Chapter 10
Homotopy Operadic Algebras

The theory of algebras up to homotopy defined by operad action is a subject whose time has come.

J.-P. May 1995, review of Hinich–Schechtman
"Homotopy Lie algebra"

When a chain complex is equipped with some compatible algebraic structure, its homology inherits this algebraic structure. The purpose of this chapter is to show that there is some hidden algebraic structure behind the scene. More precisely if the chain complex contains a smaller chain complex, which is a deformation retract, then there is a finer algebraic structure on this small complex. Moreover, the small complex with this new algebraic structure is homotopy equivalent to the starting data.

The operadic framework enables us to state explicitly this transfer of structure result as follows. Let $\mathscr{P}$ be a quadratic operad. Let A be a chain complex equipped with a $\mathscr{P}$-algebra structure. Let V be a deformation retract of A. Then the $\mathscr{P}$-algebra structure of A can be transferred into a $\mathscr{P}_\infty$-algebra structure on V, where $\mathscr{P}_\infty = \Omega\mathscr{P}^{¡}$. If $\mathscr{P}$ is Koszul, then the two objects are homotopy equivalent. Over a field, the homology can be made into a deformation retract, whence the hidden algebraic structure on the homology.

In fact this result is a particular case of a more general result, called the Homotopy Transfer Theorem, which will be stated in full in this chapter. This HTT has a long history and, in a sense, it goes back to the discovery of spectral sequences by Jean Leray and Jean-Louis Koszul in the 1940s.

This chapter is organized as follows. In Sect. 10.1, we define the notion of *homotopy $\mathscr{P}$-algebra* as an algebra over the Koszul resolution $\mathscr{P}_\infty := \Omega\mathscr{P}^{¡}$. Using the operadic bar–cobar adjunction, we give three equivalent definitions. The definition in terms of twisting morphisms is treated in details, as well as the definition in terms of square-zero coderivation on the cofree $\mathscr{P}^{¡}$-coalgebra.

Using this last definition, we define the notion of *infinity morphism*, also denoted *∞-morphism*, between homotopy $\mathscr{P}$-algebras. An ∞-morphism is not only a map

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but is made up of a collection of maps parametrized by $\mathscr{P}^{¡}$. This notion is well suited to the homotopy theory of $\mathscr{P}_\infty$-algebras.

The aforementioned Homotopy Transfer Theorem is the subject of Sect. 10.3. It states precisely that any $\mathscr{P}_\infty$-algebra structure can be transferred through a homotopy retract to produce a homotopy equivalent $\mathscr{P}_\infty$-algebra structure. It is proved by explicit formulas.

In Sect. 10.4, we study the properties of ∞-morphisms. When the first component of an ∞-morphism is invertible (respectively is a quasi-isomorphism), then it is called an *∞-isomorphism* (respectively an *∞-quasi-isomorphism*). We prove that the class of ∞-isomorphisms is the class of invertible ∞-morphisms. Any $\mathscr{P}_\infty$-algebra is shown to be decomposable into the product of a *minimal $\mathscr{P}_\infty$-algebra* and an *acyclic trivial $\mathscr{P}_\infty$-algebra*. Using this result, we prove that being ∞-quasi-isomorphic is an equivalence relation, called the *homotopy equivalence*.

In Sect. 10.5, we study the same kind of generalization "up to homotopy" but for operads this time. We introduce the notions of *homotopy operad* and *infinity morphism*, or *∞-morphism*, of homotopy operads. One key ingredient in the HTT is actually an explicit ∞-morphism between endomorphism operads. The functor from operads to Lie algebras is extended to a functor between homotopy operads to homotopy Lie algebras. This allows us to show that the relations between associative algebras, operads, pre-Lie algebra, and Lie algebras extend to the homotopy setting. Finally, we study homotopy representations of operads.

Throughout this chapter, we apply the various results to A_∞-algebras, already treated independently in the previous chapter, and to L_∞-algebras. In this chapter, the generic operad $\mathscr{P}$ is a Koszul operad.

The general study of homotopy algebras using the Koszul resolution $\mathscr{P}_\infty = \Omega\mathscr{P}^{¡}$ of $\mathscr{P}$ goes back to Ginzburg and Kapranov in [GK94] and to Getzler and Jones in [GJ94]. Many particular cases have been treated in the literature; we refer the reader to the survey given in Part I of the book [MSS02] of Markl, Shnider and Stasheff.

In this chapter, we work over a ground field $\mathbb{K}$ of characteristic 0. Notice that all the constructions and some of the results hold true without this hypothesis.

10.1 Homotopy Algebras: Definitions

In this section, we introduce the notion of homotopy $\mathscr{P}$-algebra, i.e. $\mathscr{P}_\infty$-algebra, for a Koszul operad $\mathscr{P}$. We give four equivalent definitions. We treat in detail the examples of homotopy associative algebras, or A_∞-algebras, and homotopy Lie algebras, or L_∞-algebras.

10.1.1 $\mathscr{P}_\infty$-Algebras

A *homotopy $\mathscr{P}$-algebra* is an algebra over the Koszul resolution $\Omega\mathscr{P}^{¡}$ of $\mathscr{P}$. It is sometimes called a $\mathscr{P}$-algebra up to homotopy or strong homotopy $\mathscr{P}$-algebra in the literature. We also call it a *$\mathscr{P}_\infty$-algebra*, where $\mathscr{P}_\infty$ stands for the dg operad

$\Omega\mathscr{P}^{\text{i}}$. Hence, a homotopy $\mathscr{P}$-algebra structure on a dg module A is a morphism of dg operads $\mathscr{P}_\infty = \Omega\mathscr{P}^{\text{i}} \to \text{End}_A$. The set of homotopy $\mathscr{P}$-algebra structures on A is equal to $\text{Hom}_{\mathsf{dg\ Op}}(\Omega\mathscr{P}^{\text{i}}, \text{End}_A)$.

Notice that a $\mathscr{P}$-algebra is a particular example of homotopy $\mathscr{P}$-algebra. It occurs when the structure morphism factors through $\mathscr{P}$:

$$\Omega\mathscr{P}^{\text{i}} \xrightarrow{\sim} \mathscr{P} \to \text{End}_A.$$

10.1.2 Interpretation in Terms of Twisting Morphism

Let us now make this notion explicit. We saw in Proposition 6.5.7 that a morphism of dg operads from the quasi-free operad $\Omega\mathscr{P}^{\text{i}}$ to End_A is equivalent to a twisting morphism in the convolution algebra

$$\mathfrak{g} = \mathfrak{g}_{\mathscr{P},A} := \text{Hom}_{\mathbb{S}}\big(\mathscr{P}^{\text{i}}, \text{End}_A\big).$$

Explicitly, we recall from Sect. 6.5.3 the following correspondence

$$\begin{array}{ccc} \text{Hom}_{\mathsf{Op}}\big(\Omega\mathscr{P}^{\text{i}}, \text{End}_A\big) & \cong & \text{Hom}_{\mathbb{S}}\big(\overline{\mathscr{P}}^{\text{i}}, \text{End}_A\big)_{-1} \\ \bigcup & & \bigcup \\ \text{Hom}_{\mathsf{dg\ Op}}\big(\Omega\mathscr{P}^{\text{i}}, \text{End}_A\big) & \cong & \text{Tw}\big(\mathscr{P}^{\text{i}}, \text{End}_A\big). \end{array}$$

As a direct consequence, we get the following description of homotopy algebra structures.

Proposition 10.1.1. *A homotopy $\mathscr{P}$-algebra structure on the dg module A is equivalent to a twisting morphism in* $\text{Tw}(\mathscr{P}^{i}, \text{End}_A)$.

Let us make explicit the notion of twisting morphism $\text{Tw}(\mathscr{P}^{\text{i}}, \text{End}_A)$. Suppose first that the operad $\mathscr{P}$ is homogeneous quadratic. The internal differential of $\mathscr{P}^{\text{i}}$ is trivial. For any element $\varphi \in \text{Hom}_{\mathbb{S}}(\mathscr{P}^{\text{i}}, \text{End}_A)$ and for any element $\mu^c \in \mathscr{P}^{\text{i}}$, the Maurer–Cartan equation becomes $\partial_A(\varphi(\mu^c)) + (\varphi \star \varphi)(\mu^c) = 0$, where ∂_A stands for the differential of End_A induced by the differential of A. Using Sweedler type notation of Sect. 6.1.4, we denote by $\sum(\mu^c_{(1)} \circ_i \mu^c_{(2)})^\sigma$ the image of μ^c under the infinitesimal decomposition map $\Delta_{(1)} : \mathscr{P}^{\text{i}} \to \mathscr{P}^{\text{i}} \circ_{(1)} \mathscr{P}^{\text{i}}$ of the cooperad $\mathscr{P}^{\text{i}}$. If we denote by m the image of μ^c under φ, we get the following equation in End_A:

$$\sum \pm (m_{(1)} \circ_i m_{(2)})^\sigma = \partial_A(m).$$

This formula describes the general relations satisfied by the operations of a homotopy $\mathscr{P}$-algebra.

Proposition 10.1.2. *The convolution pre-Lie algebra $\mathfrak{g}$ is endowed with a weight grading such that $\mathfrak{g} \cong \prod_{n\geq 0} \mathfrak{g}^{(n)}$.*

Proof. The Koszul dual dg cooperad $\mathscr{P}^{¡}$ is weight graded, $\mathscr{P}^{¡} = \bigoplus_{n\geq 0} \mathscr{P}^{¡\,(n)}$. Therefore the convolution pre-Lie algebra $\mathfrak{g}$ is graded by

$$\mathfrak{g}^{(n)} := \mathrm{Hom}_{\mathbb{S}}\big(\mathscr{P}^{¡\,(n)}, \mathrm{End}_A\big)$$

and the direct sum on $\mathscr{P}$ gives the product $\mathfrak{g} \cong \prod_{n\geq 0} \mathfrak{g}^{(n)}$. □

Hence, any twisting morphism φ in $\mathfrak{g}$ decomposes into a series $\varphi = \varphi_1 + \cdots + \varphi_n + \cdots$ with $\varphi_n \in \mathfrak{g}^{(n)}_{-1}$, since $I = \mathscr{P}^{¡\,(0)}$ and $\varphi_0 = 0$. Under this notation, the Maurer–Cartan equation is equivalent to

$$-\sum_{\substack{k+l=n\\ k,l<n}} \varphi_k \star \varphi_l = \partial(\varphi_n),$$

for any $n \geq 1$.

Proposition 10.1.3. *The differential of the convolution dg pre-Lie algebra* $\mathfrak{g}$ *splits into two terms* $\partial = \partial_0 + \partial_1$, *where* $\partial_0 = \partial_A$ *preserves the weight grading and where* ∂_1 *raises it by* 1.

Proof. The term ∂_1 is equal to $\partial_1(\varphi) := -(-1)^{|\varphi|}\varphi(d_{\mathscr{P}^{¡}})$. Since $d_{\mathscr{P}^{¡}}$ lowers the weight grading of the Koszul dual dg cooperad $\mathscr{P}^{¡}$ by one, ∂_1 raises the weight grading of the convolution pre-Lie algebra by one. □

Under the weight grading decomposition, the Maurer–Cartan equation reads

$$-\partial_1(\varphi_{n-1}) - \sum_{\substack{k+l=n\\ k,l<n}} \varphi_k \star \varphi_l = \partial_A(\varphi_n),$$

so the left-hand side relation holds up to the homotopy φ_n in $\mathfrak{g}^{(n)}$.

10.1.3 The Example of $\mathscr{P}$-Algebras

The $\mathscr{P}$-algebras are characterized among the $\mathscr{P}_\infty$-algebras by the following particular solutions to the Maurer–Cartan equation.

Proposition 10.1.4. *A* $\mathscr{P}_\infty$*-algebra is a* $\mathscr{P}$*-algebra if and only if its twisting morphism is concentrated in weight* 1.

Proof. Let $\mathscr{P} = \mathscr{P}(E, R)$ be a quadratic operad. A $\mathscr{P}$-algebra A is a $\mathscr{P}_\infty$-algebra whose structure map factors through $\mathscr{P}$. The map $\Omega\mathscr{P}^{¡} \twoheadrightarrow \mathscr{P}$ sends the elements of $\mathscr{P}^{¡\,(n)}$ to 0 for $n \geq 2$. So the nontrivial part under this morphism is the image of

$\mathscr{P}^{¡\,(1)} \mapsto \mathscr{P}^{(1)} = E$, that is the generating operations of $\mathscr{P}$. In this case, the only nontrivial components of the Maurer–Cartan equation are for $\mu^c \in \mathscr{P}^{¡\,(1)} \cong E$ and for $\mu^c \in \mathscr{P}^{¡\,(2)} \cong R$. The first one is equivalent, for the internal differential of A, to be a derivation with respect to the operations of E, and the second one is equivalent for these operations to satisfy the relations of R. □

The following proposition gives a first result on the algebraic structure of the homotopy $H(A)$ of a $\mathscr{P}_\infty$-algebra A.

Proposition 10.1.5. *The homotopy of a $\mathscr{P}_\infty$-algebra A, that is the homology $H(A)$ of the underlying chain complex, has a natural $\mathscr{P}$-algebra structure.*

Proof. Let A be a $\mathscr{P}_\infty$-algebra with structure map $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$. The image under φ of any element in $\mathscr{P}^{¡\,(1)}$ gives operations in End_A for which d is a derivation. Therefore these operations are stable on homology. Since the differential on $H(A)$ is null, they define a $\mathscr{P}$-algebra structure on $H(A)$. □

Considering only the $\mathscr{P}$-algebra structure on $H(A)$, we are losing a lot of data. We will see in Sect. 10.3 that we can transfer a full structure of $\mathscr{P}_\infty$-algebra on $H(A)$, which faithfully contains the homotopy type of A.

10.1.4 $\mathscr{P}_{(n)}$-Algebras

The preceding section motivates the following definition. A *$\mathscr{P}_{(n)}$-algebra* A is a homotopy $\mathscr{P}$-algebra such that the structure map $\varphi : \mathscr{P}^{¡} \to \mathrm{End}_A$ vanishes on $\mathscr{P}^{¡\,(k)}$ for $k > n$. It is equivalent to a truncated solution of the Maurer–Cartan equation in the convolution algebra $\mathfrak{g}$. Under this definition a $\mathscr{P}$-algebra is a $\mathscr{P}_{(1)}$-algebra.

10.1.5 Example: Homotopy Associative Algebras, Alias A_∞-Algebras

We pursue the study of homotopy associative algebras, started in Sec. 9.2, but in terms of twisting morphism this time.

Consider the nonsymmetric Koszul operad As, see Sect. 9.1.5. We proved in Sect. 9.2 that an algebra over $\Omega As^{¡}$, i.e. a *homotopy associative algebra* or *A_∞-algebra*, is a chain complex (A, d_A) equipped with maps $m_n : A^{\otimes n} \to A$ of degree $n-2$, for any $n \geq 2$, which satisfy

$$\sum_{\substack{p+q+r=n\\ k=p+r+1>1,\, q>1}} (-1)^{p+qr+1} m_k \circ_{p+1} m_q = \partial_A(m_n) = d_A \circ m_n - (-1)^{n-2} m_n \circ d_{A^{\otimes n}}.$$

Notice that an associative algebra is an A_∞-algebra such that the higher homotopies m_n vanish for $n \geq 3$.

The Koszul dual nonsymmetric cooperad of As is one-dimensional in each arity $As^{¡}_n = \mathbb{K}\mu^c_n$, where the degree of μ^c_n is $n-1$. The image under the infinitesimal decomposition map of μ^c_n is

$$\Delta_{(1)}\big(\mu^c_n\big) = \sum_{\substack{p+q+r=n\\ k=p+r+1>1,\, q>1}} (-1)^{r(q+1)}\big(\mu^c_k; \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p}, \mu^c_q, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r}\big).$$

Since the operad As is a nonsymmetric operad, the convolution algebra is given by $\mathrm{Hom}(As^{¡}, \mathrm{End}_A)$, without the action of the symmetric groups. It is isomorphic to the following dg module

$$\prod_{n\geq 1} \mathrm{Hom}\big(As^{¡}, \mathrm{End}_A\big)(n) \cong \prod_{n\geq 1} \mathrm{Hom}\big((sA)^{\otimes n}, sA\big) \cong \prod_{n\geq 1} s^{-n+1}\, \mathrm{Hom}\big(A^{\otimes n}, A\big).$$

The right-hand side is the direct product of the components of the nonsymmetric operad End_{sA}. Therefore, it is endowed with the pre-Lie operation of Sect. 5.9.15. For an element $f \in \mathrm{Hom}(A^{\otimes n}, A)$ and an element $g \in \mathrm{Hom}(A^{\otimes m}, A)$, it is explicitly given by

$$f \star g := \sum_{i=1}^{n} (-1)^{(i-1)(m+1)+(n+1)|g|} f \circ_i g.$$

This particular dg pre-Lie algebra was constructed by Murray Gerstenhaber in [Ger63].

Proposition 10.1.6. *The convolution dg pre-Lie algebra* $\mathfrak{g}_{As,A} = \mathrm{Hom}(As^{¡}, \mathrm{End}_A)$ *is isomorphic to the dg pre-Lie algebra* $(\prod_{n\geq 1} s^{-n+1}\, \mathrm{Hom}(A^{\otimes n}, A), \star)$, *described above.*

Proof. We denote by $\tilde{f} \in \mathrm{Hom}(As^{¡}, \mathrm{End}_A)(n)$ and by $\tilde{g} \in \mathrm{Hom}(As^{¡}, \mathrm{End}_A)(m)$ the maps which send μ^c_n to f and μ^c_m to g. Then the only nonvanishing component of the pre-Lie product $\tilde{f} \star \tilde{g}$ in the convolution algebra $\mathrm{Hom}(As^{¡}, \mathrm{End}_A)$ is equal to the composite

$$\mu^c_{n+m-1} \mapsto \sum_{i=1}^{n} (-1)^{(n-i)(m+1)}\big(\mu^c_n; \mathrm{id}, \ldots, \mathrm{id}, \underbrace{\mu^c_m}_{i\text{th place}}, \mathrm{id}, \ldots, \mathrm{id}\big)$$

$$\mapsto \sum_{i=1}^{n} (-1)^{(n-i)(m+1)+(n-1)(|g|-m+1)} f \circ_i g = f \star g. \qquad \square$$

Under this explicit description of the convolution pre-Lie algebra $\mathrm{Hom}(As^{¡}, \mathrm{End}_A)$, one can see that a twisting morphism in $\mathrm{Tw}(As^{¡}, \mathrm{End}_A)$ is exactly an A_∞-algebra structure on the dg module A.

10.1.6 Example: Homotopy Lie Algebras, Alias L_∞-Algebras

Applying Definition 10.1.1 to the operad $\mathscr{P} = Lie$, a *homotopy Lie algebra* is an algebra over the Koszul resolution $\Omega Lie^{¡} \xrightarrow{\sim} Lie$ of the operad *Lie*. It is also called an *L_∞-algebra*, or *strong homotopy Lie algebra* in the literature.

