Chapter 14
Algebras for Information Systems*

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Abstract. We present algebraic formalisms for different kinds of information systems, viz. deterministic, incomplete, and non-deterministic. Algebraic structures generated from these information systems are considered and corresponding abstract algebras are proposed. Representation theorems for these classes of abstract algebras are proved, which lead us to equational logics for deterministic, incomplete, and non-deterministic information systems.

Keywords: Information system, indiscernibility relation, similarity relation, Boolean algebra with operators.

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14.1 Introduction

With time, Pawlak's simple rough set model has seen many generalizations due to demands from different practical situations (*e.g.,* [12, 39, 32, 22, 33, 34]). As we know, the notion of an *approximation space* [29], viz. a tuple (U,R), where U is a non-empty set and R an equivalence relation, plays a crucial role in Pawlak's rough set theory. A useful natural generalization is where the relation R is not necessarily an equivalence. For instance, in [33, 17], a *tolerance approximation space* is considered, where R is a tolerance relation. The notion of lower and upper approximations of a set in these generalized approximation spaces is then defined in a natural way. In Pawlak's definition of lower and upper approximations of a subset X of the domain U in an approximation space (U,R), equivalence classes $[x]_R$ of objects are replaced by the set $R(x) := \{y \in U : (x,y) \in R\}$. That is, lower and upper approximations of a set $X(\subseteq U)$ in a generalized approximation space (U,R) are given as:

$$\underline{X}_R := \{x \in U : R(x) \subseteq X\}, \text{ and } \overline{X}_R := \{x \in U : R(x) \cap X \neq \emptyset\}.$$

There is another way to look at generalizations of Pawlak's rough set theory, viz. from the point of view of *information systems*. Most applications of rough set theory are based on these attribute-value representation models.

Definition 14.1. A *deterministic information system* (DIS) $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$, comprises a non-empty set U of objects, $\mathcal{A}$ of attributes, $\mathcal{V}_a$ of attribute values for each $a \in \mathcal{A}$, and information function $f : U \times \mathcal{A} \rightarrow \bigcup_{a\in\mathcal{A}} \mathcal{V}_a$ such that $f(x,a) \in \mathcal{V}_a$.

$\mathcal{S}_1$ and $\mathcal{S}_2$ of Table 14.1, which provide information about three patients $P1 - P3$ regarding attributes "Temperature (T)", and "Headache (H)", are examples of DISs.

Table 14.1. DISs $\mathcal{S}_1$ ans $\mathcal{S}_2$

(a) DIS $\mathcal{S}_1$

Patient	T	H
P1	very high	yes
P2	very high	no
P3	high	yes

(b) DIS $\mathcal{S}_2$

Patient	T	H
P1	very high	yes
P2	very high	no
P3	no	yes

Given a deterministic information system $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$ and a set $B \subseteq \mathcal{A}$, the *indiscernibility relation* $Ind^{\mathcal{S}}_B$ is an equivalence relation on U defined by:

$$(x,y) \in Ind^{\mathcal{S}}_B, \text{ if and only if } f(x,a) = f(y,a) \text{ for all } a \in B.$$

Thus, given a DIS $\mathcal{S}$ and a set B of attributes, we obtain an approximation space $(U, Ind^{\mathcal{S}}_B)$. For instance, in the above example, corresponding to the attribute set

$B := \{\mathrm{T}, \mathrm{H}\}$, DISs $\mathcal{S}_1$ and $\mathcal{S}_2$ give rise to the approximation spaces $(\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}, Ind_B^{\mathcal{S}_1})$ and $(\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}, Ind_B^{\mathcal{S}_2})$ respectively, where

$$Ind_B^{\mathcal{S}_1} = Ind_B^{\mathcal{S}_2} = \{(\mathrm{P1}, \mathrm{P1}), (\mathrm{P2}, \mathrm{P2}), (\mathrm{P3}, \mathrm{P3})\}.$$

One may then approximate elements of a set X with respect to an attribute set B using the notion of lower/upper approximations in the approximation space $(U, Ind_B^{\mathcal{S}})$. Note that we may have two different DISs $\mathcal{K}_1 := (U, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f_1)$ and $\mathcal{K}_2 := (U, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f_2)$ with the same domain and the same sets of attribute, attribute-values, such that $Ind_{\mathcal{A}}^{\mathcal{K}_1} = Ind_{\mathcal{A}}^{\mathcal{K}_2}$, hence generating the same approximation space with respect to the attribute set $\mathcal{A}$. This is the case, for example, with the DISs of Table 14.1.

The notion of a deterministic information system has been generalized in many ways to consider different practical situations. For instance, information regarding values of some attribute for some object may not be available (unlike the case of a DIS, where the information is *complete*). A distinguished attribute-value $*$ is used to depict such a situation.

Definition 14.2. A tuple $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f)$ is called an *information system* (IS), where $U, \mathcal{A}, Val_a, f$ are as in Definition 14.1 and $* \in \bigcap_{a \in \mathcal{A}} Val_a$. An information system which satisfies $f(x,a) = *$ for some $x \in U$ and $a \in \mathcal{A}$ will be called an *incomplete information system* (IIS).

Observe that a deterministic information system can be identified with the information system $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f)$, where $f(x,a) \neq *$ for all $x \in U$ and $a \in \mathcal{A}$.

In [20, 21], instead of an indiscernibility relation, a *similarity* relation (defined below) is considered as the distinguishability relation in the context of an incomplete information system. The assumption here is that the real value of missing attributes is one from the attribute domain.

$(x,y) \in Sim_B^{\mathcal{S}}$ if and only if , $f(x,a) = f(y,a)$ or $f(x,a) = *$, or $f(y,a) = *$, for all $a \in B$.

One could easily verify that $Sim_B^{\mathcal{S}}$ is a tolerance relation, and thus, an IIS $\mathcal{S}$ and an attribute set B give rise to a tolerance approximation space $(U, Sim_B^{\mathcal{S}})$.

DISs are deterministic in the sense that objects take a single value for each attribute. Thus, a natural generalization of DISs is obtained by allowing an object to take a *set of values* for an attribute.

Definition 14.3. A tuple $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a \in \mathcal{A}} \mathcal{V}_a, f)$ is called a *non-deterministic information system* (NIS), where $U, \mathcal{A}, Val_a$ are as in Definition 14.1 and $f : U \times \mathcal{A} \to \wp(\bigcup_{a \in \mathcal{A}} \mathcal{V}_a)$ such that $f(x,a) \subseteq \mathcal{V}_a$.

Note that an indiscernibility relation $Ind_B^{\mathcal{S}}$ for NISs can be defined in a way identical to that for DISs.

One may attach different interpretations with "$f(x,a) = V$". For instance, as exemplified in [9, 10], if a is the attribute "speaking a language", then $f(x,a) = \{\mathrm{German}, \mathrm{English}\}$ can be interpreted as (i) x speaks German and English and no other languages, (ii) x speaks German and English and possibly other languages, (iii) x speaks German or English but not both, or (iv) x speaks German or English or both. Motivated by these interpretations several relations apart from the indiscernibility relation are defined on non-deterministic information systems (*e.g.,* [28, 36, 10]). We list a few of them below.

Similarity $(x,y) \in Sim_B^S$ if and only if $f(x,a) \cap f(y,a) \neq \emptyset$ for all $a \in B$.
Inclusion $(x,y) \in In_B^S$ if and only if $f(x,a) \subseteq f(y,a)$ for all $a \in B$.
Negative similarity $(x,y) \in NSim_B^S$ if and only if $\sim f(x,a) \cap \sim f(y,a) \neq \emptyset$ for all $a \in B$, where $\sim$ is the complementation relative to $\mathcal{V}_a$.
Complementarity $(x,y) \in Com_B^S$ if and only if $f(x,a) = \sim f(y,a)$ for all $a \in B$.
Weak indiscernibility $(x,y) \in wInd_B^S$ if and only if $f(x,a) = f(y,a)$ for some $a \in B$.
Weak similarity $(x,y) \in wSim_B^S$ if and only if $f(x,a) \cap f(y,a) \neq \emptyset$ for some $a \in B$.
Weak inclusion $(x,y) \in wIn_B^S$ if and only if $f(x,a) \subseteq f(y,a)$ for some $a \in B$.
Weak negative similarity $(x,y) \in wNSim_B^S$ if and only if $\sim f(x,a) \cap \sim f(y,a) \neq \emptyset$ for some $a \in B$.
Weak complementarity $(x,y) \in wCom_B^S$ if and only if $f(x,a) = \sim f(y,a)$ for some $a \in B$.

Each of the relations defined above gives rise to a generalized approximation space, where the relation may not be an equivalence. Thus, one can approximate any subset of the domain using the lower and upper approximations defined on these generalized approximation spaces.

14.1.1 Towards an Algebra for Information Systems

In this chapter, we study classes of algebraic structures that are obtained from deterministic, incomplete and non-deterministic information systems. An algebraic approach to rough set theory was first presented by Iwiński in 1987 [14]. Since then, substantial work has been done on algebraic aspects of the theory (*e.g.,* cf. [17, 31, 4, 37]). In one direction, different representations of rough sets have been considered, and endowed with algebraic structures. It is observed that the algebras induced from approximation spaces are instances of various known as well as new algebraic structures, such as quasi-Boolean algebras, double Stone algebras, Nelson algebras, Łukasiewicz algebras, and topological quasi-Boolean algebras. A detailed survey can be found in [4]. In another direction of research, lower/upper approximations are viewed as unary operators mapping a set to its lower/upper approximations. This observation leads to a class of Boolean algebras with operators (BAO). For instance, in [38], a BAO consisting of two unary operators L and H is considered, where these operators are used to capture the lower and upper approximations. We would like to mention here that the motivation of such a Boolean algebra with operators comes from approximation operators induced by approximation spaces, where *the attribute set does not come into the picture*. However, in ISs, as evident, the notions of approximations are not absolute, but *relative to attribute sets*. In fact, a DIS $\mathcal{S} := (U, \mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a, f)$ determines an algebra $\mathcal{B}_\mathcal{S} := (\wp(U), \cap, \sim, \emptyset, \{\overline{Ind^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}})$, where $\emptyset$ denotes the empty set, $\sim$ is the operation of taking the complement of a set relative to U, $\cap$ that of taking the intersection of two sets and $\overline{Ind^{\mathcal{S}}_B}$ is a unary operator on $\wp(U)$ mapping a set $X(\subseteq U)$ to $\overline{X}_{Ind^{\mathcal{S}}_B}$. In [8], $\mathcal{B}_\mathcal{S}$ is called a *knowledge approximation algebra of type $\mathcal{A}$ derived from the DIS $\mathcal{S}$*.

In this chapter, we are interested in the following standard line of investigation in algebraic studies of classes of structures. In order to study a class $\mathcal{C}$ of structures obtained from information systems, one tries to abstract it through a class $\mathcal{F}$ of structures given by a set of axioms, such that each member of $\mathcal{C}$ is also a member of $\mathcal{F}$. Moreover, the adequacy of the class $\mathcal{F}$ of abstract structures for $\mathcal{C}$ is proved through a *representation theorem*, which involves showing that for every abstract structure $\mathfrak{A} \in \mathcal{F}$, there is a structure $\mathfrak{C} \in \mathcal{C}$, and an isomorphism/embedding (in an appropriate sense) from $\mathfrak{A}$ to $\mathfrak{C}$. In the context of frames, this is just the notion of *informational representability* [27]. It may be mentioned that informationally representable frames were first studied in [35], and a detailed study of informational representability can be found in [11].

In the above lines, the following abstract algebra is proposed in [8] corresponding to knowledge approximation algebras derived from DISs (with finite attribute set $\mathcal{A}$).

Definition 14.4. A structure $\mathfrak{B} := (\mathcal{B}, \kappa_P)_{P\subseteq\mathcal{A}}$ is a *knowledge approximation algebra of type $\mathcal{A}$* (a finite set), if $\kappa_P \in B^B$ for each $P \subseteq \mathcal{A}$ and the following axioms hold for all $x, y \in B$ and $P, Q \subseteq \mathcal{A}$.

(A_0) $\mathcal{B} := (B, \vee, \neg, 0)$ is a complete atomic Boolean algebra.
(A_1) $\kappa_P 0 = 0$.
(A_2) $\kappa_P x \geq x$.
(A_3) $\kappa_P(x \wedge \kappa_P y) = \kappa_P x \wedge \kappa_P y$.
(A_4) $x \neq 0$ implies $\kappa_\emptyset x = 1$.
(A_5) $\kappa_{P\cup Q} x = (\kappa_P x) \wedge (\kappa_Q x)$ if x is an atom of $\mathcal{B}$.

