conilpotent dga coalgebras. Since the map f and the differentials preserve the coradical filtrations, f induces a morphism of chain complexes $[f] : \mathrm{gr}\, C \to \mathrm{gr}\, C'$ between the associated graded modules. If $[f]$ is a quasi-isomorphism, then f is called a *graded quasi-isomorphism.*

Proposition 2.4.1. *For any morphism $f : C \to C'$ of conilpotent dga coalgebras which is a graded quasi-isomorphism the induced morphism $\Omega f : \Omega C \xrightarrow{\sim} \Omega C'$ is a quasi-isomorphism.*

Proof. We consider the following grading for any element c in a conilpotent coalgebra C, $\mathrm{gr}\, c := \min\{r \mid c \in F_r C\}$. We consider the filtration of the cobar construction ΩC defined by

$$F_p \Omega C := \left\{ s^{-1} c_1 \otimes \cdots \otimes s^{-1} c_n \mid \mathrm{gr}\, c_1 + \cdots + \mathrm{gr}\, c_n \leq p \right\}.$$

The increasing filtration is bounded below and exhaustive so the associated spectral sequence converges to the homology of ΩC. Its first term is equal to

$$E^0_{pq} \Omega C = (F_p \Omega C)_{p+q} / (F_{p-1} \Omega C)_{p+q} \cong (\Omega\, \mathrm{gr}\, C)^{(p)}_{p+q},$$

where

$$(\Omega\, \mathrm{gr}\, C)^{(p)} = \left\{ s^{-1} c_1 \otimes \cdots \otimes s^{-1} c_n \mid \mathrm{gr}\, c_1 + \cdots + \mathrm{gr}\, c_n = p \right\}.$$

Hence $E^0(\Omega f) = \Omega[f]$, under the preceding notation. For any fixed p, we now prove that $E^0_{p\bullet}(\Omega C) \to E^0_{p\bullet}(\Omega C')$ is quasi-isomorphism. On $E^0_{p\bullet}(\Omega C)$, we define the filtration F_k as follows: an element $s^{-1} c_1 \otimes \cdots \otimes s^{-1} c_n$ is in F_k if and only if $n \geq -k$. This filtration is increasing. Since C is conilpotent the grading gr of the elements of $\overline{C}$ is strictly greater than 0, and we have $\mathsf{F}_{-p-1} = 0$. Since it is bounded below and exhaustive, the associated spectral sequence converges by Theorem 1.5.1. The first term $\mathsf{E}^0_{k\bullet}$ is isomorphic to the sub-module of $(s^{-1}\, \mathrm{gr}\, C)^{\otimes k}$ of grading p and degree $k + \bullet$ with differential d^0 induced by the differential of $\mathrm{gr}\, C$. The morphism f being a graded quasi-isomorphism, $\mathsf{E}^0(\Omega[f])$ is also a quasi-isomorphism by Künneth formula, which concludes the proof. □

2.4.1 Weak Equivalence

Any morphism $f : C \to C'$ of dga coalgebras, such that the induced morphism $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism, is called a *weak equivalence.*

Proposition 2.4.2. *Any* weak equivalence $f : C \to C'$ *of conilpotent dga coalgebras is a quasi-isomorphism.*

Proof. Since f is a weak equivalence, Ωf is a quasi-isomorphism and by Proposition 2.2.3, the morphism of dga coalgebras $\mathrm{B}\Omega f : \mathrm{B}\Omega C \to \mathrm{B}\Omega C'$ is a quasi-isomorphism. We conclude with the following commutative diagram, where all the maps are quasi-isomorphisms by Proposition 2.3.2

$$\begin{array}{ccc} C & \xrightarrow{\upsilon_C} & \mathrm{B}\Omega C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{B}\Omega f} \\ C' & \xrightarrow{\upsilon_{C'}} & \mathrm{B}\Omega C'. \end{array}$$

□

In conclusion, the exact relationship between these notions is the following:

$$\text{graded quasi-isomorphisms} \subseteq \text{weak equivalences} \subsetneq \text{quasi-isomorphisms}.$$

