A Rigorous Error Analysis of Coupled FEM-BEM Problems with Arbitrary Many Subdomains

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Abstract. In this article, we provide a rigorous a priori error estimate for the symmetric coupling of the finite and boundary element method for the potential problem in three dimensions. Our theoretical framework allows an arbitrary number of polyhedral subdomains. Our bound is not only explicit in the mesh parameter, but also in the subdomains themselves: the bound is independent of the number of subdomains and involves only the shape regularity constants of a certain coarse triangulation aligned with the subdomain decomposition. The analysis includes the so-called BEM-based FEM as a limit case.

1 Introduction

The coupling of the finite element method (FEM) and the boundary element method (BEM) has a fruitful tradition, see e.g. [5, 7, 15, 17, 30, 38]. The computational domain is split into a finite number of subdomains. On some of the subdomains, a finite element mesh is employed, on the remaining subdomains, a boundary element mesh. Here we assume that the meshes are matching. One of the most successful coupling methods is the symmetric coupling introduced by Costabel [5]. A special case of this method is the BEM domain decomposition (DD) method introduced by Hsiao and Wendland in [13], see also [14] and [16]. An error analysis of the

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symmetric FEM-BEM coupling has been provided by Steinbach [30], see also [32] for an analysis of a non-symmetric coupling.

To our best knowledge, in all the available literature on the stability analysis of such FEM-BEM coupling or BEM-DD, it is assumed that the subdomain decomposition is fixed. When considering classes of subdomain decompositions of a fixed computational domain, the a priori error estimates depend not only on the mesh parameters, but on the subdomains themselves.

In the context of pure FEM-DD (see [34], and e.g. the FETI and FETI-DP method introduced in [8, 9]), such error estimates do not depend on the subdomain decomposition at all, because the discretization is never changed when keeping the original domain fixed. On the contrary, for the case of BEM-DD, already by splitting a single subdomain into two subdomains, we change the discretization. To our best knowledge, there is no result available for the symmetric coupling that clarifies the dependence on the subdomains, not even in the simple case where each subdomain is a simplicial coarse element of a coarse mesh. Although in most of the practical applications only a few subdomains are involved, this issue is mathematically unsatisfactory. A desirable error estimate should be explicit in both the fine and coarse mesh parameter.

The first paper towards an explicit analysis is [11], where a so-called BEM-based finite element method is analyzed for the three-dimensional Laplace problem; see also [10]. The BEM-based FEM discretization can be viewed as a special case of the BEM-DD of a domain into polygonal/polyhedral domains whose boundaries are discretized with a few boundary elements, see [3, 4, 10, 37]. If H denotes the typical subdomain diameter, we can express this fact by

$$H \to h.$$

Alternatively, the BEM-based FEM can be viewed as a local Trefftz method [35]. A diagram of all the special cases of FEM-BEM coupling mentioned above is shown in Fig. 1. It is clear that a general analysis in terms of H and h must include the limit case of BEM-based FEM.

The analysis in [11] assumes that each subdomain is the union of a few elements of an auxiliary triangulation with mesh size $H \approx h$. Also, the authors of [11] had to assume that the Poincaré and extension constants of the subdomains and related subregions are uniformly bounded. The theory in [26] yields explicit bounds for the boundary integral operators, at least for three space dimensions. Together with a few more theoretical tools, one obtains "explicit" a priori error estimates.

In the current paper, we provide an analysis for the general symmetric coupling of FEM-BEM with arbitrary subdomains for the potential equation. This includes all the cases sketched in Fig. 1. The assumptions are in their nature less restrictive than in [11]. For the case of three dimensions, we were able to remove all the assumptions on the boundedness of Poincaré and extension constants. We only need that each subdomain is the union of a few elements of a shape regular coarse triangulation and that the exterior angles of each subdomain do not degenerate. Under these

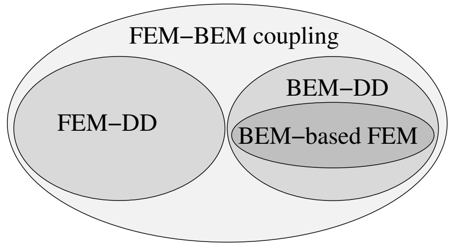

Fig. 1 Diagram of general FEM-BEM coupling and its special cases.

assumptions, we can show explicit bounds for the Poincaré and extension constants. For the bounds of the Poincaré constants we use a result from [28] which builds on [36]. To get the other necessary bounds, we construct an extension operator for polytopes in the spirit of Stein [29] and finally provide an explicit stability estimate.

On the one hand, it is surprising that it took so long to get an analysis with the above (satisfactory) properties, although there are many works available discussing fast solvers for FEM-BEM discretizations with arbitrary many subdomains, see [16, 17, 18, 19, 21, 22, 23, 24, 25, 26]. On the other hand, the analysis below requires some technical tools that were developed only recently.

In the current article, we try to be self-contained up to a certain degree. The remainder is organized as follows. Sect. 2 contains a description of our model problem, the subdomain decomposition, a survey on boundary integral operators, and the symmetric FEM-BEM coupling. In Sect. 3, we present the assumptions and statement of our main result (with the proof postponed). Explicit bounds for boundary integral operators are collected in Sect. 4. This section includes the construction of the explicit extension operator described above (see Sect. 4.2). The proof of the main result is contained in Sect. 5. We conclude with a few remarks on possible extensions.

2 Model Problem and FEM-BEM Coupling

In this section, we describe the model problem and the subdomain decomposition. On each subdomain, we define the harmonic extension operator, the Neumann trace operator, the Steklov-Poincaré operator and a Newton potential. Next, we give a survey on boundary integral operators. In particular, we write the Steklov-Poincaré operator in terms of boundary integral operators. With these ingredients, we formulate the symmetric coupling, which involves a BEM-based approximation of the continuous Steklov-Poincaré operator in the BEM subdomains, and the original bilinear form in the FEM subdomains.

2.1 Model Problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded Lipschitz polytope whose boundary $\partial\Omega$ consists of a Dirichlet boundary Γ_D with positive surface measure and a Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$. The outward unit normal vector to $\partial\Omega$ is denoted by n. We consider the weak form of the following boundary value problem. For given functions $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$, and $g_D \in H^{1/2}(\Gamma_D)$,

$$\text{find } u \in H^1(\Omega),\, u_{|\Gamma_D} = g_D: \quad a(u,v) \;=\; \langle \ell, v\rangle \qquad \forall v \in H^1_D(\Omega), \tag{1}$$

where $H^1_D(\Omega) := \{v \in H^1(\Omega) : v_{|\Gamma_D} = 0\}$ and

$$a(u,v) := \int_\Omega \alpha \nabla u \cdot \nabla v \, dx, \qquad \langle \ell, v\rangle := \int_\Omega f\, v\, dx + \int_{\Gamma_N} g_N\, v\, ds.$$

Above, $\langle\cdot,\cdot\rangle$ denotes the duality pairing. We assume that the coefficient $\alpha \in L^\infty(\Omega)$ is uniformly elliptic, i.e.,

$$\alpha(x) \geq \alpha_0 > 0 \qquad \forall x \in \Omega \text{ a.e.}$$

From these assumptions, it follows that the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is bounded, i.e.,

$$a(v,w) \;\leq\; \|\alpha\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \forall v, w \in H^1(\Omega) \tag{2}$$

and $H^1_D(\Omega)$-coercive, in particular

$$a(v,v) \;\geq\; \frac{\alpha_0}{1 + C_F^2} \|v\|^2_{H^1(\Omega)} \qquad \forall v \in H^1_D(\Omega), \tag{3}$$

where C_F is the Friedrichs constant of Ω with respect to the Dirichlet boundary Γ_D. Since $\ell \in H^1_D(\Omega)^*$, the Lax-Milgram theorem delivers the existence of a unique solution.