A representation theorem is also presented in [8] stating that every knowledge approximation algebra of type $\mathcal{A}$ is isomorphic to a knowledge approximation algebra of type $\mathcal{A}$ derived from some DIS. In order to see how the required DIS is generated from the given knowledge approximation algebra, we need a few definitions.

Given a knowledge approximation algebra $\mathfrak{B} := (\mathcal{B}, \kappa_P)_{P\subseteq\mathcal{A}}$, the *atomic structure* $\mathbf{At}(\mathbf{B})$ of $\mathfrak{B}$ is $(At(B), T_P)_{P\subseteq\mathcal{A}}$, where $At(B)$ is the set of atoms of B and for each $P \subseteq \mathcal{A}$, T_P is the equivalence relation on $At(B)$ such that

$$(x, y) \in T_P \text{ if and only if } y \leq \kappa_P x.$$

Now the atomic structure $\mathbf{At}(\mathbf{B})$, in turn, determines a DIS $\mathcal{S}(\mathbf{At}(\mathbf{B})) := (At(B), \mathcal{A}, \mathcal{V}, f)$, where $\mathcal{V} := \bigcup_{a\in\mathcal{A}} At(B)/T_{\{a\}}$ and $f(x, a) := [x]_{T_{\{a\}}}$. Finally, in the representation theorem it is shown that $\mathfrak{B} \cong \mathcal{B}_{\mathcal{S}(\mathbf{At}(\mathbf{B}))}$. Note that in this representation theorem, the atomicity property of $\mathfrak{B}$ plays a crucial role.

We note that Comer's work in [8] is confined to DISs only and does not talk about possible extensions to other types of information systems, such as incomplete and non-deterministic information systems. Moreover, the knowledge approximation algebra derived from a DIS $\mathcal{S}$ does not give a complete description of the DIS. In fact, attribute and attribute-value pairs, which are the main ingredients of a DIS, do not appear in this description. In the next section, we shall propose an algebraic formalism of DISs which captures this aspect. An abstract algebra for DISs will be proposed and the corresponding representation theorem will be proved in Section 14.3. In Section 14.4, we will see that this representation theorem also leads us to logics for DISs. In Sections 14.5, 14.6 and 14.7, we shall extend this formalism to incomplete and non-deterministic information systems as well. Section 14.8 concludes the chapter.

14.2 Algebra for Deterministic Information Systems

Let us consider the deterministic information systems $\mathcal{S}_1$ and $\mathcal{S}_2$ of Table 14.1. Table 14.2 below gives the lower and upper approximations of all the subsets of U with respect to indiscernibility relations corresponding to different sets of attributes. Observe that the two different DISs $\mathcal{S}_1$ and $\mathcal{S}_2$ generate the same knowledge approximation algebra $(\wp(U), \cap, \sim, \emptyset, \{\overline{Ind_B^{\mathcal{S}_i}}\}_{B \subseteq \mathcal{A}})$, where $\mathcal{A} := \{T, H\}$ and the operator $\overline{Ind_B^{\mathcal{S}_i}}$ is determined by Table 14.2. This fact shows that a knowledge approximation algebra does not give a complete description of the DISs. This observation leads us to the proposal of a *deterministic information system algebra* given as follows.

Table 14.2. Lower and upper approximations in the information systems $\mathcal{S}_i$, $i = 1, 2$

X	$\underline{X}_{Ind^{\mathcal{S}_i}_{\{T\}}}$	$\overline{X}_{Ind^{\mathcal{S}_i}_{\{T\}}}$	$\underline{X}_{Ind^{\mathcal{S}_i}_{\{H\}}}$	$\overline{X}_{Ind^{\mathcal{S}_i}_{\{H\}}}$	$\underline{X}_{Ind^{\mathcal{S}_i}_{\{T,H\}}}$	$\overline{X}_{Ind^{\mathcal{S}_i}_{\{T,H\}}}$
$\{P1\}$	$\emptyset$	$\{P1,P2\}$	$\emptyset$	$\{P1,P3\}$	$\{P1\}$	$\{P1\}$
$\{P2\}$	$\emptyset$	$\{P1,P2\}$	$\{P2\}$	$\{P2\}$	$\{P2\}$	$\{P2\}$
$\{P3\}$	$\{P3\}$	$\{P3\}$	$\emptyset$	$\{P1,P3\}$	$\{P3\}$	$\{P3\}$
$\{P1,P2\}$	$\{P1,P2\}$	$\{P1,P2\}$	$\emptyset$	U	$\{P1,P2\}$	$\{P1,P2\}$
$\{P1,P3\}$	$\emptyset$	U	$\{P1,P3\}$	$\{P1,P3\}$	$\{P1,P3\}$	$\{P1,P3\}$
$\{P2,P3\}$	$\{P3\}$	U	$\emptyset$	U	$\{P2,P3\}$	$\{P2,P3\}$
U	U	U	U	U	U	U
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Let us fix finite sets $\mathcal{A}$ of attributes and $\mathcal{V} := \bigcup_{a \in \mathcal{A}} \mathcal{V}_a$ of attribute values. Let $\mathcal{D}$ denote the set of all *descriptors* [30], viz. pairs (a, v), for each $a \in \mathcal{A}$, $v \in \mathcal{V}_a$. As we have already seen, given a deterministic information system $\mathcal{S} := (U, \mathcal{A}, \mathcal{V}, f)$, the upper approximations with respect to the indiscernibility relations $Ind_B^{\mathcal{S}}$, $B \subseteq \mathcal{A}$, determine unary operations $\overline{Ind_B^{\mathcal{S}}}$ on $\wp(U)$, viz.

$$\overline{Ind_B^{\mathcal{S}}}(X) := \overline{X}_{Ind_B^{\mathcal{S}}},\ X \subseteq U.$$

Similarly, one has the unary operations $\underline{Ind_B^{\mathcal{S}}}$ determined by lower approximations. Each descriptor (a,v) also determines a nullary operation (constant) $c_{(a,v)}^{\mathcal{S}}$ on $\wp(U)$:

$$c_{(a,v)}^{\mathcal{S}} := \{x \in U : f(x,a) = v\}.$$

Thus, we have the following definition. Let Ω be the tuple $(\mathcal{A}, \mathcal{V})$.

Definition 14.5. Let $\mathcal{S} := (U, \mathcal{A}, \mathcal{V}, f)$ be a deterministic information system. A *deterministic information system algebra (in brief,* DIS-algebra*) of type* Ω *generated by the deterministic information system* $\mathcal{S}$ is the structure

$$\mathcal{S}^* := (\wp(U), \cap, \sim, \emptyset, \{\underline{Ind_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{c_\gamma^{\mathcal{S}}\}_{\gamma\in\mathcal{D}}).$$

Observe that a DIS-algebra generated by a DIS $\mathcal{S}$ is actually an extension of the knowledge approximation algebra derived from $\mathcal{S}$ with a collection of nullary operations. The DIS-algebra generated by the DIS $\mathcal{S}_i$, $i = 1,2$ (cf. Table 14.1) is given by $\mathcal{S}_i{}^* := (\wp(U), \cap, \sim, \emptyset, \{\underline{Ind_B^{\mathcal{S}_i}}\}_{B\subseteq\mathcal{A}}, \{c_\gamma^{\mathcal{S}_i}\}_{\gamma\in\mathcal{D}})$, where $\mathcal{A} := \{T,H\}$, $\mathcal{D} := \{$(T, very high), (T,high), (T,no), (H,yes), (H,no)$\}$, $\underline{Ind_B^{\mathcal{S}_i}}$ is given by Table 14.2, and $c_\gamma^{\mathcal{S}_i}$ by Table 14.3 below. As expected, $\mathcal{S}_1{}^*$ and $\mathcal{S}_2{}^*$ differ only with respect to nullary operators.

Table 14.3. Nullary operators $c_\gamma^{\mathcal{S}_i}$, $i = 1,2$

γ	$c_\gamma^{\mathcal{S}_1}$	$c_\gamma^{\mathcal{S}_2}$
(T,very high)	$\{P1,P2\}$	$\{P1,P2\}$
(T, high)	$\{P3\}$	$\emptyset$
(T,no)	$\emptyset$	$\{P3\}$
(H,no)	$\{P2\}$	$\{P2\}$
(H,yes)	$\{P1,P3\}$	$\{P1,P3\}$

Notation 1. *For the elements x and y of a Boolean algebra* $(U, \wedge, \neg, 0)$*, we shall write* $x \rightarrow y$ *and* $x \leftrightarrow y$ *to denote the elements* $\neg x \vee y$ *and* $(x \rightarrow y) \wedge (y \rightarrow x)$*, respectively. Thus, in particular, for subsets X and Y of U in the power set Boolean algebra with domain* $\wp(U)$*,* $X \rightarrow Y$ *and* $X \leftrightarrow Y$*, respectively, represent the sets* $\sim X \cup Y$ *and* $(X \rightarrow Y) \cap (Y \rightarrow X)$.

The following proposition lists a few properties of DIS-algebras.

Proposition 14.1.

1. $\underline{Ind_B^{\mathcal{S}}}(X \cap Y) = \underline{Ind_B^{\mathcal{S}}}(X) \cap \underline{Ind_B^{\mathcal{S}}}(Y)$, $X \subseteq U$.
2. $\bigcup_{v\in\mathcal{V}_a} c_{(a,v)}^{\mathcal{S}} = U$.

3. $c^{\mathcal{S}}_{(a,v)} \cap c^{\mathcal{S}}_{(a,u)} = \emptyset$ *when* $v \neq u$.
4. $\underline{Ind^{\mathcal{S}}_{C}}(X) \subseteq \underline{Ind^{\mathcal{S}}_{B}}(X)$ *for* $C \subseteq B \subseteq \mathcal{A}$, *and* $X \subseteq U$.
5. $c^{\mathcal{S}}_{(a,v)} \subseteq \underline{Ind^{\mathcal{S}}_{\{a\}}}(c^{\mathcal{S}}_{(a,v)})$.
6. $c^{\mathcal{S}}_{(b,v)} \cap \underline{Ind^{\mathcal{S}}_{B\cup\{b\}}}(X) \subseteq \underline{Ind^{\mathcal{S}}_{B}}(c^{\mathcal{S}}_{(b,v)} \to X)$, $X \subseteq U$.
7. $\underline{Ind^{\mathcal{S}}_{\emptyset}}(X) \neq \emptyset$ *implies* $X = U$.
8. $\underline{Ind^{\mathcal{S}}_{\emptyset}}(U) = U$.

