Fast Domain Decomposition Algorithms for Elliptic Problems with Piecewise Variable Orthotropism

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Abstract. Second order elliptic equations are considered in the unit square, which is decomposed into subdomains by an arbitrary nonuniform orthogonal grid. For the elliptic operator we assume that the energy integral contains only squares of first order derivatives with coefficients, which are arbitrary positive finite numbers but different for each subdomain. The orthogonal finite element mesh has to satisfy only one condition: it is uniform on each subdomain. No other conditions on the coefficients of the elliptic equation and on the step sizes of the discretization and decomposition are imposed. For the resulting discrete finite element problem, we suggest domain decomposition algorithms of linear total arithmetical complexity, not depending on any of the three factors contributing to the orthotropism of the discretization on subdomains. The main problem of designing such an algorithm is the preconditioning of the inter-subdomain Schur complement, which is related in part to obtaining boundary norms for discrete harmonic functions on the shape irregular domains.

1 Introduction

The aim of this paper is to present fast DD (domain decomposition) algorithms for the discretization of partial differential equations with piecewise variable orthotropism, which is modelled by the discrete problem described below. Suppose, the domain $\Omega = (0,1) \times (0,1)$ is decomposed into subdomains

$$\Omega_j = (z_{1,j_1-1}, z_{1,j_1}) \times (z_{2,j_2-1}, z_{2,j_2}), \quad j = (j_1, j_2), \tag{1}$$

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by the rectangular *decomposition grid*

$$x_k = z_{k,j_k}, \quad j_k = 0,1,..,J_k, \quad z_{k,j_k} - z_{k,j_k-1} = H_{k,j_k} > 0, \quad z_{k,0} = 0, \quad z_{k,J_k} = 1\,. \quad (2)$$

The decomposition grid is imbedded in the nonuniform rectangular finer *source grid*

$$x_k = x_{k,i_k}, \quad i_k = 0,1,..,N_k, \quad x_{k,0} = 0, \quad x_{k,N_k} = 1, \quad (3)$$

i.e. $x_{k,\gamma_k} = z_{k,j_k}$ for some numbers $\gamma_k = \varkappa_k(j_k)$, $k = 1,2$. Assume for simplicity, that this grid is uniform on each subdomain and has sizes $h_{k,j_k} = H_{k,j_k}/n_{k,j_k}$, where n_{k,j_k} is the number of the source grid intervals on the decomposition grid interval (z_{k,j_k-1}, z_{k,j_k}).

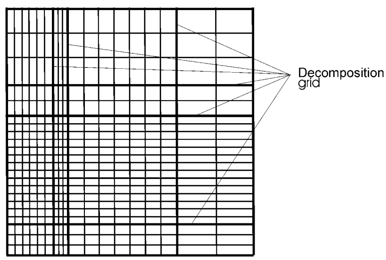

Fig. 1 Decomposition grid and subdomain wise uniform rectangular source grid.

Let $\mathring{\mathscr{V}}(\Omega)$ be the FE (finite element) space of globally continuous functions which are bilinear on each nest of the source grid and which vanish on $\partial\Omega$. We consider the problem

$$\alpha_\Omega(u,v) = \langle f, v\rangle, \quad \forall\, v \in \mathring{H}^1(\Omega), \text{ where } \alpha_\Omega(u,v) = \int_\Omega \nabla u(x)\cdot \wp(x)\,\nabla v(x)\,dx, \quad (4)$$

$\wp(x)$ is a 2×2 matrix satisfying the inequalities

$$\mu_1\rho(x) \le \wp(x) \le \mu_2\rho(x), \quad 0 \le \mu_1, \mu_2 = \text{const}, \quad (5)$$

and $\rho = \text{diag}\,[\rho_1, \rho_2]$ is a diagonal matrix with piecewise constant positive functions $\rho_k(x)$. In other words, $\rho_k(x) = \rho_{k,j} = \text{const} > 0$ for $x \in \Omega_j$, and $\rho_{k,j}$ are arbitrary positive numbers. The integral identity (4) on the space $\mathring{\mathscr{V}}(\Omega)$ is reduced to the system of linear algebraic equations

$$\mathbf{Ku} = \mathbf{f}\,. \quad (6)$$

The answer, we are looking for, is whether there exists a DD algorithm, which is robust and fast uniformly for arbitrary positive H_{k,j_k}, $\rho_{k,j}$ and h_{k,j_k}. Indeed, we suggest DD preconditioners $\mathscr{K}_{\text{DD}}$ of the Dirichlet-Dirichlet type such that

$$\text{ops}\left[\mathscr{K}_{\text{DD}}^{-1}\mathbf{f}\right] \times \left(\kappa\left[\mathscr{K}_{\text{DD}}^{-1}\mathbf{K}\right]\right)^{1/2} \leq cN_{\Omega}, \quad c = \text{const}, \tag{7}$$

where N_{Ω} is the number of unknowns in (6), $\kappa[\mathbb{A}]$ is the spectral condition number of the matrix $\mathbb{A}$, ops$[\ldots]$ is the number of arithmetical operations for performance of the operation inside the brackets, and c is an absolute constant. Therefore, the bound (7) approves that the DD preconditioner provides a solution procedure for the system (6) of a linear total arithmetical complexity. The bound retains, if the number of subdomains $J = J_1 J_2$ grows along with the number of FE unknowns, but not too fast, for instance, when $J_k \leq N_k^{1/2}/\log^{3/2}\overline{N}$, $\overline{N} = \max N_k$. It is worth to stress that the right part of (7) does not depend on any of the three factors contributing to the orthotropism of the discretization on subdomains.

The results are retained, if Ω is the union of any number of nests of the decomposition mesh, they are also retained for the respective FE discretizations by linear triangular finite elements with vertices at the nodes of the orthogonal discretization mesh. Moreover, for both types of discretizations, the preconditioners will be often defined by means of triangular elements, which provide simpler explicit representations.

In the DD algorithm, Dirichlet problems in rectangular subdomains Ω_j can be efficiently solved by numerous direct and iterative methods, including FDFT (Fast Discrete Fourier Transform). Solvers for orthotropic and some more complex types of discretizations on rectangular domains have been intensively studied. We can find respective results in Schieweck [47], Wittum [50], Dahmen [17], Griebel & Oswald [22], Grauschopf et al. [21], Cohen et al. [14], Schneider [48], Oswald [44], Pflaum [45], and many other papers. These works allow also to obtain for each subdomain optimal low energy prolongation operators and spectrally equivalent Schur complement preconditioners which are almost optimal for inversion. Therefore, at designing DD algorithm subordinate to the bound (7), the key problem is obtaining an interface preconditioner, which would not compromise this bound.

An analysis of DD interface preconditioners for isotropic elliptic equations in domains, composed of thin rectangles, can be found, e.g., in Chen et al. [13] and Nepomnyaschikh [40, 42]. In relation with some deteriorating elliptic equations, DD algorithms for discretizations with a subdomain-wise more general variation of orthotropism were studied analytically in Korneev [31] and numerically in Rytov [46] and Anufriev & Korneev [4] for 2-d and 3-d problems, respectively. Other techniques, e.g., boundary element methods, $\mathscr{H}$-matrices, and tensor-train decompositions were also attracted for obtaining efficient interface preconditioners for elliptic problems with orthotropism with a subdomain-wise variation which is subjected to some restrictions. Papers of Hsiao et al. [26], Hackbusch et al. [25], Dolgov et al. [16] are only a few representatives of this vast area of research.

The case of an assemblage of rectangular subdomains, having different arbitrary aspect ratios, accompanied by an orthotropism of the differential equation which is chaotically strongly changing from subdomain to subdomain, causes specific difficulties. They are strengthened by a jumping orthotropism of a rectangular subdomain-wise uniform, but otherwise arbitrary, finer mesh for the discretization. The results for solvers of uniformly anisotropic problems on arbitrary rectangle or corresponding Schur complement solvers are not directly applicable. The reason is that multilevel decompositions, which are efficient for each subdomain separately, are not compatible, and, therefore, in general can't be assembled to obtain an efficient interface preconditioner.

Discrete problems close to (1)-(6) with $\wp(x) = \rho(x)$ were independently addressed in Khoromskij & Wittum [27, 28], in Kwak et al. [38], Nepomnyaschikh [43], and in Korneev et al. [33, 34]. In particular [27, 28] concentrated on the interface solvers, whereas the others considered DD Dirichlet-Dirichlet solvers. There are other differences, pertaining specific components of algorithms and techniques of their analysis, which resulted in different bounds for the relative condition number of the DD preconditioner and its total computational complexity. The bounds of [27, 28, 38] depend on $\nu_{\rho,j}$, $\nu_{\rho,j}^2 = \max_{k=1,2}(\rho_{k,j}/\rho_{3-k,j})$, whereas the bounds of [33, 34] do not. In this paper, we improve the preconditioner of [33, 34] and come to estimates of linear complexity.

Compatibility of subdomain Schur complement preconditioners usually included a splitting of the vertex degrees of freedom from the rest. However, for thin rectangles this damages the relative condition number more severely, than for shape regular rectangles. One way to circumvent these obstacles is the use of an iterative preconditioner $\mathscr{S}_{1,\mathrm{it}}$, resulting from an inexact solver defined by means of two Schur complement preconditioners $\mathscr{S}_k$, $k = 1,2$. One is aimed to provide a good relative condition number and cheap matrix-vector multiplications. The other may have a not so good relative condition number, but is cheap for inversion. Seemingly, the idea of such a preconditioning was introduced by Nepomnyaschikh [40]; in Kwak et al. [38] it was implemented in the DD solver for a model problem, close to (1)-(6), but with a different, than in this paper, choice of $\mathscr{S}_k$, $k = 1,2$.

The preconditioner $\mathscr{S}_1^j$, used in this paper for one subdomain $\Omega = \Omega_j$, stems from the shape dependent boundary seminorm, which is equivalent to the $H^1(\Omega)$-seminorm for discrete harmonic functions in shape irregular rectangles. The corresponding norm was introduced in Korneev [31], see also Korneev et al. [33, 34], following the technique of Maz'ya & Poborchi [39] used for harmonic functions. The validity of this norm for discrete harmonic functions is established with the use of a Scott & Zhang [49] result on a special interpolation operator for functions from $H^1(\Omega)$. Then the shape dependent norm is simplified by means of finite-difference norms, equivalent to the $H^{1/2}$-norms on some 1-d sets. It is, by the way, worth noting that it well reflects the fact, known in applications, e.g., of boundary FE methods and referred as *absorption of singularities*. Suppose, we have a spectrally equivalent preconditioner for the boundary Schur complement for a FE discretization on a quasiuniform mesh. Then, this preconditioner retains the spectral equivalence to the Schur complement generated on any shape regular mesh, which is imbedded in the quasiuniform mesh and coincides with it on the boundary. Even more general meshes can be considered. Basic facts related to the preconditioner are presented in Subsect. 2.1 and 2.2.

There are obvious reasons to use Schur complement preconditioners, in which, apart from splitting vertices, each edge is split at least from a part of others. In the DD algorithm of this paper, $\mathscr{S}_2$ can be taken as one of the slightly different and compatible preconditioners used in Khoromskij & Wittum [27, 28], Korneev [31], and Korneev et al. [33, 34]. The proof of the bounds of its relative spectrum, independent of $\nu_{\rho,j}$, is made in a way of comparison with the sample preconditioner, which has a structure similar to $\mathscr{S}_2$, but which is more suitable for an analysis. The purpose of Subsect. 2.3 and Sect. 3 is to describe the sample preconditioner and to prepare subsidiary results for the analysis of the basic preconditioner $\mathscr{S}_2$. The derivation of the sample preconditioner for one subdomain is accomplished by means of a secondary DD technique with a secondary shape regular nonoverlapping domain decomposition. The preconditioner $\mathscr{S}_2$ is presented in Sect. 4, Sect. 5 summarizes results for separate subdomains, obtained in preceding sections, into a bound of computational cost of a DD method for the problem (1)-(6).