Recall that an n-multilinear map f is called *skew-symmetric* if it satisfies the condition $f = \operatorname{sgn}(\sigma) f^\sigma$ for any $\sigma \in \mathbb{S}_n$.

Proposition 10.1.7. *An L_∞-algebra structure on a dg module (A, d_A) is a family of skew-symmetric maps $\ell_n : A^{\otimes n} \to A$ of degree $|\ell_n| = n-2$, for all $n \geq 2$, which satisfy the relations*

$$\sum_{\substack{p+q=n+1\\ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \operatorname{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma = \partial_A(\ell_n),$$

where ∂_A is the differential in End_A *induced by d_A.*

Proof. Recall that there is a morphism of operads $Lie \to Ass$ defined by $[\ ,\ ] \mapsto \mu - \mu^{(12)}$. Its image under the bar construction functor induces a morphism of dg cooperads $\mathrm{B}Lie \to \mathrm{B}Ass$. By Proposition 7.3.1, the morphism between the syzygy degree 0 cohomology groups of the bar constructions gives a morphism between the Koszul dual cooperads $Lie^{¡} \to Ass^{¡}$. If we denote the elements of these two cooperads by $Lie^{¡}(n) \cong \mathbb{K}\,\ell_n^c \otimes \operatorname{sgn}_{\mathbb{S}_n}$ and by $Ass^{¡}(n) \cong \mathbb{K}\mu_n^c \otimes \mathbb{K}[\mathbb{S}_n]$ with $|\ell_n^c| = |\mu_n^c| = n-1$, this map is explicitly given by $\ell_n^c \mapsto \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma)(\mu_n^c)^\sigma$. Hence, the formula for the infinitesimal decomposition map of the cooperad $Ass^{¡}$ induces

$$\Delta_{(1)}(\ell_n^c) = \sum_{\substack{p+q=n+1\\ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \operatorname{sgn}(\sigma)(-1)^{(p+1)(q+1)}\left(\ell_p^c \circ_1 \ell_q^c\right)^\sigma,$$

since the (p,q)-unshuffles split the surjection $\mathbb{S}_{p+q} \twoheadrightarrow (\mathbb{S}_p \times \mathbb{S}_q)\backslash\mathbb{S}_{p+q}$, cf. Sect. 1.3.2. Let us denote by ℓ_n the image under the structure morphism $\Phi : \Omega Lie^{¡} \to \operatorname{End}_A$ of the generators $-s^{-1}\ell_n^c$. The $\mathbb{S}_n$-module $Lie^{¡}(n)$ being the one-dimensional signature representation, the map ℓ_n is skew-symmetric. The commutation of the structure morphism Φ with the differentials reads

$$\sum_{\substack{p+q=n+1\\ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \operatorname{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma = \partial_A(\ell_n).$$

□

As in the case of A_∞-algebras, we can denote the differential of A by $\ell_1 := -d_A$, and include it in the relations defining an L_∞-algebra as follows

$$\sum_{p+q=n+1} \sum_{\sigma \in Sh_{p,q}^{-1}} \operatorname{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma = 0.$$

This way of writing the definition of a homotopy Lie algebra is more compact but less explicit about the role of the boundary map $\ell_1 = -d_A$.

In the next proposition, we extend to homotopy algebras the anti-symmetrization construction of Sect. 1.1.9, which produces a Lie bracket from an associative product.

Proposition 10.1.8. [LS93] *Let $(A, d_A, \{m_n\}_{n\geq 2})$ be an A_∞-algebra structure on a dg module A. The anti-symmetrized maps $\ell_n : A^{\otimes n} \to A$, given by*

$$\ell_n := \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) {m_n}^{\sigma},$$

endow the dg module A with an L_∞-algebra structure.

Proof. It is a direct corollary of the proof of the previous proposition. The map of cooperads $Lie^{¡} \to Ass^{¡}$ induces a morphism of dg operads $\Omega Lie^{¡} \to \Omega Ass^{¡}$, given by $\ell_n^c \mapsto \sum_{\sigma\in\mathbb{S}_n} \operatorname{sgn}(\sigma)(\mu_n^c)^{\sigma}$. Hence, the pullback of a morphism $\Omega Ass^{¡} \to \operatorname{End}_A$, defining an A_∞-algebra structure on A, produces a morphism of dg operads $\Omega Lie^{¡} \to \operatorname{End}_A$, which is the expected L_∞-algebra structure on A. □

10.1.7 The Convolution Algebra Encoding L_∞-Algebras

The underlying module of the convolution pre-Lie algebra $\operatorname{Hom}_{\mathbb{S}}(Lie^{¡}, \operatorname{End}_A)$ is isomorphic to

$$\prod_{n\geq 1} \operatorname{Hom}_{\mathbb{S}}\big(Lie^{¡}, \operatorname{End}_A\big)(n) \cong \prod_{n\geq 1} \operatorname{Hom}\big(S^n(sA), sA\big) \cong \prod_{n\geq 1} s^{-n+1} \operatorname{Hom}\big(\Lambda^n A, A\big),$$

where $\Lambda^n A$ is the coinvariant space of $A^{\otimes n}$ with respect to the signature representation. Explicitly, it is the quotient of $A^{\otimes n}$ by the relations

$$a_1 \otimes \cdots \otimes a_n - \operatorname{sgn}(\sigma)\varepsilon a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

for $a_1, \ldots, a_n \in A$ and for $\sigma \in \mathbb{S}_n$ with ε the Koszul sign given by the permutation of the graded elements $a_1, \ldots, a_n$.

We endow the right-hand side with the following binary product

$$f \star g := \sum_{\substack{p+q=n+1\\ p,q>1}} \sum_{\sigma\in Sh_{p,q}^{-1}} \operatorname{sgn}(\sigma)(-1)^{(p-1)|g|}(f \circ_1 g)^{\sigma},$$

for $f \in \operatorname{Hom}(\Lambda^p A, A)$ and $g \in \operatorname{Hom}(\Lambda^q A, A)$.

This product is called the *Nijenhuis–Richardson product* from [NR66, NR67].

Proposition 10.1.9. *For any dg module A, the Nijenhuis–Richardson product endows the space $\prod_{n\geq 1} s^{-n+1}\operatorname{Hom}(\Lambda^n A, A)$ with a dg pre-Lie algebra structure, which is isomorphic to the convolution dg pre-Lie algebra* $\operatorname{Hom}_{\mathbb{S}}(Lie^{¡}, \operatorname{End}_A)$.

Proof. The proof is similar to the proof of Proposition 10.1.6 with the explicit form of the infinitesimal decomposition map of the cooperad $Lie^{¡}$ given above. We first check that the two products are sent to one another under this isomorphism. As a consequence, the Nijenhuis–Richardson product is a pre-Lie product. □

Under this explicit description of the convolution pre-Lie algebra $\operatorname{Hom}_{\mathbb{S}}(Lie^{¡}, \operatorname{End}_A)$, we leave it to the reader to verify that a twisting element is exactly an L_∞-algebra structure on the dg module A.

For other examples of homotopy algebras, we refer to Chap. 13, where examples of algebras over operads are treated in detail.

10.1.8 Equivalent Definition in Terms of Square-Zero Coderivation

In this section, we give a third equivalent definition of the notion of $\mathscr{P}_\infty$-algebra. A structure of $\mathscr{P}_\infty$-algebra can be faithfully encoded as a square-zero coderivation as follows.

By Proposition 6.3.8, we have the following isomorphisms

$$\operatorname{Hom}_{\mathbb{S}}\big(\mathscr{P}^{¡}, \operatorname{End}_A\big) \cong \operatorname{Hom}_{\mathsf{Mod}_{\mathbb{K}}}\big(\mathscr{P}^{¡}(A), A\big) \cong \operatorname{Coder}\big(\mathscr{P}^{¡}(A)\big),$$

where $\operatorname{Coder}(\mathscr{P}^{¡}(A))$ stands for the module of coderivations on the cofree $\mathscr{P}^{¡}$-coalgebra $\mathscr{P}^{¡}(A)$. Let us denote by $\varphi \mapsto d_\varphi^r$ the induced isomorphism from left to right.

Proposition 10.1.10. *The map* $\operatorname{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \operatorname{End}_A) \cong \operatorname{Coder}(\mathscr{P}^{¡}(A))$ *is an isomorphism of Lie algebras*:

$$\big[d_\alpha^r, d_\beta^r\big] = d_{[\alpha,\beta]}^r.$$

Proof. Let $\bar{\varphi}$ be the image of φ under the first isomorphism $\operatorname{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \operatorname{End}_A) \cong \operatorname{Hom}_{\mathsf{Mod}_{\mathbb{K}}}(\mathscr{P}^{¡}(A), A)$. If we denote by proj_A the canonical projection $\mathscr{P}^{¡}(A) \twoheadrightarrow A$, then Proposition 6.3.8 gives $\operatorname{proj}_A(d_\varphi^r) = \bar{\varphi}$. A direct computation shows that $\operatorname{proj}_A([d_\alpha^r, d_\beta^r]) = \overline{[\alpha, \beta]}$, which concludes the proof. □

We consider the sum

$$d_\varphi := d_{\mathscr{P}^{¡}(A)} + d_\varphi^r$$

of d_φ^r with the internal differential on $\mathscr{P}^{¡}(A)$.

Proposition 10.1.11. *A structure of $\mathscr{P}_\infty$-algebra on a dg module A is equivalent to a square-zero coderivation on the cofree $\mathscr{P}^{¡}$-coalgebra $\mathscr{P}^{¡}(A)$.*

Explicitly, an element $\varphi \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_A)$, such that $\varphi(\mathrm{id}) = 0$, satisfies the Maurer–Cartan equation $\partial(\varphi) + \varphi \star \varphi = 0$ if and only if ${d_\varphi}^2 = 0$.

Proof. Any $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$ induces a coderivation d_φ^r of degree -1 on $\mathscr{P}^{¡}(A)$. Since φ is a twisting morphism, it vanishes on the counit of $\mathscr{P}^{¡}$. As a consequence d_φ^r vanishes on $A \subset \mathscr{P}^{¡}(A)$. Under the same notation as in Proposition 10.1.10, we have

$$\begin{aligned}\mathrm{proj}_A\big({d_\varphi}^2\big) &= \mathrm{proj}_A\big(d_{\mathscr{P}^{¡}(A)} \circ d_\varphi^r + d_\varphi^r \circ d_{\mathscr{P}^{¡}(A)} + \big(d_\varphi^r\big)^2\big)\\ &= d_A \circ \bar{\varphi} + \bar{\varphi} \circ d_{\mathscr{P}^{¡}(A)} + \overline{\varphi \star \varphi}\end{aligned}$$

in $\mathrm{Hom}_{\mathsf{Mod}_{\mathbb{K}}}(\mathscr{P}^{¡}(A), A)$. We conclude with the relation

$$\overline{\partial(\varphi) + \varphi \star \varphi} = \partial(\bar{\varphi}) + \overline{\varphi \star \varphi} = d_A \circ \bar{\varphi} + \bar{\varphi} \circ d_{\mathscr{P}^{¡}(A)} + \overline{\varphi \star \varphi}. \qquad \square$$

In this case, $(\mathscr{P}^{¡}(A), d_\varphi)$ becomes a quasi-cofree $\mathscr{P}^{¡}$-coalgebra. This proposition shows that a homotopy $\mathscr{P}$-algebra structure on a dg module A is equivalent to a dg $\mathscr{P}^{¡}$-coalgebra structure on $\mathscr{P}^{¡}(A)$, where the structure maps are encoded into the coderivation. We call *codifferential* any degree -1 square-zero coderivation on a $\mathscr{P}^{¡}$-coalgebra. So the set of $\mathscr{P}_\infty$-algebra structures is equal to the set of codifferentials $\mathrm{Codiff}(\mathscr{P}^{¡}(A))$.

For example, we get the definitions of A_∞-algebras and of L_∞-algebras in terms a square-zero coderivations.

Proposition 10.1.12. *An A_∞-algebra structure on a dg module A is equivalent to a codifferential on the noncounital cofree associative coalgebra $\overline{T}^c(sA)$.*

Similarly, an L_∞-algebra structure on A is equivalent to a codifferential on the noncounital cofree cocommutative coalgebra $\overline{S}^c(sA)$.

Proof. Since the Koszul dual nonsymmetric cooperad $As^{¡}$ is isomorphic to $As^* \underset{\mathrm{H}}{\otimes} \mathrm{End}^c_{s^{-1}\mathbb{K}}$ by Sect. 7.2.3, the quasi-cofree $As^{¡}$-coalgebra $As^{¡}(A)$ is isomorphic to the desuspension of the noncounital cofree associative coalgebra $s^{-1}\overline{T}^c(sA)$.

In the same way, since the Koszul dual cooperad $Lie^{¡}$ is isomorphic to $Com^* \underset{\mathrm{H}}{\otimes} \mathrm{End}^c_{s^{-1}\mathbb{K}}$ by Sect. 7.2.3, the quasi-cofree $Lie^{¡}$-coalgebra $Lie^{¡}(A)$ is isomorphic to the desuspension of the noncounital cofree cocommutative coalgebra $s^{-1}\overline{S}^c(sA)$. $\square$

10.1.9 Rosetta Stone

Using the bar–cobar adjunction of Sect. 6.5.3, a $\mathscr{P}_\infty$-algebra structure on a dg module A is equivalently defined by a morphism of dg cooperads $\mathscr{P}^{¡} \to \mathrm{B}\,\mathrm{End}_A$.

Notice that the endomorphism operad End_A is unital but non-necessarily augmented. So by the bar construction of End_A, we mean $\mathrm{B}\,\mathrm{End}_A := \mathscr{T}^c(s\,\mathrm{End}_A)$, endowed with the same differential map as in Sect. 6.5.1. With this definition, the bar–cobar adjunction still holds.

The four equivalent definitions of homotopy $\mathscr{P}$-algebras are summed up in the following theorem.

Theorem 10.1.13 (Rosetta Stone). *The set of $\mathscr{P}_\infty$-algebra structures on a dg module A is equivalently given by*

$$\mathrm{Hom}_{\mathsf{dg\ Op}}\big(\Omega\mathscr{P}^{i}, \mathrm{End}_A\big) \cong \mathrm{Tw}\big(\mathscr{P}^{i}, \mathrm{End}_A\big) \cong \mathrm{Hom}_{\mathsf{dg\ Coop}}\big(\mathscr{P}^{i}, \mathrm{B}\,\mathrm{End}_A\big) \cong \mathrm{Codiff}\big(\mathscr{P}^{i}(A)\big).$$

10.2 Homotopy Algebras: Morphisms

In this section, we make the notion of morphism of $\mathscr{P}_\infty$-algebras explicit. Then we introduce and study the more general notion of infinity-morphism, denoted ∞-morphism, of $\mathscr{P}_\infty$-algebras, which will prove to be more relevant to the homotopy theory of $\mathscr{P}_\infty$-algebras. The data of an ∞-morphism does not consist in only one map but in a family of maps parametrized by the elements of the Koszul dual cooperad $\mathscr{P}^{¡}$. More precisely, these maps live in $\mathrm{End}^A_B := \{\mathrm{Hom}(A^{\otimes n}, B)\}_{n\in\mathbb{N}}$, the space of multilinear maps between two $\mathscr{P}_\infty$-algebras.

The examples of ∞-morphisms of A_∞-algebras and of L_∞-algebras are given.

10.2.1 Morphisms of $\mathscr{P}_\infty$-Algebras

A *morphism $f: A \to B$ between $\mathscr{P}_\infty$-algebras* is a morphism of algebras over the operad $\mathscr{P}_\infty$ as in Sect. 5.2.3.

In terms of twisting morphisms, they are described as follows. Let A and B be two $\mathscr{P}_\infty$-algebras, whose associated twisting morphisms are denoted by $\varphi : \mathscr{P}^{¡} \to \mathrm{End}_A$ and $\psi : \mathscr{P}^{¡} \to \mathrm{End}_B$ respectively. We denote by End^A_B the $\mathbb{S}$-module defined by

$$\mathrm{End}^A_B := \big\{\mathrm{Hom}\big(A^{\otimes n}, B\big)\big\}_{n\in\mathbb{N}}.$$

In other words, a morphism of $\mathscr{P}_\infty$-algebras is map $f: A \to B$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{P}^{¡} & \xrightarrow{\ \varphi\ } & \mathrm{End}_A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{End}_B & \xrightarrow{\ f^*\ } & \mathrm{End}^A_B, \end{array}$$

where f_* is the pushforward by f

$$g \in \operatorname{Hom}\big(A^{\otimes n}, A\big) \mapsto f_*(g) := fg \in \operatorname{Hom}\big(A^{\otimes n}, B\big),$$

and where f^* is the pullback by f

$$g \in \operatorname{Hom}\big(B^{\otimes n}, B\big) \mapsto f^*(g) := g(f, \ldots, f) \in \operatorname{Hom}\big(A^{\otimes n}, B\big).$$

In this case, the two homotopy $\mathscr{P}$-algebra structures strictly commute under f.

10.2.2 Infinity-Morphisms of $\mathscr{P}_\infty$-Algebras

We use the third equivalent definition of homotopy algebras to define the notion of ∞-*morphism* of homotopy algebras, which is an enhancement of the previous one.

By Proposition 10.1.11, a homotopy $\mathscr{P}$-algebra structure on A (resp. on B) is equivalent to a dg $\mathscr{P}^{¡}$-coalgebra structure on $\mathscr{P}^{¡}(A)$ (resp. on $\mathscr{P}^{¡}(B)$), with codifferential denoted by d_φ (resp. d_ψ).

By definition, an ∞-*morphism of* $\mathscr{P}_\infty$-*algebras* is a morphism

$$F : \big(\mathscr{P}^{¡}(A), d_\varphi\big) \to \big(\mathscr{P}^{¡}(B), d_\psi\big)$$

of dg $\mathscr{P}^{¡}$-coalgebras. The composite of two ∞-morphisms is defined as the composite of the associated morphisms of dg $\mathscr{P}^{¡}$-coalgebras:

$$F \circ G := \mathscr{P}^{¡}(A) \to \mathscr{P}^{¡}(B) \to \mathscr{P}^{¡}(C).$$

Therefore $\mathscr{P}_\infty$-algebras with their ∞-morphisms form a category, which is denoted by ∞-$\mathscr{P}_\infty$-alg. An ∞-morphism between $\mathscr{P}_\infty$-algebras is denoted by

$$A \rightsquigarrow B$$

to avoid confusion with the above notion of morphism.