Proof. (1) It is enough to show that $\underline{X \cap Y}_{Ind^{\mathcal{S}}_B} = \underline{X}_{Ind^{\mathcal{S}}_B} \cap \underline{Y}_{Ind^{\mathcal{S}}_B}$. Here,

$$
\begin{aligned}
& x \in \underline{X \cap Y}_{Ind^{\mathcal{S}}_B} \\
\Longleftrightarrow\ & [x]_{Ind^{\mathcal{S}}_B} \subseteq X \cap Y \\
\Longleftrightarrow\ & [x]_{Ind^{\mathcal{S}}_B} \subseteq X, \text{ and } [x]_{Ind^{\mathcal{S}}_B} \subseteq Y \\
\Longleftrightarrow\ & x \in \underline{X}_{Ind^{\mathcal{S}}_B} \cap \underline{Y}_{Ind^{\mathcal{S}}_B}.
\end{aligned}
$$

(2) We just need to prove the inclusion $U \subseteq \bigcup_{v\in\mathcal{V}_a} c^{\mathcal{S}}_{(a,v)}$. So, let us take an arbitrary $x \in U$. Then, $f(x,a) = v$ for some $v \in \mathcal{V}_a$, and hence, we obtain $x \in c^{\mathcal{S}}_{(a,v)}$.
(3) Follows directly from the definition of the operators $c^{\mathcal{S}}_{(a,v)}$, and the fact that f is a function with domain $U \times \mathcal{A}$.
(4) From the definition of the indiscernibility relation, we obtain for $C \subseteq B$, $Ind^{\mathcal{S}}_B \subseteq Ind^{\mathcal{S}}_C$, and hence $\underline{X}_{Ind^{\mathcal{S}}_C} \subseteq \underline{X}_{Ind^{\mathcal{S}}_B}$. Now using the definition of the operators $\underline{Ind^{\mathcal{S}}_B}$, and $\underline{Ind^{\mathcal{S}}_C}$, we obtain the desired result.
(5) Let $x \in c^{\mathcal{S}}_{(a,v)}$. Then, we obtain $f(x,a) = v$. Now let us consider an arbitrary y such that $(x,y) \in Ind^{\mathcal{S}}_{\{a\}}$. Then, $f(y,a) = f(x,a) = v$, and hence, $y \in c^{\mathcal{S}}_{(a,v)}$. Thus, we have shown $x \in \underline{Ind^{\mathcal{S}}_{\{a\}}}(c^{\mathcal{S}}_{(a,v)})$.
(6) Let $x \in c^{\mathcal{S}}_{(b,v)} \cap \underline{Ind^{\mathcal{S}}_{B\cup\{b\}}}(X)$. Let us take an arbitrary y such that $(x,y) \in Ind^{\mathcal{S}}_B$. We need to show $y \in c^{\mathcal{S}}_{(b,v)} \to X$, that is, $y \in \sim c^{\mathcal{S}}_{(b,v)} \cup X$. Let us assume that $y \in c^{\mathcal{S}}_{(b,v)}$, we prove $y \in X$. Note that $x,y \in c^{\mathcal{S}}_{(b,v)}$ implies $(x,y) \in Ind^{\mathcal{S}}_{\{b\}}$. This together with $(x,y) \in Ind^{\mathcal{S}}_B$ gives $(x,y) \in Ind^{\mathcal{S}}_{B\cup\{b\}}$. Thus we obtain $y \in X$ as $x \in \underline{Ind^{\mathcal{S}}_{B\cup\{b\}}}(X)$.
(7) Follows from the fact that $Ind^{\mathcal{S}}_{\emptyset} = U \times U$. (8) is obvious. □

Remark 14.1. Let us explain the above proposition. 2 and 3 say that each object takes precisely one value for each attribute. According to 4, if an object is a positive element of a set X with respect to the indiscernibility relation corresponding to an attribute set C, then it remains so with respect to indiscernibility relations corresponding to any attribute set containing C. 5-6 relate the indiscernibility relations and attribute, attribute value pairs. According to 5, if an object w takes a value v for an attribute a, then every object indiscernible with w with respect to a, also takes the value v for the attribute a. 6 says that if an object w takes the value v for an attribute

b and every object indiscernible with w with respect to attributes from $B \cup \{b\}$ belongs to the set X, then every object indiscernible with w with respect to attributes from B, and which takes the value v for b also belongs to X.

We shall see later that these properties are actually characterizing properties of DIS-algebras. Thus, we propose the following notion of an abstract DIS-algebra.

Definition 14.6. An *abstract DIS-algebra of type* Ω is a tuple

$$\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B \subseteq \mathcal{A}}, \{d_\gamma\}_{\gamma \in \mathcal{D}}),$$

where $(U, \wedge, \neg, 0)$ is a Boolean algebra and L_B and d_γ are, respectively, unary and nullary (constant) operations on U satisfying the following:

(C_0) $L_B(x \wedge y) = L_B(x) \wedge L_B(y)$;
(C_1) $\bigvee_{v \in \mathcal{V}_a} d_{(a,v)} = 1$;
(C_2) $d_{(a,v)} \wedge d_{(a,u)} = 0$ when $v \neq u$;
(C_3) $L_C(x) \leq L_B(x)$ for $C \subseteq B \subseteq \mathcal{A}$;
(C_4) $d_{(a,v)} \leq L_{\{a\}}(d_{(a,v)})$;
(C_5) $d_{(b,v)} \wedge L_{B \cup \{b\}}(x) \leq L_B(d_{(b,v)} \rightarrow x)$;
(C_6) $L_\emptyset(x) \neq 0$ implies $x = 1$.
(C_7) $L_\emptyset(1) = 1$.

As a consequence of Proposition 14.1, the DIS-algebra $\mathcal{S}^*$ generated by a DIS $\mathcal{S}$ is an abstract DIS-algebra.

Let U_B be the dual of the operator L_B, that is, $U_B(x) := \neg L_B(\neg x)$. The following proposition presents a few properties of abstract DIS-algebras.

Proposition 14.2. *1.* $L_B(x) \leq x \leq U_B(x)$.
2. For $x \leq y$, $L_B(x) \leq L_B(y)$, *and* $U_B(x) \leq U_B(y)$.
3. $L_B(x) \vee L_B(y) \leq L_B(x \vee y)$, *and* $U_B(x) \wedge U_B(y) \geq U_B(x \wedge y)$.
4. $U_B(0) = 0$ *and* $L_B(1) = 1$.
5. $U_B(U_B(x)) = U_B(x)$ *and* $L_B(L_B(x)) = L_B(x)$.
6. $U_B(x \wedge U_B(y)) = U_B(x) \wedge U_B(y)$ *and* $L_B(x \vee L_B(y)) = L_B(x) \vee L_B(y)$.
7. $x \neq 0$ *implies* $U_\emptyset x = 1$.
8. $U_B(x \vee y) = U_B(x) \vee U_B(y)$.

Proof. We only provide the proof of items (1)-(5) for L_B.
(1) Using (C_6), we obtain the result for $B = \emptyset$. Next, we prove the result for singleton B. Let $B = \{a\}$ and consider an arbitrary $v \in \mathcal{V}_a$. Then, using (C_5), and the fact that $L_\emptyset(x) \leq x$, we obtain $d_{(a,v)} \wedge L_{\{a\}}(x) \leq d_{(a,v)} \rightarrow x$, and hence, $d_{(a,v)} \wedge L_{\{a\}}(x) \leq x$. Since this is true for all $v \in \mathcal{V}_a$, we obtain $\bigvee_{v \in \mathcal{V}_a} d_{(a,v)} \wedge L_{\{a\}}(x) \leq x$. Thus, (C_1) gives $L_{\{a\}}(x) \leq x$.
Now, assuming $L_B(x) \leq x$ and following exactly the above steps, one can prove $L_{B \cup \{b\}}(x) \leq x$.

(2) Let $x \leq y$. Then, $x \wedge y = x$, and hence, using (C_0), we obtain $L_B(x) \wedge L_B(y) = L_B(x)$. Therefore, $L_B(x) \leq L_B(y)$.

(3) Since $x \leq x \vee y$, using item (2), we obtain $L_B(x) \leq L_B(x \vee y)$. Similarly, we have $L_B(y) \leq L_B(x \vee y)$, and hence, we get $L_B(x) \vee L_B(y) \leq L_B(x \vee y)$.

(4) Follows from (C_3), and (C_7).

(5) First we note that for $x = 1$, $L_\emptyset(L_\emptyset(x)) = L_\emptyset(x) = 1$ (using (C_7)), and for $x \neq 1$, $L_\emptyset(L_\emptyset(x)) = L_\emptyset(x) = 0$ (using (C_6)). Next, we prove the result for singleton B, say $B = \{a\}$. Since $L_{\{a\}}(L_{\{a\}}(x)) \leq L_{\{a\}}(x)$ (by item 1), we just need to show the reverse inequality, that is, $L_{\{a\}}(x) \leq L_{\{a\}}(L_{\{a\}}(x))$. Here,

$$\begin{aligned} d_{(a,v)} \wedge L_{\{a\}}(x) &\leq L_\emptyset(\neg d_{(a,v)} \vee x) \quad \text{(by } (C_5)) \\ &= L_\emptyset L_\emptyset(\neg d_{(a,v)} \vee x) \\ &\leq L_{\{a\}} L_{\{a\}}(\neg d_{(a,v)} \vee x) \quad \text{(by } (C_3) \text{ and item 2).} \end{aligned} \tag{14.1}$$

From (C_4) and item (2), we have $L_{\{a\}}(d_{(a,v)}) \leq L_{\{a\}} L_{\{a\}}(d_{(a,v)})$. Therefore, again using (C_4) we obtain

$$d_{(a,v)} \leq L_{\{a\}} L_{\{a\}}(d_{(a,v)}). \tag{14.2}$$

Thus, we have

$$\begin{aligned} d_{(a,v)} \wedge L_{\{a\}}(x) &\leq L_{\{a\}} L_{\{a\}}(\neg d_{(a,v)} \vee x) \wedge L_{\{a\}} L_{\{a\}}(d_{(a,v)}) \quad \text{(combining (14.1) and (14.2))} \\ &= L_{\{a\}} L_{\{a\}}((\neg d_{(a,v)} \vee x) \wedge d_{(a,v)}) \quad \text{(by } (C_0)) \\ &= L_{\{a\}} L_{\{a\}}(d_{(a,v)} \wedge x). \end{aligned} \tag{14.3}$$

Since (14.3) holds for all $v \in \mathcal{V}_a$, using item (3), we obtain

$$\bigvee_{v \in \mathcal{V}_a} d_{(a,v)} \wedge L_{\{a\}}(x) \leq L_{\{a\}} L_{\{a\}}(\bigvee_{v \in \mathcal{V}_a} d_{(a,v)} \wedge x). \tag{14.4}$$

Therefore, (C_1) gives $L_{\{a\}}(x) \leq L_{\{a\}}(L_{\{a\}}(x))$.
Next, assuming $L_B(x) \leq L_B L_B(x)$ and following exactly the above steps, we can prove $L_{B\cup\{a\}}(x) \leq L_{B\cup\{a\}}(L_{B\cup\{a\}}(x))$. This completes the proof. □

From Proposition 14.2, it is clear that U_B and L_B are, respectively, *closure* and *interior operators*. Moreover, the reduct $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}})$ is a *topological Boolean algebra* [4]. Furthermore, $(U, \wedge, \neg, 0, \{U_B\}_{B\subseteq\mathcal{A}})$ satisfies all the conditions of an abstract knowledge approximation algebra [8], except that in the latter case, the reduct $(U, \wedge, \neg, 0)$ is taken to be a complete atomic Boolean algebra, while we do not have that requirement.

Let us recall that a *cylindric algebra of dimension* $|\mathcal{A}|$ [13] is a structure

$$\mathfrak{A} := (U, \wedge, \neg, 0, \{\Lambda_a\}_{a\in\mathcal{A}}, \{\mu_{(a,b)}\}_{(a,b)\in\mathcal{A}\times\mathcal{A}}),$$

where $(U, \wedge, \neg, 0)$ is a Boolean algebra, and Λ_a, $\mu_{(a,b)}$ are, respectively, unary and nullary operations on U, such that

(L_1) $\Lambda_a(0) = 0$,
(L_2) $x \leq \Lambda_a(x)$,
(L_3) $\Lambda_a(x \wedge \Lambda_a(y)) = \Lambda_a(x) \wedge \Lambda_a(y)$,

(L_4) $\Lambda_a(\Lambda_b(x)) = \Lambda_b(\Lambda_a(x))$,
(L_5) $\mu_{(a,a)} = 1$,
(L_6) If $a \neq b, c$, then $\mu_{(b,c)} = \Lambda_a(\mu_{(b,a)} \wedge \mu_{(a,c)})$,
(L_7) If $a \neq b$, then $\Lambda_a(\mu_{(a,b)} \wedge x) \wedge \Lambda_a(\mu_{(a,b)} \wedge \neg x) = 0$.

The difference between the signature of an abstract DIS-algebra of type $(\mathcal{A}, \mathcal{V})$ and that of a cylindric algebra of dimension $|\mathcal{A}|$ is now clear. The cylindric algebra has unary and nullary operations corresponding to each element of $\mathcal{A}$, and $\mathcal{A} \times \mathcal{A}$, respectively. Whereas, in the case of abstract DIS-algebra, unary and nullary operations are indexed, respectively, over the sets $\wp(\mathcal{A})$ and $\mathcal{A} \times \mathcal{V}$. Moreover, operators U_B of an abstract DIS-algebra satisfy (L_1)–(L_3), but may fail to satisfy (L_4). (L_5)–(L_7) do not make sense in the case of abstract DIS-algebras. However, the BAO $(U, \wedge, \neg, 0, U_B)$ obtained from an abstract DIS-algebra is a cylindric algebra of dimension 1.

14.3 Representation Theorem for Abstract DIS-Algebras

The proof of the representation theorem for abstract knowledge approximation algebras given in [8] makes use of the completeness and atomicity properties of the Boolean reduct of the algebra. In fact, the embedding of an abstract knowledge approximation algebra $\mathfrak{A}$ is given in an extension of the power set algebra over the set $At(\mathfrak{A})$ of atoms of $\mathfrak{A}$. But in the case of abstract DIS-algebras, the Boolean reduct may not be complete and atomic, and hence, this technique will not work. We use *prime filters* [7] for our purpose.