We do not study solvers for the subproblem of the DD algorithm, which is related to the vertices of the subdomains. Often its dimension is much smaller than N_Ω, and at varying J_k, we assume that there is a solver, which does not compromise (7). Obviously, if $J_k \le N_k^{1/3}$, then even a direct elimination procedure will satisfy this assumption.

Let us list some notations as used in this paper. For matrices we primarily use capital letters of the styles $\mathbf{A}$, $\mathbb{A}$, $\mathscr{A}$, $\mathbf{I}$ stands for identity matrices, small boldface letters – for vectors. $(\cdot,\cdot)_\Omega$, and $\|\cdot\|_{0,\Omega}$ are the scalar product and the norm in $L^2(\Omega)$, whereas $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ are the semi-norm and the norm in the Sobolev space $H^k(\Omega)$, i.e.,

$$|v|_{k,\Omega}^2 = \sum_{|q|=k} \int_\Omega (D_x^q v)^2 dx, \quad \|v\|_{k,\Omega}^2 = \|v\|_{0,\Omega}^2 + \sum_{l=1}^{k} |v|_{l,\Omega}^2,$$

with

$$D_x^q v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2}, \quad q = (q_1, q_2), \quad q_1, q_2 \ge 0, \quad |q| = q_1 + q_2.$$

$\mathring{H}^1(\Omega)$ is the subspace of $H^1(\Omega)$ of functions having zero traces on $\partial\Omega$. For $I = (a,b)$, $\|\cdot\|_{1/2,I}$ and ${}_{00}\|\cdot\|_{1/2,I}$ are the norms in the space $H^{1/2}(I)$ and the subspace ${}_{00}H^{1/2}(I) \subset H^{1/2}(I)$ of functions having zero values at $x = a, b$, see, e.g., [1]. Expressions for these norms are

$$\|v\|_{1/2,I}^2 = \|v\|_{0,I}^2 + |v|_{1/2,I}^2, \quad |v|_{1/2,I}^2 = \int_a^b \int_a^b \left(\frac{v(x) - v(y)}{x - y} \right)^2 dxdy,$$

$${}_{00}\|v\|_{1/2,I}^2 = \|v\|_{1/2,I}^2 + \int_a^b \frac{v^2(x)}{x-a} dx + \int_a^b \frac{v^2(x)}{b-x} dx.$$

The norm $\|\cdot\|_{1/2,\gamma_i}$, when γ_i is an edge of $\sqcap = (0,1)\times(0,1)$ is defined for the traces on this edge analogously with $\|\cdot\|_{1/2,I}$. For instance, for the edge γ_i, which is on the line $x_1 = c$, $c = 0, 1$, we have

$$\|v\|^2_{1/2,\gamma_i} = \|v\|^2_{0,\gamma_i} + |v|^2_{1/2,\gamma_i}, \quad |v|^2_{1/2,\gamma_i} = \int_0^1\int_0^1 \left(\frac{v(c,t)-v(c,\tau)}{t-\tau}\right)^2 dt d\tau.$$

For a sufficiently smooth and shape regular domain Ω, the norm $\|v\|_{1/2,\partial\Omega}$ is given by the formulas

$$\|v\|^2_{1/2,\partial\Omega} = \|v\|^2_{0,\partial\Omega} + |v|^2_{1/2,\partial\Omega}, \quad |v|^2_{1/2,\partial\Omega} = \int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{v(x)-v(y)}{x-y}\right)^2 ds_x ds_y,$$

in which ds_x, ds_y are the length elements at the points $x, y \in \partial\Omega$. In the case, e.g., $\Omega = \sqcap := (0,1)\times(0,1)$ this norm is equivalent to the norm

$$\|v\|^2_{1/2,\partial\sqcap} = \|v\|^2_{0,\partial\sqcap} + |v|^2_{1/2,\partial\sqcap} \tag{8}$$

with

$$|v|^2_{1/2,\partial\sqcap} = \sum_{i=1}^4 |v|^2_{1/2,\gamma_i} + \sum_{i=1}^4 \int_0^1 \frac{\left(v_{j(i)}(t) - v_{l(i)}(t)\right)^2}{|t|} dt,$$

where $u_{j(i)}$ denotes the restriction of u onto the edge $\gamma_{j(i)}$, and t is the distance to the vertex V_i of $\sqcap$, which is common for $\gamma_{j(i)}$ and $\gamma_{l(i)}$. To each V_i, we associate a preceding edge $\gamma_{j(i)}$ and a succeeding edge $\gamma_{l(i)}$, e.g., according to a counter-clockwise orientation of the boundary. The norm and semi-norm defined in this way for the space $H^{1/2}(\partial\sqcap)$ are equivalent to $\|v\|_{1/2,\partial\sqcap} := \inf\|w\|_{1,\sqcap}$ and $|v|_{1/2,\partial\sqcap} := \inf|w|_{1,\sqcap}$ with infima taken over $w \in H^1(\sqcap)$ for which $w = v$ on $\partial\sqcap$. We refer to Grisvard [23], Ben Belgacem [7] and [39] for additional details on the introduced boundary norms.

$\mathbf{A}^+$ is the pseudo-inverse to a matrix $\mathbf{A}$, $\|\mathbf{v}\|_{\mathbf{A}} = (\mathbf{v}^\top\mathbf{A}\mathbf{v})^{1/2}$ is the norm or the seminorm, induced by a nonnegative symmetric matrix $\mathbf{A}$. If $\mathbf{A}$ is a nonnegative symmetric matrix, the notation $\mathbf{A}^{1/2}$ stands for the nonnegative symmetric matrix $\mathbf{B}$ satisfying $\mathbf{A} = \mathbf{B}\mathbf{B}$, $\ker[\mathbf{A}] = \ker[\mathbf{B}]$. The spectral condition number of a matrix $\mathbb{A}$ is denoted $\kappa[\mathbb{A}]$, $\mathrm{ops}[\cdot]$ is the number of arithmetic operations needed for the procedure in the square brackets. Symbols $\prec$, $\succ$ denote one-sided and $\asymp$ – two-sided inequalities, which hold for some, mostly absolute, constants omitted, whereas $\mathbf{A} \prec \mathbf{B}$ with nonnegative matrices $\mathbf{A}$, $\mathbf{B}$ implies $\mathbf{v}^\top\mathbf{A}\mathbf{v} \prec \mathbf{v}^\top\mathbf{B}\mathbf{v}$ for any vector $\mathbf{v}$, and similarly for signs $\succ$, $\asymp$. We write $\mathbf{v} \Leftrightarrow v$, if the vector $\mathbf{v}$ represents the FE function v in a chosen basis. Whenever we write "inversion of matrix $\mathbb{A}$" or $\mathbb{A}^{-1}\mathbf{y}$, we imply solving of the system $\mathbb{A}\mathbf{x} = \mathbf{y}$.

We avoid the use of special notations for perturbed matrices and matrices expanded by zero entries. Accordingly, sums of matrices are typically understood as topological sums, and a $n\times n$ matrix $\mathbf{A}$, initially defined for some n, is considered, when necessary, as expanded by zero entries up to a $m\times m$, $m > n$, matrix without special explanations.

2 Single Thin Rectangle

2.1 Discrete Analogues of Boundary Norms for Harmonic Functions in Thin Rectangles

Let $\Omega = (0,1)\times(0,\varepsilon)$ and ε,δ satisfy $0<\varepsilon,\ \delta\le 1$. For the traces of functions $v\in H^1(\Omega)$ on $\partial\Omega$, we consider two norms and two seminorms. Norm and seminorm of one pair, denoted by $\|\cdot\|$ and $|\cdot|$, minimize the H^1-norm and H^1-seminorm, respectively, among all functions $\phi\in H^1(\Omega)$ coinciding with a given function on the boundary:

$$\|v\|^2_{\partial\Omega} = \inf_{\phi_{|\partial\Omega}=v}\left((\delta\varepsilon^{-1})^2\|\phi\|^2_{0,\Omega}+\|\nabla\phi\|^2_{0,\Omega}\right), \quad |v|^2_{\partial\Omega} = \inf_{\phi_{|\partial\Omega}=v}\|\nabla\phi\|^2_{0,\Omega}\,. \tag{9}$$

For another norm and seminorm we use notations $]|\cdot|[_{\partial\Omega}$ and $]\cdot[_{\partial\Omega}$ and introduce them by the expressions

$$]|v|[^2_{\partial\Omega}= \delta^2\varepsilon^{-1}\|v\|^2_{0,\partial\Omega}+]v\,[^2_{\partial\Omega} \tag{10}$$

with

$$\begin{aligned}]v[^2_{\partial\Omega} &= \varepsilon^{-1}\int_0^1 (v(x_1,\varepsilon)-v(x_1,0))^2dx_1 + \int_0^1 \int_{|x_1-y_1|\le\varepsilon} \frac{(v(x_1,0)-v(y_1,0))^2}{(x_1-y_1)^2}dx_1dy_1 \\ &+\int_0^1 \int_{|x_1-y_1|\le\varepsilon} \frac{(v(x_1,\varepsilon)-v(y_1,\varepsilon))^2}{(x_1-y_1)^2}dx_1dy_1 + \int_{\Gamma_0}\int_{\Gamma_0}\frac{(v(s)-v(\bar{s}))^2}{(s-\bar{s})^2}dsd\bar{s} \\ &+\int_{\Gamma_1}\int_{\Gamma_1}\frac{(v(s)-v(\bar{s}))^2}{(s-\bar{s})^2}dsd\bar{s}. \end{aligned}$$

Here $\Gamma_0=\{x\in\partial\Omega : x_1<\varepsilon\}$, $\Gamma_1=\{x\in\partial\Omega_\varepsilon : x_1>1-\varepsilon\}$ and ds, $d\bar{s}$ are the length elements of $\partial\Omega$. The set Γ_1 is symmetric to Γ_0 with respect to the line $x_1\equiv 1/2$, see Fig.2.

Theorem 1. *For the traces of functions from $H^1(\Omega)$, the norms (9), (10) are equivalent uniformly in $\varepsilon,\delta\in(0,1]$.*

Proof. The proof can be found in Korneev et al. [33, 34]. □

We will call $]|\cdot|[_{\partial\Omega}$ and $]\cdot[_{\partial\Omega}$ the *shape dependent norm and seminorm* for boundary functions.

In this paper, discretizations on rectangular meshes are considered. Accordingly, we use the FE space $\mathscr{V}(\Omega)$ of piecewise bilinear functions on the rectangular mesh $x_k\equiv x_{k,l}$ with the steps $h_{k,l}=x_{k,l}-x_{k,l-1}$, $l=1,2,..,n_k$, satisfying the quasiuniformity conditions

$$\underline{c}h \le h_{k,l} \le \overline{c}h, \quad 0 < \underline{c}, \overline{c} = \text{const}, \tag{11}$$

and $x_{k,0} = 0$, $x_{1,n_1} = 1$, $x_{2,n_2} = \varepsilon$.

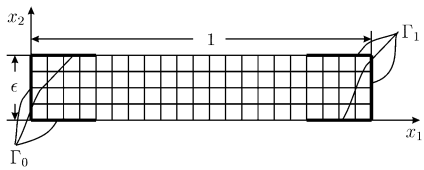

Fig. 2 High aspect ratio rectangular domain triangulated by a square mesh.

By $\mathcal{V}_{\rm tr}(\partial\Omega)$ we denote the space of traces of functions from $\mathcal{V}(\Omega)$ on $\partial\Omega$. However, most of the results hold for much more general discretizations. The mesh as described above may represent a *skeleton mesh*, while calculations are performed on a mesh, called the *source or fine mesh*, which

α) *is finer only in the interior of the domain,*
β) *has the same trace with the skeleton mesh on the boundary and*
γ) *covers the skeleton mesh, whereas the skeleton mesh itself may be a general quasiuniform quadrangular unstructured mesh with the mesh parameter h.*

For simplicity, it is convenient also to assume that there are mesh nodes at the ends of the sets Γ_k, $k = 0, 1$, and that the number of nodes on the opposite edges of Ω are equal, as in the case of an orthogonal mesh. Clearly, the skeleton and the source meshes can be as well triangular meshes, from which the former is quasiuniform.