Proposition 2.4.3. *There exist quasi-isomorphisms of dga coalgebras which are not weak equivalences.*

Proof. Let A be a unital dga algebra A, which is not acyclic. Consider its augmentation $A_+ := A \oplus \mathbb{K}1$, where 1 acts as a unit. The dga coalgebra $C := \mathrm{B}A_+ \cong T^c(sA)$ is isomorphic to $\mathbb{K} \oplus \mathrm{B}A \otimes_\pi A$. So it is quasi-isomorphic to the trivial dga coalgebra $\mathbb{K}$ by Proposition 2.2.8. But the cobar construction of $\mathbb{K}$ is acyclic, whereas the cobar construction $\Omega \mathrm{B}A_+$ is quasi-isomorphic to A_+ by Corollary 2.3.2, which is not acyclic. □

Notice that $C = \mathrm{B}A_+$ is connected but not 2-connected since C_1 contains $s1_A$, the suspension of the unit of A. So Proposition 2.2.5 does not hold for connected dga coalgebras in general. For 2-connected dga coalgebras, a quasi-isomorphism is a weak equivalence and vice versa.

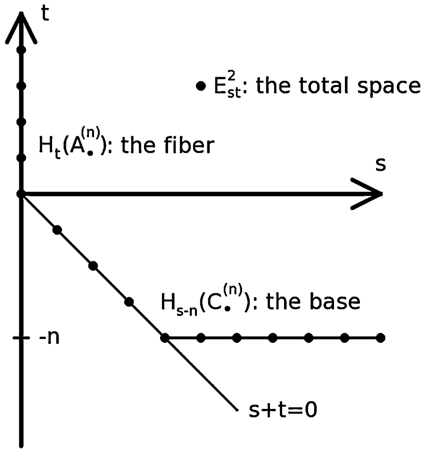
2.5 Proof of the Comparison Lemma

In this section, we prove the Comparison Lemma 2.1.5 used in the proof of the fundamental theorem of twisting morphisms (Theorem 2.3.1). We assume here that the reader is familiar with the following notions of homological algebra: long exact sequences, cones, filtrations and spectral sequences. We refer the reader to any textbook on homological algebra, for instance [ML95] by Saunders MacLane.

Lemma 2.5.1 (Comparison Lemma for twisted tensor product, Cartan [Car55]). *Let $g : A \to A'$ be a morphism of wdga connected algebras and $f : C \to C'$ be a morphism of wdga connected coalgebras. Let $\alpha : C \to A$ and $\alpha' : C' \to A'$ be two twisting morphisms, such that f and g are compatible with α and α'.*

If two morphisms among f, g and $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ (or $g \otimes f : A \otimes_\alpha C \to A' \otimes_{\alpha'} C'$) are quasi-isomorphisms, then so is the third one.

Fig. 2.1 The page E^2_{st} of the spectral sequence



Proof. Recall that the notion of weight-graded dga algebra was defined in Sect. 1.5.10. We denote by $M = \oplus_{n\geq 0} M^{(n)}$ (resp. $M' = \oplus_{n\geq 0} M'^{(n)}$) the weight-graded chain complex $C \otimes_\alpha A$ (resp. $C' \otimes_{\alpha'} A'$). We define a filtration F_s on $M^{(n)}$, where $n \in \mathbb{N}$ is the weight, by the formula

$$F_s\big(M^{(n)}\big) := \bigoplus_{d+m\leq s} \big(C_d^{(m)} \otimes A\big)^{(n)} = \bigoplus_{d+m\leq s} C_d^{(m)} \otimes A^{(n-m)}.$$

The differential d_α on $M = C \otimes_\alpha A$ is the sum of three terms $\mathrm{Id}_C \otimes d_A$, $d_C \otimes \mathrm{Id}_A$ and d^r_α. One has $\mathrm{Id}_C \otimes d_A : F_s \to F_s$, $d_C \otimes \mathrm{Id}_A : F_s \to F_{s-1}$ and $d^r_\alpha : F_s \to F_{s-2}$. Therefore, F_s is a filtration on the chain complex $M^{(n)}$. We consider the associated spectral sequence $\{E^\bullet_{st}\}_{s,t}$. One has

$$E^0_{st} = F_s\big(M^{(n)}\big)_{s+t}/F_{s-1}\big(M^{(n)}\big)_{s+t} = \bigoplus_{m=0}^{n} C^{(m)}_{s-m} \otimes A^{(n-m)}_{t+m}.$$