Recall that a *filter* of a Boolean algebra $\mathfrak{A} := (U, \wedge, \sim, 0)$ is a subset F of U such that (i) $1 \in F$, (ii) if $a, b \in F$, then $a \wedge b \in F$, (iii) if $a \in F$ and $a \leq b$, then $b \in F$. A filter is *proper* if it does not contain the smallest element 0. A proper filter is *prime* if $a \vee b \in F$ implies that at least one of a and b belongs to F. We note that for a prime filter F, we have

- $a \rightarrow b, a \in F$ implies $b \in F$, and
- $a \rightarrow b \notin F$ implies $a \in F$ and $b \notin F$.

We shall require these facts later.

Let $PF(\mathfrak{A})$ denote the set of all prime filters of $\mathfrak{A}$.

Let us consider an abstract DIS-algebra $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B \subseteq \mathcal{A}}, \{d_\alpha\}_{\alpha \in \mathcal{D}})$. $\mathfrak{A}$ determines a unique DIS $\mathfrak{A}_*$ as follows.

Consider the mapping $f_{\mathfrak{A}} : PF(\mathfrak{A}) \times \mathcal{A} \rightarrow \mathcal{V}$ such that

$$f_{\mathfrak{A}}(\Gamma, a) = v \text{ if and only if } d_{(a,v)} \in \Gamma.$$

Conditions (C_1) and (C_2) in Definition 14.6 guarantee that $f_{\mathfrak{A}}$ is a total function. Thus, we obtain the DIS $\mathfrak{A}_* := (PF(\mathfrak{A}), \mathcal{A}, \mathcal{V}, f_{\mathfrak{A}})$. $\mathfrak{A}_*$ determines the lower approximation operators $\underline{Ind_B^{\mathfrak{A}_*}}$, $B \subseteq \mathcal{A}$, on $\wp(PF(\mathfrak{A}))$.

We also recall that the reduct $(U, \wedge, \neg, 0, \{L_B\}_{B \subseteq \mathcal{A}})$ of an abstract DIS-algebra $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B \subseteq \mathcal{A}}, \{d_\alpha\}_{\alpha \in \mathcal{D}})$ determines a *complex algebra* [7] as follows.

For each $B \subseteq \mathcal{A}$, let us consider the binary relation $Q_B^{\mathfrak{A}} \subseteq PF(\mathfrak{A}) \times PF(\mathfrak{A})$:

$$(\Gamma, \Delta) \in Q_B^{\mathfrak{A}} \text{ if and only if } L_B(x) \in \Gamma \text{ implies } x \in \Delta.$$

The relations $Q_B^{\mathfrak{A}}$ are used to define the operators $m_B^{\mathfrak{A}} : \wp(PF(\mathfrak{A})) \to \wp(PF(\mathfrak{A}))$:

$$m_B^{\mathfrak{A}}(X) := \{\Gamma \in PF(\mathfrak{A}) : \text{ for all } \Delta \text{ such that } (\Gamma, \Delta) \in Q_B^{\mathfrak{A}},\ \Delta \in X\}.$$

The complex algebra corresponding to the reduct $(U, \wedge, \neg, 0, \{L_B\}_{B \subseteq \mathcal{A}})$ of the abstract DIS-algebra $\mathfrak{A}$ is given by extending the power set algebra over $PF(\mathfrak{A})$ with the operators $m_B^{\mathfrak{A}}$.

So, an abstract DIS-algebra $\mathfrak{A}$, on the one hand, determines the lower approximation operators $\underline{Ind_B^{\mathfrak{A}_*}}$. On the other hand, it gives rise to the complex algebra with operators $m_B^{\mathfrak{A}}$. Is there any relationship between the operators $m_B^{\mathfrak{A}}$, and the lower approximation operators $\underline{Ind_B^{\mathfrak{A}_*}}$? In fact, we shall now show that for each $B \subseteq \mathcal{A}$, the operators $m_B^{\mathfrak{A}}$ and $\underline{Ind_B^{\mathfrak{A}_*}}$ are the same. This result will also lead us to the desired representation theorem. Let us begin with the following proposition listing a few properties of the relations $Q_B^{\mathfrak{A}}$.

Proposition 14.3

1. *$Q_B^{\mathfrak{A}} \subseteq Q_C^{\mathfrak{A}}$ for $C \subseteq B \subseteq \mathcal{A}$.*
2. *$d_{(b,v)} \in \Gamma \cap \Delta$ for some $v \in \mathcal{V}_b$ if and only if $(\Gamma, \Delta) \in Q_{\{b\}}^{\mathfrak{A}}$.*
3. *If $(\Gamma, \Delta) \in Q_B^{\mathfrak{A}}$ and $d_{(b,v)} \in \Gamma \cap \Delta$ for some $v \in \mathcal{V}_b$, then $(\Gamma, \Delta) \in Q_{B \cup \{b\}}^{\mathfrak{A}}$.*
4. *$Q_\emptyset^{\mathfrak{A}} = PF(\mathfrak{A}) \times PF(\mathfrak{A})$.*
5. *$Q_B^{\mathfrak{A}} = \bigcap_{b \in B} Q_{\{b\}}^{\mathfrak{A}}$.*

Proof (1) is a direct consequence of (C_3). Let us prove (2). First suppose $d_{(b,v)} \in \Gamma \cap \Delta$ for some $v \in \mathcal{V}_b$, and let $L_{\{b\}}(x) \in \Gamma$. We need to show $x \in \Delta$. Using the properties of filters, we obtain $d_{(b,v)} \wedge L_{\{b\}}(x) \in \Gamma$, and hence, by (C_5) with $B = \emptyset$, we obtain $L_\emptyset(d_{(b,v)} \to x) \in \Gamma$. This shows that $L_\emptyset(d_{(b,v)} \to x) \neq 0$, and hence, by (C_6), we obtain $d_{(b,v)} \to x = 1$. Therefore, we have $d_{(b,v)} \to x \in \Delta$. Finally using the fact that $d_{(b,v)} \in \Delta$, we obtain $x \in \Delta$.

Conversely, suppose $(\Gamma, \Delta) \in Q_{\{b\}}^{\mathfrak{A}}$. By (C_1), there exists a $v \in \mathcal{V}_b$ such that $d_{(b,v)} \in \Gamma$. Therefore, using (C_4), we obtain $L_{\{b\}}(d_{(b,v)}) \in \Gamma$, and so $d_{(b,v)} \in \Delta$.

Let us now prove (3). Suppose $(\Gamma, \Delta) \in Q_B^{\mathfrak{A}}$ and $d_{(b,v)} \in \Gamma \cap \Delta$ for some $v \in \mathcal{V}_b$. Further, suppose $L_{B \cup \{b\}}(x) \in \Gamma$. We need to show $x \in \Delta$. Due to the given conditions, we obtain $d_{(b,v)} \wedge L_{B \cup \{b\}}(x) \in \Gamma$, and hence, by (C_5), $L_B(d_{(b,v)} \to x) \in \Gamma$. This gives $d_{(b,v)} \to x \in \Delta$, as $(\Gamma, \Delta) \in Q_B^{\mathfrak{A}}$. As $d_{(b,v)} \in \Delta$, $x \in \Delta$.

(4) is obvious due to (C_6). Let us now move to (5). From (1), we obtain $Q_B^{\mathfrak{A}} \subseteq \bigcap_{b \in B} Q_{\{b\}}^{\mathfrak{A}}$. It is also not difficult to see that the reverse inclusion holds when $|B| \leq 1$. To complete the proof, let us assume that the reverse inclusion holds for B, and prove it for $B \cup \{a\}$. Let $(\Gamma, \Delta) \in \bigcap_{b \in B \cup \{a\}} Q_{\{b\}}^{\mathfrak{A}}$. We need to show $(\Gamma, \Delta) \in Q_{B \cup \{a\}}^{\mathfrak{A}}$. Using

(2) and the fact that $(\Gamma,\Delta) \in Q^{\mathfrak{A}}_{\{a\}}$, we obtain $d_{(a,v)} \in \Gamma \cap \Delta$ for some v. Now (3) gives $(\Gamma,\Delta) \in Q^{\mathfrak{A}}_{B\cup\{a\}}$. □

Theorem 14.1. *Let* $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}})$ *be an abstract DIS-algebra. Then, the following hold for each* $B \subseteq \mathcal{A}$.

1. $Ind^{\mathfrak{A}_*}_B = Q^{\mathfrak{A}}_B$.
2. $\underline{Ind^{\mathfrak{A}_*}_B} = m^{\mathfrak{A}}_B$.

Proof. (2) is a direct consequence of (1). So, we only prove (1). Due to (5) of Proposition 14.3, it is enough to prove (1) for singleton B. Let $(\Gamma,\Delta) \in Ind^{\mathfrak{A}_*}_{\{b\}}$. This implies $f_{\mathfrak{A}}(\Gamma,b) = f_{\mathfrak{A}}(\Delta,b) = v$ for some v. Therefore, from the definition of $f_{\mathfrak{A}}$, we obtain $d_{(b,v)} \in \Gamma\cap\Delta$. Now using (2) of Proposition 14.3, we obtain $(\Gamma,\Delta) \in Q^{\mathfrak{A}}_{\{b\}}$ as desired. The reverse inclusion can be proved similarly. □

Theorem 14.2 (Representation theorem for abstract DIS-algebras).
Let $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}})$ *be an abstract DIS-algebra. Then the mapping* $\Psi : U \to \wp(PF(\mathfrak{A}))$ *given by*

$$\Psi(x) := \{\Gamma \in PF(\mathfrak{A}) : x \in \Gamma\},\ x \in U,$$

is an embedding of $\mathfrak{A}$ *into* $(\mathfrak{A}_*)^*$.

Proof. It is not difficult to see that $\Psi(d_\gamma) = c^{\mathfrak{A}_*}_\gamma$, $\gamma \in \mathcal{D}$. Due to item (2) of Theorem 14.1, the rest follows in the lines of the proof of Jónnson-Tarski theorem (cf. [7]). □

14.4 Logics for Deterministic Information Systems

Let us consider a language $\mathcal{L}$ consisting of a countable set $Var := \{p,q,r,\ldots\}$ of variables, a binary operator $\wedge$, unary operators $\neg$, $\mathbf{L}_B$ and constants 0, $\mathbf{d}_{(a,v)}$, where $B \subseteq \mathcal{A}$, $(a,v) \in \mathcal{D}$. The *well-formed formulae* (wffs) of $\mathcal{L}$ are defined recursively:

$$\alpha := p \in Var \mid 0 \mid \mathbf{d}_{(a,v)} \mid \neg\alpha \mid \alpha\wedge\beta \mid \mathbf{L}_B\alpha.$$

Now consider an abstract DIS-algebra $\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}})$. An *assignment* for $\mathfrak{A}$ is a map $V : Var \to U$. V can be extended to a mapping $\tilde{V}$ from the set of all $\mathcal{L}$-wffs to U in the obvious way: $0, d_{(a,v)}, L_B$ correspond, respectively, to $0, d_{(a,v)}, L_B$. An *equation* $\alpha \approx \beta$ is said to hold in $\mathfrak{A}$, denoted as $\mathfrak{A} \models \alpha \approx \beta$, if $\tilde{V}(\alpha) = \tilde{V}(\beta)$ for all V.

The notion of equivalence defined above can be used to realize laws related to DISs and approximations. For instance, one may easily verify that, for all $\mathfrak{A}$,

$$\mathfrak{A} \models ((d_{(b,v)}\wedge L_{B\cup\{b\}}(p))\to(L_B(d_{(b,v)}\to p))) \approx 1,$$

where $\to$ is the logical connective for implication defined in the usual way: $\alpha\to\beta := \neg\alpha\vee\beta$. The representation theorem also leads to the complete axiomatization for the semantic notion of equivalence in DIS-algebras generated by DISs. More formally speaking, using Birkhoff's completeness theorem for equational logic [6], one can prove that if $\alpha \approx \beta$ holds in all DIS-algebras generated by DISs, then $\alpha \approx \beta$ is derivable from the equations (C_0)–(C_7).