Proposition 10.2.1. *Let $\mathscr{C}$ be a cooperad. Any morphism $F : \mathscr{C}(V) \to \mathscr{C}(W)$ of cofree $\mathscr{C}$-coalgebras is completely characterized by its projection $\bar{f}$ onto the cogenerators $\bar{f} := \operatorname{proj}_W \circ F : \mathscr{C}(V) \to W$.*

Explicitly, the unique morphism $F : \mathscr{C}(V) \to \mathscr{C}(W)$ of $\mathscr{C}$-coalgebras which extends a map $\bar{f} : \mathscr{C}(V) \to W$ is given by the following composite

$$F = \mathscr{C}(V) = \mathscr{C} \circ V \xrightarrow{\Delta \circ \mathrm{Id}_V} \mathscr{C} \circ \mathscr{C} \circ V \xrightarrow{\mathrm{Id}_{\mathscr{C}} \circ \bar{f}} \mathscr{C} \circ W = \mathscr{C}(W).$$

Proof. The proof uses the same ideas as in Proposition 6.3.8. So it is left to the reader as an exercise. □

This proposition shows that an ∞-morphism of $\mathscr{P}_\infty$-algebras is equivalently given by a map $\bar{f} : \mathscr{P}^{¡}(A) \to B$, whose induced morphism $F : \mathscr{P}^{¡}(A) \to \mathscr{P}^{¡}(B)$ of $\mathscr{P}^{¡}$-coalgebras commutes with the differentials. Any such map $\bar{f}$ is equivalent to a map $f : \mathscr{P}^{¡} \to \mathrm{End}^A_B$. So an ∞-morphism is made out of a family of maps, from $A^{\otimes n} \to B$, parametrized by $\mathscr{P}^{¡}(n)$, for any n.

The next section makes the relation satisfied by an ∞-morphism explicit in terms of this associated map f.

10.2.3 Infinity-Morphisms in Terms of Twisting Morphisms

The module $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_A)$ with its pre-Lie convolution product $\star$ form the pre-Lie algebra $\mathfrak{g}_A := (\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_A), \star)$.

The module $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B)$ is an associative algebra with the associative product $\circledcirc$ defined by

$$\psi \circledcirc \xi := \mathscr{P}^{¡} \xrightarrow{\Delta} \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \xrightarrow{\psi \circ \xi} \mathrm{End}_B \circ \mathrm{End}_B \xrightarrow{\gamma_{\mathrm{End}_B}} \mathrm{End}_B.$$

(In general, this is not a *graded* associative algebra.) We denote this associative algebra by

$$\mathfrak{g}_B := \big(\mathrm{Hom}_{\mathbb{S}}\big(\mathscr{P}^{¡}, \mathrm{End}_B\big), \circledcirc\big).$$

Observe that the convolution product $\star$ is defined by the *infinitesimal* decomposition map $\Delta_{(1)}$, while the product $\circledcirc$ is defined by the *total* decomposition map Δ.

The composite of maps endows the $\mathbb{S}$-module End^A_B with a left module structure over the operad End_B:

$$\lambda : \mathrm{End}_B \circ \mathrm{End}^A_B \to \mathrm{End}^A_B,$$

and an infinitesimal right module structure over the operad End_A,

$$\rho : \mathrm{End}^A_B \circ_{(1)} \mathrm{End}_A \to \mathrm{End}^A_B.$$

They induce the following two actions on $\mathfrak{g}^A_B := \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}^A_B)$:

- ◇ for $\varphi \in \mathfrak{g}_A$ and $f \in \mathrm{End}^A_B$, we define $f * \varphi : \mathscr{P}^{¡} \to \mathrm{End}^A_B$ by

$$f * \varphi := \mathscr{P}^{¡} \xrightarrow{\Delta_{(1)}} \mathscr{P}^{¡} \circ_{(1)} \mathscr{P}^{¡} \xrightarrow{f \circ_{(1)} \varphi} \mathrm{End}^A_B \circ_{(1)} \mathrm{End}_A \xrightarrow{\rho} \mathrm{End}^A_B;$$

- ◇ for $\psi \in \mathfrak{g}_B$ and $f \in \mathrm{End}^A_B$, we define $\psi \circledast f : \mathscr{P}^{¡} \to \mathrm{End}^A_B$ by

$$\psi \circledast f := \mathscr{P}^{¡} \xrightarrow{\Delta} \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \xrightarrow{\psi \circ f} \mathrm{End}_B \circ \mathrm{End}^A_B \xrightarrow{\lambda} \mathrm{End}^A_B.$$

Proposition 10.2.2. *The module* $(\mathfrak{g}_B^A, *)$ *is a right module over the pre-Lie algebra* $(\mathfrak{g}_A, \star)$, *see Sect.* 1.4.4. *The module* $(\mathfrak{g}_B^A, \circledast)$ *is a left module over the associative algebra* $(\mathfrak{g}_B, \circledcirc)$.

Proof. The right action $*$ coincides with the pre-Lie subalgebra structure on $\mathfrak{g}_A \oplus \mathfrak{g}_B^A$ of the pre-Lie algebra $(\mathfrak{g}_{A\oplus B}, \star)$. In the same way, the left action $\circledast$ coincides with the associative subalgebra structure on $\mathfrak{g}_B \oplus \mathfrak{g}_B^A$ of the associative algebra $(\mathfrak{g}_{A\oplus B}, \circledcirc)$. ☐

Theorem 10.2.3. *Let* $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$ *and* $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_B)$ *be two* $\mathscr{P}_\infty$*-algebras. An* ∞*-morphism* $F : \mathscr{P}^{¡}(A) \to \mathscr{P}^{¡}(B)$ *of* $\mathscr{P}_\infty$*-algebras is equivalent to a morphism of dg* $\mathbb{S}$*-modules* $f : \mathscr{P}^{¡} \to \mathrm{End}_B^A$ *such that*

$$f * \varphi - \psi \circledast f = \partial(f)$$

in $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B^A)$:

$$\begin{array}{ccccc}
\mathscr{P}^{¡} & & \xrightarrow{\Delta} & & \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \\
\downarrow{\scriptstyle \Delta_{(1)}} & & \searrow{\scriptstyle \partial(f)} & & \downarrow{\scriptstyle \psi \circ f} \\
 & & & & \mathrm{End}_B \circ \mathrm{End}_B^A \\
 & & & & \downarrow{\scriptstyle \lambda} \\
\mathscr{P}^{¡} \circ_{(1)} \mathscr{P}^{¡} & \xrightarrow{f \circ_{(1)} \varphi} & \mathrm{End}_B^A \circ_{(1)} \mathrm{End}_A & \xrightarrow{\rho} & \mathrm{End}_B^A .
\end{array}$$

Proof. The morphism $F : \mathscr{P}^{¡}(A) \to \mathscr{P}^{¡}(B)$ of dg $\mathscr{P}^{¡}$-coalgebras commutes with the differentials d_φ and d_ψ if and only if $\mathrm{proj}_B(d_\psi \circ F - F \circ d_\varphi) = 0$. Using the explicit form of F given by the previous lemma, this relation is equivalent to the following commutative diagram

$$\begin{array}{ccccc}
\mathscr{P}^{¡}(A) & \xrightarrow{\Delta \circ \mathrm{Id}} & \mathscr{P}^{¡} \circ \mathscr{P}^{¡}(A) & \xrightarrow{\mathrm{Id} \circ \bar{f}} & \mathscr{P}^{¡}(B) \\
\downarrow{\scriptstyle d_\varphi} & & & & \downarrow{\scriptstyle d_B + \bar{\psi}} \\
\mathscr{P}^{¡}(A) & & \xrightarrow{\bar{f}} & & B .
\end{array}$$

We conclude with the explicit form of d_φ given in Proposition 6.3.8. ☐

Proposition 10.2.4. *Let* $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$, $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_B)$, *and* $\zeta \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_C)$ *be three* $\mathscr{P}_\infty$*-algebras. Let* $f \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B^A)$ *and* $g \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_C^B)$ *be two* ∞*-morphisms.*

Under the isomorphism between codifferentials on cofree $\mathscr{P}^{¡}$*-coalgebras and twisting morphisms from* $\mathscr{P}^{¡}$, *the composite of the two* ∞*-morphisms* f *and* g *is* *equal to*

$$g \circledcirc f := \mathscr{P}^{i} \xrightarrow{\Delta} \mathscr{P}^{i} \circ \mathscr{P}^{i} \xrightarrow{g \circ f} \mathrm{End}_C^B \circ \mathrm{End}_B^A \to \mathrm{End}_C^A,$$

where the last map is the natural composition of morphisms.

Proof. By the adjunction $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B^A) \cong \mathrm{Hom}(\mathscr{P}^{¡}(A), B)$, f is equivalent to a map $\bar{f} : \mathscr{P}^{¡}(A) \to B$. This latter one is equivalent to a morphism of dg $\mathscr{P}^{¡}$-coalgebras $F : \mathscr{P}^{¡}(A) \to \mathscr{P}^{¡}(B)$ by Proposition 10.2.1. Respectively, $g : \mathscr{P}^{¡} \to \mathrm{End}_C^B$ is equivalent to a morphism of dg $\mathscr{P}^{¡}$-coalgebras $G : \mathscr{P}^{¡}(B) \to \mathscr{P}^{¡}(C)$. By the formula given in Proposition 10.2.1, the projection of the composite $G \circ F$ onto the space of cogenerators C is equal to

$$\mathscr{P}^{¡}(A) \cong \mathscr{P}^{¡} \circ A \xrightarrow{\Delta \circ \mathrm{Id}_A} \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \circ A \xrightarrow{\mathrm{Id} \circ \bar{f}} \mathscr{P}^{¡}(B) \xrightarrow{\bar{g}} C.$$

We conclude by using the adjunction $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_C^A) \cong \mathrm{Hom}(\mathscr{P}^{¡}(A), C)$ once again. □

Since the cooperad $\mathscr{P}^{¡}$ is weight graded, any map $f \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B^A)$ decomposes according to this weight, $f_{(n)} : \mathscr{P}^{¡\,(n)} \to \mathrm{End}_B^A$. Since Δ preserves this weight, the square in the diagram of Theorem 10.2.3 applied to $\mathscr{P}^{¡\,(n)}$ involves only the maps $f_{(k)}$ up to $k = n-1$. Therefore, the term $f_{(n)}$ is a homotopy for the relation $f * \varphi - \psi \circledast f = \partial(f_{(n)})$ in $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡\,(n)}, \mathrm{End}_B^A)$.

The first component $f_{(0)} : \mathrm{I} \to \mathrm{Hom}(A, B)$ of an ∞-morphism is equivalent to a chain map $f_{(0)}(\mathrm{id}) : A \to B$ between the underlying chain complexes. In order to lighten the notation, we still denote this latter map by $f_{(0)}$.

10.2.4 Infinity-Isomorphism and Infinity-Quasi-isomorphism

An ∞-morphism f is called an ∞-*isomorphism* (resp. ∞-*quasi-isomorphism*) if its first component $f_{(0)} : A \to B$ is an isomorphism (resp. a quasi-isomorphism) of chain complexes. We will show later in Sect. 10.4.1 that ∞-isomorphisms are the isomorphisms of the category ∞-$\mathscr{P}_\infty$-alg.

10.2.5 Infinity-Morphisms and $\mathscr{P}$-Algebras

Proposition 10.2.5. *A morphism of $\mathscr{P}_\infty$-algebras is an ∞-morphism with only one nonvanishing component, namely the first one $f_{(0)} : A \to B$.*

Proof. Let $f : \mathscr{P}^{¡}(A) \to B$ be a morphism of dg modules such that $f_{(n)} = 0$ for $n \geq 1$. Since $\mathscr{P}^{(0)} = \mathrm{I}$, the first component $f_{(0)}$ of f is morphism of dg modules from A to B. In this particular case, the relation $\rho((f \circ_{(1)} \varphi)(\Delta_r)) - \lambda((\psi \circ f)(\Delta_l)) = \partial(f)$ applied to $\mathscr{P}^{¡\,(n)}$ for $n \geq 1$ is equivalent to $f_*(\varphi) = \psi(f^*)$. □

The category of $\mathscr{P}_\infty$-algebras with their morphisms forms a non-full subcategory of the category of $\mathscr{P}_\infty$-algebras with the ∞-morphisms.

One can also consider the category of $\mathscr{P}$-algebras with ∞-morphisms. It forms a full subcategory of ∞-$\mathscr{P}_\infty$-alg, which is denoted by ∞-$\mathscr{P}$-alg. Altogether these four categories assemble to form the following commutative diagram

$$\begin{array}{ccc} \mathscr{P}\text{-}\mathsf{alg} & \xrightarrow{\text{not full}} & \infty\text{-}\mathscr{P}\text{-}\mathsf{alg} \\ \Big\downarrow{\scriptstyle \text{f.f.}} & & \Big\downarrow{\scriptstyle \text{f.f.}} \\ \mathscr{P}_\infty\text{-}\mathsf{alg} & \xrightarrow{\text{not full}} & \infty\text{-}\mathscr{P}_\infty\text{-}\mathsf{alg}, \end{array}$$

where the vertical functors are full and faithful.

10.2.6 Infinity-Morphisms of A_∞-Algebras and L_∞-Algebras

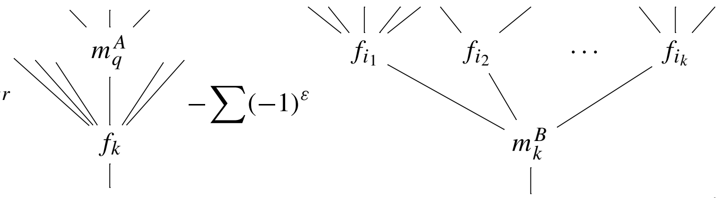
Proposition 10.2.6. *An ∞-morphism $f : A \rightsquigarrow B$ of A_∞-algebras is a family of maps $\{f_n : A^{\otimes n} \to B\}_{n\geq 1}$ of degree $n-1$ which satisfy: $d_B \circ f_1 = f_1 \circ d_A$, that is f_1 is a chain map, and for $n \geq 2$,*

$$\sum_{\substack{p+1+r=k\\ p+q+r=n}} (-1)^{p+qr} f_k \circ \big(\underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{p}, m_q^A, \underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{r}\big)$$
$$- \sum_{\substack{k\geq 2\\ i_1+\cdots+i_k=n}} (-1)^{\varepsilon} m_k^B \circ (f_{i_1}, \ldots, f_{i_k}) = \partial(f_n),$$

in $\mathrm{Hom}(A^{\otimes n}, B)$.

Under the tree representation, this relation becomes

$\partial(f_n)$

$= \sum (-1)^{p+qr}$  .

Proof. Let $\varphi \in \mathrm{Tw}(As^{¡}, \mathrm{End}_A)$ and $\psi \in \mathrm{Tw}(As^{¡}, \mathrm{End}_B)$ be two A_∞-algebra structures. Recall that $As^{¡}_n = \mathbb{K}\mu_n^c$ with $|\mu_n^c| = n-1$. We denote by $m_n^A \in \mathrm{Hom}(A^{\otimes n}, A)$ the image of μ_n^c under φ and by $m_n^B \in \mathrm{Hom}(B^{\otimes n}, B)$ the image of μ_n^c under ψ.

An ∞-morphism $f : As^{¡} \to \mathrm{End}_B^A$ between A and B is a family of maps $\{f_n : A^{\otimes n} \to B\}_{n\geq 1}$ of degree $n-1$. For $n \geq 2$, the formula of the infinitesimal decomposition map $\Delta_{(1)}$ of the cooperad $As^{¡}$ shows that the image of μ_n^c under $f * \varphi$ in End_B^A is equal to

$$(f * \varphi)\big(\mu_n^c\big) = \sum_{\substack{p+1+r=k\\ p+q+r=n}} (-1)^{p+qr} f_k \circ \big(\underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{p}, m_q^A, \underbrace{\mathrm{Id}_A, \ldots, \mathrm{Id}_A}_{r}\big).$$

On the other hand, the formula of the decomposition map Δ of the cooperad $As^{¡}$, given in Lemma 9.1.2, shows that the image of μ_n^c under $\psi \circledast f$ in End_B^A is equal to

$$(\psi \circledast f)\big(\mu_n^c\big) = \sum_{\substack{k\geq 2\\ i_1+\cdots+i_k=n}} (-1)^{\varepsilon} m_k^B \circ (f_{i_1}, \ldots, f_{i_k}),$$

where $\varepsilon = (k-1)(i_1-1)+(k-2)(i_2-1)+\cdots+2(i_{k-2}-1)+(i_{k-1}-1)$. Therefore we find the same formula as in Sect. 9.2.6. □

The case of L_∞-algebras is similar.

Proposition 10.2.7. *An ∞-morphism $f : A \rightsquigarrow B$ of L_∞-algebras, is a family of maps $\{f_n : \Lambda^n A \to B\}_{n\geq 1}$ of degree $n-1$ which satisfy: $d_A \circ f_1 = f_1 \circ d_A$, that is f_1 is a chain map, and for $n \geq 2$,*

$$\sum_{\substack{p+q=n+1\\ p,q>1}} \sum_{\sigma\in Sh_{p,q}^{-1}} \mathrm{sgn}(\sigma)(-1)^{(p-1)|q|}\big(f_p \circ_1 \ell_q^A\big)^\sigma$$
$$- \sum_{\substack{k\geq 2\\ i_1+\cdots+i_k=n}} \sum_{\sigma\in Sh_{(i_1,\ldots,i_k)}^{-1}} \mathrm{sgn}(\sigma)(-1)^{\varepsilon}\ell_k^B \circ (f_{i_1}, \ldots, f_{i_k})^\sigma = \partial(f_n),$$

in $\mathrm{Hom}(\Lambda^n A, B)$.

Proof. The proof relies on the explicit morphism of cooperads $Lie^{¡} \to Ass^{¡}$ given in the proof of Proposition 10.1.7. The results for A_∞-algebras transfer to L_∞ under this morphism. □

So far, we can see why L_∞-algebras are very close to A_∞-algebras: the Koszul dual cooperad $Lie^{¡}$ of Lie is the antisymmetrized version of $Ass^{¡}$.

10.3 Homotopy Transfer Theorem

In this section, we prove that a homotopy $\mathscr{P}$-algebra structure on a dg module induces a homotopy $\mathscr{P}$-algebra structure on any homotopy equivalent dg module, with explicit formulas. This structure is called "the" transferred $\mathscr{P}_\infty$-algebra structure. We make the examples of A_∞-algebras and L_∞-algebras explicit.

When A is a $\mathscr{P}_\infty$-algebra, we have seen in Proposition 10.1.5 that its homotopy $H(A)$ carries a natural $\mathscr{P}$-algebra structure. When working over a field, the homotopy $H(A)$ can be made into a deformation retract of A. It enables us to transfer the $\mathscr{P}_\infty$-algebra structure from A to $H(A)$. These higher operations, called the operadic Massey products, extend the $\mathscr{P}$-algebra structure of $H(A)$. They contain the full homotopy data of A, since this $\mathscr{P}_\infty$-algebra $H(A)$ is homotopy equivalent to A.

A meaningful example is given by applying the Homotopy Transfer Theorem to $\mathscr{P} = D$, the algebra of dual numbers on one generator. In this case, a D-algebra A is a bicomplex and the transferred structured on $H(A)$ corresponds to the associated spectral sequence.