For the traces of FE functions $v \in \mathcal{V}_{\rm tr}(\partial\Omega)$, the simpler norm than (9)

$$\begin{aligned} \|v\|^2_{h,\partial\Omega} &= \inf_{\phi\in\mathcal{V}(\Omega):\phi_{|\partial\Omega}=v} \left((\delta\varepsilon^{-1})^2\|\phi\|^2_{0,\Omega} + \|\nabla\phi\|^2_{0,\Omega}\right), \\ |v|^2_{h,\partial\Omega} &= \inf_{\phi\in\mathcal{V}(\Omega):\phi_{|\partial\Omega}=v} \|\nabla\phi\|^2_{0,\Omega}, \end{aligned} \tag{12}$$

can be justified, in which inf is taken only over the subspace of FE functions.

Theorem 2. *Let the FE space $\mathcal{V}(\Omega)$ be induced by the quasiuniform triangulation with the mesh parameter h or by its refinement satisfying α)-γ). Then for any $h > 0$ and $v \in \mathcal{V}_{tr}(\partial\Omega)$, the norms and seminorms (12), (10), respectively, are equivalent uniformly in $\varepsilon, \delta \in (0, 1]$.*

Proof. The proof is based, first, on Theorem 1, and, second, on the quasi-interpolation result of Lemma 2, given after the proof of the theorem.

Since $\mathcal{V}(\Omega) \subset H^1(\Omega)$, one has the inequalities

$$]|v|[_{\partial\Omega} \prec \|v\|_{\partial\Omega} \le \|v\|_{h,\partial\Omega} \quad \forall v \in \mathcal{V}_{\rm tr}(\partial\Omega), \tag{13}$$

with the first one following from Theorem 1 and the definitions of the norms $\|\cdot\|_{\partial\Omega}$ and $\|\cdot\|_{h,\partial\Omega}$. For the proof of the opposite bound

$$\|v\|_{h,\partial\Omega} \prec]|v|[_{\partial\Omega}, \quad \forall v \in \mathscr{V}_{\rm tr}(\partial\Omega) \tag{14}$$

it is sufficient to use in addition Lemma 2.

Indeed, let $\mathscr{H}(\Omega)$ be the subspace of $\mathscr{V}(\Omega)$, induced by the skeleton quasiuniform triangulation, and $v \in \mathscr{V}_{\rm tr}(\partial\Omega)$. Suppose also that $v_{\rm inf} \in H^1(\Omega)$ and $v_{d/{\rm inf}} \in \mathscr{V}(\Omega)$ are the functions on which the inf's in the first relationships of (9) and (12), respectively, are reached. Let also $\widetilde{v}$ be the interpolation of $v_{\rm inf}$ from the space $\mathscr{H}(\Omega)$, satisfying i) and ii) of Lemma 2. First of all, we note that according to Lemma 2

$$(\delta\varepsilon^{-1})^2\|\widetilde{v}\|^2_{0,\Omega} + \|\nabla\widetilde{v}\|^2_{0,\Omega} \prec (\delta\varepsilon^{-1})^2\|v_{\rm inf}\|^2_{0,\Omega} + \|\nabla v_{\rm inf}\|^2_{0,\Omega}. \tag{15}$$

Therefore, we can write

$$\begin{aligned} \|v\|^2_{h,\partial\Omega} &:= (\delta\varepsilon^{-1})^2\|v_{d/{\rm inf}}\|^2_{0,\Omega} + \|\nabla v_{d/{\rm inf}}\|^2_{0,\Omega} \\ &\prec (\delta\varepsilon^{-1})^2\|\widetilde{v}\|^2_{0,\Omega} + \|\nabla\widetilde{v}\|^2_{0,\Omega} \\ &\prec (\delta\varepsilon^{-1})^2\|v_{\rm inf}\|^2_{0,\Omega} + \|\nabla v_{\rm inf}\|^2_{0,\Omega} \prec]|v|[^2_{\partial\Omega}, \end{aligned} \tag{16}$$

where the first inequality follows by the definition of $\|v\|^2_{h,\partial\Omega}$, the second inequality – by the definition of the same norm and the inclusion $\mathscr{H}(\Omega) \subset \mathscr{V}(\Omega)$, the third inequality is simply (15), and the last one is a consequence of Theorem 1.

For the seminorms the proof is similar. □

Lemma 2, used above, is practically a corollary of a result of Scott & Zhang [49] on a special quasi-interpolation operator, which we present first.

Let $\Omega \subset \mathbb{R}^n$ be a n-dimensional domain with an arbitrary quasiuniform triangulation $\mathscr{S}_h$ with nodal points $x^{(i)}$, $i = 1, 2, \ldots, I$, and maximal edge size h. To each node $x^{(i)}$, we relate the $(n-1)$-dimensional simplex τ_i, which is the face of one of the n-dimensional simplices of the triangulation $\mathscr{S}_h$ having the vertex $x^{(i)}$. For n vertices of the simplex τ_i, we also use the notations $z^{(i)}_l$, $l = 1, 2, \ldots, n$, assuming for definiteness that $z^{(i)}_1 = x^{(i)}$. The choice of τ_i is not unique, but for $x^{(i)} \in \partial\Omega$ we always take $\tau_i \subset \partial\Omega$. By $\mathscr{V}_\Delta(\Omega)$ and $\mathscr{V}_{\rm tr}(\partial\Omega)$ we denote the space of functions, which are continuous on $\overline{\Omega}$ and linear on each simplex of the triangulation, and the space of their traces on $\partial\Omega$, respectively. Let $\theta_i \in \mathscr{P}(\tau_i)$ be the function satisfying

$$\int_{\tau_i} \theta_i \lambda^{(i)}_l \, dx = \delta_{1,l}, \quad l = 1, 2, .., n,$$

where $\lambda^{(i)}_l$ are the barycentric coordinates in τ_i related to its vertices $z^{(i)}_l$, and $\delta_{i,l}$ is the Kronecker symbol. If $\phi_i \in \mathscr{V}_\Delta(\Omega)$ are Galerkin FE basis functions such that $\phi_i(x_j) = \delta_{i,j}$, $i, j = 1, 2, \ldots, I$, then for each $v \in H^1(\Omega)$ the quasi-interpolation $\mathscr{I}_h v \in \mathscr{V}_\Delta(\Omega)$ is defined as

$$\mathscr{I}_h v = \sum_{i=1}^{I} \left(\int_{\tau_i} \theta_i v \, dx \right) \phi_i(x).$$

A triangulation $\mathscr{S}_h$ by simplices is called quasiuniform with respect to some mesh parameter $h > 0$ in the usual sense, see Ciarlet [15] and Korneev [29]. In two dimensions, quasiuniformity of quadrangular meshes is controlled by the conditions that lengths of edges and angles at vertices of quadrangles belong to intervals $(\alpha^{(1)}h, h)$ and $(\theta, \pi - \theta)$, respectively, with $0 < \alpha^{(1)}$, $\theta = \text{const}$.

Lemma 1. *The quasi-interpolation operator $\mathscr{I}_h$ satisfies*

a) $\mathscr{I}_h v : H^1(\Omega) \mapsto \mathscr{V}_\Delta(\Omega)$, *and, if* $v \in \mathscr{V}_\Delta(\Omega)$, *then* $\mathscr{I}_h v = v$,
b) $(v - \mathscr{I}_h v) \in \mathring{H}^1(\Omega)$, *if* $v_{|\partial\Omega} \in \mathscr{V}_{\text{tr}}(\partial\Omega)$,
c) $\|v - \mathscr{I}_h v\|_{t,\Omega} \leq c_{\text{int}} h^{s-t} \|v\|_{s,\Omega}$ *for* $t = 0, 1$, *and* $s = 1, 2$,
d) $|\mathscr{I}_h v|_{1,\Omega} \leq c_{\text{int}} |v|_{1,\Omega}$ *and* $\|\mathscr{I}_h v\|_{1,\Omega} \leq c_{\text{int}} \|v\|_{1,\Omega}$ *for all* $v \in H^1(\Omega)$, *where* c_{int} *is a constant, depending only on* $\alpha^{(1)}$, θ *from the quasiuniformity conditions.*

Proof. The proof was given by Scott & Zhang [49] and can be found also in Xu & Zou [51]. □

The operator $\mathscr{I}_h$, obviously, is a projection onto the space $v \in \mathscr{V}_\Delta(\Omega)$.

Now we will formulate for $n = 2$ the interpolation result used in the proof. Suppose, $\mathscr{V}(\Omega)$ is the FE space induced by first order quadrangular finite elements. We define a triangulation of Ω by subdividing each quadrangular nest of the mesh in two triangles by one of the diagonals of the nest and then denote by $\mathscr{V}_\Delta(\Omega)$ the space of continuous piecewise linear functions, induced by the obtained triangulation. For each $v \in H^1(\Omega)$, we define $\Pi_h v$ by the equality $(\Pi_h v)(x^{(i)}) = (\mathscr{I}_h v)(x^{(i)})$ for all vertices $x^{(i)}$ of the triangulation. It is easy to note that Π_h is not a projection operator, but it retains some other useful properties of the operator $\mathscr{I}_h$.

Lemma 2. *For any $v \in H^1(\Omega)$ the interpolation $\Pi_h v \in \mathscr{V}(\Omega)$ is such that*

i) if $v_{|\partial\Omega} \in \mathscr{V}_{\text{tr}}(\partial\Omega)$, then the traces of $\Pi_h v$ and v on the boundary $\partial\Omega$ coincide,
ii) the interpolation satisfies the stability estimates

$$|\Pi_h v|_{1,\Omega} \prec |v|_{1,\Omega}, \quad \|\Pi_h v\|_{1,\Omega} \prec \|v\|_{1,\Omega}, \tag{17}$$

iii) and approximation estimates

$$\|v - \Pi_h v\|_{t,\Omega} \prec h^{s-t} \|v\|_{s,\Omega}, \quad t = 0, 1, \; s = 1, 2. \tag{18}$$

Proof. We omit the proof, which in part can be found in Korneev et al. [33, 34]. □

Obviously, the norm (10) defines a matrix $\mathbb{B}_{\text{KPS}}$ such that $\|\mathbf{v}\|_{\mathbb{B}_{\text{KPS}}} =]v[_{\partial\Omega}, \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}_{\text{tr}}(\partial\Omega)$. Let the FE matrix $\mathbf{A}$ be induced by the Dirichlet integral over Ω and the FE space $\mathscr{V}(\Omega)$ with the nodal basis functions. Representing it in the block form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_I & \mathbf{A}_{I,B} \\ \mathbf{A}_{B,I} & \mathbf{A}_B \end{pmatrix}, \tag{19}$$

we denote by **B** the Schur complement

$$\mathbf{B} = \mathbf{A}_B - \mathbf{A}_{B,I}\mathbf{A}_I^{-1}\mathbf{A}_{I,B}, \tag{20}$$

where the lower indices I and B are related to the degrees of freedom at the nodes, living in the interior of the domain Ω and on its boundary, respectively.

Corollary 1. $\mathbb{B}_{\text{KPS}} \prec \mathbf{B} \prec \mathbb{B}_{\text{KPS}}$ *uniformly in h.*

Thus, $\mathbb{B}_{\text{KPS}}$ can be used as a spectrally equivalent preconditioner for the Schur complement **B**. In the next subsection, we introduce a simpler boundary seminorm for discrete harmonic functions, which is more convenient for matrix vector multiplications.

2.2 Finite-Difference Shape Dependent Boundary Norm for Finite Element Functions

Let us turn to a simpler case of the skeleton mesh, whose trace on $\partial\Omega$ coincides with the trace of an auxiliary orthogonal quasiuniform grid satisfying (11). We denote by γ_k for $k = 0,1$ the left and the right vertical edges of Ω, and by γ_k, $k = 2,3$, the horizontal lower and upper edges, respectively.