The study of the differential d_α on the filtration F_s of M shows that $d_0 = \mathrm{Id}_C \otimes d_A$ and that $d_1 = d_C \otimes \mathrm{Id}_A$. It follows that

$$E^2_{st} = \bigoplus_{m=0}^{n} H_{s-m}\big(C^{(m)}_\bullet\big) \otimes H_{t+m}\big(A^{(n-m)}_\bullet\big).$$

Since A and C are weight graded and connected, the part $m = 0$ is concentrated in $s = 0$ and $t \geq 0$, where it is equal to $E^2_{0t} = H_t(A^{(n)}_\bullet)$. The part $m = n$ is concentrated in $t = -n$ and $s \geq n$, where it is equal to $E^2_{s-n} = H_{s-n}(C^{(n)}_\bullet)$. For any $0 < m < n$, the nonvanishing part of $H_{s-m}(C^{(m)}_\bullet) \otimes H_{t+m}(A^{(n-m)}_\bullet)$ is in $s \geq 1$ and $t \geq -n+1$. See Fig. 2.1.

The filtration F_s is exhaustive $M^{(n)} = \bigcup_{s\geq 0} F_s(M^{(n)})$ and bounded below $F_{-1}(M^{(n)}) = \{0\}$, so the spectral sequence converges to the homology of $M^{(n)}$ by the classical convergence theorem 1.5.1:

$$E^{\infty}_{st}\big(M^{(n)}\big) \cong F_s\big(H_{s+t}\big(M^{(n)}\big)\big)/F_{s-1}\big(H_{s+t}\big(M^{(n)}\big)\big).$$

We consider the same filtration on M' and we denote by Φ the morphism of chain complexes $\Phi := f \otimes g$. We treat the three cases one after the other.

(1) *If f and g are quasi-isomorphisms, then $\Phi = f \otimes g$ is a quasi-isomorphism.*

For every s, t and n, the maps $E^2_{st}(M^{(n)}) \xrightarrow{H_\bullet(f)\otimes H_\bullet(g)} E^2_{st}(M'^{(n)})$ are isomorphisms. By the convergence of the two spectral sequences, the maps

$$E^{\infty}_{st}\big(M^{(n)}\big) \xrightarrow{\sim} E^{\infty}_{st}\big(M'^{(n)}\big)$$

are again isomorphisms. So the map Φ is a quasi-isomorphism.

(2) *If $\Phi = f \otimes g$ and g are quasi-isomorphisms, then f is a quasi-isomorphism.*

Let us work by induction on the weight n. When $n = 0$, the map $f^{(0)} : \mathbb{K} \to \mathbb{K}$, which is the identity, is a quasi-isomorphism. Suppose now that the result is true up to weight $n-1$. We consider the mapping cone of $\Phi^{(n)}$: $cone(\Phi^{(n)}) := s^{-1}M^{(n)} \oplus M'^{(n)}$ and the associated filtration $F_s(cone(\Phi^{(n)})) := F_{s-1}(M^{(n)}) \oplus F_s\big(M'^{(n)}\big)$, which satisfies $E^1_{\bullet t}(cone(\Phi^{(n)})) = cone(E^1_{\bullet t}(\Phi^{(n)}))$. The long exact sequence of the mapping cone reads

$$\cdots \to H_{s+1}\big(cone\,\big(E^1_{\bullet t}\big(\Phi^{(n)}\big)\big)\big) \to H_s\big(E^1_{\bullet t}\big(M^{(n)}\big)\big)$$
$$\xrightarrow{H_s(E^1_{\bullet t}(\Phi^{(n)}))} H_s\big(E^1_{\bullet t}\big(M'^{(n)}\big)\big) \to H_s\big(cone\big(E^1_{\bullet t}\big(\Phi^{(n)}\big)\big)\big) \to \cdots .$$