2.2 Subdomain Decomposition

Let $\{\Omega_i\}_{i=1}^N$ be a non-overlapping decomposition of Ω into open Lipschitz polytopes such that

$$\overline{\Omega} \;=\; \bigcup_{i=1}^N \overline{\Omega}_i, \qquad \Omega_j \cap \Omega_j \;=\; \emptyset \qquad \text{for } i \neq j. \tag{4}$$

The *skeleton* Γ_S is given by

$$\Gamma_S := \bigcup_{i=1}^{N} \partial\Omega_i .$$

Fig. 2 shows a sample domain $\Omega \subset \mathbb{R}^2$ (with two holes) and a subdomain decomposition.

For each subdomain Ω_i, let n_i denote the outward unit normal vector on $\partial\Omega_i$. We assume that the coefficient is piecewise constant with respect to the subdomain decomposition, i.e.,

$$\alpha_{|\Omega_i} = \alpha_i = \text{const} \qquad \forall i = 1,\dots,N.$$

Thanks to the assumptions on f and g_N, we have the splitting property

$$a(u,v) \;=\; \sum_{i=1}^{N} a_i(u_{|\Omega_i}, v_{|\Omega_i}), \qquad \langle \ell, v\rangle \;=\; \sum_{i=1}^{N} \langle \ell_i, v_{|\Omega_i}\rangle, \tag{5}$$

where $a_i : H^1(\Omega_i) \times H^1(\Omega_i) \to \mathbb{R}$ and $\ell_i \in H^1(\Omega_i)^*$ are given by

$$a_i(u,v) \;=\; \alpha_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx, \qquad \langle \ell_i, v\rangle \;=\; \int_{\Omega_i} f\, v\, dx + \int_{\partial\Omega_i \cap \Gamma_N} g_N\, v\, ds.$$

Note that the theory below can be generalized without any problems to a general functional $\ell \in H^1(\Omega)^*$ that obeys a splitting of the form (5).

2.3 *Operators Associated to the Potential Equation*

Definition 1 (harmonic extension). For each $i = 1,\dots,N$, let $\mathscr{H}_i : H^{1/2}(\partial\Omega_i) \to H^1(\Omega_i)$ denote the harmonic extension operator such that for $v \in H^{1/2}(\partial\Omega_i)$,

$$(\mathscr{H}_i v)_{|\partial\Omega_i} = v, \qquad a_i(\mathscr{H}_i v, w) = 0 \quad \forall w \in H_0^1(\Omega_i).$$

Due to the Ritz minimum principle, we have that

$$\mathscr{H}_i v \;=\; \operatorname{argmin}\big\{ a_i(\widetilde{v}, \widetilde{v}) : \widetilde{v} \in H^1(\Omega_i),\, \widetilde{v}_{|\partial\Omega_i} = v \big\}. \tag{6}$$

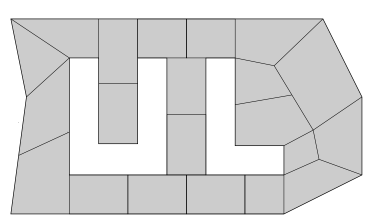

Fig. 2 Example of a subdomain decomposition of a non-convex domain.

Definition 2 (Neumann trace). Let $H_\Delta(\Omega_i) := \{v \in H^1(\Omega_i) : \Delta v \in L^2(\Omega_i)\}$, where Δ is the distributional Laplace operator, and let $\gamma_i^1 : H_\Delta(\Omega_i) \to H^{-1/2}(\partial\Omega_i)$ denote the Neumann trace operator, given by

$$\langle \gamma_i^1 u, v\rangle \;=\; a_i(u, \mathscr{H}_i v) + (\Delta u, \mathscr{H}_i v)_{L^2(\Omega_i)} \qquad \text{for } v \in H^{1/2}(\partial\Omega_i).$$

Note that $\gamma_i^1 u = \alpha_i \frac{\partial u}{\partial n_i}$ for smooth functions u, and that $\Delta \mathscr{H}_i v = 0$ for all functions $v \in H^{1/2}(\partial\Omega_i)$.

Definition 3 (Steklov-Poincaré operator). Let $S_i : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$ denote the Steklov-Poincaré operator, given by $S_i := \gamma_i^1 \mathscr{H}_i$.

We have the relation

$$\langle S_i v, w\rangle \;=\; a_i(\mathscr{H}_i v, \mathscr{H}_i w) \qquad \forall v, w \in H^{1/2}(\partial\Omega_i). \tag{7}$$

Definition 4 (Newton potential). For a functional $\psi \in H^1(\Omega_i)^*$, let $u_\psi \in H_0^1(\Omega_i)$ denote the unique solution of

$$a_i(u_\psi, v) \;=\; \langle \psi, v\rangle \qquad \forall v \in H_0^1(\Omega_i).$$

The Newton potential $N_i : H^1(\Omega_i)^* \to H^{-1/2}(\partial\Omega_i)$ is defined by the relation

$$\langle N_i \psi, v_{|\partial\Omega_i}\rangle \;=\; \langle \psi, v\rangle - a_i(u_\psi, v) \qquad \forall v \in H^1(\Omega_i),$$

see also [20].

For any $u \in H^1(\Omega_i)$ and $\psi \in H^1(\Omega_i)^*$ with $a_i(u, v) = \langle \psi, v\rangle$ for all $v \in H_0^1(\Omega_i)$, we have Green's identity

$$a_i(u, v) - \langle \psi, v\rangle \;=\; \langle S_i u_{|\partial\Omega_i} - N_i \psi, v_{|\partial\Omega_i}\rangle \qquad \forall v \in H^1(\Omega_i), \tag{8}$$

such that $S_i u_{|\partial\Omega_i} - N_i \psi$ is the (generalized) conormal derivative of u.

2.4 Boundary Integral Operators

The fundamental solution of the Laplace operator is given by

$$U^*(x, y) \;=\; \begin{cases} -\frac{1}{2\pi} \log|x-y| & \text{if } d = 2, \\ \frac{1}{4\pi} |x-y|^{-1} & \text{if } d = 3. \end{cases}$$

Following, e.g., [31], we define the four boundary integral operators

$$\begin{aligned} &V_i : H^{-1/2}(\partial\Omega_i) \to H^{1/2}(\partial\Omega_i), \qquad && K_i : H^{1/2}(\partial\Omega_i) \to H^{1/2}(\partial\Omega_i), \\ &K_i' : H^{-1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i), \qquad && D_i : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i), \end{aligned}$$

called in turn single layer potential, double layer potential, adjoint double layer potential, and hypersingular operator. For smooth functions, they obey the integral representations

$$(V_i w)(x) = \int_{\partial\Omega_i} U^*(x,y)\,w(y)\,ds_y, \qquad (K_i v)(x) = \int_{\partial\Omega_i} \frac{\partial U^*}{\partial n_{i,y}}(x,y)\,v(y)\,ds_y,$$
$$(D_i v)(x) = -\frac{\partial}{\partial n_{i,x}} \int_{\partial\Omega_i} \frac{\partial U^*}{\partial n_{i,y}}(x,y)\big(v(y)-v(x)\big)\,ds_y.$$

Note also that V_i and D_i are self-adjoint and K_i' is the adjoint of K_i. We assume throughout the paper that $\operatorname{diam}(\Omega) \le 1$ if $d = 2$, which ensures that the single layer potential operator is elliptic (see e.g. [12, 31]). From the Caldéron identities (cf. [31, Sect. 6.6]), we get

$$S_i = V_i^{-1}(\tfrac{1}{2}I + K_i) = D_i + (\tfrac{1}{2}I + K_i')V_i^{-1}(\tfrac{1}{2}I + K_i). \tag{9}$$