The auxiliary *coarse (quasiuniform) grid*, by its definition, is the coarsest rectangular imbedded quasiuniform grid. It has rectangular nests as much as possible close to the square $\varepsilon \times \varepsilon$ and is obtained by subdividing the domain Ω by vertical lines $x_1 = t_{1,i}$ in such a way that $t_{1,0} = 0$, $t_{1,n_\varepsilon} = 1$ for some integer $n_\varepsilon \geq 1$ and sizes $\eta_{1,i} := t_{1,i} - t_{1,i-1}$, $i = 1,2,\dots,n_\varepsilon$, satisfy the inequalities

$$\varepsilon \leq \eta_{1,i} \leq \overline{c}_\circ \varepsilon, \tag{21}$$

with $\overline{c}_\circ = \text{const} \leq 2$. The notation $\mathscr{V}_{\text{c}}(\Omega)$ will stand for the space of functions which are continuous on Ω and bilinear on each nest of the coarse quasiuniform mesh. This space will be called the *coarse finite element space*.

Simultaneously, we have defined overlapping intervals

$$\tau_0 = (0, t_{1,1}), \quad \tau_i = (t_{1,i-1}, t_{1,i+1}), \quad i = 1,2,\dots,n_\varepsilon - 1, \quad \tau_{n_\varepsilon} = (t_{1,n_\varepsilon - 1}, 1),$$

and intersections $\gamma_{k,i}$ of τ_i, $i = 1,2,\dots,n_\varepsilon - 1$, with the edges γ_k, $k = 2,3$. We use the notations $\gamma_0 = \Gamma_0$, $\gamma_{n_\varepsilon} = \Gamma_1$ and for simplicity we assume that $\overline{\tau}_0 \cap \partial\Omega = \overline{\Gamma}_0$ and $\overline{\tau}_{n_\varepsilon} \cap \partial\Omega = \overline{\Gamma}_1$ and that the numbers of nodes on these sets are the same and equal to ν.

Let

$$\Delta_{1/2,k} = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & \mathbb{O} & \\ & & \ddots & \ddots & \ddots & \\ & \mathbb{O} & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}^{1/2}, \quad k=0,1, \tag{22}$$

be $\nu \times \nu$ matrices, acting on vectors of degrees of freedom corresponding to the nodes on $\overline{\gamma}_k$. If ν_i is the number of the nodes on $\overline{\tau}_i$, then $\Delta_{1/2,k,i}$ is the, similar to (22), $\nu_i \times \nu_i$ matrix related to the segment $\overline{\gamma}_{k,i}$, $k = 2,3$. In the $(n_1+1)\times(n_1+1)$ matrix

$$\nabla = \frac{h}{\varepsilon}\begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix},$$

the unity matrices $\mathbf{I}$ on the diagonal correspond to the nodes on the edges $\overline{\gamma}_2, \overline{\gamma}_3$, respectively.

Let us remind that, as it was stated in the introduction, matrices $\Delta_{1/2,}$, $\Delta_{1/2,k,i}$ and ∇ are considered as defined on degrees of freedom at the nodes of the sets $\overline{\Gamma}_0$, $\overline{\Gamma}_1$, $\overline{\gamma}_{2,i}$, $\overline{\gamma}_{3,i}$ and $\overline{\gamma}_2$, $\overline{\gamma}_3$, respectively, and continued by zeroes on the degrees of freedom of the remaining nodes of the boundary, when necessary. This implies that sums like (23) below should be understood as *topological* sums.

Lemma 3. *The matrix*

$$\mathbf{C} = \nabla + \Delta_{1/2,0} + \Delta_{1/2,1} + \sum_{k=2,3}\sum_{i=1}^{n_\varepsilon - 1} \Delta_{1/2,k,i} \tag{23}$$

is spectrally equivalent to the matrix $\mathbb{B}_{\mathrm{KPS}}$ *uniformly in* h *and* $\varepsilon \in (0,1]$. *Besides, for any vector* $\mathbf{v}_B$ *the arithmetical costs of the matrix-vector multiplication* $\mathbf{C}\mathbf{v}_B$ *are* $ops\,[\mathbf{C}\mathbf{v}_B] = \mathscr{O}((n_1+n_2)(1+\log n_2))$.

Proof. We outline the proof, omitting details, which is completed in two steps. First, we introduce the seminorm $\rfloor \cdot \lfloor_{\partial\Omega}$ by the expression

$$\begin{aligned} \rfloor v \lfloor^2_{\partial\Omega} &= \varepsilon^{-1}\int_0^1 (v(x_1,\varepsilon) - v(x_1,0))^2 dx_1 \\ &+ \sum_{i=1}^{n_\varepsilon - 1}\int_{\tau_i}\int_{\tau_i}\left[\frac{(v(x_1,0)-v(y_1,0))^2}{(x_1-y_1)^2} + \frac{(v(x_1,\varepsilon)-v(y_1,\varepsilon))^2}{(x_1-y_1)^2})\right] dx_1 dy_1 \\ &+ \int_{\Gamma_0}\int_{\Gamma_0}\frac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} + \int_{\Gamma_1}\int_{\Gamma_1}\frac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} \end{aligned} \tag{24}$$

and show that it is equivalent to the seminorm $]v[_{\partial\Omega}$, i.e.,

$$\underline{\gamma}_0 \rfloor v \lfloor_{\partial\Omega} \leq]v[_{\partial\Omega} \leq 3 \rfloor v \lfloor_{\partial\Omega}, \quad 0 < \underline{\gamma}_0 = \underline{\gamma}_0(\overline{c}) = \mathrm{const}, \quad \forall v \in H^1(\Omega). \tag{25}$$

Result of the second step are the inequalities

$$\underline{\gamma}_C \mathbf{v}^\top \mathbf{C}\mathbf{v} \le \rfloor v \lfloor^2_{\partial\Omega} \le \overline{\gamma}_C \mathbf{v}^\top \mathbf{C}\mathbf{v}, \quad \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}(\Omega), \tag{26}$$

which hold with some constants $\underline{\gamma}_C$, $\overline{\gamma}_C > 0$. These bounds follow from the equivalences

$$\varepsilon^{-1}\int_0^1 (v(x_1,\varepsilon)-v(x_1,0))^2 dx_1 \asymp \mathbf{v}^\top \nabla \mathbf{v} \quad \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}_{\mathrm{tr}}(\gamma_2 \cup \gamma_3),$$

$$\int_{\tau_i}\int_{\tau_i} \frac{(v(x_1,0)-v(y_1,0))^2}{(x_1-y_1)^2} dx_1 dy_1 \asymp \mathbf{v}^\top \Delta_{1/2,2,i}\mathbf{v}, \quad \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}_{\mathrm{tr}}(\gamma_2),$$

$$\int_{\tau_i}\int_{\tau_i} \frac{(v(x_1,\varepsilon)-v(y_1,\varepsilon))^2}{(x_1-y_1)^2} dx_1 dy_1 \asymp \mathbf{v}^\top \Delta_{1/2,3,i}\mathbf{v}, \quad \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}_{\mathrm{tr}}(\gamma_3),$$

$$\int_{\Gamma_k}\int_{\Gamma_k} \frac{(v(s)-v(\overline{s}))^2}{(s-\overline{s})^2} ds d\overline{s} \asymp \mathbf{v}^\top \Delta_{1/2,k}\mathbf{v}, \quad \forall \mathbf{v} \Leftrightarrow v \in \mathscr{V}_{\mathrm{tr}}(\Gamma_k),$$

where $\mathscr{V}_{\mathrm{tr}}(\gamma_2 \cup \gamma_3)$, $\mathscr{V}_{\mathrm{tr}}(\gamma_2)$, $\mathscr{V}_{\mathrm{tr}}(\gamma_3)$ and $\mathscr{V}_{\mathrm{tr}}(\Gamma_k)$, $k = 0,1$, are the trace spaces of the FE space on the corresponding subsets of the boundary. Bounds of the first line follow by the spectral equivalence of the FE 1-d mass matrix to its diagonal. Lines 1-3 express another known fact. Suppose, some interval τ is subdivided by a quasiuniform grid with ν intervals and $\mathscr{H}(\tau)$ is the corresponding space of continuous piecewise linear functions. Then the matrix of the quadratic form $|v|^2_{1/2,\tau}$ on the space $\mathscr{H}(\tau)$ is spectrally equivalent to the matrix $\Delta_{1/2}$ of the form (22). We found an early proof of this fact in Andreev [2, 3].

Combining (25) and (26), one comes to

$$\underline{\beta}_C \mathbf{C} \prec \mathbb{B}_{\mathrm{KPS}} \prec \overline{\beta}_C \mathbf{C} \tag{27}$$

with positive constants $\underline{\beta}_C$, $\overline{\beta}_C > 0$ depending only on the constants from the quasi-uniformity conditions (11).

The matrix-vector multiplication $\nabla \mathbf{v}$ requires $6n_2 + 1$ arithmetical operations. The matrix-vector multiplications by each matrix $\Delta_{1/2,k}$ or $\Delta_{1/2,k,i}$ can be completed by FDFT and requires $\mathscr{O}(n_2 \log n_2)$ arithmetical operations. Hence, the vector-matrix multiplication by the topological sum of these matrices requires in total $\mathscr{O}((n_1 + n_2)(1+\log n_2))$ arithmetical operations. This approves the estimate of the arithmetic work for the multiplication $\mathbf{C}\mathbf{v}$ given in the Lemma. □

By Theorem 2, the definition of the matrix $\mathbb{B}_{\partial\Omega}$, and Lemma 3, it follows that:

Corollary 2. *For the matrix* $\mathbf{C}$ *and the Schur complement* $\mathbf{B}$ *of (20), it holds* $\mathbf{B} \asymp \mathbf{C}$.

Similar to (23) we can define preconditioners in the case of orthotropic discretizations, e.g., by the mesh, which is quasiuniform in each direction and has characteristic sizes h_1, h_2. We cover such a discretization mesh by a finer mesh, called *condensed mesh*, which has nests as close as possible in the shape to the square $h \times h$, $h = \min(h_1, h_2)$ and define the matrix (23) for this mesh, which we denote $\mathbf{C}_{\mathrm{cond}}$. After that, we restrict this matrix to the set of nodes corresponding to the space $\mathscr{V}_{\mathrm{tr}}(\partial\Omega)$ denoting the new matrix $\mathbf{C}_{\mathrm{ff}}$. At that time, we represent it in the form

$$\mathbf{C}_{\mathrm{ff}} = \nabla + \Delta^{(0)}_{1/2,\mathrm{ff}} + \Delta^{(1)}_{1/2,\mathrm{ff}} + \sum_{k=2,3} \sum_{i=1}^{n_\varepsilon - 1} \Delta_{1/2,k,i}, \tag{28}$$

where the matrices ∇, $\Delta_{1/2,k,i}$ are defined as above on the discretization mesh, whereas $\Delta^{(k)}_{1/2,\mathrm{ff}}$, $k = 1, 2$, are defined as the respective matrices $\Delta_{1/2,k,i}$, but for the condensed mesh with understanding that added nodes are treated as hanging nodes.

Corollary 3. *Let the discretization mesh be quasiuniform in each direction with the characteristic sizes h_1, h_2, $\mathbf{B}$ be the Schur complement, generated by the corresponding FE space $\mathscr{V}(\Omega)$. Then $\mathbf{B} \asymp \mathbf{C}_{\mathrm{ff}}$ and*

$$ops\,[\mathbf{C}_{\mathrm{ff}}\mathbf{v}] \prec (n_1 + n_2) \log \max(n_2, n_1/n_\varepsilon) \prec \overline{n} \log \overline{n}, \quad \overline{n} = \max(n_1, n_2), \quad \forall \mathbf{v},$$

uniformly in $h_1 \in (0,1)$, $h_2 \in (0, \varepsilon)$.