Therefore there is a long exact sequence (ξ_t)

$$(\xi_t) \qquad \cdots \to E^2_{s+1t}\big(cone\big(\Phi^{(n)}\big)\big) \to E^2_{st}\big(M^{(n)}\big)$$
$$\xrightarrow{E^2_{st}(\Phi^{(n)})} E^2_{st}\big(M'^{(n)}\big) \to E^2_{st}\big(cone\big(\Phi^{(n)}\big)\big) \to \cdots$$

where $E^2_{st}(\Phi^{(n)})$ is given by $H_\bullet(f) \otimes H_\bullet(g)$.

When $t > -n$, we have seen that only $C^{(m)}$ (and $C'^{(m)}$) with $m < n$ are involved in E^2_{st}. In that case, since $E^2_{st}(M^{(n)}) = \bigoplus_{m=0}^{n-1} H_{s-m}(C^{(m)}_\bullet) \otimes H_{t+m}(A^{(n-m)}_\bullet)$, the induction hypothesis implies that

$$E^2_{st}\big(M^{(n)}\big) \xrightarrow{H_\bullet(f)\otimes H_\bullet(g)} E^2_{st}\big(M'^{(n)}\big)$$

is an isomorphism for every s and every $t > -n$. Using the long exact sequence (ξ_t) for $t > -n$, it gives $E^2_{st}\big(cone(\Phi^{(n)})\big) = 0$ for every s and every $t \neq -n$. The collapsing of the spectral sequence $E^\bullet_{st}(cone(\Phi^{(n)}))$ at rank 2 implies the equality $E^\infty_{st}(cone(\Phi^{(n)})) = E^2_{st}(cone(\Phi^{(n)}))$. The convergence of the spectral sequence $E^\bullet_{st}(cone(\Phi^{(n)}))$ shows that

$$E^2_{st}\big(cone\big(\Phi^{(n)}\big)\big) = F_s\big(H_{s+t}\big(cone\big(\Phi^{(n)}\big)\big)\big)/F_{s-1}\big(H_{s+t}\big(cone\big(\Phi^{(n)}\big)\big)\big) = 0$$

since $\Phi^{(n)}$ is a quasi-isomorphism. Since $E^2_{s-n}(cone(\Phi^{(n)})) = 0$, the long exact sequence (ξ_{-n}) gives the isomorphism

$$H_{s-n}\big(C^{(n)}_\bullet\big) = E^2_{s-n}\big(M^{(n)}\big) \xrightarrow{H_\bullet(f)} E^2_{s-n}\big(M'^{(n)}\big) = H_{s-n}\big(C'^{(n)}_\bullet\big),$$

for every s. So f is a quasi-isomorphism as expected.

(3) *If $\Phi = f \otimes g$ and f are quasi-isomorphisms, then g is a quasi-isomorphism.*

Once again, we work by induction on the weight n. For $n = 0$, the map $g^{(0)} : \mathbb{K} \to \mathbb{K}$ is an isomorphism. Suppose that the result is true up to weight $n-1$. When $s \geq 1$, we have seen that only $A^{(n-m)}$ (and $A'^{(n-m)}$) with $m > 0$ are involved in E^2_{st},

$$E^2_{st}\big(M^{(n)}\big) = \bigoplus_{m=1}^{n} H_{s-m}\big(C^{(m)}_\bullet\big) \otimes H_{t+m}\big(A^{(n-m)}_\bullet\big).$$