Recall that the particular case $\mathscr{P} = As$ has been treated independently in Sect. 9.4. It serves as a paradigm for the general theory developed here.

The Homotopy Transfer Theorem for A_∞-algebras and L_∞-algebras has a long history in mathematics, often related to the Perturbation Lemma. We refer the reader to the survey of Jim Stasheff [Sta10] and references therein. Its extension to the general operadic setting has been studied in the PhD thesis of Charles Rezk [Rez96]. One can find in the paper [Bat98] of Michael Batanin the case of algebras over non-symmetric simplicial operads. A version of HTT was recently proved for algebras over the bar–cobar construction $\Omega \mathrm{B}\mathscr{P}$ by Joseph Chuang and Andrey Lazarev in [CL10] and by Sergei Merkulov in [Mer10a]. Using a generalization of the Perturbation Lemma, it was proved for $\mathscr{P}_\infty$-algebras by Alexander Berglund in [Ber09]. The existence part of the theorem can also be proved by model category arguments, see Clemens Berger and Ieke Moerdijk [BM03a] and Benoit Fresse [Fre09b].

10.3.1 The Homotopy Transfer Problem

Let (V, d_V) be a homotopy retract of (W, d_W):

$$h \circlearrowright (W, d_W) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V),$$

$$\mathrm{Id}_W - ip = d_W h + h d_W,$$

where the chain map i is a quasi-isomorphism.

The transfer problem is the following one: given a structure of $\mathscr{P}_\infty$-algebra on W, does there exist a $\mathscr{P}_\infty$-algebra structure on V such that i extends to an ∞-quasi-isomorphism of $\mathscr{P}_\infty$-algebras? We will show that this is always possible and we say that the $\mathscr{P}_\infty$-algebra structure of W has been transferred to V.

Theorem 10.3.1 (Homotopy Transfer Theorem). *Let $\mathscr{P}$ be a Koszul operad and let (V, d_V) be a homotopy retract of (W, d_W). Any $\mathscr{P}_\infty$-algebra structure on W can be transferred into a $\mathscr{P}_\infty$-algebra structure on V such that i extends to an ∞-quasi-isomorphism.*

Proof. To prove this theorem, we use the third definition of a $\mathscr{P}_\infty$-algebra given in the Rosetta Stone (Theorem 10.1.13):

$$\mathrm{Hom}_{\mathsf{dg\ Op}}\big(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big) \cong \mathrm{Tw}\big(\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big) \cong \mathrm{Hom}_{\mathsf{dg\ Coop}}\big(\mathscr{P}^{\text{¡}}, \mathrm{B}\,\mathrm{End}_A\big).$$

The plan of the proof is the following one. First, we show in Proposition 10.3.2 that the homotopy retract data between V and W induces a morphism of dg cooperads $\mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$. Since a $\mathscr{P}_\infty$-algebra structure on W is equivalently given by a morphism of dg cooperads $\mathscr{P}^{\text{¡}} \to \mathrm{B}\,\mathrm{End}_W$, the composite

$$\mathscr{P}^{\text{¡}} \to \mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$$

defines a $\mathscr{P}_\infty$-algebra structure on V.

We give an explicit formula for this transferred structure in Theorem 10.3.3. The extension of i into an ∞-quasi-isomorphism is provided through an explicit formula in Theorem 10.3.6. □

10.3.2 The Morphism of DG Cooperads $\mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$

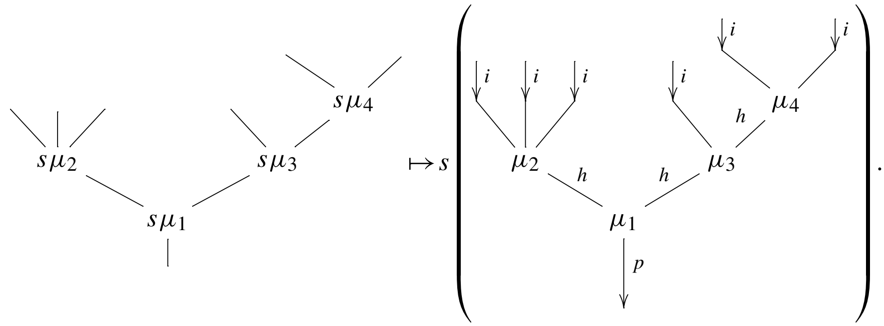
Let (V, d_V) be a homotopy retract of (W, d_W). We consider the map defined by $\mu \in \mathrm{End}_W(n) \mapsto p\mu i^{\otimes n} \in \mathrm{End}_V(n)$. Since i and p are morphisms of dg modules, this map is a morphism of dg $\mathbb{S}$-modules. But it does not commute with the operadic composition maps in general. For $\mu_1 \in \mathrm{End}_W(k)$ and $\mu_2 \in \mathrm{End}_W(l)$, with $k + l - 1 = n$, and for $1 \le j \le k$, we have

$$\big(p\mu_1 i^{\otimes k}\big) \circ_j \big(p\mu_2 i^{\otimes l}\big) = p\big(\mu_1 \circ_j (ip\mu_2)\big) i^{\otimes n},$$

which is not equal to $p(\mu_1 \circ_j \mu_2) i^{\otimes n}$ because ip is not equal to Id_W. Since ip is homotopic to Id_W, we will show that this morphism commutes with the operadic composition maps only up to homotopy. In the previous example, we have to consider the homotopy $\mu_1 \circ_j (h\mu_2)$ to get

$$p\big(\mu_1 \circ_j \big(\partial(h)\mu_2\big)\big) i^{\otimes n} = p(\mu_1 \circ_j \mu_2) i^{\otimes n} - p\big(\mu_1 \circ_j (ip\mu_2)\big) i^{\otimes n}.$$

Therefore the idea for defining the morphism of dg cooperads $\Psi : \mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$ is to label the internal edges by the homotopy h as follows. A basis of $\mathscr{T}^c(s\,\mathrm{End}_W)$ is given by trees labeled by elements of $s\,\mathrm{End}_W$. Let $t := t(s\mu_1, \ldots, s\mu_k)$ be such a tree, where the vertices $1, \ldots, k$ are read for bottom to top and from left to right. The image of $\mathscr{T}$ under Ψ is defined by the suspension of the following composite: we label every leaf of the tree $t(\mu_1,\ldots,\mu_k)$ with $i : V \to W$, every internal edge by h and the root by p.



This composite scheme defines a map in $s\,\mathrm{End}_V$. Since $\mathscr{T}^c(s\,\mathrm{End}_V)$ is a conilpotent cofree cooperad, this map $\mathscr{T}^c(s\,\mathrm{End}_W) \to s\,\mathrm{End}_V$ extends to a unique morphism of cooperads $\Psi : \mathscr{T}^c(s\,\mathrm{End}_W) \to \mathscr{T}^c(s\,\mathrm{End}_V)$. Since the degree of h is $+1$, the degree of Ψ is 0. The next result states that this morphism of cooperads commutes with the differentials.

Proposition 10.3.2. [vdL03] *Let (V, d_V) be a homotopy retract of (W, d_W). The map $\Psi : \mathrm{B\,End}_W \to \mathrm{B\,End}_V$, defined above, is a morphism of dg cooperads.*

Proof. In this proof, by a slight abuse of notation, we denote the above defined map $\mathscr{T}^c(s\,\mathrm{End}_W) \to s\,\mathrm{End}_V$ by Ψ. By Proposition 10.5.3, we have to check that

$$(\xi): \quad \partial(\Psi) - \Psi\,\mathscr{T}^c(\mathrm{Id};\gamma_{End_W})\Delta' + \gamma_{End_V}\,\mathscr{T}^c(\Psi)\bar{\Delta} = 0,$$

where the map

$$\gamma_{End_V} : \mathscr{T}^c(s\mathrm{End}_V) \twoheadrightarrow \mathscr{T}^c(s\mathrm{End}_V)^{(2)} \to s\mathrm{End}_V$$

(respectively γ_{End_W}) is given by the partial compositions of the operad End_V (respectively End_W), see Sect. 10.5.1. So it vanishes on $\mathscr{T}^c(s\,\mathrm{End}_V)^{(\geq 3)}$ (respectively on $\mathscr{T}^c(s\,\mathrm{End}_W)^{(\geq 3)}$). We apply Equation (ξ) to a tree $t = t(s\mu_1,\ldots,s\mu_k)$.

1. The first term $\partial(\Psi)(t)$ is equal to the sum over the internal edges e of t of trees $st(\mu_1,\ldots,\mu_k)$, where every internal edge is labeled by h except e, which is labeled by $\partial(h) = d_W h + h d_W$.
2. In the second term, one singles out a subtree with two vertices out of t, composes it in End_W and then one applies Ψ to the resulting tree. Therefore it is equal to the sum over the internal edges e of t of trees $st(\mu_1,\ldots,\mu_k)$, where every internal edge is labeled by h except e, which is labeled by Id_W.
3. The third term consists in splitting the tree t into two parts, applying Ψ to the two induced subtrees and then composing the two resulting images in End_V. Hence it is equal to the sum over all internal edges e of t of trees $st(\mu_1,\ldots,\mu_k)$, where every internal edge is labeled by h except e, which is labeled by ip.

Finally, Equation (ξ) applied to the tree t is equal to the sum, over all internal edges e of t, of trees $st(\mu_1, \ldots, \mu_k)$, where every internal edge is labeled by h except e, which is labeled by

$$d_W h + h d_W - \mathrm{Id}_W + ip = 0.$$

It concludes the proof. □

10.3.3 Transferred Structure

Let $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_W)$ be a $\mathscr{P}_\infty$-algebra structure on W. We define a transferred structure of $\mathscr{P}_\infty$-algebra $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_V)$ on V as follows.

By Sect. 6.5.4, the twisting morphism φ is equivalent to a morphism of dg cooperads $f_\varphi : \mathscr{P}^{¡} \to \mathrm{B}\,\mathrm{End}_W$. We compose it with the morphism of dg cooperads $\Psi : \mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$. The resulting composite Ψf_φ is a morphism of dg cooperads, which gives the expected twisting morphism $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_V)$ by Sect. 6.5.4 again.

$$\begin{array}{ccc} \mathscr{P}^{¡} & \xrightarrow{f_\varphi} & \mathrm{B}\,\mathrm{End}_W \\ & {\scriptstyle f_\psi} \searrow & \downarrow {\scriptstyle \Psi} \\ & & \mathrm{B}\,\mathrm{End}_V\,. \end{array}$$

The associated twisting morphism $\psi : \mathscr{P}^{¡} \to \mathrm{End}_V$ is equal to the projection of Ψf_φ on End_V. By a slight abuse of notation, we still denote it by

$$\psi = \Psi f_\varphi : \mathscr{P}^{¡} \to \mathrm{End}_V.$$

Theorem 10.3.3 (Explicit formula). *Let $\mathscr{P}$ be a Koszul operad, let $\varphi \in \mathrm{Tw}(\mathscr{P}^{i}, \mathrm{End}_W)$ be a $\mathscr{P}_\infty$-algebra structure on W, and let (V, d_V) be a homotopy retract of (W, d_W).*

The transferred $\mathscr{P}_\infty$-algebra structure $\psi \in \mathrm{Tw}(\mathscr{P}^{i}, \mathrm{End}_V)$, defined above, on the dg module V is equal to the composite

$$\mathscr{P}^{i} \xrightarrow{\Delta_{\mathscr{P}^{i}}} \mathscr{T}^c\big(\overline{\mathscr{P}}^{i}\big) \xrightarrow{\mathscr{T}^c(s\varphi)} \mathscr{T}^c(s\mathrm{End}_W) \xrightarrow{\Psi} \mathrm{End}_V,$$

where $\Delta_{\overline{\mathscr{P}}^{i}}$ is the structure map corresponding to the combinatorial definition of the cooperad $\mathscr{P}^{i}$, see Sect. 5.8.8.

Proof. By Proposition 5.8.6, the unique morphism of dg cooperads $f_\varphi : \mathscr{P}^{¡} \to \mathrm{B}\,\mathrm{End}_W = \mathscr{T}^c(s\,\mathrm{End}_W)$, which extends $s\varphi : \mathscr{P}^{¡} \to s\,\mathrm{End}_W$, is equal to

$$\mathscr{P}^{¡} \xrightarrow{\Delta_{\mathscr{P}^{¡}}} \mathscr{T}^c\big(\overline{\mathscr{P}}^{¡}\big) \xrightarrow{\mathscr{T}^c(s\varphi)} \mathscr{T}^c(s\mathrm{End}_W).$$ □

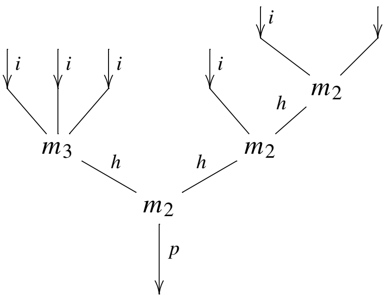
So the transferred structure given here is the composite of three distinct maps. The first map depends only on the cooperad $\mathscr{P}^{¡}$, that is on the type of algebraic structure we want to transfer. The second map depends only the starting $\mathscr{P}_\infty$-algebra structure. And the third map depends only on the homotopy retract data.

10.3.4 Examples: A_∞ and L_∞-Algebras Transferred

In the case of A_∞-algebras, we recover the formulas given in Sect. 9.4.

Theorem 10.3.4. *Let $\{m_n : W^{\otimes n} \to W\}_{n\geq 2}$ be an A_∞-algebra structure on W. The transferred A_∞-algebra structure $\{m'_n : V^{\otimes n} \to V\}_{n\geq 2}$ on a homotopy retract V is equal to*

$$m'_n = \sum_{PT_n} \pm$$



,

where the sum runs over the set PT_n of planar rooted trees with n leaves.

Proof. The combinatorial definition of the (nonsymmetric) cooperad $As^{¡}$ is given by

$$\Delta_{As^{¡}} : \mu_n^c \mapsto \sum_{t\in \mathscr{P}\mathscr{T}_n} \pm t\left(\mu^c\right) \in \mathscr{T}^c\left(\overline{As}^{¡}\right),$$

where the sum runs over planar rooted trees t with n leaves and whose vertices with k inputs are labeled by μ_k^c. We conclude with Theorem 10.3.3. □

Theorem 10.3.5. *Let $\{\ell_n : W^{\otimes n} \to W\}_{n\geq 2}$ be an L_∞-algebra structure on W. The transferred L_∞-algebra structure $\{l_n : V^{\otimes n} \to V\}_{n\geq 2}$ on a homotopy retract V is equal to*

$$l_n = \sum_{t\in RT_n} \pm p t(\ell, h) i^{\otimes n},$$

where the sum runs over rooted trees t with n leaves and where the notation $t(\ell, h)$ stands for the n-multilinear operation on V defined by the composition scheme t with vertices labeled by the ℓ_k and internal edges labeled by h.

Proof. By Sect. 7.2.3, the Koszul dual cooperad $\mathit{Lie}^{¡}$ is isomorphic to $\mathrm{End}^c_{s^{-1}\mathbb{K}} \underset{\mathrm{H}}{\otimes} \mathit{Com}^*$. Therefore, the decomposition map of the combinatorial definition of the cooperad $\mathit{Lie}^{¡}$ is given, up to signs, by the one of Com^*, which is made up of nonplanar rooted trees. □

10.3.5 Infinity-Quasi-isomorphism

We define a map $\widetilde{\Psi} : \mathcal{T}^c(s\mathrm{End}_W) \to \mathrm{End}^V_W$ by the same formula as Ψ except for the root, which is labeled by the homotopy h and not by p this time. We consider the map $i_\infty : \mathcal{P}^{¡} \to \mathrm{End}^V_W$ defined by the following composite:

$$i_\infty : \overline{\mathcal{P}^{¡}} \xrightarrow{\Delta_{\mathcal{P}^{¡}}} \mathcal{T}\big(\overline{\mathcal{P}^{¡}}\big) \xrightarrow{\mathcal{T}(s\varphi)} \mathcal{T}(s\mathrm{End}_W) \xrightarrow{\widetilde{\Psi}} \mathrm{End}^V_W$$

and by $\mathrm{id} \in \mathrm{I} \mapsto i \in \mathrm{Hom}(V, W) \subset \mathrm{End}^V_W$.

Theorem 10.3.6. *Let $\mathcal{P}$ be a Koszul operad, let (W, φ) be a $\mathcal{P}_\infty$-algebra, and let (V, d_V) be a homotopy retract of (W, d_W).*

The map $i_\infty : \mathcal{P}^{i} \to \mathrm{End}^V_W$ is an ∞-quasi-isomorphism between the $\mathcal{P}_\infty$-algebra (V, ψ), with the transferred structure, and the $\mathcal{P}_\infty$-algebra (W, φ).

Proof. Using Theorem 10.2.3, we have to prove that $i_\infty * \psi - \varphi \circledast i_\infty = \partial(i_\infty)$.

The first term $i_\infty * \psi$ is equal to

$$\big(\big(\widetilde{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}\big) \circ_{(1)} \big(\Psi\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}\big)\big)\Delta_{(1)} + i_*\Psi\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}.$$

It is equal to the composite $\widehat{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}$, where $\widehat{\Psi}$ is defined as $\widetilde{\Psi}$, except that either one internal edge or the root is labeled by ip instead of h. To prove this, we use the formula of $\Delta_{\mathcal{P}^{¡}}$ given in Sect. 5.8 in terms of the iterations of $\tilde{\Delta}$.

The second term $-\varphi \circledast i_\infty$ is equal to $-(\varphi \circ (\widetilde{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}))\Delta$. It is equal to $-\breve{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}}$, where $\breve{\Psi}$ is defined as Ψ, except for the root, which is labeled by the identity of W.

The right-hand side $\partial(i_\infty)$ is equal to $d_{\mathrm{End}^V_W}\, i_\infty - i_\infty\, d_{\mathcal{P}^{¡}}$. The latter term $i_\infty\, d_{\mathcal{P}^{¡}}$ is equal to $\widetilde{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}} d_{\mathcal{P}^{¡}}$. Since $d_{\mathcal{P}^{¡}}$ is a coderivation of the cooperad $\mathcal{P}^{¡}$, we get $i_\infty\, d_{\mathcal{P}^{¡}} = \widetilde{\Psi}\,\mathcal{T}^c(s\varphi; s\varphi\, d_{\mathcal{P}^{¡}})\Delta_{\mathcal{P}^{¡}}$, where the notation $\mathcal{T}^c(f; g)$ was introduced in Sect. 6.3.2. The other term $d_{\mathrm{End}^V_W}\, i_\infty$ is equal to

$$-\widetilde{\Psi}\,\mathcal{T}^c(s\varphi; sd_{\mathrm{End}_W}\varphi)\Delta_{\mathcal{P}^{¡}} - \mathring{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}} + i_*\Psi\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}} - \breve{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}},$$

where $\mathring{\Psi}$ is defined as $\widetilde{\Psi}$, except that one internal edge is labeled by $[d_W, h]$ instead of h. Since φ is a twisting morphism, $\varphi \in \mathrm{Tw}(\mathcal{P}^{¡}, \mathrm{End}_W)$, it satisfies the Maurer–Cartan equation $-d_{\mathrm{End}_W}\varphi - \varphi\, d_{\mathcal{P}^{¡}} = (\varphi \circ_{(1)} \varphi)\Delta_{(1)}$. Therefore

$$-\widetilde{\Psi}\,\mathcal{T}^c(s\varphi; sd_{\mathrm{End}_W}\varphi) - \widetilde{\Psi}\,\mathcal{T}^c(s\varphi; s\varphi\, d_{\mathcal{P}^{¡}}) = \overline{\Psi}\,\mathcal{T}^c(s\varphi)\Delta_{\mathcal{P}^{¡}},$$

where $\overline{\Psi}$ is defined as $\widetilde{\Psi}$ except that one internal edge is labeled by the identity of W instead of h.