We end this section with the remark that abstract DIS-algebras are actually an algebraic counterpart of the logic LIS of deterministic information systems proposed in [15]. In order to see it, recall that LIS-wffs are given by the scheme

$$(a,v) \in \mathcal{D} \mid p \in Var \mid \neg\alpha \mid \alpha \wedge \beta \mid [I(B)]\alpha.$$

Clearly, LIS-wffs are identifiable with the $\mathcal{L}$-wffs by the bijection θ mapping p to p, $\gamma \in \mathcal{D}$ to d_γ and $[I(B)]\alpha$ to $L_B(\theta(\alpha))$. Using Theorem 14.2, one can then show that $\vdash_{LIS} \alpha$ if and only if $\theta(\alpha) \approx 1$ holds in all abstract DIS-algebras.

14.5 Algebra for Incomplete Information Systems

We recall the definitions (Definition 14.2) of information systems and incomplete information systems. As mentioned in Section 14.1, in the case of (incomplete) information systems, a similarity relation is used as the distinguishability relation, rather than the indiscernibility relation. As in the case of the indiscernibility relation, the similarity relation $Sim_B^{\mathcal{S}}$ determines the unary operation $\underline{Sim_B^{\mathcal{S}}}$ on $\wp(U)$ mapping a set X to $\underline{X}_{Sim_B^{\mathcal{S}}}$. Thus, we extend the notion of DIS-algebra to incorporate the similarity relation and define *IS-algebra of type* Ω *generated by the information system* $\mathcal{S}$ as the structure

$$\mathcal{S}^* := (\wp(U), \cap, \sim, \emptyset, \{\underline{Ind_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{\underline{Sim_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{c_\gamma^{\mathcal{S}}\}_{\gamma\in\mathcal{D}}).$$

Moreover, an *abstract IS-algebra* of type Ω is a tuple

$$\mathfrak{A} := (U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}}, \{S_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}}),$$

where $(U, \wedge, \neg, 0, \{L_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}})$ is an abstract DIS-algebra and $\mathcal{S}_B, B \subseteq \mathcal{A}$ are unary operations on U such that

(C_8) $S_C(x) \leq S_B(x)$ for $C \subseteq B \subseteq \mathcal{A}$,
(C_9) $d_{(a,v)} \leq S_{\{a\}}(d_{(a,v)} \vee d_{(a,*)})$, where $v \neq *$,
(C_{10}) $d_{(b,v)} \wedge S_{B\cup\{b\}}(x) \leq S_B((d_{(b,v)} \vee d_{(b,*)}) \to x)$, where $v \neq *$,
(C_{11}) $d_{(b,*)} \wedge S_{B\cup\{b\}}(x) \leq S_B(x)$.

If the abstract IS-algebra satisfies the additional condition

(C_{12}) $\bigvee_{\alpha\in\mathcal{D}'} d_\gamma \neq 1$, where $\mathcal{D}' := \{(a,v) : a \in \mathcal{A}, v \in \mathcal{V}_a \setminus \{*\}\}$,

then it will be called an *abstract IIS-algebra*.

Similarly to Theorem 14.2, we obtain the following representation theorem.

Theorem 14.3 (Representation theorem for abstract (I)IS-algebras)
Every abstract IS-algebra (IIS-algebra) is isomorphic to a subalgebra of $\mathcal{S}^*$ *corresponding to some IS (IIS) S.*

Using the above representation theorem, as in the case of DIS, one can show that the abstract IS-algebra is an algebraic counterpart of the logic proposed in [16] for information systems. Moreover, the theorem also gives an equational logic for ISs consisting of the equations $(C_0)-(C_{11})$.

14.6 Algebra for Non-deterministic Information Systems

In this section, our aim is to propose an algebraic formalism for NISs that would also capture the notion of set approximations with respect to different relations defined on NISs. For the moment, we restrict ourselves to indiscernibility, similarity, and inclusion relations (cf. Section 14.1). In Section 14.5, we have seen that the results obtained in Sections 14.2-14.4 for DISs can be extended in a natural way to obtain an algebra for ISs. The situation is not so simple for NISs. In fact, we need axioms which relate approximations relative to different sets of attributes and attribute-value pairs. In the case of DISs, axioms $(C_3)-(C_5)$ serve the purpose, but axioms (C_4), (C_5) are not sound (as will be illustrated in Example 14.1) when we move to NISs. Therefore, we need replacements for these axioms and for this purpose, we shall take the help of unary operators which provide *names* to objects. These operators will help to reason about the equality of objects. Thus, we have the following notion of an algebra generated from a NIS.

As before, let us consider finite sets $\mathcal{A}$ and $\mathcal{V}_a$, $a \in \mathcal{A}$, of attributes and attribute values and let Ω be the tuple $(\mathcal{A}, \bigcup_{a\in\mathcal{A}} \mathcal{V}_a)$. Recall that the attribute-value pairs $(a,v) \in \mathcal{D}$ represent the collection of objects taking the value v for the attribute a. As objects take *a set of* attribute values in the case of NISs, we consider the set $\mathcal{D}^a := \{(a,V) : V \subseteq \mathcal{V}_a\}$, for each $a \in \mathcal{A}$. Thus, each element (a,v) in $\mathcal{D}$ may be viewed as the element $(a,\{v\})$ of $\mathcal{D}^a$. Observe that $|\prod_{a\in\mathcal{A}} \mathcal{D}^a|$ is finite. Moreover, for any NIS $\mathcal{S} := (U,\mathcal{A},\mathcal{V},f)$, $|U/Ind_{\mathcal{A}}^{\mathcal{S}}| \leq |\prod_{a\in\mathcal{A}} \mathcal{D}^a|$. Let $\Theta := \{i, j, \ldots\}$ be a finite set of "nominals" with $|\Theta| = |\prod_{a\in\mathcal{A}} \mathcal{D}^a|$.

Definition 14.7. Let $\mathcal{S} := (U,\mathcal{A},\mathcal{V},f)$ be a non-deterministic information system. A *non-deterministic information system algebra (in brief,* NIS-algebra*) of type* Ω *generated by the non-deterministic information system* $\mathcal{S}$ is a structure

$$\mathcal{S}^* := (\wp(U), \cap, \sim, \emptyset, \{\underline{Ind_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{\underline{Sim_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{\underline{In_B^{\mathcal{S}}}\}_{B\subseteq\mathcal{A}}, \{c_\gamma^{\mathcal{S}}\}_{\gamma\in\mathcal{D}}, \{c_i^{\mathcal{S}}\}_{i\in\Theta}),$$

where $\underline{Ind_B^{\mathcal{S}}}$, $\underline{Sim_B^{\mathcal{S}}}$, $\underline{In_B^{\mathcal{S}}}$ are operators on $\wp(U)$ mapping a set X to $\underline{X}_{Ind_B^{\mathcal{S}}}$, $\underline{X}_{Sim_B^{\mathcal{S}}}$, and $\underline{X}_{In_B^{\mathcal{S}}}$ respectively, for $\gamma := (a,v) \in \mathcal{D}$, $c_\gamma^{\mathcal{S}}$ is the nullary operation (constant) given by the subset $\{x \in U : f(x,a) = v\}$ of U, and $c_i^{\mathcal{S}}$ are nullary operations on $\wp(U)$ satisfying the following.

(N1) $U/Ind^{\mathcal{S}}_{\mathcal{A}} \subseteq \{c^{\mathcal{S}}_i : i \in \Theta\}$.
(N2) $c^{\mathcal{S}}_i \cap c^{\mathcal{S}}_j = \emptyset$, for $i \neq j$.
(N3) $c^{\mathcal{S}}_i \in U/Ind^{\mathcal{S}}_{\mathcal{A}} \cup \{\emptyset\}$.
(N4) If $(x,y) \notin Ind^{\mathcal{S}}_{\mathcal{A}}$, $[x]_{Ind^{\mathcal{S}}_{\mathcal{A}}} = c^{\mathcal{S}}_i$ and $[y]_{Ind^{\mathcal{S}}_{\mathcal{A}}} = c^{\mathcal{S}}_j$, then $i \neq j$.

From conditions (N1)–(N4), it is clear that the nullary operators $c^{\mathcal{S}}_i$ are used to name the equivalence classes of $Ind^{\mathcal{S}}_{\mathcal{A}}$ such that different equivalence classes are provided with different names. This could also be viewed as providing names to elements of the set U such that elements belonging to the same equivalence class of the relation $Ind^{\mathcal{S}}_{\mathcal{A}}$ are provided with the same name and elements belonging to different equivalence classes of $Ind^{\mathcal{S}}_{\mathcal{A}}$ have different names. Observe that due to size of the set Θ, we have enough nominals to achieve this task. Also note that the reduct $(\wp(U), \cap, \sim, \emptyset, \{\underline{Ind^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{\underline{Sim^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{\underline{In^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{c^{\mathcal{S}}_\gamma\}_{\gamma\in\mathcal{D}})$ of $\mathcal{S}^*$ is determined uniquely, and thus, a NIS $\mathcal{S}$ can generate two distinct NIS-algebras which can differ only with respect to naming of the objects, that is, with respect to nullary operators corresponding to the elements from the set Θ.

We would like to mention here that the above idea of naming objects (elements of the domain) is not new. In fact, it is the main idea of hybrid logics (cf. [7]). The idea of naming objects is also used by Konikowska [18, 19] in the proposals of modal logics for information systems and rough set theory. The main difference between these and our way of naming is that we are providing the same name to the elements belonging to the same equivalence class of $Ind^{\mathcal{S}}_{\mathcal{A}}$ and different names to objects belonging to different classes. In other words, as mentioned above, we are effectively providing names to the equivalence classes instead of individual elements.

The following proposition lists a few properties of NIS-algebras.

Proposition 14.4. *Let X be a subset of the domain U.*

1. $L_C(X) \subseteq L_B(X)$ *for* $C \subseteq B \subseteq \mathcal{A}$, $L \in \{Ind^{\mathcal{S}}, Sim^{\mathcal{S}}, In^{\mathcal{S}}\}$.
2. $c^{\mathcal{S}}_{(a,v)} \subseteq L_{\{a\}}(c^{\mathcal{S}}_{(a,v)})$, $L \in \{Ind^{\mathcal{S}}, In^{\mathcal{S}}\}$.
3. $\sim c^{\mathcal{S}}_{(a,v)} \subseteq Ind^{\mathcal{S}}_{\{a\}}(\sim c^{\mathcal{S}}_{(a,v)})$.
4. $c^{\mathcal{S}}_i \subseteq Sim^{\mathcal{S}}_{\{a\}}(\bigcup_{v\in\mathcal{V}_a}(c^{\mathcal{S}}_{(a,v)} \cap Ind^{\mathcal{S}}_{\emptyset}(c^{\mathcal{S}}_i \to c^{\mathcal{S}}_{(a,v)})))$.
5. $c^{\mathcal{S}}_i \cap Ind^{\mathcal{S}}_{B\cup\{b\}}(X) \subseteq Ind^{\mathcal{S}}_B\left(\bigcap_{v\in\mathcal{V}_b}\left(c^{\mathcal{S}}_{(b,v)} \leftrightarrow Ind^{\mathcal{S}}_{\emptyset}(c^{\mathcal{S}}_i \to c^{\mathcal{S}}_{(b,v)})\right) \to X\right)$.
6. $c^{\mathcal{S}}_{(b,v)} \cap Sim^{\mathcal{S}}_{B\cup\{b\}}(X) \subseteq Sim^{\mathcal{S}}_B(c^{\mathcal{S}}_{(b,v)} \to X)$.
7. $c^{\mathcal{S}}_i \cap In^{\mathcal{S}}_{B\cup\{b\}}(X) \subseteq In^{\mathcal{S}}_B\left(\bigcap_{v\in\mathcal{V}_b}\left(Ind^{\mathcal{S}}_{\emptyset}(c^{\mathcal{S}}_i \to c^{\mathcal{S}}_{(b,v)}) \to c^{\mathcal{S}}_{(b,v)}\right) \to X\right)$.
8. $Ind^{\mathcal{S}}_{\emptyset}(X) = Sim^{\mathcal{S}}_{\emptyset}(X) = In^{\mathcal{S}}_{\emptyset}(X)$.
9. $Ind^{\mathcal{S}}_{\emptyset}(X) \neq \emptyset$ *implies* $X = U$.
10. $c^{\mathcal{S}}_i \cap c^{\mathcal{S}}_{(a,v)} \subseteq Ind^{\mathcal{S}}_{\emptyset}(c^{\mathcal{S}}_i \to c^{\mathcal{S}}_{(a,v)})$.
11. $c^{\mathcal{S}}_i \cap \sim c^{\mathcal{S}}_{(a,v)} \subseteq Ind^{\mathcal{S}}_{\emptyset}(c^{\mathcal{S}}_i \to \sim c^{\mathcal{S}}_{(a,v)})$.
12. $\bigcup_{i\in\Theta} c^{\mathcal{S}}_i = U$.
13. $\sim c^{\mathcal{S}}_i \cup \sim c^{\mathcal{S}}_j = U$ *for* $i \neq j$.