In this case, the induction hypothesis implies that $E^2_{st}(M^{(n)}) \xrightarrow{H_\bullet(f)\otimes H_\bullet(g)} E^2_{st}(M'^{(n)})$ is an isomorphism for every $s \geq 1$ and every t. The long exact sequence (ξ_t) shows that $E^2_{st}(cone(\Phi^{(n)})) = 0$ for $s \geq 2$ and every t. The spectral sequence of the cone of $\Phi^{(n)}$ converges to its homology, which is null since $\Phi^{(n)}$ is a quasi-isomorphism. Therefore, $E^2_{1,t-1}(cone(\Phi^{(n)})) = E^2_{0,t}(cone(\Phi^{(n)})) = 0$ for every t. This implies $E^2_{st}(cone(\Phi^{(n)})) = 0$ for every t and s. Finally, the beginning $(s = 0)$ of the exact sequence (ξ_t) gives the isomorphism

$$H_t\big(A^{(n)}_\bullet\big) = E^2_{0t}\big(M^{(n)}\big) \xrightarrow{H_\bullet(g)} E^2_{0t}\big(M'^{(n)}\big) = H_t\big(A'^{(n)}_\bullet\big).$$

So g is a quasi-isomorphism as expected. □

2.5.1 Comparison with Algebraic Topology

The Comparison Lemma is the algebraic avatar of the following result in algebraic topology. Let

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow{\scriptstyle f_F} & & \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_B} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array}$$

be a morphism between two fibrations of simply-connected spaces. If two of the morphisms f_F, f_X, f_B are isomorphisms in homology, then so is the third. Using the Whitehead theorem it can be proved as follows: homology isomorphism is equivalent to homotopy isomorphism for simply-connected CW-complexes. When two of the morphisms are homotopy isomorphisms, then so is the third by the long

Serre exact homotopy sequence. Since a homotopy isomorphism is also a homology isomorphism, we are done.

In order to translate this result into homological algebra one needs some extra idea since the trick of passing to homotopy is not available anymore. The idea goes back to the Cartan seminar [Car55] and was later generalized to any first quadrant spectral sequence by Zeeman [Zee57]. This latter one applies to Leray–Serre spectral sequence of fiber spaces, whence the name *base* for the x-axis terms E^2_{s0} (E^2_{s-n} in the present proof) and *fiber* for the y-axis terms E^2_{0t}. More precisely, there is a twisting morphism between the singular chain complex $S(B)$ of the base space, which is a dg coalgebra, and the singular chain complex $S(F)$ of the fiber space which is a module over the algebra of the singular chain complex $S(\Omega B)$ of the loops of B. The induced twisted tensor product is shown to be quasi-isomorphic to the singular chain complex $S(X)$ of the total space, under certain hypotheses, by E.H. Brown in [Bro59]. The spectral sequence introduced in the core of this proof is an algebraic analog of the Leray–Serre spectral sequence.

2.6 Résumé

2.6.1 Twisting Morphism and Twisted Tensor Products

Convolution dga algebra: C dga coalgebra and A dga algebra:

$$\big(\mathrm{Hom}(C, A), \star, \partial\big),$$
$$f \star g = \mu \circ (f \otimes g) \circ \Delta, \qquad \partial(f) = d^A f - (-1)^{|f|} f d^C.$$

Twisting morphism, $\mathrm{Tw}(C, A)$*:* Solution of degree -1 to the Maurer–Cartan equation

$$\partial(\alpha) + \alpha \star \alpha \equiv \partial(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0.$$

Any $\alpha \in \mathrm{Tw}(C, A)$ induces

- ▹ a twisted differential $\partial_\alpha := \partial + [\alpha, -]$ in $\mathrm{Hom}(C, A)$,
- ▹ a differential $d_\alpha := d_{C \otimes A} + d^r_\alpha$ on the tensor product $C \otimes A$ defining the right twisted tensor product $C \otimes_\alpha A$,
- ▹ a differential $d_\alpha := d_{A \otimes C} + d^l_\alpha$ on the tensor product $A \otimes C$ defining the left twisted tensor product $A \otimes_\alpha C$.

Table 2.1 summarizes this hierarchy of notions.