We define the subspaces

$$H_*^{-1/2}(\partial\Omega_i) := \{w \in H^{-1/2}(\partial\Omega_i) : \langle w, 1\rangle = 0\},$$
$$H_*^{1/2}(\partial\Omega_i) := \{v \in H^{1/2}(\partial\Omega_i) : \langle V_i^{-1}v, 1\rangle = 0\},$$

cf. [31, Sect. 6.6.1]. Following [33], we have the contraction property

$$(1-c_{K,i})\,\|v\|_{V_i^{-1}} \le \|(\tfrac{1}{2}I + K_i)v\|_{V_i^{-1}} \le c_{K,i}\,\|v\|_{V_i^{-1}} \qquad \forall v \in H_*^{1/2}(\partial\Omega_i), \tag{10}$$

with the norm $\|v\|_{V_i^{-1}} := \sqrt{\langle V_i^{-1}v, v\rangle}$ and the contraction constant

$$c_{K,i} = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_{0,i}} \in \big(\tfrac{1}{2}, 1\big), \quad \text{where} \quad c_{0,i} = \inf_{v \in H_*^{1/2}(\partial\Omega_i)} \frac{\langle D_i v, v\rangle}{\langle V_i^{-1}v, v\rangle} \in \big(0, \tfrac{1}{4}\big).$$

2.5 Continuous Domain-Skeleton Formulation

Let $I_{\mathrm{BEM}} \subset \{1,\dots,N\}$ denote the subset of subdomain indices where we want to discretize with the boundary element method, and set $I_{\mathrm{FEM}} = \{1,\dots,N\} \setminus I_{\mathrm{BEM}}$. We define two subspaces of partially harmonic functions

$$V_S := \{v \in H^1(\Omega) : \forall i \in I_{\mathrm{BEM}} : v_{|\Omega_i} = \mathscr{H}_i(v_{|\partial\Omega_i})\},$$
$$V_{S,D} := \{v \in V_S : v_{|\Gamma_D} = 0\}.$$

Equipped with the usual H^1-norm, these spaces are Hilbert spaces. We see that the values on $\Gamma_S \cup \big(\bigcup_{i \in I_{\mathrm{FEM}}} \Omega_i\big)$ already determine a function in V_S. Moreover, we have the a-orthogonal splitting

$$H^1(\Omega) \;=\; V_S \oplus \bigcup_{i\in I_{\rm BEM}} H_0^1(\Omega_i).$$

We consider the variational formulation

$$\text{find } u_S \in V_S,\, u_{S|\Gamma_D} = g_D: \quad a_S(u_S, v) \;=\; \langle \ell_S, v\rangle \qquad \forall v \in V_{S,D}, \tag{11}$$

where

$$\begin{aligned} a_S(u,v) \;&=\; \sum_{i\in I_{\rm BEM}} \langle S_i u_{|\partial\Omega_i}, v_{|\partial\Omega_i}\rangle + \sum_{i\in I_{\rm FEM}} a_i(u_{|\Omega_i}, v_{|\Omega_i}),\\ \langle \ell_S, v\rangle \;&=\; \sum_{i\in I_{\rm BEM}} \langle N_i \ell_i, v_{|\partial\Omega_i}\rangle + \sum_{i\in I_{\rm FEM}} \langle \ell_i, v_{|\Omega_i}\rangle. \end{aligned}$$

Since V_S and $V_{S,D}$ are subspaces of $H^1(\Omega)$ and $H_D^1(\Omega)$, it follows immediately that the bilinear form $a_S(\cdot,\cdot): V_S \times V_S \to \mathbb{R}$ is bounded and $V_{S,D}$-coercive. The following lemma follows from Green's identity (8).

Lemma 1. *Let u_S be the unique solution of* (11)*, and for $i \in I_{\rm BEM}$, let $u_i \in H_0^1(\Omega_i)$ be the unique solution of*

$$a_i(u_i, v) \;=\; \langle \ell_i, v\rangle - \langle S_i u_{S|\partial\Omega_i}, v_{|\partial\Omega_i}\rangle \qquad \forall v \in H_0^1(\Omega_i).$$

Then problem (1) *is solved by*

$$u_S + \sum_{i\in I_{\rm BEM}} u_i.$$

In other words, $u_{S|\Omega_i} + u_i$ solves the Dirichlet problem on Ω_i with Dirichlet data $u_{S|\partial\Omega_i}$.

2.6 Symmetric FEM-BEM Coupling

Let $\mathscr{T}^h(\Gamma_S) = \{\gamma\}$ be a simplicial triangulation of the skeleton Γ_S into line segments if $d = 2$ and into triangular faces if $d = 3$. For each $i \in I_{\rm FEM}$, let $\mathscr{T}^h(\Omega_i) = \{\tau\}$ be a simplicial triangulation of Ω_i (into triangles if $d = 2$ and tetrahedra if $d = 3$) that matches with $\mathscr{T}^h(\Gamma_S)$ on $\partial\Omega_i$. Our discretization space is given by

$$V_S^h := \Big\{ v \in V_S:\; \begin{array}{ll} v_{|\gamma} \in P_1 & \forall \gamma \in \mathscr{T}^h(\Gamma_S),\\ v_{|\tau} \in P_1 & \forall \tau \in \mathscr{T}^h(\Omega_i) \quad \forall i \in I_{\rm FEM} \end{array} \Big\},$$

where P_1 are the polynomials of total degree ≤ 1. Functions in V_S^h are piecewise linear on the skeleton. Restricted to a FEM subdomain Ω_i, they are piecewise linear with respect to $\mathscr{T}^h(\Omega_i)$.

Assumption 1. *The Dirichlet data g_D is piecewise linear with respect to the skeleton triangulation.*

Assumption 1 can always be fulfilled by interpolating or projecting the Dirichlet data. The Galerkin discretization of (11) reads

$$\text{find } u_S^h \in V_S^h,\, u^h_{S|\Gamma_D} = g_D: \quad a_S(u_S^h, v^h) \;=\; \langle \ell_S, v^h \rangle \quad \forall v^h \in V^h_{S,D}, \tag{12}$$

where

$$V^h_{S,D} \;:=\; \left\{ v^h \in V_S^h : v^h{}_{|\Gamma_D} = 0 \right\}.$$

With Céa's lemma,

$$\|u_S - u_S^h\|_{H^1(\Omega)} \;\leq\; \frac{\|\alpha\|_{L^\infty(\Omega)}}{\alpha_0}(1+C_F^2) \inf_{v^h \in V_S^h} \|u_S - v^h\|_{H^1(\Omega)}.$$

However, computing the stiffness matrix associated to S_i is in general not possible: although we can express S_i via boundary integral operators, we would need the exact inverse V_i^{-1} that appears in the two representations (9).

For $i \in I_{\text{BEM}}$, we use the following approximation of S_i in terms of the boundary integral operators, see [30, Sect. 3.4] and also [5]. Let the space Z_i^h of piecewise constant functions be given by

$$Z_i^h \;:=\; \{ z \in L^2(\partial\Omega_i) : z_{|\gamma} \in P_0 \quad \forall \gamma \in \mathscr{T}^h(\partial\Omega_i) \} \;\subset\; H^{-1/2}(\partial\Omega_i), \tag{13}$$

where $\mathscr{T}^h(\partial\Omega_i)$ is the restriction of $\mathscr{T}^h(\Gamma_S)$ to $\partial\Omega_i$.