14. $c_i^{\mathcal{S}} \subseteq Ind_{\mathcal{A}}^{\mathcal{S}} c_i^{\mathcal{S}}$.
15. $Ind_{\emptyset}^{\mathcal{S}}(U) = U$.
16. $L_B(X \cap Y) = L_B(X) \cap L_B(Y)$ *for* $L \in \{Ind^{\mathcal{S}}, Sim^{\mathcal{S}}, In^{\mathcal{S}}\}$.

Proof. We only provide the proof of items (4) and (5).
(4) Let $x \in c_i^{\mathcal{S}}$. We need to show $x \in Sim_{\{a\}}^{\mathcal{S}}(\bigcup_{v \in \mathcal{V}_a}(c_{(a,v)}^{\mathcal{S}} \cap Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(a,v)}^{\mathcal{S}})))$. Let y be such that $(x,y) \in Sim_{\{a\}}^{\mathcal{S}}$, that is, there exists a $v \in f(x,a) \cap f(y,a)$. In order to prove the result, it is enough to show that $y \in c_{(a,v)}^{\mathcal{S}} \cap Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(a,v)}^{\mathcal{S}})$. Since $v \in f(y,a)$, we obtain $y \in c_{(a,v)}^{\mathcal{S}}$. In order to prove $y \in Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(a,v)}^{\mathcal{S}})$, let us take an arbitrary $z \in c_i^{\mathcal{S}}$, and we prove $z \in c_{(a,v)}^{\mathcal{S}}$. Since $x, z \in c_i^{\mathcal{S}}$, by the condition (N3) of Definition 14.7, we obtain $z \in [x]_{Ind_{\mathcal{A}}^{\mathcal{S}}}$. Since $v \in f(x,a)$, we obtain $v \in f(z,a)$, and hence $z \in c_{(a,v)}^{\mathcal{S}}$.

(5) Let $x \in c_i^{\mathcal{S}} \cap Ind_{B \cup \{b\}}^{\mathcal{S}}(X)$. Thus, by the condition (N3) of Definition 14.7, we obtain

$$c_i^{\mathcal{S}} = [x]_{Ind_{\mathcal{A}}^{\mathcal{S}}}. \tag{14.5}$$

Let us consider an arbitrary y such that $(x,y) \in Ind_B^{\mathcal{S}}$, and

$$y \in \bigcap_{v \in \mathcal{V}_b} \left(c_{(b,v)}^{\mathcal{S}} \leftrightarrow Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(b,v)}^{\mathcal{S}}) \right). \tag{14.6}$$

We need to show $y \in X$. Since $x \in Ind_{B \cup \{b\}}^{\mathcal{S}}(X)$, it is enough to show that $(x,y) \in Ind_{\{b\}}^{\mathcal{S}}$, that is, $v \in f(x,b)$ if and only if $v \in f(y,b)$.
First suppose $v \in f(x,b)$. Then we have $x \in c_{(b,v)}^{\mathcal{S}}$. Therefore, from (14.5), we obtain $c_i^{\mathcal{S}} = [x]_{Ind_{\mathcal{A}}^{\mathcal{S}}} \subseteq [x]_{Ind_{\{b\}}^{\mathcal{S}}} \subseteq c_{(b,v)}^{\mathcal{S}}$, and hence $Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(b,v)}^{\mathcal{S}}) = U$. Therefore, (14.6) gives $y \in c_{(b,v)}^{\mathcal{S}}$, that is, $v \in f(y,b)$.
Now suppose $v \in f(y,b)$, that is, $y \in c_{(b,v)}^{\mathcal{S}}$. Then from (14.6), we obtain $y \in Ind_{\emptyset}^{\mathcal{S}}(c_i^{\mathcal{S}} \to c_{(b,v)}^{\mathcal{S}})$. This implies that for all $z \in c_i^{\mathcal{S}}$, we have $z \in c_{(b,v)}^{\mathcal{S}}$. Since $x \in c_i^{\mathcal{S}}$, we obtain $x \in c_{(b,v)}^{\mathcal{S}}$. □

Remark 14.2. Note that 8, 9 and 15 list the properties of lower approximations with respect to indiscernibility, similarity and inclusion relations relative to the empty set of attributes. The lower approximation of a proper subset of the domain U with respect to any of these relations relative to the empty set of attributes is empty and that of the domain U is U itself. 10–14 give the rules followed in naming the objects. According to 10, 11, objects with the same name take the same values for each attribute. Properties 12, 13 guarantee that each object is assigned precisely one name. According to 14, objects belonging to the same equivalence class with respect to the indiscernibility relation relative to $\mathcal{A}$ have the same names. Apart from these, there are properties which relate approximations relative to different sets of attributes and attribute-value pairs. We have 1–3 and 5 serving this purpose for the indiscernibility relation. For the similarity relation, we have 1, 4 and 6. 1, 2 and 7 serve the purpose for the inclusion relation.

2 is similar to 4 of Proposition 14.1. According to it, if an object w takes the value v for an attribute a, then every object indiscernible to w relative to a, also takes the value v for a. This is also true for the inclusion relation, but not true for the similarity relation, as will be illustrated by Example 14.1 below. In the case of indiscernibility, we have more: if an object w does not take the value v for an attribute a, then every object indiscernible to w relative to a, also does not take the value v for a. This is precisely what 3 says. 3 is also not true for the similarity relation, and we have 4 for it. According to 4, if an object w is named i, then for every object w' similar to w relative to an attribute a, there exists an attribute value v such that w' and every object named i take the value v for a. Thus, it means there exists an attribute value v such that w and w' take this value for a. 6 is similar to 5 of Proposition 14.1, which is explained in Remark 14.1. The fact captured by 6 is not true for indiscernibility and inclusion relations defined on NISs – this is illustrated by Example 14.1 below. Instead, we have 5 and 7 for these relations. According to 5, if an object w is named i and every object indiscernible with w relative to the attribute set $B \cup \{b\}$ belongs to X, then every object w' such that (i) w' is indiscernible with w relative to the attribute set B and (ii) w' takes precisely those values for b which are taken by the objects named i for b, also belongs to X. The interpretation of 7 for inclusion relation is very similar to the above interpretation of 5 except that in (ii) we have a weaker condition. It says that if an object w is named i and for every w^0 with $(w, w^0) \in In^{S}_{B\cup\{b\}}$, we have $w^0 \in X$, then, every object w' such that (i) $(w, w') \in In^{S}_{B}$ and (ii) w' takes a value v for the attribute b whenever an object named i does so, also belongs to X.

We shall find later that properties 1-16 given in Proposition 14.4 are actually characterizing properties of NIS-algebras. Thus, we propose the following abstract algebra for NISs.

Definition 14.8. An *abstract NIS-algebra of type* Ω is a tuple

$$\mathfrak{A} := (U, \wedge, \neg, 0, \{I_B\}_{B\subseteq \mathcal{A}}, \{S_B\}_{B\subseteq \mathcal{A}}, \{N_B\}_{B\subseteq \mathcal{A}}, \{d_\gamma\}_{\gamma\in \mathcal{D}}, \{d_i\}_{i\in\Theta}),$$

where $(U, \wedge, \neg, 0)$ is a Boolean algebra, I_B, S_B, N_B are unary operations, and d_γ, d_i are nullary (constant) operations on U satisfying the following.

(N_1) $L_C(x) \leq L_B(x)$ for $C \subseteq B \subseteq \mathcal{A}, L \in \{I, S, N\}$.
(N_2) $d_{(a,v)} \leq L_{\{a\}}(d_{(a,v)}), L \in \{I, N\}$.
(N_3) $\neg d_{(a,v)} \leq I_{\{a\}}(\neg d_{(a,v)})$.
(N_4) $d_i \leq S_{\{a\}}(\bigvee_{v\in\mathcal{V}_a}(d_{(a,v)} \wedge I_\emptyset(d_i \to d_{(a,v)})))$.
(N_5) $d_i \wedge I_{B\cup\{b\}}(x) \leq I_B\left(\bigwedge_{v\in\mathcal{V}_b}\left(d_{(b,v)} \leftrightarrow I_\emptyset(d_i \to d_{(b,v)})\right) \to x\right)$.
(N_6) $d_{(b,v)} \wedge S_{B\cup\{b\}}(x) \leq S_B(d_{(b,v)} \to x)$.
(N_7) $d_i \wedge N_{B\cup\{b\}}(x) \leq N_B\left(\bigwedge_{v\in\mathcal{V}_b}\left(I_\emptyset(d_i \to d_{(b,v)}) \to d_{(b,v)}\right) \to x\right)$.

(N_8) $I_\emptyset(x) = S_\emptyset(x) = N_\emptyset(x)$.
(N_9) $I_\emptyset(x) \neq 0$ implies $x = 1$.
(N_{10}) $d_i \wedge d_{(a,v)} \leq I_\emptyset(d_i \rightarrow d_{(a,v)})$.
(N_{11}) $d_i \wedge \neg d_{(a,v)} \leq I_\emptyset(d_i \rightarrow \neg d_{(a,v)})$.
(N_{12}) $\bigvee_{i\in\Theta} d_i = 1$.
(N_{13}) $\neg d_i \vee \neg d_j = 1$ for $i \neq j$.
(N_{14}) $d_i \leq I_{\mathcal{A}} d_i$.
(N_{15}) $I_\emptyset(1) = 1$.
(N_{16}) $L_B(x \wedge y) = L_B(x) \wedge L_B(y)$ for $L \in \{I, S, N\}$.

We note that a NIS-algebra $\mathcal{S}^*$ generated by a NIS $\mathcal{S}$ satisfies the axioms $(N_1) - (N_{16})$, and hence, every NIS-algebra generated by the NISs are abstract NIS-algebra.

Example 14.1. Let us consider the NIS $\mathcal{S}$ of Table 14.4, which is a modified form of the one given in [3].

Table 14.4. NIS $\mathcal{S}$

	Languages (L)	Sports (S)
Ann	{Arabic, Bulgarian}	{athletics, basketball}
Bob	{Arabic, Dutch}	{athletics, basketball}
Cindy	{German, Dutch}	{cycling}

Here $\mathcal{A} := \{L, S\}$, $\mathcal{V}_L :=$ {Arabic, Bulgarian, Dutch, German}, and $\mathcal{V}_S :=$ {athletics, basketball, cycling}. Thus, $|\mathcal{D}^L| = 16$ and $|\mathcal{D}^S| = 8$. Let us take $\Theta := \{1, 2, \ldots, 128\}$. A NIS-algebra generated by NIS $\mathcal{S}$ is given by

$$\mathcal{S}^* := (\wp(U), \cap, \sim, \emptyset, \{\underline{Ind^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{\underline{Sim^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{\underline{In^{\mathcal{S}}_B}\}_{B\subseteq\mathcal{A}}, \{c^{\mathcal{S}}_\gamma\}_{\gamma\in\mathcal{D}}, \{c^{\mathcal{S}}_i\}_{i\in\Theta}),$$

where $U :=$ {Ann, Bob, Cindy}, and the operators are given by Tables 14.5 and 14.6.
We note that

$$c^{\mathcal{S}}_{(\text{L,Bulgarian})} \not\subseteq \underline{Sim^{\mathcal{S}}_{\{L\}}}(c^{\mathcal{S}}_{(\text{L,Bulgarian})}), \tag{14.7}$$

$$c^{\mathcal{S}}_{(\text{L,Arabic})} \cap \underline{In^{\mathcal{S}}_{\{L,S\}}}(\{Bob\}) \not\subseteq \underline{In^{\mathcal{S}}_{\{S\}}}(c^{\mathcal{S}}_{(\text{L,Arabic})} \rightarrow \{Bob\}), \tag{14.8}$$

$$c^{\mathcal{S}}_{(\text{L,Arabic})} \cap \underline{Ind^{\mathcal{S}}_{\{L,S\}}}(\{Bob\}) \not\subseteq \underline{Ind^{\mathcal{S}}_{\{S\}}}(c^{\mathcal{S}}_{(\text{L,Arabic})} \rightarrow \{Bob\}). \tag{14.9}$$