We conclude by using $[d_W, h] = \mathrm{Id}_W - ip$. □

This theorem provides a homotopy control of the transferred structure: the starting $\mathscr{P}_\infty$-algebra and the transferred one are related by an explicit ∞-quasi-isomorphism. Therefore the two $\mathscr{P}_\infty$-algebras are homotopy equivalent, see Sect. 10.4.4.

Theorem 10.3.7. *Let $\mathscr{P}$ be a Koszul operad and let $i : V \xrightarrow{\sim} W$ be a quasi-isomorphism. Any $\mathscr{P}_\infty$-algebra structure on W can be transferred into a $\mathscr{P}_\infty$-algebra structure on V such that i extends to an ∞-quasi-isomorphism.*

Proof. Since we work over a field, any quasi-isomorphism $i : V \xrightarrow{\sim} W$ extends to a homotopy retract

$$h \circlearrowright (W, d_W) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V).$$

One shows this fact by refining the arguments of Lemma 9.4.4. So this result is equivalent to Theorem 10.3.1. □

10.3.6 Operadic Massey Products

In this section, we suppose the characteristic of the ground field $\mathbb{K}$ to be 0. Let (A, d) be a chain complex. Recall from Lemma 9.4.4, that, under a choice of sections, the homology $(H(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} \big(H(A), 0\big).$$

Lemma 10.3.8. *By construction, these maps also satisfy the following* side conditions:

$$h^2 = 0, \qquad ph = 0, \qquad hi = 0.$$

Proof. It is a straightforward consequence of the proof of Lemma 9.4.4. □

As a consequence, when A carries a $\mathscr{P}_\infty$-algebra structure, its homotopy $H(A)$ is endowed with a $\mathscr{P}_\infty$-algebra structure, such that the map i extends to an ∞-quasi-isomorphism, by Theorem 10.3.1. In this case, we can prove the same result for the map p as follows.

To the homotopy h relating Id_A and ip, we associate the degree one map $h^n : A^{\otimes n} \to A^{\otimes n}$ defined by

$$h^n := \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} h^\sigma,$$

where

$$\begin{aligned} h^\sigma := &\ \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \underbrace{h}_{\sigma(1)} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \\ &+ \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \underbrace{ip}_{\sigma(1)} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \underbrace{h}_{\sigma(2)} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \cdots \\ &+ i \circ p \otimes \cdots \otimes ip \otimes \underbrace{h}_{\sigma(n)} \otimes ip \otimes \cdots \otimes ip. \end{aligned}$$

The map h^n is a symmetric homotopy relating $\mathrm{Id}_A^{\otimes n}$ and $(ip)^{\otimes n}$, that is

$$\partial(h^n) = \mathrm{Id}_A^{\otimes n} - (ip)^{\otimes n} \quad \text{and} \quad h^n \sigma = \sigma h^n, \quad \forall \sigma \in \mathbb{S}_n.$$

We denote by H the sum $\mathrm{H} := \sum_{n \geq 1} h^n : \overline{T}(A) \to \overline{T}(A)$.

We define the map Δ^{lev} as follows. To any element $\mu^c \in \mathscr{P}^{¡}$, its image under $\Delta_{\mathscr{P}^{¡}}$ is a sum of trees. To any of these trees, we associate the sum of all the leveled trees obtained by putting one and only one nontrivial vertex per level. (Notice that this operation might permute vertices and therefore it might yield signs.) The image of μ^c under Δ^{lev} is the sum of all these leveled trees.

Proposition 10.3.9. *Let $\mathbb{K}$ be a field of characteristic 0. Let $\mathscr{P}$ be a Koszul operad and let (A, φ) be a $\mathscr{P}_\infty$-algebra. The chain map $p : A \to H(A)$ extends to an ∞-quasi-isomorphism p_∞ given by the formula*

$$p_\infty := p_* \mathscr{T}^c_{\mathrm{lev}}(\varphi, H) \Delta^{\mathrm{lev}}$$

on $\overline{\mathscr{P}^{i}}$ and by $\mathrm{id} \in \mathscr{P}^{i} \mapsto p$. The map $\mathscr{T}^c_{\mathrm{lev}}(\varphi, H)$ first labels the vertices of a leveled tree by φ and the levels (including the leaves) by H and then composes the associated maps in $\mathrm{End}^A_{H(A)}$.

Proof. The map p_∞ given by this formula is well defined thanks to the conilpotency of the Koszul dual cooperad $\mathscr{P}^{¡}$. Let us denote by ψ the transferred $\mathscr{P}_\infty$-algebra structure on $H(A)$. By Theorem 10.2.3, we have to prove that $p_\infty * \varphi - \psi \circledast p_\infty = \partial(p_\infty)$. The arguments are similar to the arguments used in the proofs of Theorem 10.3.6 and Theorem 10.4.1 but use the side conditions of Lemma 10.3.8. The computations are left to the reader as a good exercise. □

Theorem 10.3.10 (Higher structures). *Let $\mathbb{K}$ be a field of characteristic 0. Let $\mathscr{P}$ be a Koszul operad and let A be a $\mathscr{P}_\infty$-algebra.*

◇ *There is a $\mathscr{P}_\infty$-algebra structure on the homology $H(A)$ of the underlying chain complex of A, which extends its $\mathscr{P}$-algebra structure.*
◇ *The embedding $i : H(A) \rightarrowtail A$ and the projection $p : A \twoheadrightarrow H(A)$, associated to the choice of sections for the homology, extend to ∞-quasi-isomorphisms of $\mathscr{P}_\infty$-algebras.*
◇ *The $\mathscr{P}_\infty$-algebra structure on the homotopy $H(A)$ is independent of the choice of sections of $H(A)$ into A in the following sense: any two such transferred structures are related by an ∞-isomorphism, whose first map is the identity on $H(A)$.*

Proof. The explicit form of the transferred $\mathscr{P}_\infty$-algebra structure on $H(A)$, given in Theorem 10.3.3, proves that it extends the $\mathscr{P}$-algebra structure given in Proposition 10.1.5.

The embedding $H(A) \rightarrowtail A$ extends to an ∞-quasi-isomorphism by Theorem 10.3.6. The projection $A \twoheadrightarrow H(A)$ extends to an ∞-quasi-isomorphism by Proposition 10.3.9.

Let (i, p) and (i', p') be the maps associated to two decompositions of the chain complex A. They induce two $\mathscr{P}_\infty$-algebra structures on $H(A)$ such that i, i', p and p' extend to ∞-quasi-isomorphisms by Theorem 10.3.6 and Proposition 10.3.9. The composite $p'_\infty i_\infty$ defines an ∞-quasi-isomorphism, from $H(A)$ with the first transferred structure to $H(A)$ with the second transferred structure, such that the first component is equal to $p'i = \mathrm{Id}_{H(A)}$. □

These higher $\mathscr{P}_\infty$-operations on the homotopy of a $\mathscr{P}_\infty$-algebra are called the *operadic Massey products*.

EXAMPLES. The case of the operad *As* has already been treated in Sect. 9.4.5. The terminology "Massey products" comes from the example given by the singular cochains $C^\bullet_{\mathrm{sing}}(X)$ of a topological space X endowed with its associative cup product [Mas58]. The case of the operad *Lie* was treated by Retakh in [Ret93].

Though the differential on $H(A)$ is equal to 0, the $\mathscr{P}_\infty$-algebra structure on $H(A)$ is *not* trivial in general. In this case, the relations satisfied by the $\mathscr{P}_\infty$-algebra operations on $H(A)$ do not involve any differential. Hence the operations of weight 1 satisfy the relations of a $\mathscr{P}$-algebra. But the higher operations exist and contain the homotopy data of A.

10.3.7 An Example: HTT for the Dual Numbers Algebra

The homotopy transfer theorem (HTT) can be applied to reduced operads which are concentrated in arity 1, that is to unital associative algebras. Recall that for such an operad, an algebra over it is simply a left module. The algebra of dual numbers $D := \mathbb{K}[\varepsilon]/(\varepsilon^2 = 0)$ is obviously Koszul and its Koszul dual coalgebra is the free coalgebra on one cogenerator $D^{¡} := \mathbb{K}[s\varepsilon]$. Observe that the element $(s\varepsilon)^n$ is in degree n and that the coproduct is given by

$$\Delta\big((s\varepsilon)^n\big) = \sum_{\substack{i+j=n\\ i\geq 0, j\geq 0}} (s\varepsilon)^i (s\varepsilon)^j .$$

From Sect. 2.2.2 we can compute $D_\infty := \Omega D^{¡}$. It follows that a D_∞-module is a chain complex (A, d) equipped with linear maps

$$t_n : A \to A, \quad \text{for } n \geq 1, \quad |t_n| = n-1,$$

such that for any $n \geq 1$ the following identities hold

$$\partial(t_n) = -\sum_{\substack{i+j=n\\ i\geq 1, j\geq 1}} (-1)^i t_i t_j .$$

Observe that, denoting $t_0 := d$, this identity becomes

$$\sum_{\substack{i+j=n\\ i\geq 0, j\geq 0}} (-1)^i t_i t_j = 0, \quad \text{for any } n \geq 0.$$

Such a structure $(A, t_0, \ldots, t_n, \ldots)$ is called a *chain multicomplex*. As expected a D-module is a particular case of chain multicomplex for which $t_n = 0$ for $n \geq 2$.

The HTT can be written for any homotopy retract whose big chain complex is a chain multicomplex $(A, \{t_n\}_{n\geq 0})$ and it gives a chain multicomplex structure on the small chain complex $(V, \{t'_n\}_{n\geq 0})$. The explicit formulas are as follows:

$$t'_n := p\Big(\sum \pm t_{j_1} h t_{j_2} h \cdots h t_{j_k}\Big) i,$$

for any $n \geq 1$, where the sum runs over all the families $(j_1, \ldots, j_k)$ such that $j_1 + \cdots + j_k = n$.

SPECTRAL SEQUENCE. Let us look at a particular case. Any first quadrant bicomplex $(C_{\bullet,\bullet}, d^v, d^h)$ gives rise to a chain complex (A, d), which is a left module over D by declaring that $A_n := \bigoplus_p C_{p,n}$, $d := d^v$ and the action of ε is induced by d^h. More precisely, since $d^h d^v + d^h d^v = 0$, the restriction of ε to A_n is $(-1)^n d^h$.

It is well known that any first quadrant bicomplex gives rise to a spectral sequence $\{(E^n, d^n)\}_{n\geq 1}$ where $E^1 = H_\bullet(C, d^v)$ and $E^n = H_\bullet(E^{n-1}, d^{n-1})$. We claim that, after choosing sections which make $(E^1, 0)$ into a deformation retract of (C, d^v), cf. Lemma 9.4.4, the chain multicomplex structure of E^1 gives the spectral sequence. More precisely the map d^n is induced by t'_n.

The advantage of this point of view on bicomplexes, versus spectral sequences, is that the HTT can be applied to bicomplexes equipped with a deformation retract whose boundary map is not necessarily trivial.

For instance the cyclic bicomplex of a unital associative algebra, which involves the boundary maps b, b' and the cyclic operator, cf. [LQ84, Lod98], admits a deformation retract made up of the columns involving only b. Applying the HTT to it gives a chain multicomplex for which

$$t'_0 = b, \qquad t'_1 = 0, \qquad t'_2 = B, \qquad t'_n = 0, \quad \text{for } n \geq 3.$$

So, we get automatically Connes' boundary map B and we recover the fact that, in cyclic homology theory, the (b, B)-bicomplex is quasi-isomorphic to the cyclic bicomplex.

The details for this section can be found in [LV12], where direct explicit proofs are given. Similar results based simplicial technics can be found in [Mey78] and based on the perturbation lemma in [Lap01].

10.4 Inverse of ∞-Isomorphisms and ∞-Quasi-isomorphisms

In this section, we first prove that the ∞-isomorphisms are the invertible ∞-morphisms in the category ∞-$\mathscr{P}_\infty$-alg. We give the formula for the inverse of an ∞-morphism. Then we show that any $\mathscr{P}_\infty$-algebra is ∞-isomorphic to the product of a $\mathscr{P}_\infty$-algebra whose internal differential is null, with a $\mathscr{P}_\infty$-algebra whose structure operations are null and whose underlying chain complex is acyclic. Applying these two results, we prove that any ∞-quasi-isomorphism admits an ∞-quasi-isomorphism in the opposite direction. So, being ∞-quasi-isomorphic defines an equivalence relation among $\mathscr{P}_\infty$-algebras, which is called the homotopy equivalence.

10.4.1 Inverse of Infinity-Isomorphisms

Here we find the formula for the inverse of an ∞-isomorphism. We use the maps $\hat{\Delta}^k : \mathscr{P}^{\text{i}} \to (\mathscr{P}^{\text{i}})^{\circ(k+1)}$ introduced in Sect. 5.8.5.

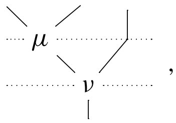
Theorem 10.4.1. *Let $\mathscr{P}$ be a Koszul operad and let A and B be two $\mathscr{P}_\infty$-algebras. Any ∞-isomorphism $f : A \rightsquigarrow B$ admits a unique inverse in the category* ∞-$\mathscr{P}_\infty$-alg. *When f is expressed in terms of $f : \mathscr{P}^{i} \to \mathrm{End}^A_B$, its inverse is given by the formula $(f^{-1})_{(0)} := (f_{(0)})^{-1} : B \to A$ and by*

$$f^{-1} := \sum_{k=0}^{\infty} (-1)^{k+1} \big(f_{(0)}^{-1}\big)_* \big(\big(f_{(0)}^{-1}\big)^*(f)\big)^{\circ(k+1)} \hat{\Delta}^k,$$

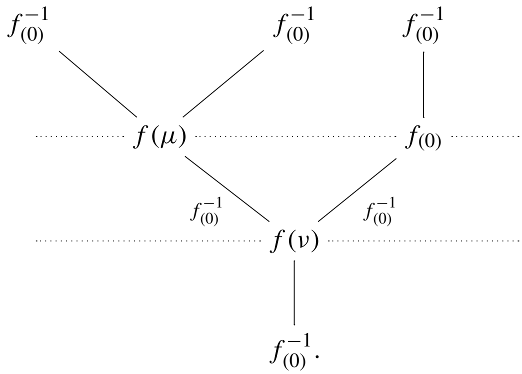
on $\overline{\mathscr{P}^{i}}$, where the right-hand side is equal to the composite

$$\mathscr{P}^{i} \xrightarrow{\hat{\Delta}^k} \big(\mathscr{P}^{i}\big)^{\circ(k+1)} \xrightarrow{((f_{(0)}^{-1})^*(f))^{\circ(k+1)}} (\mathrm{End}_B)^{\circ(k+1)} \to \mathrm{End}_B \xrightarrow{(f_{(0)}^{-1})_*} \mathrm{End}^B_A.$$

For example, when $\hat{\Delta}$ produces an element of the form



the associated composite in End_A^B is



Proof. Let us denote by $g : \mathscr{P}^{¡} \to \mathrm{End}_A^B$ the above defined map. It is well defined thanks to the conilpotency (Sect. 5.8.5) of the Koszul dual cooperad $\mathscr{P}^{¡}$.

We first show that g is an ∞-morphism. Let us denote by $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$ and by $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_B)$ the respective $\mathscr{P}_\infty$-algebra structures. By Theorem 10.2.3, we prove now that $g * \psi - \varphi \circledast g = \partial(g)$ in $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_A^B)$. By Proposition 5.8.6, the map g is equal to the composite

$$\mathscr{P}^{¡} \xrightarrow{\Delta_{\mathscr{P}^{¡}}} \mathscr{T}^c\big(\overline{\mathscr{P}^{¡}}\big) \xrightarrow{\mathscr{T}^c(f)} \mathscr{T}^c\big(\mathrm{End}_B^A\big) \xrightarrow{\Theta} \mathrm{End}_A^B,$$

where the map Θ amounts to labeling the leaves, the internal edges and the root of the trees by f_0^{-1} and to composing all the maps along the tree scheme. It also multiplies the elements by $(-1)^k$, where k is the minimal number of levels on which the tree can be put.