Definition 5 (Approximate Steklov-Poincaré operator). The approximate Steklov-Poincaré operator

$$\widetilde{S}_i : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$$

is defined by

$$\widetilde{S}_i v \;:=\; D_i v + (\tfrac{1}{2} I + K_i) w_i^h(v),$$

where $w_i^h(v) \in Z_i^h$ is the unique solution of the variational problem

$$\langle z^h, V_i w_i^h(v) \rangle \;=\; \langle z^h, (\tfrac{1}{2} I + K_i) v \rangle \qquad \forall z^h \in Z_i^h.$$

Let $w_i(v) \in H^{-1/2}(\partial\Omega_i)$ be given by

$$w_i(v) \;:=\; V_i^{-1}(\tfrac{1}{2} I + K_i) v \;=\; S_i v.$$

By the Galerkin orthogonality and an energy argument,

$$\langle \widetilde{S}_i v, v \rangle \;=\; \langle D_i v, v \rangle + \langle w_i^h(v), V_i w_i^h(v) \rangle \;\leq\; \langle D_i v, v \rangle + \langle w_i(v), V_i w_i(v) \rangle \;=\; \langle S_i v, v \rangle.$$

Using Cauchy's inequality and the contraction properties (10), we obtain that for $v \in H_*^{1/2}(\partial\Omega_i)$,

$$\begin{aligned} \langle S_i v, v \rangle &= \langle V_i^{-1}(\tfrac{1}{2}I + K_i)v, v \rangle \le \|(\tfrac{1}{2}I + K_i)v\|_{V_i^{-1}} \|v\|_{V_i^{-1}} \le c_{K,i} \|v\|_{V_i^{-1}}^2 \\ &\le c_{K,i} c_{0,i}^{-1} \langle D_i, v, v \rangle \le c_{K,i} c_{0,i}^{-1} \Big(\langle D_i, v, v \rangle + \langle V_i w_i^h(v), w_i^h(v) \rangle \Big). \end{aligned}$$

Since the first and last term are invariant to adding a constant, we can summarize that

$$\frac{c_{0,i}}{c_{K,i}} \langle S_i v, v \rangle \le \langle \widetilde{S}_i v, v \rangle \le \langle S_i v, v \rangle \qquad \forall v \in H^{1/2}(\partial \Omega_i), \tag{14}$$

see also [6], [30], and [25, Lemma 1.33]. Using the approximations $\widetilde{S}_i \approx S_i$ for $i \in I_{\mathrm{BEM}}$, we define the modified bilinear form

$$\widetilde{a}_S(v, w) := \sum_{i \in I_{\mathrm{BEM}}} \langle \widetilde{S}_i v, w \rangle + \sum_{i \in I_{\mathrm{FEM}}} a_i(v, w) \qquad \text{for } v, w \in V_S.$$

For simplicity, we assume that there are no volume sources given in the BEM subdomains.

Assumption 2. *For all $i \in I_{\mathrm{BEM}}$, we have $f_{|\Omega_i} = 0$.*

Under Assumption 2, the evaluation of the Newton potential $N_i \ell_i$ simplifies to integrating g_N against a test function over $\partial \Omega_i \cap \Gamma_N$, and so no approximation of N_i is necessary.

The inexact Galerkin formulation corresponding to (11) reads

$$\text{find } u_S^h \in V_S^h,\, u_{S|\Gamma_D}^h = g_D : \quad \widetilde{a}_S(u_S^h, v^h) = \langle \ell_S, v^h \rangle \qquad \forall v^h \in V_{S_D}^h. \tag{15}$$

3 Main Result

In this section, we state our main result: an a-priori error estimate for the formulation (15). Not only will this estimate be explicit in the discretization parameters, but it will in a certain sense be independent of the subdomain decomposition. In order to parameterize the subdomain decomposition, we could assume that each subdomain is an element of a coarse mesh. To be more general and to allow at least for subdomains that are polytopes, we use the following assumption which is standard in the theory of iterative substructuring methods, cf. [34, Assumption 4.3].

Assumption 3. *Each subdomain Ω_i is the union of a few simplicial elements of a global shape regular triangulation $\mathscr{T}^H(\Omega)$ such that the number of coarse elements per subdomain is uniformly bounded.*

Let $H_i = \operatorname{diam}(\Omega_i)$ denote the subdomain diameters. The above assumption implies that $H_i \simeq H_j$ if $\partial \Omega_i \cap \partial \Omega_j \neq \emptyset$, and that each subdomain boundary $\partial \Omega_i$ splits into a uniformly bounded number of coarse facets (cf. [11, Assumption 4.4]).

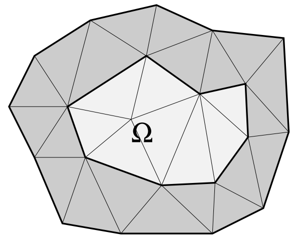

Fig. 3 Sketch of triangulation $\mathscr{T}^H(\widehat{\Omega})$. In dark: $\widehat{\Omega}\setminus\Omega$, thin lines indicate the coarse elements of $\mathscr{T}^H(\widehat{\Omega})$.

The next assumption essentially states that the exterior angles of all subdomains (including those touching the outer boundary $\partial\Omega$) are bounded away from zero, see also Sect. 6.

Assumption 4. *The coarse triangulation $\mathscr{T}^H(\Omega)$ from Assumption 3 can be extended to a shape regular triangulation $\mathscr{T}^H(\widehat{\Omega})$ of a larger domain $\widehat{\Omega}\supset\overline{\Omega}$.*

For an illustration see Fig. 3. Our final assumption concerns the fine triangulations used for the FEM and BEM.

Assumption 5. *The triangulations $\mathscr{T}^h(\Gamma_S)$ and $\mathscr{T}^h(\Omega_i)$, $i\in I_{\mathrm{FEM}}$, are shape regular.*

We define the local mesh parameters

$$h_i := \begin{cases} \max\limits_{\gamma\in\mathscr{T}^h(\partial\Omega_i)} \operatorname{diam}(\gamma) & \text{if } i\in I_{\mathrm{BEM}},\\ \max\limits_{\tau\in\mathscr{T}^h(\Omega_i)} \operatorname{diam}(\tau) & \text{if } i\in I_{\mathrm{FEM}}, \end{cases}$$

and set $h := \max_{i=1}^N h_i$.

Theorem 1. *Let $d=3$, let Assumptions 1–5 hold, and suppose that the solution u of* (1) *satisfies $u\in H^2(\Omega)$. Then for the solution u_S^h of* (15)*,*

$$\|u_S - u_S^h\|_{H^1(\Omega)} \le C\Big(\sum_{i=1}^N h_i^2\,|u|^2_{H^2(\Omega_i)}\Big)^{1/2} \le C\,h\,|u|_{H^2(\Omega)}\,.$$

The constant C depends only on the coefficient α, on the Friedrichs constant C_F, and on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$, $\mathscr{T}^h(\Gamma_S)$ and $\mathscr{T}^h(\Omega_i)$, $i\in I_{\mathrm{FEM}}$.

Proof. The proof is postponed to Sect. 5.4. □

4 Explicit Bounds for the Constants $c_{0,i}$

In this subsection, we work out an explicit lower bound for the constants $c_{0,i}$ from Sect. 2.4 in three dimensions which depends only on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$. We heavily use the results from [27], where a series of constants related to the boundary integral operators V_i and D_i are bounded in terms of Poincaré and extension constants. Throughout the rest of the paper, C denotes a generic constant.