2.6.2 Bar and Cobar Constructions

Bar construction:

$$\mathrm{B}A := \big(T^c(s\bar{A}), d_1 + d_2\big), \qquad d_2(sx \otimes sy) = (-1)^{|x|} s(xy).$$

Table 2.1 Hierarchy of notions

$\alpha \in$	Determines:
$\mathrm{Hom}(C, A)_{-1}$	$d_\alpha : C \otimes A \to C \otimes A$
$\bigcup$	
$\mathrm{Tw}(C, A)$	chain complex $C \otimes_\alpha A$, $d_\alpha^2 = 0$
$\bigcup$	
$\mathrm{Kos}(C, A)$	acyclicity of $C \otimes_\alpha A$

Cobar construction:

$$\Omega C := \left(T\left(s^{-1}\overline{C}\right), d_1 + d_2\right), \qquad d_2\left(s^{-1}x\right) = -\sum (-1)^{|x_{(1)}|} s^{-1}x_{(1)} \otimes s^{-1}x_{(2)}.$$

Summary of Theorem 2.2.6 (second row) and Theorem 2.3.1 (third row):

$$\begin{array}{ccccc}
\mathrm{Hom}_{\text{ga alg}}\left(T\left(s^{-1}\overline{C}\right), A\right) & \cong & \mathrm{Hom}(\overline{C}, \bar{A})_{-1} & \cong & \mathrm{Hom}_{\text{ga coalg}}\left(C, T^c(s\bar{A})\right) \\
\bigcup & & \bigcup & & \bigcup \\
\mathrm{Hom}_{\text{dga alg}}(\Omega C, A) & \cong & \mathrm{Tw}(C, A) & \cong & \mathrm{Hom}_{\text{dga coalg}}(C, \mathrm{B}A) \\
\bigcup & & \bigcup & & \bigcup \\
\text{q-Iso}_{\text{dga alg}}(\Omega C, A) & \cong & \mathrm{Kos}(C, A) & \cong & \text{q-Iso}_{\text{dga coalg}}(C, \mathrm{B}A) .
\end{array}$$

With $C = \mathrm{B}A$, we get

$$\Omega \mathrm{B}A \xrightarrow{\varepsilon} A \leftrightarrow \mathrm{B}A \xrightarrow{\pi} A \leftrightarrow \mathrm{B}A \xrightarrow{\mathrm{Id}} \mathrm{B}A,$$

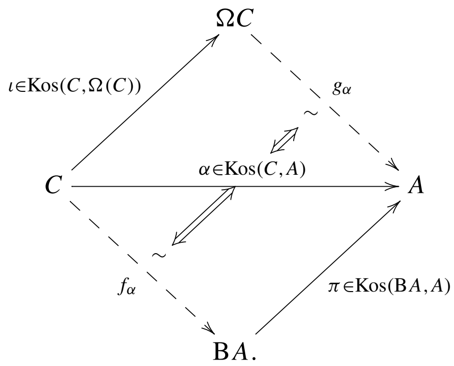
and with $A = \Omega C$, we get

$$\Omega C \xrightarrow{\mathrm{Id}} \Omega C \leftrightarrow C \xrightarrow{\iota} \Omega C \leftrightarrow C \xrightarrow{\upsilon} \mathrm{B}\Omega C.$$

2.6.3 Universal Twisting Morphisms and Fundamental Theorem

Universal twisting morphisms: $\iota : C \to \Omega C$ and $\pi : \mathrm{B}A \to A$, which are Koszul.

Factorization of any twisting morphism $\alpha : C \to A$:



- ▹ $g_\alpha : \Omega C \to A$ morphism of dg algebras,
- ▹ $f_\alpha : C \to \mathrm{B}A$ morphism of dg coalgebras.

Twisting Morphisms Fundamental Theorem. *The following assertions are equivalent*

- ▹ *a twisting morphism $\alpha : C \to A$ is Koszul,*
- ▹ *the morphism of dg algebras $g_\alpha : \Omega C \xrightarrow{\sim} A$ is a quasi-isomorphism,*
- ▹ *the morphism of dg coalgebras $f_\alpha : C \xrightarrow{\sim} \mathrm{B}A$ is a quasi-isomorphism.*

Corollary. $\varepsilon : \Omega \mathrm{B}A \xrightarrow{\sim} A$ *and* $\upsilon : C \xrightarrow{\sim} \mathrm{B}\Omega C$.