The derivative $\partial(g)$ is equal to

$$\partial(g) = d_{\mathrm{End}_A^B}\, \Theta\, \mathscr{T}^c(f) \Delta_{\mathscr{P}^{¡}} - \Theta\, \mathscr{T}^c(f) \Delta_{\mathscr{P}^{¡}}\, d_{\mathscr{P}^{¡}}.$$

Since $d_{\mathscr{P}^{¡}}$ is a coderivation for the cooperad $\mathscr{P}^{¡}$, we get

$$\partial(g) = \Theta\, \mathscr{T}^c(f; d_{\mathrm{End}_B^A} f - f d_{\mathscr{P}^{¡}}) \Delta_{\mathscr{P}^{¡}}.$$

By Theorem 10.2.3, since the map f is an ∞-morphism, it satisfies $\partial(f) = f * \varphi - \psi \circledast f$. So we get

$$\partial(g) = \Theta\mathscr{T}^c(f; f * \varphi)\Delta_{\mathscr{P}^{\text{!`}}} - \Theta\mathscr{T}^c(f; \psi \circledast f)\Delta_{\mathscr{P}^{\text{!`}}}.$$

In $f * \varphi$, there are two kinds of terms involving either $f_{(0)}$ or $f_{(>1)}$. Therefore the term $\Theta\mathscr{T}^c(f; f * \varphi)\Delta_{\mathscr{P}^{\text{!`}}}$ splits into two sums on the trees produced by $\Delta_{\mathscr{P}^{\text{!`}}}$. Because of the sign based on the number of levels, almost all the terms cancel. Only remains the trees with φ labeling the vertex above the root. Hence, we get

$$\Theta\mathscr{T}^c(f; f * \varphi)\Delta_{\mathscr{P}^{\text{!`}}} = -\varphi \circledast g.$$

Using the same kind of arguments, one proves that $\Theta\mathscr{T}^c(f; \psi \circledast f)\Delta_{\mathscr{P}^{\text{!`}}}$ is made up of trees with ψ labeling any vertex at the top of the tree, that is

$$\Theta\mathscr{T}^c(f; \psi \circledast f)\Delta_{\mathscr{P}^{\text{!`}}} = g * \psi.$$

By Proposition 10.2.4, it is enough to prove that $f \circledcirc g = \mathrm{Id}_B$ and that $g \circledcirc f = \mathrm{Id}_A$. Since $g_{(0)} := (f_{(0)})^{-1}$, these two relations are satisfied on $\mathrm{I} = \mathscr{P}^{\text{!`}\,(0)}$. Higher up, since $\Delta(\mu) = \tilde{\Delta}(\mu) + (\mathrm{id}; \mu)$, for any $\mu \in \mathscr{P}^{\text{!`}}$, we have

$$\begin{aligned}(f \circledcirc g)(\mu) = &\sum_{k=0}^{\infty} (-1)^k \left(\left(f_{(0)}^{-1}\right)^*(f)\right)^{\circ(k+1)} \hat{\Delta}^k(\mu) \\ &+ \sum_{k=0}^{\infty} (-1)^{k+1} \left(\left(f_{(0)}^{-1}\right)^*(f)\right)^{\circ(k+1)} \hat{\Delta}^k(\mu) = 0.\end{aligned}$$

In the same way, since $\Delta(\mu) = \bar{\Delta}(\mu) + (\mathrm{id}; \mu) + (\mu; \mathrm{id}, \ldots, \mathrm{id})$, for any $\mu \in \mathscr{P}^{\text{!`}}$, we have

$$\begin{aligned}(g \circledcirc f)(\mu) = &\sum_{k=1}^{\infty} (-1)^k \left(\left(f_{(0)}^{-1}\right)_*(f)\right)^{\circ(k+1)} \hat{\Delta}^k(\mu) \\ &+ \left(f_{(0)}^{-1}\right)_*(f)(\mu) \\ &+ \sum_{k=0}^{\infty} (-1)^{k+1} \left(\left(f_{(0)}^{-1}\right)_*(f)\right)^{\circ(k+1)} \hat{\Delta}^k(\mu) = 0.\end{aligned}$$

□

REMARK. The formula which gives the inverse of an ∞-isomorphism is related to the inverse under composition of power series as follows. Let us consider the nonsymmetric cooperad As^* and the $\mathbb{K}$-modules $A = B = \mathbb{K}$. There is a bijection between the power series $f(x) = a_0 x + a_1 x^2 + \cdots$ with coefficients in $\mathbb{K}$ and the elements of $\mathrm{Hom}(As^*, \mathrm{End}_{\mathbb{K}})$, given by $\tilde{f} := \mu_n^c \mapsto a_{n-1} 1_n$, where μ_n^c is the generating element of $As^*(n)$ and where 1_n is the generating element of $\mathrm{End}_{\mathbb{K}}(n)$. This map is an isomorphism of associative algebras: $\widetilde{g \circ f} = \tilde{g} \circledcirc \tilde{f}$. So a power series is invertible for the composition if and only if a_0 is invertible. This condition is equivalent to $\tilde{f}_{(0)}$ invertible in $\mathrm{Hom}(\mathbb{K}, \mathbb{K})$. When $a_1 = 1$, the formula given in Theorem 10.4.1

induces the formula for the inverse of the power series f. For yet another approach to this formula, see Sect. 13.11.7.

10.4.2 Decomposition: Minimal ⊕ Acyclic Trivial

By definition, a $\mathscr{P}_\infty$-algebra (A, d_A, φ) is called

- ◇ *minimal* when $d_A = 0$;
- ◇ *acyclic* when the underlying chain complex (A, d_A) is acyclic;
- ◇ *trivial* when the structure map is trivial: $\varphi = 0$.

Lemma 10.4.2. *Let* $(H, 0, \varphi)$ *be a minimal* $\mathscr{P}_\infty$*-algebra and let* $(K, d_K, 0)$ *be an acyclic trivial* $\mathscr{P}_\infty$*-algebra. Their product in the category* ∞-$\mathscr{P}_\infty$-alg *exists and its underlying chain complex is the direct sum* $H \oplus K$.

Proof. We consider the following $\mathscr{P}_\infty$-structure on $H \oplus K$:

$$\mathscr{P}^{¡} \xrightarrow{\varphi} \mathrm{End}_H \rightarrowtail \mathrm{End}_{H\oplus K}.$$

It satisfies the Maurer–Cartan equation in $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_{H\oplus K})$, since φ satisfies the Maurer–Cartan equation in $\mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_H)$. To any $\mathscr{P}_\infty$-algebra B with two ∞-morphisms, $f \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}^B_H)$ from B to H and $g \in \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}^B_K)$ from B to K respectively, we associate the following morphism

$$\mathscr{P}^{¡} \xrightarrow{f+g} \mathrm{End}^B_H \oplus \mathrm{End}^B_K \cong \mathrm{End}^B_{H\oplus K}.$$

We leave it to the reader to check that this composite is an ∞-morphism, which satisfies the universal property of products. □

Theorem 10.4.3 (Minimal model for $\mathscr{P}_\infty$-algebras). *Let* $\mathbb{K}$ *be a field of characteristic* 0 *and let* $\mathscr{P}$ *be a Koszul operad. In the category of* $\mathscr{P}_\infty$*-algebras with* ∞*-morphisms, any* $\mathscr{P}_\infty$*-algebra is* ∞*-isomorphic to the product of a minimal* $\mathscr{P}_\infty$*-algebra, given by the transferred structure on its homotopy, with an acyclic trivial* $\mathscr{P}_\infty$*-algebra.*

Proof. Let (A, d_A, φ) be a $\mathscr{P}_\infty$-algebra. As in Lemma 9.4.4, we decompose the chain complex A with respect to its homology and boundary: $A_n \cong B_n \oplus H_n \oplus B_{n-1}$. We denote by $K_n := B_n \oplus B_{n-1}$ the acyclic sub-chain complex of A, so that A is the direct sum of the two dg modules $A \cong H(A) \oplus K$. By Theorems 10.3.1 and 10.3.10, the homotopy $H(A)$ is endowed with a minimal $\mathscr{P}_\infty$-algebra structure and we consider the trivial $\mathscr{P}_\infty$-algebra structure on K.

We define an ∞-morphism f from A to K as follows. Let q denote the projection from A to K and let of $f_{(0)}$ be equal to q. Higher up, f is given by the composite $f = (qh)_* \varphi$

$$f : \overline{\mathscr{P}^{¡}} \xrightarrow{\varphi} \mathrm{End}_A \xrightarrow{(qh)_*} \mathrm{End}_K^A .$$

Since the $\mathscr{P}_\infty$-algebra structure on K is trivial, we only have to check the equality $f * \varphi = \partial(f)$, by Theorem 10.2.3. This equation reads on $\overline{\mathscr{P}^{¡}}$:

$$(qh)_* (\varphi \circ_{(1)} \varphi)\Delta_{(1)} + q_* \varphi = d_{\mathrm{End}_K^A} (q\,h)_* \varphi - (qh)_* \varphi\, d_{\mathscr{P}^{¡}} .$$

Since μ is a twisting morphism, we have

$$(qh)_* (\varphi \circ_{(1)} \varphi)(\Delta_{(1)}) = -(qh)_* d_{\mathrm{End}_A} \varphi - (qh)_* \varphi\, d_{\mathscr{P}^{¡}} .$$

We conclude by using the equality $q(h \circ d_A + d_A \circ h) = q$.

The ∞-morphism p_∞ from A to $H(A)$ of Proposition 10.3.9 together with the ∞-morphism f from A to K induce an ∞-morphism from A to $H(A) \oplus K$, since this latter space is the product of $H(A)$ and K by Lemma 10.4.2. The first component of this ∞-morphism is equal to $p + q : A \cong H(A) \oplus K$, which is an isomorphism. □

10.4.3 Inverse of Infinity-Quasi-isomorphisms

Theorem 10.4.4. *Let $\mathscr{P}$ be a Koszul operad and let A and B be two $\mathscr{P}_\infty$-algebras. If there exists an ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$, then there exists an ∞-quasi-isomorphism in the opposite direction $B \overset{\sim}{\rightsquigarrow} A$, which is the inverse of $H(A) \xrightarrow{\cong} H(B)$ on the level on holomogy.*

Proof. Let $f : A \overset{\sim}{\rightsquigarrow} B$ denote an ∞-quasi-isomorphism. By Theorem 10.3.6 and Proposition 10.3.9, the following composite g of ∞-quasi-isomorphisms

$$H(A) \overset{i_\infty^A}{\rightsquigarrow} A \overset{f}{\rightsquigarrow} B \overset{p_\infty^B}{\rightsquigarrow} H(B)$$

is an ∞-isomorphism. It admits an inverse ∞-isomorphism $g^{-1} : H(B) \rightsquigarrow H(A)$ by Theorem 10.4.1. The ∞-quasi-isomorphism $B \overset{\sim}{\rightsquigarrow} A$ is given by the following composite of ∞-quasi-isomorphisms

$$B \overset{p_\infty^B}{\rightsquigarrow} H(B) \overset{g^{-1}}{\rightsquigarrow} H(A) \overset{i_\infty^A}{\rightsquigarrow} A .$$ □

10.4.4 Homotopy Equivalence

We define the following relation among $\mathscr{P}_\infty$-algebras: *a $\mathscr{P}_\infty$-algebra A is homotopy equivalent to a $\mathscr{P}_\infty$-algebra B* if there exists an ∞-quasi-isomorphism from

A to B. The previous section shows that it is an equivalence relation. We denote it by $A \sim B$.

Under this terminology, the Higher Structure Theorem 10.3.10 implies that in the homotopy class of any $\mathscr{P}_\infty$-algebra, there is a minimal $\mathscr{P}_\infty$-algebra.

10.5 Homotopy Operads

In this section, we relax the notion of operad, up to homotopy, thereby defining homotopy operads. As for associative algebras and homotopy associative algebras, the relations satisfied by the partial compositions of an (nonunital) operad are relaxed up to a full hierarchy of higher homotopies. We introduce the notion of ∞-*morphism* for homotopy operads. We have already used this notion, without saying it, in Proposition 10.3.2, where the morphism Ψ is an ∞-morphism of operads.

We describe a functor from homotopy operads to homotopy Lie algebras.

Finally, we show that a homotopy representation of an operad, that is a homotopy morphism from $\mathscr{P}$ to End_A, is equivalent to a $\Omega\mathrm{B}\mathscr{P}$-algebra structure on A.

The notions of homotopy operad and ∞-morphism come from the work of Pepijn Van der Laan [vdL02, vdL03].

10.5.1 Definition

A *homotopy operad* is a graded $\mathbb{S}$-module $\mathscr{P}$ with a square-zero coderivation d of degree -1 on the cofree conilpotent cooperad $\mathscr{T}^c(s\mathscr{P})$. By extension, we call the dg cooperad $(\mathscr{T}^c(s\mathscr{P}), d)$ the *bar construction* of the homotopy operad $\mathscr{P}$ and we denote it by $\mathrm{B}\mathscr{P}$. Hence any nonunital operad $\mathscr{P}$ is a homotopy operad and the associated bar construction coincides with the classical bar construction of Sect. 6.5.1.

For any graded $\mathbb{S}$-module M, recall from Proposition 6.3.7 that any coderivation d_γ on the cofree cooperad $\mathscr{T}^c(M)$ is completely characterized by its projection onto the space of cogenerators $\gamma = \mathrm{proj} \circ d_\gamma : \overline{\mathscr{T}}^c(M) \to M$.

Let α and β be maps in $\mathrm{Hom}_{\mathbb{S}}(\overline{\mathscr{T}}^c(M), M)$. Their *convolution product* $\alpha \star \beta$ is defined by the composite

$$\alpha \star \beta := \overline{\mathscr{T}}^c(M) \xrightarrow{\Delta'} \overline{\mathscr{T}}^c\big(M; \overline{\mathscr{T}}^c(M)\big) \xrightarrow{\overline{\mathscr{T}}^c(\mathrm{Id}_M;\beta)} \overline{\mathscr{T}}^c(M; M) \to \overline{\mathscr{T}}^c(M) \xrightarrow{\alpha} M,$$

where the first map Δ' singles out every nontrivial subtree of a tree whose vertices are indexed by M, see Sect. 6.3.8.

Lemma 10.5.1. *For any map γ of degree -1 in* $\mathrm{Hom}_{\mathbb{S}}(\overline{\mathscr{T}}^c(M), M)$, *the associated coderivation d_γ on the cofree cooperad $\mathscr{T}^c(M)$ satisfies*

$$(d_\gamma)^2 = d_{\gamma\star\gamma}.$$

Proof. Since γ has degree -1, the composite $d_\gamma \circ d_\gamma$ is equal to $\frac{1}{2}[d_\gamma, d_\gamma]$; so it is a coderivation of $\overline{\mathscr{T}}^c(M)$. By Proposition 6.3.7, it is completely characterized by its projection onto M: $\mathrm{proj}((d_\gamma)^2) = \gamma \star \gamma$. □

Any degree -1 square-zero coderivation d on the cofree cooperad $\mathscr{T}^c(s\mathscr{P})$ is equal to the sum $d = d_1 + d_\gamma$, where d_1 is the coderivation which extends an internal differential $d_{\mathscr{P}}$ on $\mathscr{P}$ and where d_γ is the unique coderivation which extends the restriction $\gamma := \mathrm{proj}(d) : \mathscr{T}^c(s\mathscr{P})^{(\geq 2)} \to s\mathscr{P}$.

Proposition 10.5.2. *Let $(\mathscr{P}, d_{\mathscr{P}})$ be a dg $\mathbb{S}$-module. A structure of homotopy operad on $\mathscr{P}$ is equivalently defined by a map $\gamma : \mathscr{T}^c(s\mathscr{P})^{(\geq 2)} \to s\mathscr{P}$ of degree -1 such that*

$$\partial(\gamma) + \gamma \star \gamma = 0$$

in $\mathrm{Hom}_{\mathbb{S}}(\overline{\mathscr{T}}^c(s\mathscr{P}), s\mathscr{P})$.

Proof. Any coderivation $d : \mathscr{T}^c(s\mathscr{P}) \to \mathscr{T}^c(s\mathscr{P})$ defining a structure of homotopy operad satisfies $d^2 = 0$, which is equivalent to $d_1 d_\gamma + d_\gamma d_1 + d_\gamma d_\gamma = 0$. By projecting onto the space of cogenerators, this relation is equivalent to $d_{s\mathscr{P}} \gamma + \gamma d_1 + \gamma \star \gamma = 0$ in $\mathrm{Hom}_{\mathbb{S}}(\overline{\mathscr{T}}^c(s\mathscr{P}), s\mathscr{P})$. □

Hence a structure of homotopy operad on a dg $\mathbb{S}$-module $\mathscr{P}$ is a family of maps $\{\gamma_n : \mathscr{T}^c(s\mathscr{P})(n) \to s\mathscr{P}\}_{n\geq 2}$, which "compose" any tree with n vertices labeled by elements of $s\mathscr{P}$. The map $\gamma \star \gamma$ composes first any nontrivial subtree of a tree with γ and then composes the remaining tree with γ once again.

When the $\mathbb{S}$-module $\mathscr{P}$ is concentrated in arity 1, a homotopy operad structure on $\mathscr{P}$ is nothing but a homotopy associative algebra on $\mathscr{P}(1)$. If the structure map γ vanishes on $\mathscr{T}^c(s\mathscr{P})^{(\geq 3)}$, then the only remaining product

$$\gamma_2 : \mathscr{T}^c(s\mathscr{P})^{(2)} \cong s\mathscr{P} \circ_{(1)} s\mathscr{P} \to s\mathscr{P}$$

satisfies the same relations as the partial compositions (Sect. 5.3.4) of an operad. In this case $\mathscr{P}$ is a nonunital operad.

One can translate this definition in terms of operations $\{\mathscr{T}^{(n)}(\mathscr{P}) \to \mathscr{P}\}_{n\geq 2}$, without suspending the $\mathbb{S}$-module $\mathscr{P}$. This would involve extra signs as usual.

10.5.2 Infinity-Morphisms of Homotopy Operads

Let $(\mathscr{P}, \gamma)$ and $(\mathscr{Q}, \nu)$ be two homotopy operads. By definition, an ∞*-morphism of homotopy operads* between $\mathscr{P}$ and $\mathscr{Q}$ is a morphism

$$F : \mathrm{B}\mathscr{P} := \big(\mathscr{T}^c(s\mathscr{P}), d\big) \to \mathrm{B}\mathscr{Q} := \big(\mathscr{T}^c(s\mathscr{Q}), d'\big)$$

of dg cooperads. We denote it by $\mathscr{P} \rightsquigarrow \mathscr{Q}$. Homotopy operads with their ∞-morphisms form a category, which is denoted by $\infty\text{-}\mathsf{Op}_\infty$.

For any $\mathbb{S}$-module M, we consider the morphism of $\mathbb{S}$-modules

$$\Delta' : \mathscr{T}^c(M) \to \mathscr{T}^c\big(M; \mathscr{T}^c(M)^{(\geq 2)}\big),$$

which singles out one subtree with at least two vertices. We also consider the morphism of $\mathbb{S}$-modules

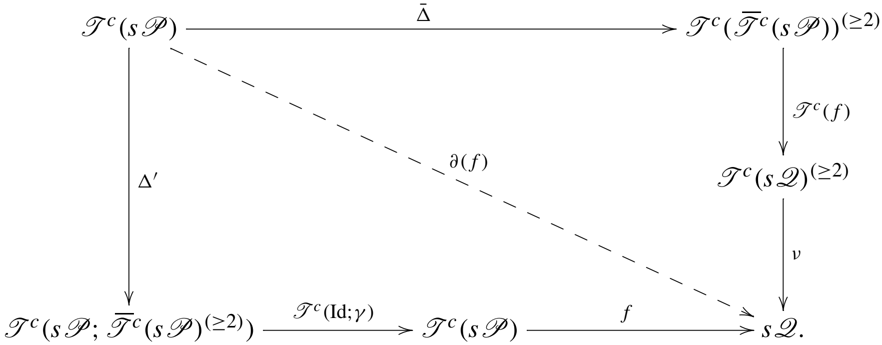
$$\bar{\Delta} : \mathscr{T}^c(M) \to \mathscr{T}^c\big(\overline{\mathscr{T}}^c(M)\big)^{(\geq 2)}$$

which is defined by the projection of $\Delta(M) : \mathscr{T}^c(M) \to \mathscr{T}^c(\overline{\mathscr{T}}^c(M))$, see Sect. 6.3.8, onto $\mathscr{T}^c(\overline{\mathscr{T}}^c(M))^{(\geq 2)}$. In words, it splits a tree into all partitions of subtrees with at least two nontrivial subtrees.