4.1 Explicit Bounds for Poincaré Constants

Definition 6. For a bounded Lipschitz domain $D \subset \mathbb{R}^3$, the Poincaré constant is defined as the smallest constant $C_P(D)$ such that

$$\|v - \overline{v}^D\|_{L^2(D)} \leq C_P(D)\,\mathrm{diam}(D)\,|v|_{H^1(D)} \qquad \forall v \in H^1(D),$$

where $\overline{v}^D = |D|^{-1} \int_D v\,dx$ is the mean value of v.

The following lemma is a direct consequence of [28, Lemma 4.1], see also [36].

Lemma 2. *Let Assumption 3 hold and let m be a fixed integer. Then there exists a constant C that depends only on m and on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$ such that for any connected union D of at most m coarse elements of $\mathscr{T}^H(\widehat{\Omega})$,*

$$C_P(D) \leq C.$$

4.2 An Extension Operator for Polytopes

In this subsection, we define a Sobolev extension operator for Lipschitz polytopes in the spirit of Stein [29] and provide an explicit estimate in terms of shape regularity constants only.

Let D be the connected union of a few elements from $\mathscr{T}^H(\Omega)$ and let its open surrounding D' be defined by

$$\overline{D'} = \bigcup \{\overline{T} : T \in \mathscr{T}^H(\widehat{\Omega}),\, T \not\subset D,\, \overline{T} \cap \partial D \neq \emptyset\}, \tag{16}$$

see Fig. 4 (right). Let $\mathscr{V}_{\partial D} = \{p\}$ be the set of coarse vertices of $\mathscr{T}^H(\widehat{\Omega})$ that lie on ∂D. For each such coarse vertex, we define the vertex patch ω_p by

$$\overline{\omega}_p = \bigcup \{\overline{T} : T \in \mathscr{T}^H(\widehat{\Omega}),\, p \in \overline{T}\},$$

and

$$\omega_p^{\mathrm{int}} := \omega_p \cap D, \qquad \omega_p^{\mathrm{ext}} := \omega_p \cap D',$$

cf. Fig. 4 (right). Without loss of generality, we assume that ω_p^{int} and ω_p^{ext} each contain at least one coarse node that does not lie on ∂D. This condition can always be fulfilled by formally subdividing some of the coarse elements.

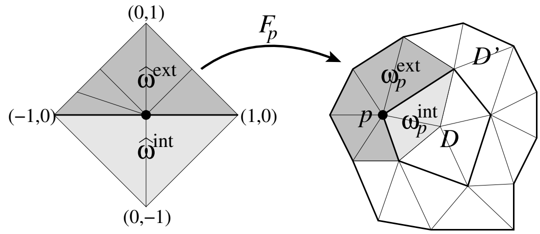

Fig. 4 Mapping of a node patch ω_p in two dimensions.

We define the reference patch

$$\widehat{\omega} := \begin{cases} \operatorname{conv}^\circ(\{(-1,0),(1,0),(0,1),(0,-1)\}) & \text{if } d=2, \\ \operatorname{conv}^\circ(\{(-1,0,0),(1,1,0),(1,-1,0),(0,0,1),(0,0,-1)\}) & \text{if } d=3, \end{cases}$$

where $\operatorname{conv}^\circ(S)$ denotes the interior of the convex hull of the set S. Furthermore, we define the subsets

$$\widehat{\omega}^{\text{int}} := \widehat{\omega} \cap \{x : x_d < 0\}, \qquad \widehat{\omega}^{\text{ext}} := \widehat{\omega} \cap \{x : x_d > 0\},$$

where x_d refers to the d-th component of x.

Let $\mathscr{T}_p(\widehat{\omega})$ be a shape regular simplicial triangulation of $\widehat{\omega}$ such that there exists a bijective continuous mapping $F_p : \widehat{\omega} \to \omega_p$ with the following properties.

- For each element $T \in \mathscr{T}_p(\widehat{\omega})$, the restricted mapping $F_{p|T}$ is affine linear,
- $F_p(0) = p$,
- $F_p(\widehat{\omega} \cap \{x : x_d = 0\}) = \omega_p \cap \partial D$,
- $F_p(\widehat{\omega}^{\text{int}}) = \omega_p^{\text{int}}$ and $F_p(\widehat{\omega}^{\text{ext}}) = \omega_p^{\text{ext}}$,
- for each element $T \in \mathscr{T}_p(\widehat{\omega})$,

$$c_1 H_D^d \le \det(F'_{p|T}) \le c_2 H_D^d,$$
$$\|F'_{p|T}\|_{\ell^2} \le c_3 H_D, \qquad \|(F'_{p|T})^{-1}\|_{\ell^2} \le c_4 H_D^{-1},$$

where $H_D := \operatorname{diam}(D)$ and the constants c_1, c_2, c_3, and c_4 only depend on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$.

For an illustration in two dimensions, see Fig. 4. Under the conditions on $\mathscr{T}^H(\widehat{\Omega})$ stated in Assumption 4, such a triangulation and mapping exists for every coarse vertex $p \in \mathscr{V}_{\partial D}$.

On the reference patch we define

$$\widehat{E} : H^1(\widehat{\omega}^{\text{int}}) \to H^1(\widehat{\omega}^{\text{ext}}), \qquad (\widehat{E}w)(x_1,\ldots,x_d) := w(x_1,\ldots,x_{d-1},-x_d),$$

i.e., the reflection of v across the hyperplane $\{x : x_d = 0\}$, where the above definition first applies to C^∞ functions and is then completed by density (which indeed leads to a bounded operator). For each coarse node $p \in \mathscr{V}_{\partial D}$ we define

$$E_p : H^1(\omega_p^{\text{int}}) \to H^1(\omega_p^{\text{ext}}), \qquad E_p v := \big(\widehat{E}(v \circ F_p)\big) \circ F_p^{-1}.$$

Since F_p is continuous and piecewise affine linear, $E_p v$ is indeed in H^1. Furthermore, we have by construction that

$$(E_p v)_{|\omega_p \cap \partial D} \;=\; v_{|\omega_p \cap \partial D}\,.$$

Finally, we define the extension operator

$$\mathscr{E}_D : H^1(D) \to H^1(\mathbb{R}^d), \qquad (\mathscr{E}_D v)_{|D} := v, \quad (\mathscr{E}_D v)_{|D'} := \sum_{p \in \mathscr{V}_{\partial D}} \varphi_p \cdot E_p v,$$

where φ_p is the nodal finite element basis function on $\mathscr{T}^H(\widehat{\Omega})$ associated with the coarse node p.

Lemma 3. *Let Assumptions 3 and 4 hold, let D be the connected union of a few elements from $\mathscr{T}^H(\Omega)$, and let the extension operator $\mathscr{E}_D$ be defined as above. Then $\mathscr{E}_D$ indeed maps into $H^1(\mathbb{R}^d)$. Let $\mathscr{D} = \{D\}$ be a collection of subregions of Ω such that every $D \in \mathscr{D}$ is the connected union of at most m elements of $\mathscr{T}^H(\Omega)$. Then there exists a constant $C_{\mathscr{E}}$ depending only on m and on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$ such that for all $D \in \mathscr{D}$,*

$$|\mathscr{E}_D v|^2_{H^1(\mathbb{R}^d)} + H_D^{-2}\,\|\mathscr{E}_D v\|^2_{L^2(\mathbb{R}^d)} \;\leq\; C_{\mathscr{E}}\left(|v|^2_{H^1(D)} + H_D^{-2}\,\|v\|^2_{L^2(D)}\right) \quad \forall v \in H^1(D).$$

Proof. Let $v \in H^1(D)$ be arbitrary but fixed. For each $p \in \mathscr{V}_{\partial D}$, the function $\varphi_p \cdot E_p v$ vanishes on $\mathbb{R}^d \setminus (\overline{D} \cup \omega_p^{\text{ext}})$. Hence,

$$(\mathscr{E}_D v)_{|\mathbb{R}^d \setminus \overline{D}} \;\in\; H^1(\mathbb{R}^d \setminus \overline{D}).$$

Since

$$\sum_{p \in \mathscr{V}_{\partial D}} \varphi_p(x) \;=\; 1 \qquad \forall x \in \partial D,$$

we have $(\mathscr{E}_D v)_{|\partial D} = v_{|\partial D}$ and hence $\mathscr{E}_D v \in H^1(\mathbb{R}^d)$. With standard finite element techniques (see e.g. [1, 2]), one shows that

$$|E_p v|_{H^1(\omega_p^{\text{ext}})} \;\leq\; C\,|v|_{H^1(\omega_p^{\text{int}})}, \qquad \|E_p v\|_{L^2(\omega_p^{\text{ext}})} \;\leq\; C\,\|v\|_{L^2(\omega_p^{\text{int}})}.$$

The constant C depends only on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$ because there is only a small number of different triangulations $\mathscr{T}_p(\widehat{\omega})$.