Note that for matrix vector multiplications by $\mathbf{C}_{\mathrm{ff}}$, the FDFT can be used and that for $h_1 \ll h_2$ or $h_2 \ll h_1$ we have $\dim \Delta_{1/2,k,i} \le \dim \Delta^{(l)}_{1/2,\mathrm{ff}} \approx 3 \max(n_2, n_1/n_\varepsilon)$, $l = 0, 1$.

2.3 Preconditioning by Nonoverlapping Domain Decomposition

In this paragraph we consider a thin rectangle and derive a preconditioner for the boundary Schur complement by an implementation of the DD procedure with nonoverlapping subdomains. It allows to split degrees of freedom of each vertical edge and degrees of freedom of the pair of longest edges from other degrees of freedom. Then we additionally split the degrees of freedom at the vertices of the rectangle from all other ones.

The coarse grid, introduced in the preceding subsection, defines a nonoverlapping domain decomposition of the rectangle Ω into subdomains

$$\Omega^i = (t_{1,i-1}, t_{1,i}) \times (0, \varepsilon), \quad i = 1, 2, \ldots, n_\varepsilon.$$

For their edges, we use the notations Γ^i_k, $k = 0, 1, 2, 3$, and order them counterclockwise starting from the lower edge of Ω^i. The FE space can be represented by the direct sum

$$\mathscr{V}(\Omega) = \mathscr{V}_{\mathrm{c}}(\Omega) \oplus \mathscr{W}_{\mathrm{r}}(\Omega), \tag{29}$$

where $\mathscr{V}_{\mathrm{c}}(\Omega)$ is the space of continuous functions which are bilinear on each subdomain Ω^i. The second subspace in (29) is supplied by the index ”r”, because in what follows it will play the role of the subspace, induced by the rarefied mesh. Notations $\mathscr{V}_{\mathrm{c}}(\Omega^i)$ and $\mathscr{W}_{\mathrm{r}}(\Omega^i)$ will stand for restrictions to Ω^i of the spaces $\mathscr{V}_{\mathrm{c}}(\Omega)$ and $\mathscr{W}_{\mathrm{r}}(\Omega)$.

We consider any $v = (v_c + v_W) \in \mathscr{V}(\Omega^i)$, such that $v_0 \in \mathscr{V}_c(\Omega^i)$ and $v_W \in \mathscr{W}_r(\Omega^i)$ is discrete harmonic on Ω^i. Subdomains Ω^i are shape regular, and according to a result of Bramble et al. [9]–[12]

$$\frac{1}{(1+\log n_2)^2}\left(|v_c|^2_{1,\Omega^i} + \sum_{k=0}^{3} {}_{00}|v_W|^2_{1/2,\Gamma_k^i}\right) \prec |v|^2_{1,\Omega^i} \prec |v_c|^2_{1,\Omega^i} + \sum_{k=0}^{3} {}_{00}|v_W|^2_{1/2,\Gamma_k^i}. \tag{30}$$

From here, for a FE function $v \in \mathscr{V}(\Omega)$ which is discrete harmonic in Ω, we directly come to the inequalities

$$\frac{1}{(1+\log n_2)^2}\left(|v_c|^2_{1,\Omega} + {}_{00}|v_W|^2_{1/2,\Gamma_3^1} + {}_{00}|v_W|^2_{1/2,\Gamma_1^{n\varepsilon}} + \sum_{i=1}^{n_\varepsilon}\sum_{k=0,2} {}_{00}|v_W|^2_{1/2,\Gamma_k^i}\right)$$
$$\prec |v|^2_{1,\Omega} \prec |v_c|^2_{1,\Omega} + {}_{00}|v_W|^2_{1/2,\Gamma_3^1} + {}_{00}|v_W|^2_{1/2,\Gamma_1^{n\varepsilon}} + \sum_{i=1}^{n_\varepsilon}\sum_{k=0,2} {}_{00}|v_W|^2_{1/2,\Gamma_k^i}. \tag{31}$$

Let $\mathscr{B}_W$ be the matrix which is spectrally equivalent to the matrix of the quadratic form (31) on the subspace $\mathscr{W}(\Omega)$, i.e.,

$$\mathbf{v}_W^\top \mathscr{B}_W \mathbf{v}_W \prec {}_{00}|v_W|^2_{1/2,\Gamma_3^1} + {}_{00}|v_W|^2_{1/2,\Gamma_k^{n\varepsilon}} + \sum_{i=1}^{n_\varepsilon}\sum_{k=0,2} {}_{00}|v_W|^2_{1/2,\Gamma_k^i} \prec \mathbf{v}_W^\top \mathscr{B}_W \mathbf{v}_W. \tag{32}$$

Then (31) is equivalent to the inequalities

$$\frac{1}{(1+\log n_2)^2}\mathbf{v}^\top \mathbb{C}_{\rm hi}\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}_{\rm hi}\mathbf{v} \prec \mathbf{v}^\top \mathbb{C}_{\rm hi}\mathbf{v}, \tag{33}$$

where

$$\mathbb{C}_{\rm hi} = \begin{pmatrix} \mathscr{B}_W & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_c \end{pmatrix}$$

and the notation $\mathbf{B}_{\rm hi}$ for the Schur complement reflects that it is written in the two level basis, corresponding to the representation $\mathscr{V}(\Omega) = \mathscr{V}_c(\Omega) \oplus \mathscr{W}_r(\Omega)$ of the FE space. The matrix $\mathbf{B}_c$ is the block, corresponding to the subspace $\mathscr{V}_c(\Omega)$.

Let the notation $\mathbf{B}_{\rm Hi}$ stand for the Schur complement corresponding to the three level representation of the FE space

$$\mathscr{V}(\Omega) = \mathscr{W}_r(\Omega) \oplus \mathscr{W}_c(\Omega) \oplus \mathscr{V}_0(\Omega),$$

and let $\mathbf{B}^{(\rm w)}$ and $\mathbf{B}^{(\rm v)}$ be the notations for the blocks on the diagonal of $\mathbf{B}_{\rm Hi}$, corresponding to the subspaces $\mathscr{W}_c(\Omega)$ and $\mathscr{V}_0(\Omega)$, respectively, and $\underline{n} = \min(n_1, n_2)$, $\overline{n} = \max(n_1, n_2)$. With a slightly changing reasoning, one also can get

$$\min\left(\frac{1}{n_\varepsilon(1+\log \underline{n})}, \frac{1}{(1+\log \underline{n})^2}\right)\mathbf{v}^\top \mathscr{C}_{\rm Hi}\mathbf{v} \prec \mathbf{v}^\top \mathbf{B}_{\rm Hi}\mathbf{v} \prec \mathbf{v}^\top \mathscr{C}_{\rm Hi}\mathbf{v}, \tag{34}$$

where

$$\mathscr{C}_{\mathrm{Hi}} = \begin{pmatrix} \mathscr{B}_{\mathrm{W}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{(\mathrm{w})} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}^{(\mathrm{v})} \end{pmatrix}. \tag{35}$$

We reorder the sets $\Gamma_k^i \subset \partial\Omega$ consecutively counter clockwise, starting from Γ_0^1, introduce for them the notations $\mathscr{T}^i$, $i = 1,2,\ldots,2n_\varepsilon+2$, and by ν_i the number of the intervals of the source mesh on $\mathscr{T}^i$. Estimates (32) and therefore (33), (34) hold for

$$\mathscr{B}_{\mathrm{W}} = \mathrm{diag}[\Delta_{1/2,i}]_{i=1}^{2(n_\varepsilon+1)} \tag{36}$$

with $\Delta_{1/2,i}^{1/2} = \mathrm{tridiag}\,[-1,2,-1]_1^{\nu_i-1}$. Thus, we have proved:

Lemma 4. *With $\mathscr{B}_{\mathrm{W}}$ defined in (36), the estimates (34) hold for all $\varepsilon \in (0,1]$ and $h \le (0,\varepsilon]$.*

The preconditioner $\mathscr{C}_{\mathrm{hi}}$ is sufficiently simple. In particular, it is given explicitly, is easily invertible, and, therefore, can be used for assembling the Schur complement preconditioner for a domain which is decomposed into rectangular subdomains. The system of algebraic equations with the preconditioner $\mathscr{C}_{\mathrm{hi}}$ of Lemma 4 for the matrix is solved in $\mathscr{O}(n_\varepsilon \underline{n} \log \underline{n}) = \mathscr{O}(\overline{n} \log \underline{n})$ arithmetical operations, where for the case under consideration $\overline{n} = n_1$, $\underline{n} = n_2$. However, the subspaces $\mathscr{V}_0$ and $\mathscr{W}_{\mathrm{r}}$ depend on the aspect ratio of the rectangle Ω, and, therefore, even the assembling procedure of the interface Schur complement preconditioner can be not simple, not to speak about its inversion.

3 Orthotropic Discretization with Arbitrary Aspect Ratio on Thin Rectangles

3.1 Finite Element Space Decomposition

A more complicated situation arises, when we consider a heat conduction problem in a slim domain with different heat conduction coefficients along different axes, and a uniform rectangular mesh is used for discretization. No restrictions are imposed on the aspect ratios of conductivity coefficients and sizes of the mesh, except that they are finite. Therefore, the model problem, we turn here to, is

$$\alpha_\Omega(u,v) = \langle f,v\rangle, \quad \alpha_\Omega(u,v) = \int_\Omega \nabla u(x)\cdot\rho(x)\nabla v(x)\,dx, \quad \forall v \in H^1(\Omega), \tag{37}$$

in a slim rectangle $\Omega = (0,1)\times(0,\varepsilon)$. Now, $\rho = \mathrm{diag}\,[\rho_1,\rho_2]$ with an arbitrary constant $\rho_k > 0$. For simplicity, we restrict ourselves to a uniform rectangular mesh of arbitrary sizes $h_1,h_2 > 0$.

For the ease of future references, we use, different from the previously used notations, $\mathbf{Q}$, $\mathbf{Y}$ for the FE stiffness matrix and its boundary Schur complement

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_I & \mathbf{Q}_{I,B} \\ \mathbf{Q}_{B,I} & \mathbf{Q}_B \end{pmatrix}, \quad \mathbf{Y} = \mathbf{Q}_B - \mathbf{Q}_{B,I}\mathbf{Q}_I^{-1}\mathbf{Q}_{I,B}. \tag{38}$$

The derivation of a good preconditioner for the Schur complement will be completed in three steps. At step 1), we change variables and reduce the problem (37) to a transformed isotropic problem on some domain Ω_ξ. At step 2), we introduce the rarefied transformed mesh, which is the finest and, if possible, quasiuniform mesh imbedded in the transformed source mesh. It is obtained by rarefication of the transformed source mesh in one direction, corresponding to the smallest size of the latter mesh. Then the block diagonal preconditioner for the FE matrix $\mathbf{Q}$ is introduced, containing two independent blocks on the diagonal, one of which is the FE stiffness matrix, induced by the rarefied transformed mesh. In turn, the preconditioner for the FE matrix $\mathbf{Q}$ allows us to obtain the block diagonal preconditioner for the Schur complement $\mathbf{Y}$ with two independent blocks. At step 3), a further decoupling is accomplished. The block of the Schur complement preconditioner, corresponding to the transformed rarefied mesh, obviously, can be handled as in the preceding section. Another block does not require an additional treatment, because it itself is a block diagonal matrix with simple explicitly written down blocks, specified on the unknowns, subjected to rarefication.