Proposition 10.5.3. *Let $(\mathscr{P}, \gamma)$ and $(\mathscr{Q}, \nu)$ be two homotopy operads. An ∞-morphism of homotopy operads between $\mathscr{P}$ and $\mathscr{Q}$ is equivalently given by a morphism $f : \overline{\mathscr{T}}^c(s\mathscr{P}) \to s\mathscr{Q}$ of graded* $\mathbb{S}$*-modules, which satisfies*

$$f\mathscr{T}^c(\mathrm{Id}; \gamma)\Delta' - \nu\mathscr{T}^c(f)\bar{\Delta} = \partial(f)$$

in $\mathrm{Hom}_{\mathbb{S}}(\overline{\mathscr{T}}^c(s\mathscr{P}), s\mathscr{Q})$:



Proof. The universal property of cofree conilpotent cooperads states that every morphism $F : \mathscr{T}^c(s\mathscr{P}) \to \mathscr{T}^c(s\mathscr{Q})$ of cooperads is completely characterized by its projection onto the space of the cogenerators $f : \overline{\mathscr{T}}^c(s\mathscr{P}) \to s\mathscr{Q}$. Explicitly, the unique morphism of cooperads F which extends a map $f : \overline{\mathscr{T}}^c(s\mathscr{P}) \to s\mathscr{Q}$ is equal to the composite

$$F : \mathscr{T}^c(s\mathscr{P}) \xrightarrow{\Delta(s\mathscr{P})} \mathscr{T}^c\big(\overline{\mathscr{T}}^c(s\mathscr{P})\big) \xrightarrow{\mathscr{T}^c(f)} \mathscr{T}^c(s\mathscr{Q}).$$

The map f defines an ∞-morphism of homotopy operads if and only if the map F commutes with the differentials $d_1 + d_\gamma$ on $\mathscr{T}^c(s\mathscr{P})$ and $d_1' + d_\nu$ on $\mathscr{T}^c(s\mathscr{Q})$ respectively. Since F is a morphism of cooperads and since $d_1 + d_\gamma$ and $d_1' + d_\nu$

are coderivations, the relation $(d_1' + d_\nu)F = F(d_1 + d_\gamma)$ holds if and only if $\mathrm{proj}((d_1' + d_\nu)F - F(d_1 + d_\gamma)) = 0$. By the aforementioned universal property of cofree cooperads and by Proposition 6.3.7, we have

$$\mathrm{proj}\big((d_1' + d_\nu)F - F(d_1 + d_\gamma)\big) = \partial(f) + \nu \mathscr{T}^c(f)\bar{\Delta} - f \mathscr{T}^c(\mathrm{Id}; \gamma)\Delta'. \qquad \square$$

Therefore an ∞-morphism $\mathscr{P} \rightsquigarrow \mathscr{Q}$ of homotopy operads is a family of maps, which associate to any tree t labeled by elements of $s\mathscr{P}$ an element of $s\mathscr{Q}$. Since an operad is a particular case of homotopy operad, one can consider ∞-morphism of operads. Proposition 10.3.2 gives such an example.

Proposition 10.5.4. *A morphism of operads is an ∞-morphism with only one nonvanishing component, namely the first one*:

$$s\mathscr{P} \cong \overline{\mathscr{T}}^c(s\mathscr{P})^{(1)} \to s\mathscr{Q}.$$

Proof. By straightforward application of the definitions. $\square$

As for homotopy algebras, one can define four categories by considering either operads or homotopy operads for the objects and morphisms or infinity morphisms for the maps.

$$\begin{array}{ccc} \mathsf{Op} & \xrightarrow{\text{not full}} & \infty\text{-}\mathsf{Op} \\ \downarrow{\scriptstyle \text{f.f.}} & & \downarrow{\scriptstyle \text{f.f.}} \\ \mathsf{Op}_\infty & \xrightarrow{\text{not full}} & \infty\text{-}\mathsf{Op}_\infty \end{array}$$

(A morphism of homotopy operads is an ∞-morphism with nonvanishing components except for the first one.)

10.5.3 From Homotopy Operads to Homotopy Lie Algebras

We define a functor from homotopy operads to L_∞-algebras, which extends the functor from operads to *Lie*-algebras constructed in Proposition 5.4.3

$$\begin{array}{ccc} \mathsf{Op} & \longrightarrow & \mathit{Lie}\text{-}\mathsf{alg} \\ \big\downarrow & & \big\downarrow \\ \infty\text{-}\mathsf{Op}_\infty & \dashrightarrow & \infty\text{-}L_\infty\text{-}\mathsf{alg}. \end{array}$$

Let $\mathscr{P}$ be a dg $\mathbb{S}$-module. We consider either the direct sum of its components $\bigoplus_n \mathscr{P}(n)$ or the direct product $\prod_n \mathscr{P}(n)$. By a slight abuse of notation, we still denote it by $\mathscr{P}$ in this section.

By Proposition 10.1.12, an L_∞-algebra structure on $\mathscr{P}$ is equivalent to a degree -1 square-zero coderivation on the cofree cocommutative coalgebra $\overline{S}^c(s\mathscr{P})$. Recall that its underlying space is the space of invariant elements of the cofree coalgebra $\overline{T}^c(s\mathscr{P})$ under the permutation action. We denote its elements with the symmetric tensor notation:

$$\overline{S}^c(s\mathscr{P})^{(n)} := s\mu_1 \odot \cdots \odot s\mu_n \in (s\mathscr{P})^{\odot n}.$$

Let t be a tree with n vertices and let $\mu_1, \ldots, \mu_n$ be n elements of $\mathscr{P}$. We denote by $t(s\mu_1, \ldots, s\mu_n)$ the sum of all the possible ways of labeling the vertices of $\mathscr{T}$ with $s\mu_1, \ldots, s\mu_n$ according to the arity. We consider the following morphism

$$\Theta : \overline{S}^c(s\mathscr{P}) \to \overline{\mathscr{T}}^c(s\mathscr{P}); \qquad \Theta_n(s\mu_1 \odot \cdots \odot s\mu_n) := \sum_t t(s\mu_1, \ldots, s\mu_n),$$

where t runs over the set of n-vertices trees.

Proposition 10.5.5. *Let $(\mathscr{P}, \gamma)$ be a homotopy operad. The maps $\ell_n : S^c(s\mathscr{P})^{(n)} \to s\mathscr{P}$ of degree -1, defined by the composite $\ell_n := \gamma \circ \Theta_n$, endow the dg modules $\bigoplus_n \mathscr{P}(n)$, respectively $\prod_n \mathscr{P}(n)$, with an L_∞-algebra structure.*

Proof. We consider the "partial" coproduct δ' on the cofree cocommutative coalgebra $\overline{S}^c(s\mathscr{P})$ defined by

$$\delta' : \overline{S}^c(s\mathscr{P}) \to \overline{S}^c\big(s\mathscr{P}; \overline{S}^c(s\mathscr{P})\big),$$
$$s\mu_1 \odot \cdots \odot s\mu_n \mapsto \sum_{p=1}^{n-1} \sum_{\sigma \in Sh_{p,q}} \pm(s\mu_{\sigma(1)} \odot \cdots \odot s\mu_{\sigma(p)}) \odot s\mu_{\sigma(p+1)} \odot \cdots \odot s\mu_{\sigma(n)},$$

where the sign comes from the permutation of the graded elements, as usual. The unique coderivation on $\overline{S}^c(s\mathscr{P})$, which extends $d_{s\mathscr{P}} + \ell$ is equal to the following composite

$$d_1 + d_\ell := \overline{S}^c(s\mathscr{P}) \xrightarrow{\delta'} \overline{S}^c\big(s\mathscr{P}, \overline{S}^c(s\mathscr{P})\big) \xrightarrow{\overline{S}^c(\mathrm{Id}, d_{s\mathscr{P}}+\ell)} \overline{S}^c(s\mathscr{P}, s\mathscr{P}) \to \overline{S}^c(s\mathscr{P}).$$

Under the isomorphism $s\mathscr{P} \cong \mathscr{P}$, the map ℓ defines an L_∞-algebra structure on $\mathscr{P}$ if and only if this coderivation squares to zero.

The following commutative diagram

$$\begin{array}{ccccccc}
\overline{S}^c(s\mathscr{P}) & \xrightarrow{\delta'} & \overline{S}^c(s\mathscr{P}, S^c(s\mathscr{P})^{(\geq 2)}) & \xrightarrow{\overline{S}^c(\mathrm{Id}, \ell)} & \overline{S}^c(s\mathscr{P}, s\mathscr{P}) & \longrightarrow & \overline{S}^c(s\mathscr{P}) \\
\downarrow \Theta & & & & & & \downarrow \Theta \\
\overline{\mathscr{T}}^c(s\mathscr{P}) & \xrightarrow{\Delta'} & \overline{\mathscr{T}}^c(s\mathscr{P}, \mathscr{T}^c(s\mathscr{P})^{(\geq 2)}) & \xrightarrow{\overline{\mathscr{T}}^c(\mathrm{Id}, \gamma)} & \overline{\mathscr{T}}^c(s\mathscr{P}; s\mathscr{P}) & \longrightarrow & \overline{\mathscr{T}}^c(s\mathscr{P})
\end{array}$$

proves that Θ commutes with the coderivations d_ℓ and d_γ. Hence Θ commutes with the full coderivations $d_1 + d_\ell$ and $d_1 + d_\gamma$. Since $d_1 + d_\ell$ is a coderivation, it squares to zero if and only if the projection of $(d_1 + d_l)^2$ onto the space of cogenerators $s\mathscr{P}$ vanishes. This projection is equal to the projection of $(d_1 + d_\gamma)^2\Theta$ on $s\mathscr{P}$, which is equal to zero, by the definition of a homotopy operad. □

This proposition includes and generalizes Proposition 10.1.8. If $\mathscr{P}$ is concentrated in arity 1, then it is an A_∞-algebra. In this case, the class of trees considered are only ladders. We recover the formula of Proposition 10.1.8, which associates an L_∞-algebra to an A_∞-algebra.

Proposition 10.5.6. *Let $(\mathscr{P}, \gamma)$ and $(\mathscr{Q}, \nu)$ be two homotopy operads and let $F : \mathrm{B}\mathscr{P} \to \mathrm{B}\mathscr{Q}$ be an ∞-morphism. The unique morphism of cocommutative coalgebras, which extends*

$$\overline{S}^c(s\mathscr{P}) \xrightarrow{\Theta} \overline{\mathscr{T}}^c(s\mathscr{P}) \xrightarrow{F} \overline{\mathscr{T}}^c(s\mathscr{Q}) \twoheadrightarrow s\mathscr{Q},$$

commutes with the differentials. In other words, it defines an ∞-morphism of L_∞-algebras. So there is a well-defined functor ∞-$\mathsf{Op}_\infty \to \infty$-$L_\infty$-$\mathsf{alg}$.

Proof. Let us denote by $\widetilde{F} : \overline{S}^c(s\mathscr{P}) \to \overline{S}^c(s\mathscr{Q})$ this unique morphism of cocommutative coalgebras. We first prove that the map Θ commutes with the morphisms F and $\widetilde{F}$ respectively. Let us introduce the structure map $\delta : \overline{S}^c(s\mathscr{P}) \to \overline{S}^c(\overline{S}^c(s\mathscr{P}))$, defined by

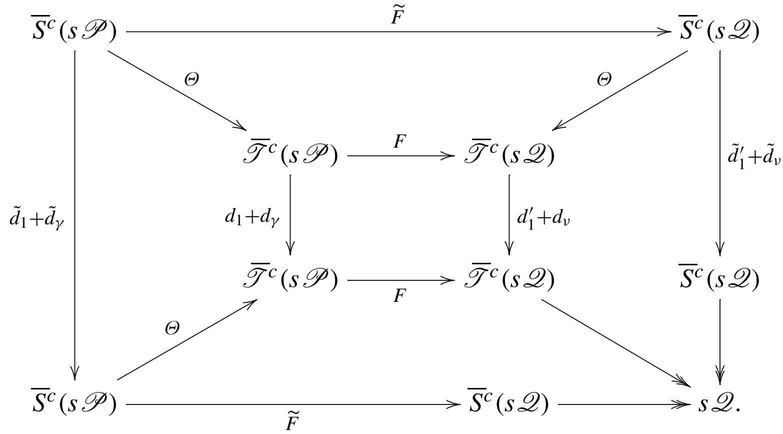
$$\begin{aligned}&\delta(s\mu_1 \odot \cdots \odot s\mu_n)\\ &\quad := \sum \pm (s\mu_{\sigma(1)} \odot \cdots \odot s\mu_{\sigma(i_1)}) \odot \cdots \odot (s\mu_{\sigma(i_1+\cdots+i_{n-1}+1)} \odot \cdots \odot s\mu_{\sigma(n)}),\end{aligned}$$

where the sum runs over $k \geq 1$, $i_1 + \cdots + i_k = n$ and $\sigma \in Sh_{i_1,\ldots,i_k}$. If we denote by f the projection of F onto the space of cogenerators, then the unique morphism of cocommutative coalgebras $\widetilde{F}$ extending $f\Theta$ is equal to $\widetilde{F} = \overline{S}^c(f\Theta)\delta$. Since the morphism F is equal to the composite $F = \overline{\mathscr{T}}^c(f)\Delta$, the following diagram is commutative

$$\begin{array}{ccccc}
 & & \overset{\widetilde{F}}{\curvearrowright} & & \\
\overline{S}^c(s\mathscr{P}) & \xrightarrow{\ \delta\ } & \overline{S}^c(\overline{S}^c(s\mathscr{P})) & \xrightarrow{\overline{S}^c(f)} & \overline{S}^c(s\mathscr{Q}) \\
\downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta(\Theta)} & & \downarrow{\scriptstyle \Theta} \\
\overline{\mathscr{T}}^c(s\mathscr{P}) & \xrightarrow{\ \Delta\ } & \overline{\mathscr{T}}^c(\overline{\mathscr{T}}^c(s\mathscr{P})) & \xrightarrow{\overline{\mathscr{T}}^c(f)} & \overline{\mathscr{T}}^c(s\mathscr{Q}). \\
 & & \underset{F}{\smile} & &
\end{array}$$

Let us denote by $\tilde{d}_\gamma$, and by $\tilde{d}_\nu$ respectively, the induced square-zero coderivations on $\overline{S}^c(s\mathscr{P})$, and on $\overline{S}^c(s\mathscr{Q})$ respectively. Since $\widetilde{F}$ is a morphism of cocommutative coalgebras, to prove that it commutes with the coderivations $\tilde{d}_1 + \tilde{d}_\gamma$ and

$\tilde{d}'_1 + \tilde{d}_\nu$ respectively, it is enough to prove by projecting onto the space $s\mathscr{Q}$ of cogenerators. To this end, we consider the following commutative diagram.

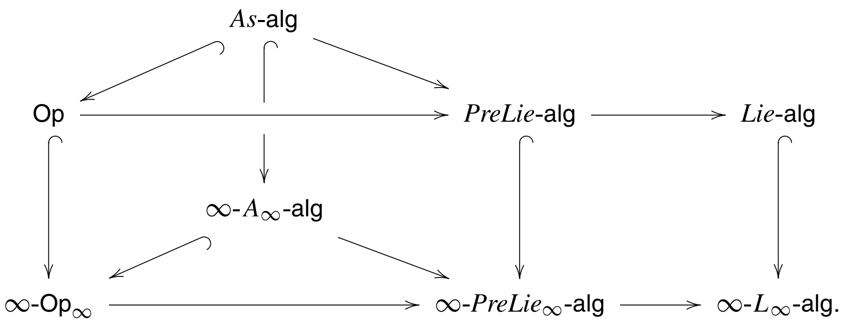


Since the internal diagram is commutative, the external one also commutes, which concludes the proof. □

10.5.4 From Homotopy Operads to Homotopy pre-Lie Algebras

The two aforementioned Propositions 10.5.5 and 10.5.6 extend from *Lie*-algebras and L_∞-algebras to *preLie*-algebras and $preLie_\infty$-algebras respectively, see Sect. 13.4.

Finally, we have the following commutative diagram of categories, which sums up the relations between the various algebraic structures encountered so far.



10.5.5 Homotopy Algebra Structures vs ∞-Morphisms of Operads

Since a $\mathscr{P}$-algebra structure on a dg module A is given by a morphism of dg operads $\mathscr{P} \to \mathrm{End}_A$, it is natural to ask what ∞-morphisms $\mathscr{P} \rightsquigarrow \mathrm{End}_A$ between $\mathscr{P}$ and

End_A do model, cf. [Lad76, vdL03]. The next proposition shows that a homotopy representation of an operad $\mathscr{P}$ is a $\Omega \mathrm{B}\mathscr{P}$-algebra.

Proposition 10.5.7. *For any operad $\mathscr{P}$ and any dg module A, there is a natural bijection between ∞-morphisms from $\mathscr{P}$ to End_A and $\Omega\mathrm{B}\mathscr{P}$-algebra structures on A:*

$$\mathrm{Hom}_{\infty\text{-}\mathsf{Op}}(\mathscr{P}, \mathrm{End}_A) \cong \mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega\mathrm{B}\mathscr{P}, \mathrm{End}_A).$$

Proof. By definition, the first set is equal to $\mathrm{Hom}_{\mathsf{dg\ Coop}}(\mathrm{B}\mathscr{P}, \mathrm{B}\,\mathrm{End}_A)$. The natural bijection with $\mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega\mathrm{B}\mathscr{P}, \mathrm{End}_A)$ given by the bar–cobar adjunction of Theorem 6.5.7 concludes the proof. □

By pulling back along the morphism of dg operads $\Omega\mathscr{P}^{\text{¡}} \to \Omega\mathrm{B}\mathscr{P}$, any $\Omega\mathrm{B}\mathscr{P}$-algebra A determines a $\mathscr{P}_\infty$-algebra. So an ∞-morphism of operads from $\mathscr{P}$ to End_A induces a $\mathscr{P}_\infty$-algebra structure on A.

Any $\mathscr{P}_\infty$-algebra structure on A is a morphism of dg operads $\Omega\mathscr{P}^{\text{¡}} \to \mathrm{End}_A$, which is a particular ∞-morphism of operads from $\Omega\mathscr{P}^{\text{¡}}$ to End_A by Proposition 10.5.4.

$$\begin{aligned}\mathrm{B} : \mathrm{Hom}_{\mathsf{dg\ Op}}\big(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big) &\to \mathrm{Hom}_{\mathsf{dg\ coOp}}\big(\mathrm{B}\Omega\mathscr{P}^{\text{¡}}, \mathrm{B}\,\mathrm{End}_A\big)\\ &\cong \mathrm{Hom}_{\infty\text{-}\mathsf{Op}}\big(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big).\end{aligned}$$

Again, what do ∞-morphisms $\Omega\mathscr{P}^{\text{¡}} \rightsquigarrow \mathrm{End}_A$ model? Any ∞-morphism of operads $\mathrm{B}\Omega\mathscr{P}^{\text{¡}} \to \mathrm{B}\,\mathrm{End}_A$ induces a $\mathscr{P}_\infty$-algebra structure on A by pulling back along the unit of adjunction $\upsilon_{\mathscr{P}^{\text{¡}}} : \mathscr{P}^{\text{¡}} \to \mathrm{B}\Omega\mathscr{P}^{\text{¡}}$ and by using the bar–cobar adjunction. Let us denote this map by

$$\upsilon^* : \mathrm{Hom}_{\infty\text{-}\mathsf{Op}}\big(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big) \to \mathrm{Hom}_{\mathsf{dg\ Op}}\big(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A\big).$$

So the set of $\mathscr{P}_\infty$-algebra structures on A is a "retract" of the set of ∞-morphisms from $\mathscr{P}_\infty = \Omega\mathscr{P}^{\text{¡}}$ to End_A.