Since $\|\varphi_p\|_{L^\infty} = 1$, it follows from the above that

$$\|\varphi_p \cdot E_p v\|^2_{L^2(\omega_p^{\text{ext}})} \;\leq\; C\,\|v\|^2_{L^2(\omega_p^{\text{int}})}.$$

Since $\|\nabla\varphi_p\|_{L^\infty} \leq C H_D^{-1}$, we can conclude from the product rule that

$$\begin{aligned} |\varphi_p \cdot E_p v|^2_{H^1(\omega_p^{\mathrm{ext}})} &\leq C\left(|E_p v|^2_{H^1(\omega_p^{\mathrm{ext}})} + H_D^{-2}\,\|E_p v\|^2_{L^2(\omega_p^{\mathrm{ext}})}\right) \\ &\leq C\left(|v|^2_{H^1(\omega_p^{\mathrm{int}})} + H_D^{-2}\,\|v\|^2_{L^2(\omega_p^{\mathrm{int}})}\right). \end{aligned}$$

Since the number of coarse elements and coarse nodes in $\overline{D}$ is bounded in terms of m, the desired estimate follows by summing the above estimate over $p \in \mathscr{V}_{\partial D}$. □

Let the operator

$$\mathscr{E}_{D'} : H^1(D') \to H^1(\overline{D} \cup D')$$

be defined analogously to $\mathscr{E}_D$, but exchanging the roles of D and D'.

Lemma 4. *Let $\mathscr{D} = \{D\}$ as in Lemma 3 and let D' denote the surroundings of D as defined in* (16)*. Then there exists a uniform constant $C_{\mathscr{E}'}$ depending only on m and on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$ such that*

$$|\mathscr{E}_{D'} v|^2_{H^1(D)} \leq C_{\mathscr{E}'}\,|v|^2_{H^1(D')} \qquad \forall v \in H^1(D').$$

Proof. The proof follows by combining the proof of Lemma 3 with the Poincaré inequality in D, see Lemma 2. □

4.3 Explicit Bounds for Boundary Integral Operators

Definition 7. For each subdomain Ω_i, we define the seminorm and norm

$$\begin{aligned} |v|_{\star,H^{1/2}(\partial\Omega_i)} &:= |\mathscr{H}_i v|_{H^1(\Omega_i)}, \\ \|v\|_{\star,H^{1/2}(\partial\Omega_i)} &:= \left(|\mathscr{H}_i v|^2_{H^1(\Omega_i)} + \frac{1}{\mathrm{diam}(\Omega_i)^2}\,\|\mathscr{H}_i v\|^2_{L^2(\Omega_i)}\right)^{1/2} \end{aligned}$$

(see [27]), which is equivalent to the Sobolev-Slobodeckii norm $\|\cdot\|_{H^{1/2}(\partial\Omega)}$, and the associated dual norm

$$\|w\|_{\star,H^{-1/2}(\partial\Omega_i)} := \sup_{v \in H^{1/2}(\partial\Omega_i)} \frac{\langle w, v\rangle}{\|v\|_{\star,H^{1/2}(\partial\Omega_i)}}.$$

Above and in the following we silently exclude $v = 0$ from the supremum.

In the sequel, we state ellipticity and boundedness results for the boundary integral operators V_i and D_i. In several of the lemmas below, we have to assume that $d = 3$. The two-dimensional case is harder and not further considered in the article at hand. See also [27, Remark 4] and Sect. 6.

Lemma 5. *Let $d = 3$ and let Assumptions 3–4 hold. Then, for each $i = 1, \ldots, N$,*

$$\langle w, V_i w\rangle \geq \tfrac{1}{2} C_{\mathscr{E}}^{-2}\,\|w\|^2_{\star,H^{-1/2}(\partial\Omega_i)} \qquad \forall w \in H^{-1/2}(\partial\Omega_i),$$

i.e., the operators V_i are uniformly elliptic with respect to the norms $\|\cdot\|_{\star,H^{-1/2}(\partial\Omega_i)}$ and the ellipticity constant depends only on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$.

Proof. The statement follows from [27, Lemma 6.1, Corollary 6.2]. The proof there uses the Jones extension, but remains valid for the extension operator $\mathscr{E}_{\Omega_i}$ constructed in Sect. 4.2. □

Lemma 6. *Let Assumptions 3–4 hold and let Ω_i' be the surrounding of Ω_i as defined in* (16)*. Then*

$$\langle D_i v, v\rangle \;\geq\; \tfrac{1}{2}\,C_{\mathscr{E}'}^{-2}\,|v|^2_{\star,H^{1/2}(\partial\Omega_i)} \qquad \forall v \in H^{1/2}(\partial\Omega_i).$$

Proof. See [27, Lemma 3.8, Lemma 6.4]. □

Lemma 7. *Let $d=3$ and let Assumptions 3–4 hold. Then*

$$H_i^{-2}\,\|\mathscr{H}_i v\|^2_{L^2(\Omega_i)} \;\leq\; C_P^*\,|\mathscr{H}_i v|^2_{H^1(\Omega_i)} \qquad \forall v \in H_*^{1/2}(\partial\Omega_i),$$

where the constant C_P^* depends only on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$.

Proof. See [27, Lemma 6.7]. □

Lemma 8. *Let $d=3$ and let Assumptions 3–4 hold. Then*

$$\|V w\|_{H^{-1/2}(\partial\Omega_i)} \;\leq\; C_V^*\,\|w\|_{\star,H^{-1/2}(\partial\Omega_i)} \qquad \forall v \in H^{-1/2}(\partial\Omega_i),$$

where the constant C_V^* depends only on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$.

Proof. See [27, Lemma 6.8]. □

Lemma 9. *For $d=3$, and each subdomain Ω_i, we have*

$$c_{0,i} \;\geq\; \tfrac{1}{4}\,(C_{\mathscr{E}})^{-2}\,(C_{\mathscr{E}'})^{-2}\,(1+C_P^*)^{-1},$$

i.e., there is a uniform lower bound for the constants $c_{0,i}$ just in terms of the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$.

Proof. See [27, Corollary 6.10]. □

5 Error Analysis

This section contains the proof of our main theorem. First, we formulate a lemma à la Strang which bounds the total error in terms of the approximation error of the Dirichlet data on the skeleton and the H^1 approximation error in the FEM subdomains, and the approximation error of the Neumann data in the norm induced by the local single layer potentials. Both terms can be estimated explicitly in the fine and coarse mesh parameter.

Since the original domain Ω is fixed, we assume without loss of generality that $\operatorname{diam}(\Omega) = 1$.