The domain Ω represents one subdomain $\Omega = \Omega_j$ of the decomposition. The described process is based on the sequence of the FE spaces, which is related to the sequence of the image spaces

$$\mathbb{V} = \mathbb{V}_{\mathrm{r}} \oplus \mathbb{W}, \quad \mathbb{V}_{\mathrm{r}} = \mathbb{V}_{\mathrm{c}} \oplus \mathbb{W}_{\mathrm{r}}, \quad \mathbb{V}_{\mathrm{c}} = \mathbb{V}_0 \oplus \mathbb{W}_{\mathrm{c}}, \tag{39}$$

with

$$\mathbb{V} = \mathbb{W} \oplus \mathbb{W}_{\mathrm{r}} \oplus \mathbb{W}_{\mathrm{c}} \oplus \mathbb{V}_0, \quad \mathbb{V}_0 \subseteq \mathbb{V}_{\mathrm{c}} \subseteq \mathbb{V}_{\mathrm{r}} \subseteq \mathbb{V},$$

defined for the transformed problem and its discretization on the transformed subdomain $\Omega_\xi = \Omega_{j,\xi}$. In other words, the spaces $\mathbb{V}(\Omega_\xi)$, $\mathbb{V}_{\mathrm{r}}(\Omega_\xi)$, $\mathbb{V}_{\mathrm{c}}(\Omega_\xi)$, $\mathbb{V}_0(\Omega_\xi)$ are induced by the transformed source FE mesh, rarefied and coarse meshes, imbedded in the transformed source mesh, and by the space of bilinear functions on Ω_ξ, respectively. The spaces $\mathbb{V}_{\mathrm{r}}(\Omega_\xi),\ldots,\mathbb{W}_{\mathrm{c}}(\Omega_\xi)$ define preimage spaces denoted by $\mathsf{V}_{\mathrm{r}}(\Omega),\ldots,\mathsf{W}_{\mathrm{c}}(\Omega)$. In the preceding sections, the space $\mathscr{V}(\Omega)$ played the role of $\mathbb{V}_{\mathrm{r}}(\Omega_\xi)$.

3.2 Reducing to Isotropic Discretization

The change of variables $\xi_1 = x_1$, $\xi_2 = \sqrt{\rho_1/\rho_2}\,x_2$ transforms the bilinear form $\alpha_\Omega(\cdot,\cdot)$ into

$$\alpha_\Omega(u,v) = \sqrt{\rho_1\rho_2}\,\widetilde{\alpha}(u,v), \quad \widetilde{\alpha}(u,v) = \int_{\Omega_\xi} \nabla_\xi u \cdot \nabla_\xi v \, d\xi, \tag{40}$$

with the notations ∇_ξ for the gradient in the variables ξ, $\Omega_\xi = (0,1)\times(0,\widetilde{\varepsilon})$ for the new domain and $\widetilde{\varepsilon} = \varepsilon\sqrt{\rho_1/\rho_2}$. With this the FE space $\mathscr{V}(\Omega)$ is transformed into the space $\mathbb{V}(\Omega_\xi)$ of piecewise bilinear functions on the rectangular transformed source mesh of the sizes $\hbar_1 = h_1$, $\hbar_2 = \sqrt{\rho_1/\rho_2}\,h_2$ with the mesh lines $\xi_k \equiv \xi_{k,l} = l\hbar_k$. We have $\mathbf{Q} = \sqrt{\rho_1\rho_2}\,\mathbb{Q}$, $\mathbf{Y} = \sqrt{\rho_1\rho_2}\,\mathbb{Y}$,

$$\mathbb{Y} = \mathbb{Q}_B - \mathbb{Q}_{B,I}\mathbb{Q}_I^{-1}\mathbb{Q}_{I,B} \tag{41}$$

where $\mathbb{Q}_I$, $\mathbb{Q}_{I,B}$, $\mathbb{Q}_B$ are blocks of the stiffness matrix $\mathbb{Q}$ generated by the bilinear form $\widetilde{\alpha}(u,v)$ on the space $\mathbb{V}(\Omega_\xi)$. Therefore, the preconditioning of $\mathbf{Y}$ is reduced to the preconditioning of the Schur complement $\mathbb{Y}$.

We can restrict ourselves to the consideration of the case $\widetilde{\varepsilon} < 1$, since the case $\widetilde{\varepsilon} > 1$ is reduced to the former one by the interchange of variables. Under the condition $\widetilde{\varepsilon} \le 1$, three cases can be distinguished:

$$i) \quad \hbar_2 \le \hbar_1 \le \widetilde{\varepsilon}, \qquad ii) \quad \hbar_2 \le \widetilde{\varepsilon} \le \hbar_1, \qquad iii) \quad \hbar_1 \le \hbar_2 \le \widetilde{\varepsilon}, \tag{42}$$

and we start with $i)$. Under the stated conditions, the embedded rarefied quasiuniform rectangular grid

$$\xi_k \equiv \widetilde{\xi}_{k,i}, \quad k = 1,2, \quad \text{with the steps} \quad \eta_{k,i} = \widetilde{\xi}_{k,i} - \widetilde{\xi}_{k,i-1},$$

is introduced by coarsening only in one direction ξ_2. In other words, it is the same uniform grid in the direction ξ_1 with $\eta_{1,i} \equiv \hbar_1 \equiv h_1$ and nonuniform in the direction ξ_2 with the sizes $\eta_{2,j}$ as much close as possible to $\hbar_1$. The mesh lines $\xi_2 \equiv \widetilde{\xi}_{2,j}$ can be defined as follows. We find $m_2 = \text{integer}[\widetilde{\varepsilon}/h_1]$, then define the uniform mesh $\zeta_{2,j} = j\widetilde{\varepsilon}/m_2$, $j = 0,1,\dots,m_2$, and then shift the lines of this uniform nonembedded coarse mesh $\xi_2 = \zeta_{2,j}$ to the nearest lines $\xi_2 \equiv \xi_{2,l} = l\hbar_2$ of the transformed source mesh of the size $\hbar_2$ in the direction ξ_2. We retain the notation m_2 for the number of the rarefied mesh intervals in the direction ξ_2 whereas the number of the rarefied mesh intervals in the direction ξ_1 is $m_1 = n_1$. Obviously, the sizes of this mesh satisfy inequalities

$$\underline{c}h_1 \le \eta_{k,i} \le \overline{c}h_1, \quad \underline{c} > 0, \quad k = 1,2, \tag{43}$$

with positive constants, for which we retain the notations as in (11).

The space $\mathbb{V}(\Omega_\xi)$ may be represented by the direct sum

$$\mathbb{V}(\Omega_\xi) = \mathbb{V}_{\mathrm{r}}(\Omega_\xi) \oplus \mathbb{W}(\Omega_\xi), \tag{44}$$

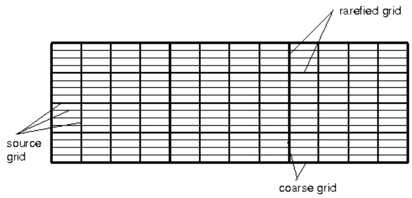

Fig. 3 Transformed rectangular domain and the source, rarefied and coarse grids.

where $\mathbb{V}_r(\Omega_\xi)$ is the space of FE functions which are continuous on $\overline{\Omega}_\xi$ and bilinear on each nest of the rarefied grid. Obviously, the space $\mathbb{W}(\Omega_\xi)$ contains FE functions, which vanish on the lines $\xi_2 \equiv \widetilde{\xi}_{2,j}$ of the rarefied grid. Let $\mathbb{V}_{tr}(\partial\Omega_\xi)$ be the space of traces of functions from $\mathbb{V}(\Omega_\xi)$ on $\partial\Omega_\xi$. For $u \in \mathbb{V}(\Omega_\xi)$ and $v \in \mathbb{V}_{tr}(\partial\Omega_\xi)$, respectively, we introduce the norms

$$|u|_{1,\Omega_\xi} = \|u\|_{\Omega_\xi} = (\widetilde{\alpha}(u,u))^{1/2}, \quad \|v\|_{h,\partial\Omega_\xi} = \inf_{\phi \in \mathbb{V}(\Omega_\xi):\phi_{|\partial\Omega}=v} \|\phi\|_{\Omega_\xi}. \tag{45}$$

The matrix $\mathbb{Q}$ can be represented in the block form

$$\mathbb{Q} = \begin{pmatrix} \mathbb{Q}_s & \mathbb{Q}_{sr} \\ \mathbb{Q}_{rs} & \mathbb{Q}_r \end{pmatrix}, \tag{46}$$

with blocks $\mathbb{Q}_s$ and $\mathbb{Q}_r$ corresponding to subspaces $\mathbb{W}(\Omega_\xi)$ and $\mathbb{V}_r(\Omega_\xi)$, respectively. Let $\nu_{2,j}$ denote the number of the fine mesh intervals on the rarefied mesh interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ and

$$\Delta_{2,j} = \mathrm{tridiag}\,[-1,2,-1]_1^{\nu_{2,j}-1}. \tag{47}$$

An intermediate preconditioner $\mathscr{Q}^*$ for $\mathbb{Q}$ may be defined in the following block form:

$$\mathscr{Q}^* = \mathrm{diag}\,[\mathscr{Q}_s, \mathbb{Q}_r], \quad \mathscr{Q}_s^2 = \mathrm{diag}\,[\underbrace{\Delta_{2,j}, \Delta_{2,j}, \ldots, \Delta_{2,j}}_{(n_1+1)\text{ times}}]_{j=1}^{m_2}. \tag{48}$$

Note that $\ker \mathscr{Q}^* = \ker \mathbb{Q}_r$. For a given j, the i-th block $\Delta_{2,j}$ in the square brackets is related to the nodes on the interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ of the mesh line $\xi_1 \equiv \widetilde{\xi}_{1,i}$.

In the case *ii*), the rarefied quasiuniform mesh of the characteristic size $\widetilde{\varepsilon}$ does not exist, and we introduce the *transformed rarefied uniform rectangular mesh* of the characteristic sizes h_1, $\widetilde{\varepsilon}$. Therefore, we have only one layer of n_1 cells $h_1 \times \widetilde{\varepsilon}$, meaning $m_2 = 1$, and similarly to (48) we can set

$$\mathscr{Q}^* = \text{diag}\,[\mathscr{Q}_{\text{s}}, \mathbb{Q}_{\text{r}}], \quad \mathscr{Q}_{\text{s}}^2 = \text{diag}\,[\underbrace{\Delta_2, \Delta_2, \ldots, \Delta_2}_{(n_1+1)\text{ times}}],$$

with the $(n_2-1)\times(n_2-1)$ blocks Δ_2. The matrix $\mathbb{Q}_{\text{r}}$ is defined on the uniform rectangular coarse transformed grid, which coincides with the rarefied transformed grid and has all nodes on $\partial\Omega_\xi$. This matrix is block-tridiagonal with the blocks 2×2 and does not require preconditioning.

If we have *iii*), the mesh parameter for the quasiuniform rectangular coarse grid is $\hbar_2$ and it satisfies

$$\underline{c}\hbar_2 \le \eta_{k,i} \le \overline{c}\hbar_2, \quad \underline{c} > 0, \quad k = 1, 2. \tag{49}$$

At a proper ordering of unknowns, we have again $\mathscr{Q}^* = \text{diag}\,[\mathscr{Q}_{\text{s}}, \mathbb{Q}_{\text{r}}]$, but

$$\mathscr{Q}_{\text{s}}^2 = \text{diag}\,[\underbrace{\Delta_{1,i}, \Delta_{1,i}, \ldots, \Delta_{1,i}}_{(n_2+1)\text{ times}}]_{j=1}^{m_1}, \quad \Delta_{1,i} = \text{tridiag}\,[-1, 2, -1]_1^{\nu_{1,i}-1}, \tag{50}$$

where $\nu_{1,i}$ is the number of the fine mesh intervals on the coarse mesh interval $(\widetilde{\xi}_{1,i-1}, \widetilde{\xi}_{1,i})$. For a given i, the j-th block $\Delta_{1,i}$ in the square brackets is related to the nodes on the interval $(\widetilde{\xi}_{1,i-1}, \widetilde{\xi}_{1,i})$ of the mesh line $\xi_2 \equiv \widetilde{\xi}_{2,j}$.