We sum up the hierarchy of homotopy notions in the following table.

$$\begin{array}{ccc}
\{\mathscr{P}\text{-algebra structures on } A\} & = & \mathrm{Hom}_{\mathsf{Op}}(\mathscr{P}, \mathrm{End}_A)\\
\Big\downarrow & & \Big\downarrow\\
 & & \mathrm{Hom}_{\infty\text{-}\mathsf{Op}}(\mathscr{P}, \mathrm{End}_A)\\
 & & \Big\downarrow \cong\\
\{\Omega\mathrm{B}\mathscr{P}\text{-algebra structures on } A\} & = & \mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega\mathrm{B}\mathscr{P}, \mathrm{End}_A)\\
\Big\downarrow & & \Big\downarrow\\
\{\mathscr{P}_\infty\text{-algebra structures on } A\} & = & \mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A)\\
 & & \upsilon^*\Big\uparrow\ \Big\downarrow\\
 & & \mathrm{Hom}_{\infty\text{-}\mathsf{Op}}(\Omega\mathscr{P}^{\text{¡}}, \mathrm{End}_A)
\end{array}$$

10.6 Résumé

10.6.1 Homotopy $\mathscr{P}$-Algebras

Homotopy $\mathscr{P}$-algebra: algebra over $\mathscr{P}_\infty := \Omega\mathscr{P}^{¡}$.

ROSETTA STONE. The set of $\mathscr{P}_\infty$-algebra structures on A is equal to

$$\begin{aligned}\mathrm{Hom}_{\mathsf{dgOp}}\big(\Omega\mathscr{P}^{¡}, \mathrm{End}_A\big) &\cong \mathrm{Tw}\big(\mathscr{P}^{¡}, \mathrm{End}_A\big)\\ &\cong \mathrm{Hom}_{\mathsf{dgCoop}}\big(\mathscr{P}^{¡}, \mathrm{B}\,\mathrm{End}_A\big) \cong \mathrm{Codiff}\big(\mathscr{P}^{¡}(A)\big).\end{aligned}$$

10.6.2 Infinity-Morphisms

Let (A, φ) and (B, ψ) be two $\mathscr{P}_\infty$-algebras.

Infinity-morphism or ∞-morphism $A \rightsquigarrow B$: morphism of dg $\mathscr{P}^{¡}$-coalgebras

$$F : \big(\mathscr{P}^{¡}(A), d_\varphi\big) \to \big(\mathscr{P}^{¡}(B), d_\psi\big) \quad \Longleftrightarrow \quad f : \mathscr{P}^{¡} \to \mathrm{End}^A_B,$$
$$\text{such that } f * \varphi - \psi \circledcirc f = \partial(f) \text{ in } \mathrm{Hom}_{\mathbb{S}}\big(\mathscr{P}^{¡}, \mathrm{End}^A_B\big).$$

Category of $\mathscr{P}_\infty$-algebras with ∞-morphisms denoted ∞-$\mathscr{P}_\infty$-alg.

$$\begin{array}{ccc}\mathscr{P}\text{-alg} & \xrightarrow{\text{not full}} & \infty\text{-}\mathscr{P}\text{-alg}\\ \big\downarrow{\scriptstyle \text{f.f.}} & & \big\downarrow{\scriptstyle \text{f.f.}}\\ \mathscr{P}_\infty\text{-alg} & \xrightarrow{\text{not full}} & \infty\text{-}\mathscr{P}_\infty\text{-alg}\end{array}$$

∞-isomorphism: when $f_{(0)} : A \to B$ is an isomorphism.

Theorem. *∞-isomorphisms are the isomorphisms of the category ∞-$\mathscr{P}_\infty$-alg.*

∞-quasi-isomorphism: when $f_{(0)} : A \to B$ is a quasi-isomorphism.

10.6.3 Homotopy Transfer Theorem

Homotopy data: let (V, d_V) be a homotopy retract of (W, d_W)

$$h \circlearrowright (W, d_W) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V).$$

Proposition. *Any homotopy retract gives rise to a morphism $\Psi : \mathrm{B}\,\mathrm{End}_W \to \mathrm{B}\,\mathrm{End}_V$ of dg cooperads.*

Algebraic data: let φ be a $\mathscr{P}_\infty$-algebra structure on W.

HOMOTOPY TRANSFER THEOREM. There exists a $\mathscr{P}_\infty$-algebra structure on V such that i extends to an ∞-quasi-isomorphism.

$$\begin{array}{ccc} \mathscr{P}^{\text{¡}} & \xrightarrow{f_\varphi} & \mathrm{B}\,\mathrm{End}_W \\ & \searrow^{f_\psi} & \downarrow \Psi \\ & & \mathrm{B}\,\mathrm{End}_V. \end{array}$$

EXPLICIT TRANSFERRED STRUCTURE.

$$\mathscr{P}^{\text{¡}} \xrightarrow{\Delta_{\mathscr{P}^{\text{¡}}}} \mathscr{T}^c\big(\overline{\mathscr{P}}^{\text{¡}}\big) \xrightarrow{\mathscr{T}^c(s\varphi)} \mathscr{T}^c(s\mathrm{End}_W) \xrightarrow{\Psi} \mathrm{End}_V.$$

OPERADIC MASSEY PRODUCTS. They are the higher operations in the particular case: $W = A$, a $\mathscr{P}_\infty$-algebra, and $V = H(A)$.

CHAIN MULTICOMPLEX. Particular case where $\mathscr{P} = D := \mathbb{K}[\varepsilon]/(\varepsilon^2)$.

$$\begin{array}{rcl} D\text{-algebra on } A & \longleftrightarrow & \text{bicomplex}, \\ \text{transferred } D_\infty\text{-algebra on } H(A) & \longleftrightarrow & \text{spectral sequence}. \end{array}$$

10.6.4 Homotopy Theory of Homotopy Algebras

DECOMPOSITION: MINIMAL $\oplus$ ACYCLIC TRIVIAL. Any $\mathscr{P}_\infty$-algebra A is ∞-isomorphic to a product

$$A \overset{\cong}{\rightsquigarrow} M \oplus K$$

in ∞-$\mathscr{P}_\infty$-alg, where M is minimal, i.e. $d_M = 0$, and where K is acyclic trivial, i.e. acyclic underlying chain complex and trivial $\mathscr{P}_\infty$-algebra structure.

HOMOTOPY EQUIVALENCE. If there exists an ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$, then there exists an ∞-quasi-isomorphism $B \overset{\sim}{\rightsquigarrow} A$.

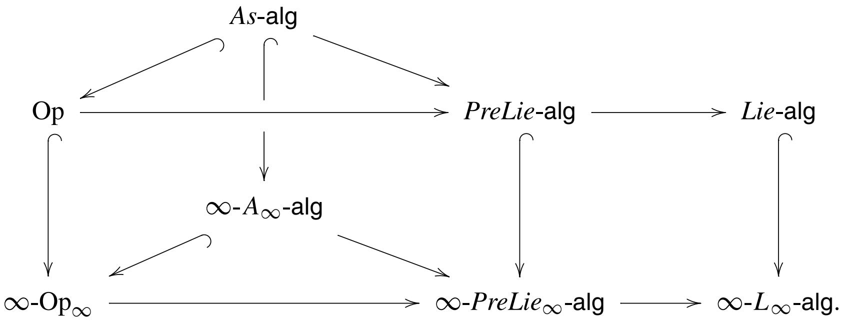
10.6.5 Homotopy Operads

Homotopy operad: degree -1 square-zero coderivation on $\mathscr{T}^c(s\mathscr{P})$

$$\iff \quad \gamma : \mathscr{T}^c(\mathscr{P})^{(\geq 2)} \to \mathscr{P}, \quad \text{such that } \partial(\gamma) + \gamma \star \gamma = 0.$$

Infinity-morphisms of homotopy operads: morphism of dg cooperads

$$\mathrm{B}\mathscr{P} = \big(\mathscr{T}^c(s\mathscr{P}), d_1 + d_\gamma\big) \to \mathrm{B}\mathscr{Q} = \big(\mathscr{T}^c(s\mathscr{Q}), d_1' + d_\nu\big),$$



HOMOTOPY REPRESENTATION OF OPERAD.

$$\begin{aligned}\mathrm{Hom}_{\infty\text{-}\mathsf{Op}}(\mathscr{P}, \mathrm{End}_A) &\cong \mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega \mathrm{B}\mathscr{P}, \mathrm{End}_A)\\ &= \{\Omega \mathrm{B}\mathscr{P}\textit{-algebra structures on } A\}.\end{aligned}$$

10.7 Exercises

Exercise 10.7.1 (Homotopy $\mathscr{P}$-algebra concentrated in degree 0). Let A be a $\mathbb{K}$-module. We consider it as a dg module concentrated in degree 0 with trivial differential. Prove that a $\mathscr{P}_\infty$-algebra structure on A is a $\mathscr{P}$-algebra structure.

Exercise 10.7.2 (Universal enveloping algebra of an L_∞-algebra [LM95]). In Proposition 10.1.8, we introduced a functor from A_∞-algebras to L_∞-algebras, which is the pullback functor f^* associated to the morphism of operads $f : \Omega Lie^{¡} \to \Omega Ass^{¡}$, see Sect. 5.2.12.

Show that this functor admits a left adjoint functor provided by the *universal enveloping algebra of an* L_∞*-algebra* $(A, d_A, \{\ell_n\}_{n\geq 2})$:

$$U(A) := Ass_\infty(A)/I,$$

where I is the ideal generated, for $n \geq 2$, by the elements

$$\sum_{\sigma\in\mathbb{S}_n} \mathrm{sgn}(\sigma)\varepsilon\big(\mu_n^c; a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}\big) - \ell_n(a_1, \ldots, a_n),$$

where ε is the sign induced by the permutation of the graded elements $a_1, \ldots, a_n \in A$.

HINT. It is a direct consequence of Sect. 5.2.12 and Exercise 5.11.26, where $U(A) = f_!(A)$.

Exercise 10.7.3 (Homotopy pre-Lie-algebra). Make explicit the notion of homotopy pre-Lie algebra, see Sect. 13.4, together with the convolution algebra $\mathfrak{g}_{preLie,A}$, which controls it.

HINT. The Koszul dual operad of *preLie* is *Perm*, which admits a simple presentation, see Sect. 13.4.6.

Exercise 10.7.4 (Equivalent Maurer–Cartan equation ★). Let $\mathscr{C}$ be a coaugmented cooperad, with coaugmentation $\eta : \mathrm{I} \to \mathscr{C}$, and let (A, d_A) be a dg module. To any morphism $\alpha : \mathscr{C} \to \mathrm{End}_A$ of $\mathbb{S}$-modules, such that $\alpha \circ \eta = 0$, we associate the morphism of $\mathbb{S}$-modules $\tilde{\alpha} : \mathscr{C} \to \mathrm{End}_A$ defined by $\tilde{\alpha} \circ \eta(\mathrm{id}) := d_A$ and $\tilde{\alpha} := \alpha$ otherwise. If α has degree -1, then $\tilde{\alpha}$ has also degree -1.

1. Prove that α satisfies the Maurer–Cartan equation $\partial(\alpha) + \alpha \star \alpha = 0$ if and only if $\tilde{\alpha}$ squares to zero, $\tilde{\alpha} \star \tilde{\alpha} = 0$.

Let (B, d_B) be another chain complex and let $\varphi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_A)$ and $\psi \in \mathrm{Tw}(\mathscr{P}^{¡}, \mathrm{End}_B)$ be two $\mathscr{P}_\infty$-algebra structures on A and B respectively.

2. Show that any morphism of dg $\mathbb{S}$-modules $f : \mathscr{P}^{¡} \to \mathrm{End}^A_B$ is an ∞-morphism if and only if $f * \tilde{\varphi} = \tilde{\psi} \circledast f$.

Let $f : \mathscr{P}^{¡} \to \mathrm{End}^A_B$ and $g : \mathscr{P}^{¡} \to \mathrm{End}^B_C$ be two ∞-morphisms.

3. Show directly that the composite

$$g \circledcirc f = \mathscr{P}^{¡} \xrightarrow{\Delta} \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \xrightarrow{g \circ f} \mathrm{End}^B_C \circ \mathrm{End}^A_B \to \mathrm{End}^A_C$$

   of Proposition 10.2.4 is an ∞-morphism.

Exercise 10.7.5 (Action of the convolution algebra ★). Let A and B be two dg modules and let $\mathscr{P}$ be a Koszul operad. We denote by $\mathfrak{g}_B := \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_B)$ the convolution pre-Lie algebra and we consider $\mathfrak{g}^A_B = \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}^A_B)$, as in Sect. 10.2.3. We defined the action $\psi \circledast f$, for $\psi \in \mathfrak{g}_B$ and $f \in \mathfrak{g}^A_B$ and the action $\psi \circledcirc \xi$ on $\mathfrak{g}_B$ by the following composite

$$\psi \circledcirc \xi := \mathscr{P}^{¡} \xrightarrow{\Delta} \mathscr{P}^{¡} \circ \mathscr{P}^{¡} \xrightarrow{\psi \circ \xi} \mathrm{End}_B \circ \mathrm{End}_B \xrightarrow{\gamma_{\mathrm{End}_B}} \mathrm{End}_B .$$

1. Show that $\circledcirc$ defines an associative algebra structure on $\mathfrak{g}_B$, where the unit is the composite of the coaugmentation of $\mathscr{P}^{¡}$ followed by the unit of End_B: $\mathscr{P}^{¡} \to \mathrm{I} \to \mathrm{End}_A$.
2. Show that $\circledast$ defines a left module action of the associative algebra $(\mathfrak{g}_B, \circledcirc)$ on $\mathfrak{m}$.
3. In the same way, show that the action $*$ of $\mathfrak{g}_A := \mathrm{Hom}_{\mathbb{S}}(\mathscr{P}^{¡}, \mathrm{End}_A)$ on $\mathfrak{m}$ is a right pre-Lie action.

Exercise 10.7.6 (Kleisli category [HS10]).

1. Let $\alpha : \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism between a dg cooperad $\mathscr{C}$ and a dg operad $\mathscr{P}$. Show that one can define a comonad K_α in the category of $\mathscr{P}$-algebras by setting

$$K_\alpha(A) := \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha A .$$

(We refer to Sect. 11.3 for more details about the construction $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha A = \Omega_\alpha \mathrm{B}_\alpha A$. Notice that the comonad $K_\alpha = \Omega_\alpha \mathrm{B}_\alpha$ is the comonad arising from the bar–cobar adjunction.)

2. Let C be a category and let (K, Δ, ε) be a comonad on C. Show that one can define a category C_K on the following data. The objects of the category C_K are the ones of the category C. The morphisms in the category C_K between two objects A, A' are the morphisms in the category C between $K(A)$ and A':

$$\mathrm{Hom}_{\mathsf{C}_K}\big(A, A'\big) := \mathrm{Hom}_{\mathsf{C}}\big(K(A), A'\big).$$

This category is called the *Kleisli category* C_K associated to the comonad K.

3. Prove that there is an isomorphism of categories

$$\mathsf{C}_{K_\kappa} \cong \infty\text{-}\mathscr{P}\text{-}\mathsf{alg}.$$

4. Prove that there is an isomorphism of categories

$$\mathsf{C}_{K_\iota} \cong \infty\text{-}\mathscr{P}_\infty\text{-}\mathsf{alg}.$$

For more details, we refer to the paper [HS10] of K. Hess and J. Scott.

Exercise 10.7.7 (Inverse of ∞-isomorphisms ★). Make explicit the inverse of ∞-isomorphisms given in Theorem 10.4.1 in the particular cases where the operad $\mathscr{P}$ is the ns operad *As*, the operad *Com*, and the operad *PreLie*.

Exercise 10.7.8 (Homotopy Transfer Theorem, Solution II). This exercise proposes another proof to the Homotopy Transfer Theorem.

1. Let $\mathscr{C}$ be a dg cooperad and let $\mathscr{P}$ and $\mathscr{Q}$ be two dg operads. Show that any ∞-morphism $\Phi : \mathscr{P} \rightsquigarrow \mathscr{Q}$ between the dg operads $\mathscr{P}$ and $\mathscr{Q}$ naturally induces an ∞-morphism

$$\Phi_* : \mathrm{Hom}(\mathscr{C}, \mathscr{P}) \rightsquigarrow \mathrm{Hom}(\mathscr{C}, \mathscr{Q})$$

between the associated convolution operads.

By Proposition 10.5.6, the ∞-morphism Φ_* induces an ∞-morphism between the dg Lie algebras $\prod_n \mathrm{Hom}(\mathscr{C}(n), \mathscr{P}(n))$ and $\prod_n \mathrm{Hom}(\mathscr{C}(n), \mathscr{Q}(n))$.

2. Show that this ∞-morphism passes to invariant elements, i.e. it induces an ∞-morphism ϕ between the convolution dg Lie algebras $\mathrm{Hom}_{\mathbb{S}}(\mathscr{C}, \mathscr{P})$ and $\mathrm{Hom}_{\mathbb{S}}(\mathscr{C}, \mathscr{Q})$.

Hence, the ∞-morphism $\Psi : \mathrm{End}_W \to \mathrm{End}_V$ of operads of Proposition 10.3.2 induces a natural ∞-morphism

$$\psi_\bullet : \mathfrak{g}_W = \mathrm{Hom}_{\mathbb{S}}\big(\mathscr{P}^{\mathrm{i}}, \mathrm{End}_W\big) \to \mathfrak{g}_V = \mathrm{Hom}_{\mathbb{S}}\big(\mathscr{P}^{\mathrm{i}}, \mathrm{End}_V\big)$$

between the associated convolution dg Lie algebras.

3. Let α be a Maurer–Cartan element in $\mathfrak{g}_W$, which vanishes on the coaugmentation of $\mathscr{P}^{¡}$. Show that $\sum_{n=1}^{\infty} \frac{1}{n!}\psi(\alpha, \ldots, \alpha)$ defines a Maurer–Cartan element in $\mathfrak{g}_V$, which vanishes on the coaugmentation of $\mathscr{P}^{¡}$.
4. Compare this proof of the Homotopy Transfer Theorem with the one given in Proposition 10.3.3.