So (14.7)–(14.9) of Example 14.1 show that in a NIS, (N_2) may not be satisfied by the lower approximation operators corresponding to similarity relations and (N_6) may not be satisfied by the lower approximation operators corresponding to indiscernibility and inclusion relations. It is thus that we have used (N_4) as the replacement of (N_2) for the similarity relation, and (N_5), (N_7) as the replacements of (N_6)

Table 14.5. Lower approximation operators generated by the NIS $\mathcal{S}$

	$\underline{Ind^{\mathcal{S}}_{\{L\}}}$, $\underline{Ind^{\mathcal{S}}_{\{L,S\}}}$, $\underline{In^{\mathcal{S}}_{\{L\}}}$, $\underline{In^{\mathcal{S}}_{\{L,S\}}}$	$\underline{Sim^{\mathcal{S}}_{\{L\}}}$	$\underline{Ind^{\mathcal{S}}_{\{S\}}}$, $\underline{In^{\mathcal{S}}_{\{S\}}}$, $\underline{Sim^{\mathcal{S}}_{\{S\}}}$, $\underline{Sim^{\mathcal{S}}_{\{L,S\}}}$
{Ann}	{Ann}	$\emptyset$	$\emptyset$
{Bob}	{Bob}	$\emptyset$	$\emptyset$
{Cindy}	{Cindy}	$\emptyset$	{Cindy}
{Ann, Bob}	{Ann, Bob}	{Ann}	{Ann, Bob}
{Ann, Cindy}	{Ann, Cindy}	$\emptyset$	{Cindy}
{Bob, Cindy}	{Bob, Cindy}	{Cindy}	{Cindy}
U	U	U	U
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 14.6. Nullary operators $c^{\mathcal{S}}_{\gamma}$ and $c^{\mathcal{S}}_{i}$

(a) Nullary operators $c^{\mathcal{S}}_{\gamma}$

γ	$c^{\mathcal{S}}_{\gamma}$
(L,Arabic)	{Ann, Bob}
(L,Bulgarian)	{Ann}
(L,Dutch)	{Bob, Cindy}
(L,German)	{Cindy}
(S,athletics)	{Anna, Bob}
(S,basketball)	{Anna, Bob}
(S,cycling)	{Cindy}

(b) Nullary operators $c^{\mathcal{S}}_{i}$

$i \in \Theta$	$c^{\mathcal{S}}_{i}$
1	{Ann}
2	{Bob}
3	{Cindy}
$i \in \Theta \setminus \{1,2,3\}$	$\emptyset$

for indiscernibility and inclusion relations, respectively. Observe that these replacements make use of the unary operators d_i.

14.7 Representation Theorem for Abstract NIS-Algebras and Equational Logic for NISs

In this section, we shall prove the representation theorem for abstract NIS-algebras, which will also lead us to an equational logic for NISs. We proceed as in the case of abstract DIS-algebras and for each $B \subseteq \mathcal{A}$, we consider the relations $L^{\mathfrak{A}}_B \subseteq PF(\mathfrak{A}) \times PF(\mathfrak{A})$, $L \in \{I, S, N\}$, generated by an abstract NIS-algebra

$$\mathfrak{A} := (U, \wedge, \neg, 0, \{I_B\}_{B\subseteq\mathcal{A}}, \{S_B\}_{B\subseteq\mathcal{A}}, \{N_B\}_{B\subseteq\mathcal{A}}, \{d_\gamma\}_{\gamma\in\mathcal{D}}, \{d_i\}_{i\in\Theta}),$$

such that

$$(\Gamma, \Delta) \in L^{\mathfrak{A}}_B \text{ if and only if } L_B(x) \in \gamma \text{ implies } x \in \Delta.$$

For a $\Gamma \in PF(\mathfrak{A})$, and $b \in \mathcal{A}$, let Γ_b denote the set $\{d_{(b,v)} : d_{(b,v)} \in \Gamma\}$. The following proposition presents a few properties of these operators.

Proposition 14.5

1. *$L_B^{\mathfrak{A}} \subseteq L_C^{\mathfrak{A}}$ for $C \subseteq B \subseteq \mathcal{A}$, $L \in \{I,S,N\}$.*
2. *$\Gamma_b = \Delta_b$ if and only if $(\Gamma,\Delta) \in I_{\{b\}}^{\mathfrak{A}}$.*
3. *$\Gamma_b \cap \Delta_b \neq \emptyset$ if and only if $(\Gamma,\Delta) \in S_{\{b\}}^{\mathfrak{A}}$.*
4. *$\Gamma_b \subseteq \Delta_b$ if and only if $(\Gamma,\Delta) \in N_{\{b\}}^{\mathfrak{A}}$.*
5. *If $(\Gamma,\Delta) \in I_B^{\mathfrak{A}}$ and $\Gamma_b = \Delta_b$, then $(\Gamma,\Delta) \in I_{B\cup\{b\}}^{\mathfrak{A}}$.*
6. *If $(\Gamma,\Delta) \in S_B^{\mathfrak{A}}$ and $\Gamma_b \cap \Delta_b \neq \emptyset$, then $(\Gamma,\Delta) \in S_{B\cup\{b\}}^{\mathfrak{A}}$.*
7. *If $(\Gamma,\Delta) \in N_B^{\mathfrak{A}}$ and $\Gamma_b \subseteq \Delta_b$, then $(\Gamma,\Delta) \in N_{B\cup\{b\}}^{\mathfrak{A}}$.*
8. *$L_\emptyset^{\mathfrak{A}} = PF(\mathfrak{A}) \times PF(\mathfrak{A})$, $L \in \{I,S,N\}$.*
9. *$L_B^{\mathfrak{A}} = \bigcap_{b\in B} L_{\{b\}}^{\mathfrak{A}}$, $L \in \{I,S,N\}$.*

Proof We only provide the proofs of (2) and (3).
(2) First suppose $\Gamma_b = \Delta_b$, and let $I_{\{b\}}(x) \in \Gamma$. We need to show $x \in \Delta$. From (N_{12}), we obtain $d_i \in \Gamma$ for some $i \in \Theta$. Therefore, $d_i \wedge I_{\{b\}}(x) \in \Gamma$. Now using (N_5) for $B = \emptyset$, we obtain

$$I_\emptyset\left(\bigwedge_{v\in\mathcal{V}_b}\left(d_{(b,v)} \leftrightarrow I_\emptyset(d_i \to d_{(b,v)})\right) \to x\right) \in \Gamma.$$

Therefore, (N_9) gives

$$\bigwedge_{v\in\mathcal{V}_b}\left(d_{(b,v)} \leftrightarrow I_\emptyset(d_i \to d_{(b,v)})\right) \to x = 1,$$

and hence

$$\bigwedge_{v\in\mathcal{V}_b}\left(d_{(b,v)} \leftrightarrow I_\emptyset(d_i \to d_{(b,v)})\right) \to x \in \Delta.$$

If possible, let $x \notin \Delta$. Then, there exists a $v \in \mathcal{V}_b$ such that either $d_{(b,v)} \to I_\emptyset(d_i \to d_{(b,v)}) \notin \Delta$, or $I_\emptyset(d_i \to d_{(b,v)}) \to d_{(b,v)} \notin \Delta$. First suppose, $d_{(b,v)} \to I_\emptyset(d_i \to d_{(b,v)}) \notin \Delta$. Then, $d_{(b,v)} \in \Delta$, and $I_\emptyset(d_i \to d_{(b,v)}) \notin \Delta$. Now using the fact that $\Gamma_b = \Delta_b$, we obtain $d_{(b,v)} \in \Gamma$, and hence, $d_i \wedge d_{(b,v)} \in \Gamma$. Therefore, from ($N_{10}$, we obtain $I_\emptyset(d_i \to d_{(b,v)}) \in \Gamma$. Using (N_9), this gives $d_i \to d_{(b,v)} = 1$, and hence, by (N_{15}), $I_\emptyset(d_i \to d_{(b,v)}) = 1$. This implies $I_\emptyset(d_i \to d_{(b,v)}) \in \Delta$, a contradiction. Similarly, $I_\emptyset(d_i \to d_{(b,v)}) \to d_{(b,v)} \notin \Delta$ will also lead us to a contradiction.

Conversely, suppose $(\Gamma,\Delta) \in I_{\{b\}}^{\mathfrak{A}}$. We need to show $d_{(b,v)} \in \Gamma$ if and only if $d_{(b,v)} \in \Delta$. First let $d_{(b,v)} \in \Gamma$. Then, from (N_2), we obtain $I_{\{b\}}d_{(b,v)} \in \Gamma$, and hence $d_{(b,v)} \in \Delta$. Now suppose $d_{(b,v)} \in \Delta$. If $d_{(b,v)} \notin \Gamma$, then using (N_3), we obtain $I_{\{b\}}(\neg d_{(b,v)}) \in \Gamma$, and hence, $\neg d_{(b,v)} \in \Delta$, a contradiction.

(3) First suppose $\Gamma_b \cap \Delta_b \neq \emptyset$ and we show $(\gamma,\Delta) \in S_{\{b\}}^{\mathfrak{A}}$. Let $d_{(b,v)} \in \gamma_b \cap \Delta_b$. Let $S_{\{b\}}(x) \in \gamma$. We need to show $x \in \Delta$. We have $d_{(b,v)} \wedge S_{\{b\}}(x) \in \Gamma$, and hence by

(N_6), $S_\emptyset(d_{(b,v)} \to x) \in \Gamma$. This gives $I_\emptyset(d_{(b,v)} \to x) \in \Gamma$ (due to (N_8)), and therefore, we obtain $d_{(b,v)} \to x = 1$ (by (N_9)). This gives us $x \in \Delta$ as $d_{(b,v)} \in \Delta$.
Conversely, suppose $(\Gamma,\Delta) \in S^{\mathfrak{A}}_{\{b\}}$, and we prove $\Gamma_b \cap \Delta_b \neq \emptyset$. (N_{12}) guarantees the existence of a $i \in \Theta$ such that $d_i \in \Gamma$. Therefore, by (N_4), we obtain $S_{\{b\}}(\bigvee_{v\in\mathcal{V}_b}(d_{(b,v)} \wedge I_\emptyset(d_i \to d_{(b,v)}))) \in \Gamma$, and hence, $\bigvee_{v\in\mathcal{V}_b}(d_{(b,v)} \wedge I_\emptyset(d_i \to d_{(b,v)})) \in \Delta$. Therefore, for some $v \in \mathcal{V}_b$, $d_{(b,v)} \wedge I_\emptyset(d_i \to d_{(b,v)}) \in \Delta$. Using (N_9) and the fact that $d_i \in \Gamma$, this gives us $d_{(b,v)} \in \gamma \cap \Delta$, and hence, $\Gamma_b \cap \Delta_b \neq \emptyset$. □

Let us consider an abstract NIS-algebra $\mathfrak{A} := (U, \wedge, \neg, 0, \{I_B\}_{B\subseteq\mathcal{A}}, \{S_B\}_{B\subseteq\mathcal{A}}, \{N_B\}_{B\subseteq\mathcal{A}}, \{d_\alpha\}_{\alpha\in\mathcal{D}}, \{d_i\}_{i\in\Theta})$. As in the case of abstract DIS algebra, abstract NIS-algebra $\mathfrak{A}$ determines a unique NIS $\mathfrak{A}_* := (PF(\mathfrak{A}), \mathcal{A}, \mathcal{V}, f_{\mathfrak{A}})$, where

$$f_{\mathfrak{A}}(\Gamma,a) = \{v : d_{(a,v)} \in \Gamma\}.$$

The following proposition relates the abstract NIS-algebra $\mathfrak{A}$ and NIS $\mathfrak{A}_*$.

Theorem 14.4. *The following hold for each $B \subseteq \mathcal{A}$.*

1. a. $Ind^{\mathfrak{A}_*}_B = I^{\mathfrak{A}}_B$.
   b. $Sim^{\mathfrak{A}_*}_B = S^{\mathfrak{A}}_B$.
   c. $In^{\mathfrak{A}_*}_B = N^{\mathfrak{A}}_B$.
2. a. $\underline{Ind^{\mathfrak{A}_*}_B} = m_{I^{\mathfrak{A}}_B}$.
   b. $\underline{Sim^{\mathfrak{A}_*}_B} = m_{S^{\mathfrak{A}}_B}$.
   c. $\underline{In^{\mathfrak{A}_*}_B} = m_{N^{\mathfrak{A}}_B}$.