Lemma 5. *For any positive h_1, h_2, ρ_1, ρ_2 and ε,*

$$\frac{1}{1+\log\delta}\,\mathscr{Q}^* \prec \mathbb{Q} \prec \mathscr{Q}^*, \quad \delta = \max_k \min\left(n_k, \frac{h_{3-k}\sqrt{\rho_k}}{h_k\sqrt{\rho_{3-k}}}\right). \tag{51}$$

Proof. In the case *i*), we consider the transformed discretization mesh and add mesh lines subdividing each interval $(\widetilde{\xi}_{1,i-1}, \widetilde{\xi}_{1,i})$ in $\nu_2 = \text{integer}[h_1/\hbar_2]$ parts and come to a shape regular orthogonal mesh. To FE spaces on this mesh, we can apply results of Bramble *et. al* [9], which allow us to write

$$\frac{1}{1+\log\nu_2}\,\mathscr{Q}^* \prec \mathbb{Q} \prec c_2\,\mathscr{Q}^*. \tag{52}$$

For *ii)*, we have

$$\nu_2 = \min(n_2, \overline{c}h_1/\hbar_2) = \min\left(n_2, \frac{h_1\sqrt{\rho_2}}{h_2\sqrt{\rho_1}}\right), \tag{53}$$

which in the general case should be replaced by δ. □

We will now use Lemma 5 for defining some preconditioner for the Schur complement $\mathbb{Y}$, see (41), restricting ourselves for simplicity to the case *i)*. Taking into account, similar to (19) and (20), representations for the matrices $\mathbb{Q}_{\text{r}}$ and $\mathscr{Q}_{\text{s}}$

$$\mathbb{Q}_{\mathrm{r}} = \begin{pmatrix} \mathbb{Q}^{\mathrm{r}}_{I} & \mathbb{Q}^{\mathrm{r}}_{I,B} \\ \mathbb{Q}^{\mathrm{r}}_{B,I} & \mathbb{Q}^{\mathrm{r}}_{B} \end{pmatrix}, \qquad \mathscr{Q}_{\mathrm{s}} = \begin{pmatrix} \mathscr{Q}^{\mathrm{s}}_{I} & \mathscr{Q}^{\mathrm{s}}_{I,B} \\ \mathscr{Q}^{\mathrm{s}}_{B,I} & \mathscr{Q}^{\mathrm{s}}_{B} \end{pmatrix},$$
$$\mathbb{Y}_{\mathrm{r}} = \mathbb{Q}^{\mathrm{r}}_{B} - \mathbb{Q}^{\mathrm{r}}_{B,I}(\mathbb{Q}^{\mathrm{r}}_{I})^{-1}\mathbb{Q}^{\mathrm{r}}_{I,B}, \quad \mathscr{G}_{\mathrm{s}} = \mathscr{Q}^{\mathrm{s}}_{B} - \mathscr{Q}^{\mathrm{s}}_{B,I}(\mathscr{Q}^{\mathrm{s}}_{I})^{-1}\mathscr{Q}^{\mathrm{s}}_{I,B}, \tag{54}$$

we conclude that the Schur complement $\mathscr{G}^*$ for $\mathscr{Q}^*$ has the form

$$\mathscr{G}^* = \mathrm{diag}\,[\mathscr{G}_{\mathrm{s}}, \mathbb{Y}_{\mathrm{r}}]. \tag{55}$$

According to (48), in the matrix $\mathscr{Q}_{\mathrm{s}}$ internal degrees of freedom are not coupled with the boundary degrees of freedom, i.e., $\mathscr{Q}^{\mathrm{s}}_{I,B} = (\mathscr{Q}^{\mathrm{s}}_{B,I})^{\top} = \mathbf{0}$. Therefore,

$$\mathscr{G}_{\mathrm{s}} = \mathscr{Q}^{\mathrm{s}}_{B} = \frac{h_1}{\hbar_2}\,\mathrm{diag}\,[\Delta_{2,j}, \Delta_{2,j}]_{j=1}^{m_2}, \tag{56}$$

where the two matrices in the square brackets correspond to nodes in the interval $(\widetilde{\xi}_{2,j-1}, \widetilde{\xi}_{2,j})$ on the left and right vertical edges of $\partial\Omega_{\xi}$, respectively.

Corollary 4. *Let* $\mathscr{G}^* = diag\,[\mathscr{Q}^{\mathrm{s}}_{\mathrm{B}}, \mathbb{Y}_{\mathrm{r}}]$. *Then for all positive* h_1, h_2, ρ_1, ρ_2, *and* ε *we have*

$$\frac{1}{1+\log\delta}\,\mathscr{G}^* \prec \mathbb{Y} \prec \mathscr{G}^*. \tag{57}$$

For the reason of a complete analogy between the Schur complements $\mathbf{B}_{\mathrm{Hi}}$ and $\mathbb{Y}_{\mathrm{r}}$, Corollary 4 reduces the Schur complement preconditioning to the case, considered in the preceding section. There, the FE space $\mathscr{V}(\Omega) = \mathscr{W}_{\mathrm{r}}(\Omega) \oplus \mathscr{W}_{\mathrm{c}}(\Omega) \oplus \mathscr{V}_0(\Omega)$ plays here the role of the rarefied transformed space $\mathbb{V}_{\mathrm{r}}(\Omega_{\xi}) = \mathbb{W}_{\mathrm{r}}(\Omega_{\xi}) \oplus \mathbb{W}_{\mathrm{c}}(\Omega_{\xi}) \oplus \mathbb{V}_0(\Omega_{\xi})$. Hence, we can introduce the preconditioner for $\mathbb{Y}_{\mathrm{r}}$

$$\mathscr{C}_{\mathrm{r}} = \begin{pmatrix} \mathscr{B}_{\mathrm{W_r}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{W_{\mathrm{c}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_0 \end{pmatrix} \tag{58}$$

in a completely similar way to the preconditioner $\mathscr{C}_{\mathrm{Hi}}$ of (35) for $\mathbf{B}_{\mathrm{Hi}}$ and the preconditioner for the Schur complement $\mathbb{Y}$

$$\mathscr{G} = \mathrm{diag}\,[\mathscr{Q}^{\mathrm{s}}_{B}, \mathscr{C}_{\mathrm{r}}] = \mathrm{diag}\,[\mathscr{Q}^{\mathrm{s}}_{B}, \mathscr{B}_{W_{\mathrm{r}}}, \mathbf{B}_{W_{\mathrm{c}}}, \mathbf{B}_0]. \tag{59}$$

For the cases (42), the proof of the bounds

$$\frac{1}{(1+\log\delta)(1+\log\underline{m})}\min\left(\frac{1}{m_{\varepsilon}}, \frac{1}{(1+\log\underline{m})}\right)\mathscr{G} \prec \mathbb{Y} \prec \mathscr{G}, \tag{60}$$

where $\underline{m} = \min(m_1, m_2)$ and $m_{\varepsilon} = 1/\widetilde{\varepsilon}$, follows by combining the bounds of Lemma 5 and Lemma 4.

For a general rectangle $\Omega = H_1 \times H_2$, in the same way we introduce the rarefied source and the coarsest meshes, while the role of ε is played by $\min_k (H_k/H_{3-k})$. The above form of the preconditioner is retained, if after the transformation to the isotropic problem the shortest edge is directed along the axis x_2. In general, with the notation

$$\theta = \max_k \min\left(m_k, \frac{H_k\sqrt{\rho_{3-k}}}{H_{3-k}\sqrt{\rho_k}} \right),$$

the counterpart of (60) for all positive $H_k, h_k \le H_k, \rho_k$ is

$$\underline{\mu}\mathscr{G} \prec \mathbb{Y} \prec \mathscr{G} \tag{61}$$

with

$$\underline{\mu} = \frac{1}{(1+\log\delta)(1+\log\underline{m})} \min\left(\frac{1}{\theta}, \frac{1}{(1+\log\underline{m})} \right). \tag{62}$$

For slim rectangular domains Ω, a Schur complement preconditioner with a less degree of decoupling can be introduced. It is based on the FE space decomposition

$$\mathbb{V}(\Omega_\xi) = \mathbb{W}_{\rm d}(\Omega_\xi) \oplus \mathbb{V}_{\rm c}(\Omega_\xi), \quad \mathbb{V}_{\rm c}(\Omega_\xi) = \mathbb{W}_{\rm c}(\Omega_\xi) \oplus \mathbb{V}_0(\Omega_\xi),$$

where $\mathbb{W}_d(\Omega_\xi)$ is the space of functions $v \in \mathbb{V}(\Omega_\xi)$ with zero values at the nodes of the coarse mesh. The preconditioner, for which we retain the notation $\mathscr{G}$, gets the form

$$\mathscr{G} = \mathrm{diag}\,[\mathscr{B}_{W_{\rm d}}, \mathbf{B}_{W_{\rm c}}, \mathbf{B}_0] \tag{63}$$

with the same $\mathbf{B}_{W_{\rm c}}$, $\mathbf{B}_0$ as in (59). The block $\mathscr{B}_{W_{\rm d}}$ looks like $\mathscr{B}_W$ in (36), i.e.,

$$\mathscr{B}_{W_{\rm d}} = \mathrm{diag}\,[\Delta_{1/2,i}]_{i=1}^{2(n_\varepsilon+1)}, \quad \Delta_{1/2,i}^{1/2} = \mathrm{tridiag}\,[-1,2,-1]_1^{\nu_i-1},$$

but now ν_i denotes the number of the discretization mesh on the corresponding coarse mesh interval belonging to $\partial\Omega$. In the proof of the relative spectrum bounds results of Bramble *et. al* [9] are applied to the domain decomposition mesh by the coarse mesh. For this, FE functions are considered as elements of the space $\mathscr{V}_{\rm ff}(\Omega)$, induced by the condensed transformed discretization mesh, which is the coarsest shape regular orthogonal mesh, covering the transformed discretization mesh. In the resulting inequalities (61)

$$\underline{\mu} = \frac{1}{(1+\log n_{\rm ff})} \min\left(\frac{1}{\theta}, \frac{1}{(1+\log n_{\rm ff})} \right), \quad n_{\rm ff} = \max_{k=1,2} \frac{n_k}{\max[1, \frac{H_k\sqrt{\rho_{3-k}}}{H_{3-k}\sqrt{\rho_k}}]}. \tag{64}$$

Thus, we have proved:

Theorem 3. *For all positive H_k, $h_k \le H_k$, ρ_k, the Schur complement preconditioners $\mathscr{G}$ of (59) and (63) satisfy (61) with $\underline{\mu}$ from (62), (64), respectively.*

Solving systems $\mathscr{G}\mathbf{v}_B = \mathbf{f}_B$ requires not more than $\mathscr{O}(\overline{n}\log\underline{n})$ arithmetical operations, but the relative condition depends on $\theta \le m_k$. At the same time this Schur complement preconditioner, as others previously considered, has an obvious drawback, if we turn to the problem (1)-(5). Let $\mathscr{G} = \mathscr{G}_j$ be such a preconditioner for the subdomain Ω_j and $\mathscr{S}$ be the preconditioner for the interface Schur complement $\mathbf{S} = \mathbf{K}_B - \mathbf{K}_{B,I}\mathbf{K}_I^{-1}\mathbf{K}_{I,B}$ of the matrix $\mathbf{K}$, assembled from preconditioners $\sqrt{\rho_1\rho_2}\mathscr{G}_j$. For different subdomains Ω_j, decompositions $\mathscr{V}(\Omega_j) = \mathsf{W}(\Omega_j) \oplus \mathsf{W}_{\mathrm{r}}(\Omega_j) \oplus \mathsf{W}_{\mathrm{c}}(\Omega_j) \oplus \mathsf{V}_0(\Omega_j)$ are not compatible and, therefore, a fast solving procedure for systems with such a matrix $\mathscr{S}$ requires a new consideration.

4 Compatible Schur Complement Preconditioner

Khoromskij & Wittum [27, 28], Korneev [31], Korneev et al. [33, 34] and Rytov [46] used in relation with the problem (37) slightly different compatible subdomain edge Schur complement preconditioners. They were obtained from the Schur complement $\mathbf{Y}$ by decoupling some of its blocks on the diagonal. In [31], the preconditioner has on the diagonal the independent block of vertex degrees of freedom and two independent blocks each related to a pair of parallel edges of the rectangle Ω_ξ, whereas in [27, 28] two short edges were additionally decoupled. In this section, we present bounds of the relative condition numbers of these preconditioners, which follow from (61).