5.1 A Lemma à la Strang

Lemma 10. *Let $u_S \in V_S$ and $u_S^h \in V_S^h$ be the solutions of* (11) *and* (15). *For $i \in I_{\mathrm{BEM}}$, let $w_i(u_S) \in H^{-1/2}(\partial\Omega_i)$ be given by*

$$w_i(u_S) := V_i^{-1}(\tfrac{1}{2}I + K_i)u_{S|\partial\Omega_i} = S_i\, u_{S|\partial\Omega_i}.$$

Then, we have the error estimate

$$\|u_S - u_S^h\|_{H^1(\Omega)} \le \delta \left[\inf_{v^h \in V_S^h} \|u_S - v^h\|_{H^1(\Omega)} + \Big(\sum_{i \in I_{\mathrm{BEM}}} \inf_{z^h \in Z_i^h} \|w_i(u_S) - z^h\|_{V_i}^2 \Big)^{1/2} \right],$$

where

$$\delta = \max(1+\beta, \beta\, \|\alpha\|_{L^\infty(\Omega)}) \max\Big(1, \max_{i \in I_{\mathrm{BEM}}} \frac{c_{K,i}}{\sqrt{1-c_{K,i}}}\Big),$$

and

$$\beta = \frac{1+C_F^2}{\alpha_0} \max\Big(1, \max_{i \in I_{\mathrm{BEM}}} \frac{c_{K,i}}{c_{0,i}}\Big).$$

Proof. First, we homogenize (11) and (15). Let $g \in V_S^h$ be an arbitrary but fixed extension of the Dirichlet datum g_D (i.e., $g_{|\Gamma_D} = g_D$). Then $u_S = g + u_{S,0}$ and $u_{S,h} = g + u_{S,0}^h$ where

$$\begin{aligned} u_{S,0} \in V_{S,D}:\quad & a_S(u_{S,0}, v) = \langle \ell_S, v\rangle - a_S(g, v) && \forall v \in V_{S,D}, \\ u_{S,0}^h \in V_{S,D}^h:\quad & \widetilde{a}_S(u_{S,0}^h, v^h) = \langle \ell_S, v^h\rangle - \widetilde{a}_S(g, v^h) && \forall v^h \in V_{S,D}^h. \end{aligned}$$

From (14), (7), (2), and (3) it follows that

$$\begin{aligned} \widetilde{a}_S(v, v) &\ge \frac{\alpha_0}{1+C_F^2} \min\Big(1, \min_{i \in I_{\mathrm{BEM}}} \frac{c_{0,i}}{c_{K,i}}\Big) \|v\|_{H^1(\Omega)}^2 && \forall v \in V_{S,D}, \\ \widetilde{a}_S(v, w) &\le \|\alpha\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} && \forall v, w \in V_S. \end{aligned}$$

The Strang lemma from [11, Lemma 4.1] implies that

$$\begin{aligned} \|u_{S,0} - u_{S,0}^h\|_{H^1(\Omega)} \le{}& \max(1+\beta, \beta\, \|\alpha\|_{L^\infty(\Omega)}) \times \\ & \left[\inf_{v^h \in V_{S,D}^h} \|u_{S,0} - v^h\|_{H^1(\Omega)} + \sup_{v^h \in V_{S,D}^h} \frac{\widetilde{a}_S(u_S, v_h) - \langle \ell_S, v_h\rangle}{\|v_h\|_{H^1(\Omega)}} \right]. \end{aligned}$$

Using that $\langle \ell_S, v_h \rangle = a_S(u_S, v_h)$, we obtain

$$\widetilde{a}_S(u_S, v_h) - \langle \ell_S, v_h \rangle = \sum_{i \in I_{\text{BEM}}} \langle \widetilde{S}_i u_S - S_i u_S, v_h \rangle.$$

Following the proof of [11, Lemma 4.2], we get

$$\langle \widetilde{S}_i u_S - S_i u_S, v_h \rangle \le \frac{c_{K,i}}{\sqrt{1 - c_{K,i}}} |v_h|_{H^1(\Omega_i)} \, \|w_i(u_S) - w_i^h(u_S)\|_{V_i}.$$

The rest of the proof follows from Cauchy's inequality and the fact that $u_S - u_{S,0} = g \in V_S^h$. □

5.2 Error Estimate for the Dirichlet Data

Theorem 2. *Let Assumptions 1–3 and Assumption 5 hold. Assume further that the solution u of* (1) *satisfies $u \in H^2(\Omega)$. Then there exists a constant C only depending on the shape regularity constants of $\mathscr{T}^h(\Gamma_S)$, $\mathscr{T}^H(\Omega)$, and $\mathscr{T}^h(\Omega_i)$, $i \in I_{\text{FEM}}$, such that*

$$\inf_{v^h \in V_S^h} \|u_S - v^h\|_{H^1(\Omega)} \le C \Big(\sum_{i=1}^N h_i^2 \, |u|^2_{H^2(\Omega_i)} \Big)^{1/2} \le C h |u|_{H^2(\Omega)}.$$

Proof. The proof is analogous to [11, Theorem 4.8]. First, recall that due to Assumption 2, $f_{|\Omega_i} = 0$ for $i \in I_{\text{BEM}}$, and so

$$u_S = u.$$

From Assumption 3 and Assumption 5 it follows that for each $i \in I_{\text{BEM}}$, the triangulation $\mathscr{T}^h(\partial\Omega_i)$ can be extended to an auxiliary triangulation $\widetilde{\mathscr{T}}^h(\Omega_i)$ with mesh parameter h_i, such that the shape regularity constants of $\widetilde{\mathscr{T}}^h(\Omega_i)$ are bounded in terms of the shape regularity constants of $\mathscr{T}^h(\Gamma_S)$ and $\mathscr{T}^H(\Omega)$. This implies a global triangulation $\widetilde{\mathscr{T}}^h(\Omega)$ of the entire domain Ω. Let

$$\widetilde{V}^h(\Omega) := \{ v \in H^1(\Omega) : v_{|T} \in P_1 \quad \forall T \in \widetilde{\mathscr{T}}^h(\Omega) \},$$

and let $I^h u_S \in \widetilde{V}^h(\Omega)$ denote the nodal interpolant of $u_S \in H^2(\Omega)$. Due to the minimizing property (6) of the harmonic extension and a standard interpolation result (see [2]), we obtain

$$\begin{aligned} \inf_{v^h \in V_S^h} \|u_S - v^h\|_{H^1(\Omega)} &\le \inf_{v^h \in \widetilde{V}^h(\Omega)} \|u_S - v^h\|_{H^1(\Omega)} \\ &\le \|u_S - I^h u_S\|_{H^1(\Omega)} \le C \Big(\sum_{i=1}^N h_i^2 \, |u_S|^2_{H^2(\Omega_i)} \Big)^{1/2}, \end{aligned}$$

where C depends only on the mentioned shape regularity constants. □

5.3 Error Estimate for the Neumann Data

Throughout this subsection, assume that $d = 3$ and that Assumptions 3–5 hold. Let $\mathscr{F}_i = \{F\}$ denote the set of triangular coarse faces on $\partial\Omega_i$ (cf. Assumption 3). We define the face seminorms

$$|v|_{H^{1/2}_{\sim}(F)} := \left(\int_F \int_F \frac{|v(x) - v(y)|^2}{|x-y|^3} \, ds_x \, ds_y \right)^{1/2} \qquad \text{for } v \in H^{1/2}(F),\ F \in \mathscr{F}_i,$$

and the piecewise seminorm

$$|v|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)} := \Big(\sum_{F \in \mathscr{F}_i} |v|^2_{H^{1/2}_{\sim}(F)} \Big)^{1/2}.$$

The space $H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)$ is the subspace of $L^2(\partial\Omega_i)$ where the above seminorm is bounded.