2.6.4 Quasi-isomorphisms Under Bar and Cobar Constructions

Proposition. *The bar construction* B *preserves quasi-isomorphisms between dga algebras.*

Proposition. *The cobar construction* Ω *preserves quasi-isomorphisms between* 2*-connected dga coalgebras.*

Weak equivalence: $f : C \to C'$ such that $\Omega f : \Omega C \xrightarrow{\sim} \Omega C'$.

$$\text{graded quasi-isomorphisms} \subseteq \text{weak equivalences} \subsetneq \text{quasi-isomorphisms.}$$

2.7 Exercises

Exercise 2.7.1 (Convolution dga algebra). Draw a picture proof of Proposition 2.1.1, as in Proposition 1.6.2.

Exercise 2.7.2 (Bar construction as an algebra). We know that the cofree coalgebra can be endowed with a commutative algebra structure through the shuffle product, cf. Sect. 1.3.2. Show that the bar construction of a dga algebra is a dg commutative Hopf algebra.

Exercise 2.7.3 (Application of the sign rule). Let $\mathbb{K}s$ be the vector space of degree 1 equipped with the degree -1 product $s \otimes s \mapsto s$. Show that the transpose of the product on the linear dual $\mathbb{K}s^*$ is given by $\Delta(s^*) = -s^* \otimes s^*$.

Exercise 2.7.4 (Universal twisting morphism). Verify directly that $\iota : C \to \Omega C$ is a twisting morphism.

Exercise 2.7.5 (Functoriality). Prove that $\mathrm{Tw} : \mathsf{dga\ coalg}^{\mathsf{op}} \times \mathsf{dga\ alg} \to \mathsf{Set}$ is a bifunctor.

Exercise 2.7.6 (Cotangent complex). Let A be a dga algebra, C a dga coalgebra and let $\alpha : C \to A$ be a twisting morphism. We consider the following twisted differential on $A \otimes C \otimes A$, the free A-bimodule on C:

$$d_\alpha := d_{A\otimes C\otimes A} + \mathrm{Id}_A \otimes d^r_\alpha - d^l_\alpha \otimes \mathrm{Id}_A,$$

where

$$d^r_\alpha := (\mathrm{Id}_C \otimes \mu) \circ (\mathrm{Id}_C \otimes \alpha \otimes \mathrm{Id}_A) \circ (\Delta \otimes \mathrm{Id}_A),$$

and

$$d^l_\alpha := (\mu \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_A \otimes \alpha \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_A \otimes \Delta).$$

◇ Prove that ${d_\alpha}^2 = 0$.

We denote this chain complex by

$$A \otimes_\alpha C \otimes_\alpha A := (A \otimes C \otimes A, d_\alpha).$$

◇ Show that there is an isomorphism of chain complexes

$$\big(\mathrm{Hom}^\alpha(C, A), \partial_\alpha\big) \cong \big(\mathrm{Hom}_{A-\mathrm{biMod}}(A \otimes_\alpha C \otimes_\alpha A, A), \partial\big).$$

◇ Show that the following composite

$$\xi \ : \ A \otimes C \otimes A \xrightarrow{\mathrm{Id}\otimes\varepsilon\otimes\mathrm{Id}} A \otimes \mathbb{K} \otimes A \cong A \otimes A \xrightarrow{\mu} A$$

is a morphism of dg A-bimodules.

◇ Under the same weight grading assumptions as in Theorem 2.3.1, prove that $\xi : A \otimes_\alpha C \otimes_\alpha A \xrightarrow{\sim} A$ is a quasi-isomorphism if and only if α is a Koszul morphism.

Exercise 2.7.7 (Naturality). Prove that the bijections given in Theorem 2.2.6 are functorial in A and C.

Exercise 2.7.8 (Fundamental Theorem). Using the Comparison Lemma 2.1.5, prove directly the equivalence (2) $\Longleftrightarrow$ (3) of Theorem 2.3.1.

Exercise 2.7.9 (Unit of adjunction). Use the same kind of filtrations as in the proof of Proposition 2.4.1 to prove that the unit of adjunction $\upsilon : C \to \mathrm{B}\Omega C$ is a quasi-isomorphism, when C is a conilpotent dga coalgebra.