Proof. We only prove (1) for singleton B. First suppose $(\Gamma,\Delta) \in Ind^{\mathfrak{A}_*}_{\{b\}}$. Therefore, we obtain $f_{\mathfrak{A}}(\Gamma,b) = f_{\mathfrak{A}}(\Delta,b)$. This implies $\Gamma_b = \Delta_b$, and hence, by item (2) of Proposition 14.5, we obtain $(\Gamma,\Delta) \in I^{\mathfrak{A}}_{\{b\}}$, as desired. Conversely, let $(\Gamma,\Delta) \in I^{\mathfrak{A}}_{\{b\}}$. Then by item (2) of Proposition 14.5, we obtain $\Gamma_b = \Delta_b$. This gives $f_{\mathfrak{A}}(\Gamma,b) = f_{\mathfrak{A}}(\Delta,b)$ and hence $(\Gamma,\Delta) \in Ind^{\mathfrak{A}_*}_{\{b\}}$. One can prove 1(b) and 1(c) in the same way. □

For each $i \in \Theta$, let us consider the nullary operators $c^{\mathfrak{A}_*}_i$ defined as

$$c^{\mathfrak{A}_*}_i := \{\Gamma \in PF(\mathfrak{A}) : d_i \in \Gamma\}.$$

Then we obtain the following theorem.

Theorem 14.5

1. *$PF(\mathfrak{A})/Ind^{\mathfrak{A}_*}_{\mathcal{A}} \subseteq \{c^{\mathfrak{A}_*}_i : i \in \Theta\}$.*
2. *$c^{\mathfrak{A}_*}_i \cap c^{\mathfrak{A}_*}_j = \emptyset$, for $i \neq j$.*
3. *$c^{\mathfrak{A}_*}_i \in U/Ind^{\mathfrak{A}_*}_{\mathcal{A}} \cup \{\emptyset\}$.*
4. *If $(\Gamma,\Delta) \notin Ind^{\mathfrak{A}_*}_{\mathcal{A}}$, $[\Gamma]_{Ind^{\mathfrak{A}_*}_{\mathcal{A}}} = c^{\mathfrak{A}_*}_i$ and $[\Delta]_{Ind^{\mathfrak{A}_*}_{\mathcal{A}}} = c^{\mathfrak{A}_*}_j$, then $i \neq j$.*

Proof (1) Let $[\Gamma]_{Ind_{\mathcal{A}}^{\mathfrak{A}_*}} \in U/Ind_{\mathcal{A}}^{\mathfrak{A}_*}$. From ($N_{12}$), we obtain $d_i \in \Gamma$ for some $i \in \Theta$. Also, due to (N_{13}), $d_j \notin \Gamma$ for all j distinct from i. We claim that $[\Gamma]_{Ind_{\mathcal{A}}^{\mathfrak{A}_*}} = c_i^{\mathfrak{A}_*}$. In order to see it, first suppose $(\Gamma,\Delta) \in Ind_{\mathcal{A}}^{\mathfrak{A}_*}$. We need to show $d_i \in \Delta$. But since $d_i \in \Gamma$, we obtain, using (N_{14}), $I_{\mathcal{A}} d_i \in \Gamma$. Now using the facts that $(\Gamma,\Delta) \in Ind_{\mathcal{A}}^{\mathfrak{A}_*}$ and $Ind_{\mathcal{A}}^{\mathfrak{A}_*} = I_{\mathcal{A}}^{\mathfrak{A}}$, we obtain $d_i \in \Delta$. Next, suppose $\Delta \in c_i^{\mathfrak{A}_*}$, and we prove $(\Gamma,\Delta) \in Ind_{\mathcal{A}}^{\mathfrak{A}_*}$. For this, by item (2) of Proposition 14.5, it is enough to show that for each $a \in \mathcal{A}$, and $v \in \mathcal{V}_a$, $d_{(a,v)} \in \Gamma$ if and only if $d_{(a,v)} \in \Delta$. First suppose $d_{(a,v)} \in \Gamma$. Then using (N_{10}), we obtain $I_{\emptyset}(d_i \to d_{(a,v)}) \in \Gamma$. Thus using ($N_9$), we obtain $d_i \to d_{(a,v)} \in \Delta$, and hence $d_{(a,v)} \in \Delta$, as $d_i \in \Delta$. Similarly, using (N_{11}), one can show that if $d_{(a,v)} \in \Delta$, then $d_{(a,v)} \in \Gamma$.

(2) Follows from (N_{13}).

(3) Let us consider $c_i^{\mathfrak{A}_*} \neq \emptyset$. Then there exists $\gamma \in c_i^{\mathfrak{A}_*}$. Now giving argument similar to (1), one can show that $[\Gamma]_{Ind_{\mathcal{A}}^{\mathfrak{A}_*}} = c_i^{\mathfrak{A}_*}$.

(4) If possible, let $(\Gamma,\Delta) \notin Ind_{\mathcal{A}}^{\mathfrak{A}_*}$, $[\Gamma]_{Ind_{\mathcal{A}}^{\mathfrak{A}_*}} = [\Delta]_{Ind_{\mathcal{A}}^{\mathfrak{A}_*}} = c_i^{\mathfrak{A}_*}$. Since $(\Gamma,\Delta) \notin Ind_{\mathcal{A}}^{\mathfrak{A}_*}$, without loss of generality we assume the existence of an $a \in \mathcal{A}$ and v such that $v \in f_{\mathfrak{A}}(\Gamma,a)$, but $v \notin f_{\mathfrak{A}}(\Delta,a)$. That is, $d_{(a,v)} \in \Gamma$ and $d_{(a,v)} \notin \Delta$. Therefore, we obtain $d_i \wedge d_{(a,v)} \in \Gamma$ and by (N_{10}), $I_{\emptyset}(d_i \to d_{(a,v)}) \in \Gamma$. This implies $I_{\emptyset}(d_i \to d_{(a,v)}) \neq 0$ and hence by (N_9), $d_i \to d_{(a,v)} = 1$. But this contradicts the fact that $d_i \wedge \neg d_{(a,v)} \in \Delta$. □

Finally, we consider the NIS algebra $(\mathfrak{A}_*)^*$ generated by $\mathfrak{A}_*$ by taking nullary operators corresponding to elements of Θ as $c_i^{\mathfrak{A}_*}$. That is,

$$(\mathfrak{A}_*)^* := (\wp(PF(\mathfrak{A})), \cap, \sim, \emptyset, \{\underline{Ind_B^{\mathfrak{A}_*}}\}_{B\subseteq\mathcal{A}}, \{\underline{Sim_B^{\mathfrak{A}_*}}\}_{B\subseteq\mathcal{A}}, \{\underline{In_B^{\mathfrak{A}_*}}\}_{B\subseteq\mathcal{A}}, \{c_\gamma^{\mathfrak{A}_*}\}_{\gamma\in\mathcal{D}}, \{c_i^{\mathfrak{A}_*}\}_{i\in\Theta}).$$

Now, one can prove the following representation theorem for abstract NIS-algebras, using Theorem 14.4.

Theorem 14.6 (Representation theorem for abstract NIS-algebras)
Let $\mathfrak{A} := (U, \wedge, \neg, 0, \{I_B\}_{B\subseteq\mathcal{A}}, \{S_B\}_{B\subseteq\mathcal{A}}, \{N_B\}_{B\subseteq\mathcal{A}}, \{d_\alpha\}_{\alpha\in\mathcal{D}}, \{d_i\}_{i\in\Theta})$ *be an abstract NIS-algebra. Then, the mapping* $\Psi : U \to \wp(PF(\mathfrak{A}))$ *given by*

$$\Psi(x) := \{\Gamma \in PF(\mathfrak{A}) : x \in \Gamma\},\ x \in U,$$

is an embedding of $\mathfrak{A}$ *into* $(\mathfrak{A}_*)^*$.

As in the case of DISs, the above representation theorem gives us an equational logic for NISs consisting of the axioms $(N_1) - (N_{16})$.

14.7.1 Extension to Other Types of Relations Defined on NISs

So far in our study of NISs, we have restricted ourselves to indiscernibility, similarity and inclusion relations. But, one can extend the scheme of this work to other types of relations defined on NISs as well. In fact, as mentioned in Remark 14.2, the main task is to come up with axioms relating the approximations (with respect to the relation considered) relative to different sets of attributes and attribute-value pairs. We list below the axioms serving this purpose for the relations defined in Section 14.1. Let R_B, $B \subseteq \mathcal{A}$, be the operators corresponding to the relation considered.

Negative similarity relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_C(x) \leq R_B(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_i \leq R_{\{a\}}(\bigvee_{v \in \mathcal{V}_a}(\neg d_{(a,v)} \wedge I_\emptyset(d_i \rightarrow \neg d_{(a,v)})))$.
- $\neg d_{(b,v)} \wedge R_{B \cup \{b\}}(x) \leq R_B(\neg d_{(b,v)} \rightarrow x)$.
- $R_\emptyset = I_\emptyset$.

Complementarity relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_C(x) \leq R_B(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_{(a,v)} \leq R_{\{a\}}(\neg d_{(a,v)})$.
- $\neg d_{(a,v)} \leq R_{\{a\}}(d_{(a,v)})$.
- $d_i \wedge R_{B \cup \{b\}}(x) \leq R_B\left(\bigwedge_{v \in \mathcal{V}_b}\left(\neg d_{(b,v)} \leftrightarrow I_\emptyset(d_i \rightarrow d_{(b,v)})\right) \rightarrow x\right)$.
- $R_\emptyset = I_\emptyset$.

Weak indiscernibility relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_B(x) \leq R_C(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_i \leq R_B\left(\bigvee_{b \in B} \bigwedge_{v \in \mathcal{V}_b}\left(d_{(b,v)} \leftrightarrow I_\emptyset(d_i \rightarrow d_{(b,v)})\right)\right)$.

Weak similarity relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_B(x) \leq R_C(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_i \leq R_B\left(\bigvee_{b \in B} \bigvee_{v \in \mathcal{V}_b}\left(d_{(b,v)} \wedge I_\emptyset(d_i \rightarrow d_{(b,v)})\right)\right)$.

Weak inclusion relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_B(x) \leq R_C(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_i \leq R_B\left(\bigvee_{b \in B} \bigwedge_{v \in \mathcal{V}_b}\left(I_\emptyset(d_i \rightarrow d_{(b,v)}) \rightarrow d_{(b,v)}\right)\right)$.

Weak negative similarity relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_B(x) \leq R_C(x)$ for $C \subseteq B \subseteq \mathcal{A}$.

- $d_i \leq R_B\left(\bigvee_{b\in B}\bigvee_{v\in \mathcal{V}_b}\left(\neg d_{(b,v)} \wedge I_\emptyset(d_i \rightarrow \neg d_{(b,v)})\right)\right)$.

Weak complementarity relation:

- $R_B(x \wedge y) = R_B(x) \wedge R_B(y)$.
- $R_B(x) \leq R_C(x)$ for $C \subseteq B \subseteq \mathcal{A}$.
- $d_i \leq R_B\left(\bigvee_{b\in B}\bigwedge_{v\in \mathcal{V}_b}\left(\neg d_{(b,v)} \leftrightarrow I_\emptyset(d_i \rightarrow d_{(b,v)})\right)\right)$.

Thus we need to consider the axioms listed above, in addition to the axioms (N_9), (N_{15}) for $I_\emptyset$, and (N_{10})–(N_{14}) for d_i. Then one can obtain the counterparts of Theorem 14.4 for these relations. As a consequence of this, we would obtain the desired representation theorems for abstract NIS-algebras that include any of the above relations.

14.8 Conclusions

Classes of algebras induced by information systems – deterministic, incomplete or non-deterministic, are considered. These algebras are also able to capture the notion of approximations defined on these information systems. Abstract algebras are proposed, which model such classes of algebras. Corresponding representation theorems are proved. The representation theorems also lead us to equational logics for the respective information systems. In the process, it is also established that the proposed classes of abstract algebras for DISs and ISs constitute the algebraic counterparts of the logics for information systems studied in [15, 16].

A search for a suitable logic for information systems and rough set approximations remains the main issue of many research articles (*e.g.,* [23, 28, 24, 25, 26, 30, 2, 1, 15, 16], c.f. [11, 5]). In [16], the logics for deterministic/incomplete information systems are extended to propose dynamic logics for information systems, which can capture a formalization of the notion of information and information update in the context of information systems. A natural question would be to extend this work and propose a dynamic logic for non-deterministic information systems. A first step in this direction would be to translate the equational logic for NISs obtained in this chapter, into a modal logic. An extension of the latter to a dynamic logic for NISs may then be thought of, where we can capture the notion of information flow and information update for NISs.

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