Definition 8. For each $i \in I_{\mathrm{BEM}}$, the L^2-projector $Q_i^h : L^2(\partial\Omega_i) \to Z_i^h$ is given by

$$(Q_i^h v, z^h)_{L^2(\partial\Omega)} = (v, z^h)_{L^2(\partial\Omega_i)} \qquad \forall z^h \in Z_i^h,$$

with the space Z_i^h from (13).

Of course, the above equation can be localized and

$$(Q_i^h v)_{|\gamma} = \frac{1}{|\gamma|} \int_\gamma v \, ds \qquad \text{for } \gamma \in \mathscr{T}^h(\partial\Omega_i).$$

Lemma 11. *The operator Q_i^h satisfies, for all $w \in H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)$, the approximation properties*

$$\|w - Q_i^h w\|_{L^2(\partial\Omega_i)} \le C\, h_i^{1/2}\, |w|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)},$$

$$\|w - Q_i^h w\|_{\star, H^{-1/2}(\partial\Omega_i)} \le C\, h_i\, |w|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)},$$

where the constant C depends only on the shape regularity constants of $\mathscr{T}^H(\Omega)$ and $\mathscr{T}^h(\Gamma_S)$.

Proof. First, we split the local boundary $\partial\Omega_i$ into the (plane) triangular faces $F \in \mathscr{F}_i$. Each such face can be mapped to a reference face. Applying [31, Theorem 10.2] to each face and summing over the faces, we obtain the first estimate (the proof of that theorem is constructed by interpolating estimates in the L^2- and H^1-seminorm at $1/2$).

The second estimate is shown along the lines of [31, Corollary 10.3]: Using the definition of the dual norm, the projection property of Q_i^h, Cauchy's inequality, and the first estimate of the current lemma, we obtain

$$\begin{aligned}\|w - Q_i^h w\|_{\star,H^{-1/2}(\partial\Omega_i)} &= \sup_{v\in H^{1/2}(\partial\Omega_i)} \frac{(w - Q_i^h w, v)_{L^2(\partial\Omega_i)}}{\|v\|_{\star,H^{1/2}(\partial\Omega_i)}} \\ &= \sup_{v\in H^{1/2}(\partial\Omega_i)} \frac{(w - Q_i^h w, v - Q_i^h v)_{L^2(\partial\Omega_i)}}{\|v\|_{\star,H^{1/2}(\partial\Omega_i)}} \\ &\leq \|w - Q_i^h w\|_{L^2(\partial\Omega_i)} \sup_{v\in H^{1/2}(\partial\Omega_i)} \frac{\|v - Q_i^h v\|_{L^2(\partial\Omega_i)}}{\|v\|_{\star,H^{1/2}(\partial\Omega_i)}} \\ &\leq C h_i^{1/2} |w|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)} C h_i^{1/2} \sup_{v\in H^{1/2}(\partial\Omega_i)} \frac{|v|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)}}{\|v\|_{\star,H^{1/2}(\partial\Omega_i)}}.\end{aligned}$$

Using (A9) and (A12) from [11], we can conclude that

$$|v|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)} \leq C \|v\|_{\star,H^{1/2}(\partial\Omega_i)} \qquad \forall v \in H^{1/2}(\partial\Omega_i).$$

The (generic) constants in both estimates depend only on the shape regularity constants of $\mathscr{T}^H(\Omega)$. □

Our last prerequisite is a Neumann trace inequality. For a proof see [11, Theorem 4.10 and Sect. A.2].

Lemma 12. *There exists a constant C depending only on the shape regularity constants of $\mathscr{T}^H(\Omega)$ such that*

$$|\gamma_i^1 v|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)} \leq C |v|_{H^2(\Omega_i)} \qquad \forall v \in H^2(\Omega_i).$$

Combining the tools and estimates above we get the following error estimate.

Theorem 3. *Let $d = 3$ and let Assumptions 3–5 hold. Then there exists a constant C only depending on the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$ and $\mathscr{T}^h(\Gamma_S)$ such that*

$$\inf_{z^h\in Z_i^h} \|\gamma_i^1 v - z^h\|_{V_i} \leq C h_i |v|_{H^2(\Omega_i)} \qquad \forall v \in H^2(\Omega_i).$$

Proof. Using Lemma 8 and Lemma 11, we obtain

$$\begin{aligned}\inf_{z^h\in Z_i^h} \|\gamma_i^1 v - z^h\|_{V_i} &\leq C_V^* \inf_{z^h\in Z_i^h} \|\gamma_i^1 v - z^h\|_{\star,H^{-1/2}(\partial\Omega_i)} \\ &\leq C_V^* \|\gamma_i^1 v - Q_i^h \gamma_i^1 v\|_{\star,H^{-1/2}(\partial\Omega_i)} \leq C_V^* C h_i |\gamma_i^1 v|_{H^{1/2}_{\sim\mathrm{pw}}(\partial\Omega_i)}.\end{aligned}$$

An application of Lemma 12 concludes the proof. □

5.4 *Proof of Theorem 1*

Noticing that $w_i(u_S) = \gamma_i^1 u$ and combining Lemma 10 and Theorem 2, we obtain

$$\|u_S - u_S^h\|_{H^1(\Omega)} \leq C\left[\Big(\sum_{i=1}^{N} h_i^2\, |u|_{H^2(\Omega_i)}^2\Big)^{1/2} + \Big(\sum_{i\in I_{\mathrm{BEM}}} \inf_{z^h\in Z_i^h} \|\gamma_i^1 u - z^h\|_{V_i}^2\Big)^{1/2}\right]$$

Because of Lemma 9 and because $c_{K,i}$ depends monotonically decreasingly on $c_{0,i}$, the constant C above is bounded only in terms of the shape regularity constants of $\mathscr{T}^H(\widehat{\Omega})$, $\mathscr{T}^h(\Gamma_S)$, and $\mathscr{T}^h(\Omega_i)$, $i \in I_{\mathrm{FEM}}$. Applying Theorem 3 on each BEM subdomain concludes the proof of Theorem 1.

6 Conclusion and Extensions

First, we would like to note that we can relax Assumption 4 to the weaker assumption that there exists a shape regular coarse triangulation for the neighborhood of each subdomain (with uniform shape regularity constants). This way, small exterior angles of the computational domain Ω are allowed as long as there are no small exterior angles of the subdomains themselves.

We believe that with careful effort, the above theory can be extended to the two-dimensional case, see [27, Remark 4]. Also, it should be possible to drop Assumption 2 and incorporate an approximation of the Newton potential, see [30].

Using the explicit bounds for the boundary integral operators, it is possible to lift the results in [16, 17] on BETI and coupled FETI/BETI methods to the current setting. Hence, the convergence of these solvers does not depend on the subdomains, but only on the shape regularity of the subdomain decomposition.

For the case of reduced regularity ($u_S \notin H^2(\Omega)$), we first show a stability result. By choosing $v^h = 0$ and $z^h = 0$ in the infima in the statement of Lemma 10, and using [27, Lemma 5.4], one can show that

$$\|u_S - u_S^h\|_{H^1(\Omega)} \leq C\, |u_S|_{H^1(\Omega)},$$

under the minimal assumption that $u_S \in H^1(\Omega)$. Interpolating the H^2 and H^1 error estimate, we immediately get that

$$\|u_S - u_S^h\|_{H^1(\Omega)} \leq C h^s\, \|u_S\|_{H^{1+s}(\Omega)}$$

if $u_S \in H^{1+s}(\Omega)$.

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