We call by the source triangulation the one obtained by subdivision of each rectangular nest of the source mesh in two triangles by the diagonal of the same direction and denote $\mathscr{U}_\Delta(\Omega)$ the space of continuous functions which are linear on each triangle. It is represented by the direct sums $\mathscr{U}(\Omega) = \mathscr{U}_I(\Omega) \oplus \mathscr{U}^{\mathrm{B}}(\Omega)$, $\mathscr{U}^{\mathrm{B}}(\Omega) = \mathscr{U}^{\mathrm{E}}(\Omega) \oplus \mathscr{U}^{\mathrm{V}}(\Omega)$, spanned over internal, boundary, edge and vertex FE functions, which are a nodal basis for the source triangulation. Respectively, the stiffness matrix induced by the space $\mathscr{U}(\Omega)$, which is denoted $\mathbf{L}$, is represented in the block forms

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_I & \mathbf{L}_{IB} \\ \mathbf{L}_{BI} & \mathbf{L}_B \end{pmatrix} = \begin{pmatrix} \mathbf{L}_I & \mathbf{L}_{IE} & \mathbf{L}_{IV} \\ \mathbf{L}_{EI} & \mathbf{L}_E & \mathbf{L}_{EV} \\ \mathbf{L}_{VI} & \mathbf{L}_{VE} & \mathbf{L}_V \end{pmatrix}, \tag{65}$$

and $\mathbf{L}_{IV} = \mathbf{L}_{VI}^\top = \mathbf{0}$. Therefore, the Schur complement $\mathscr{L} = \mathbf{L}_B - \mathbf{L}_{BI}\mathbf{L}_I^{-1}\mathbf{L}_{IB}$ has the form

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_E & \mathbf{L}_{EV} \\ \mathbf{L}_{VE} & \mathbf{L}_V \end{pmatrix}, \quad \mathscr{L}_E = \mathbf{L}_E - \mathbf{L}_{EI}\mathbf{L}_I^{-1}\mathbf{L}_{IE},$$

and, due to the spectral equivalence $\mathbf{L} \prec \mathbf{Q} \prec \mathbf{L}$, we have

$$\mathscr{L} \prec \mathbf{Y} \prec \mathscr{L}\,.$$

Let us represent $\mathscr{L}_E$ in the 4×4 block form $\mathscr{L}_E = \left\{\mathscr{L}^{\mathrm{E}}_{k,l}\right\}_{k,l=0}^{3}$ with the blocks corresponding to the edges γ_k and note that all 16 blocks are nonzero, see, e.g., Korneev [31] and Rytov [46]. If θ is not big, say $\theta \le 2$, it is possible to use the preconditioner

$$\mathscr{Y}^{\mathrm{E}} = \mathrm{diag}\,[\mathscr{L}^{\mathrm{E}}_{0,0}, \mathscr{L}^{\mathrm{E}}_{1,1}, \mathscr{L}^{\mathrm{E}}_{2,2}, \mathscr{L}^{\mathrm{E}}_{3,3}]\,.$$

For $\theta > 2$ any of the two preconditioners

$$\mathscr{Y}^{\mathrm{E}} = \begin{pmatrix} \mathscr{L}^{\mathrm{E}}_{0,0} & \mathscr{L}^{\mathrm{E}}_{0,1} & \mathbf{0} & \mathbf{0} \\ \mathscr{L}^{\mathrm{E}}_{1,0} & \mathscr{L}^{\mathrm{E}}_{1,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathscr{L}^{\mathrm{E}}_{2,2} & \mathscr{L}^{\mathrm{E}}_{2,3} \\ \mathbf{0} & \mathbf{0} & \mathscr{L}^{\mathrm{E}}_{3,2} & \mathscr{L}^{\mathrm{E}}_{3,3} \end{pmatrix}, \quad \mathscr{Y}^{\mathrm{E}} = \begin{pmatrix} \mathscr{L}^{\mathrm{E}}_{0,0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathscr{L}^{\mathrm{E}}_{1,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathscr{L}^{\mathrm{E}}_{2,2} & \mathscr{L}^{\mathrm{E}}_{2,3} \\ \mathbf{0} & \mathbf{0} & \mathscr{L}^{\mathrm{E}}_{3,2} & \mathscr{L}^{\mathrm{E}}_{3,3} \end{pmatrix} \tag{66}$$

can be used. The first one is obtained by decoupling adjacent edges, and the second one additionally assumes a decoupling the pair of parallel edges, which became shortest after mapping to Ω_ξ and are the edges γ_0, γ_1 in the above expression. Edge Schur complement preconditioners can be defined similarly by means of the matrix $\mathbf{Q}$ or $\widetilde{\mathbf{Q}}$ with the latter obtained from $\mathbf{Q}$ by setting $\mathbf{Q}_{IV} = \mathbf{Q}_{VI}^{\top} = \mathbf{0}$.

The Schur complement preconditioner

$$\mathscr{Y} = \mathrm{diag}\,[\mathscr{Y}^{\mathrm{E}}, \mathscr{Y}_0], \quad \mathscr{Y}_0 = \sqrt{\rho_1\rho_2}\,\mathbf{B}_0 \tag{67}$$

corresponds to the two-level decomposition $\mathscr{U}^{\mathrm{B}}(\partial\Omega) = \mathscr{U}^{\mathrm{E}}(\partial\Omega) \oplus \mathscr{U}_0(\partial\Omega)$, of the boundary FE space, written for the traces of the spaces entering the decomposition $\mathscr{U}^{\mathrm{B}}(\Omega) = \mathscr{U}^{\mathrm{E}}(\Omega) \oplus \mathscr{U}_0(\Omega)$. Here $\mathbf{B}_0$, is the matrix generated by the space $\mathscr{U}_0(\Omega)$ of continuous functions which are linear on each of the two triangles, having vertices at the vertices of Ω. This matrix may be also generated by the subspace $\mathscr{V}_0(\Omega)$ of bilinear polynomials on Ω.

Theorem 4. *For all positive ρ_k, H_k, and $h_k \le H_k$ the preconditioners $\mathscr{Y}^{\mathrm{E}}$, $\mathscr{Y}$ satisfy the inequalities*

$$\underline{\beta}_{\mathrm{E}}\,\mathscr{Y}^{\mathrm{E}} \prec \mathbf{Y}^{\mathrm{E}} \prec \mathscr{Y}^{\mathrm{E}}, \tag{68}$$

$$\underline{\mu}\,\mathscr{Y} \prec \mathbf{Y} \prec \mathscr{Y}, \tag{69}$$

with $\underline{\mu}$, defined by the maximum of the values (62), (64) and

$$\underline{\beta}_{\mathrm{E}} = \max\left[\frac{1}{(1+\log\delta)(1+\log\underline{m})^2}, \frac{1}{(1+\log n_{\mathrm{ff}})^2}\right].$$

Proof. We will consider the case of $\theta > 2$ and the preconditioner $\mathscr{Y}^{\mathrm{E}}$ defined by the second expression in (66). Suppose, that for some positive $\underline{\mu}$, $\overline{\mu}$ the preconditioner $\Upsilon = \mathrm{diag}\,[\Upsilon_E, \Upsilon_0]$ satisfies

$$\underline{\mu}\,\Upsilon \le \mathbf{Y} \le \overline{\mu}\,\Upsilon, \tag{70}$$

where Υ_E has the structure, similar to $\mathscr{Y}^{\mathrm{E}}$, i.e.,

$$\Upsilon_E = \begin{pmatrix} \Upsilon_{E,0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Upsilon_{E,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Upsilon_{E,2} & \Upsilon_{E,23} \\ \mathbf{0} & \mathbf{0} & \Upsilon_{E,32} & \Upsilon_{E,3} \end{pmatrix}. \tag{71}$$

Then

$$(\underline{\mu}/\overline{\mu})\,\mathscr{Y} \prec \mathbf{Y} \prec 4\overline{\mu}\,\mathscr{Y}. \tag{72}$$

Indeed, let V_0, V_{E_0}, V_{E_1}, V_{E_2,E_3} be the vector spaces of degrees of freedom corresponding to the independent blocks on the diagonal of the matrix $\mathscr{Y}$. From (70), considered on these subspaces, and (66), (71) it follows that

$$\underline{\mu}\,\Upsilon_E \prec \mathscr{Y}^{\mathrm{E}} \prec \overline{\mu}\,\Upsilon_E, \quad \underline{\mu}\,\Upsilon_0 \prec \mathscr{Y}_0 \prec \overline{\mu}\,\Upsilon_0, \tag{73}$$

and combining (70) and (73) we get

$$\mathbf{Y} \geq \underline{\mu}\,\mathrm{diag}\,[\Upsilon_E, \Upsilon_0] \geq (\underline{\mu}/\overline{\mu})\,\mathrm{diag}\,[\mathscr{Y}^{\mathrm{E}}, \mathscr{Y}_0] = (\underline{\mu}/\overline{\mu})\,\mathscr{Y}.$$

This proves the left inequality (72). The right inequality (72) follows from the inequality of Cauchy and the last one in (73).

Let us turn now to the block $\mathscr{G}_E = \mathrm{diag}\,[\mathscr{Q}_B^{\mathrm{s}}, \mathscr{B}_{W_{\mathrm{r}}}, \mathbf{B}_{W_{\mathrm{c}}}]$, of the preconditioner $\mathscr{G}$ in Theorem 3. It has the same structure as $\mathscr{Y}^{\mathrm{E}}$. If we repeat the derivation of (60), however omitting steps related to the splitting of vertices, we come to the bounds

$$\underline{\beta}_E\,\mathscr{G}_E \prec \mathbf{Y}_E \prec \mathscr{G}_E, \tag{74}$$

where without change of the notation it is implied that $\mathscr{G}$ is transformed to the basis common with $\mathbf{Y}$. Therefore, one can take $\sqrt{\rho_1\rho_2}\,\mathscr{G}_E$ for Υ_E and obtain (68).

Similarly, on the basis (72) and Theorem 3, inequalities (69) are proved, including the case of the use of the preconditioner given by first expression (66). □

Several facts are important for numerical implementations of the preconditioners (66). For each subdomain, we have introduced the discretization, transformed rarefied and transformed coarse imbedded meshes, and in general all these meshes can be nonuniform. However, in the DD preconditioner they all can be replaced by uniform orthogonal non-imbedded meshes, what is assumed in what follows. This does not influence the asymptotic computational cost. FDFT, applied edge-wise to the first of the preconditioners (66), makes the preconditioner a block diagonal matrix with 2×2 blocks. Obviously, the matrix of FDFT, which is designated $\mathscr{F}_E$, is the block diagonal matrix with the identical blocks for the opposite among edges γ_k, $k = 0,1,2,3$:

$$\mathscr{F}_E = [\mathscr{F}_0, \mathscr{F}_0, \mathscr{F}_2, \mathscr{F}_2]. \tag{75}$$

For the problem (37) the block diagonal matrix $\Lambda := \mathscr{F}_E^{\top}\mathscr{Y}^{\mathrm{E}}\mathscr{F}_E$ has the form

$$\Lambda = \mathrm{diag}\left[\mathrm{diag}\,[\Lambda_i^{(0)}]_{i=1}^{n_2-1}, \mathrm{diag}\,[\Lambda_k^{(2)}]_{k=1}^{n_1-1}\right],$$

$$\Lambda = \mathrm{diag}\left[\mathrm{diag}\,[\mathrm{diag}\,[\Lambda_{0,i}, \Lambda_{0,i}]]_{i=1}^{n_2-1}, \mathrm{diag}\,[\Lambda_k^{(2)}]_{k=1}^{n_1-1}\right],$$

respectively to the first and second expressions (66). Each block $\Lambda_i^{(0)}$ couples a pair of opposite nodes $(0, x_{2,i})$ and $(1, x_{2,i})$ on the vertical edges and each block $\Lambda_k^{(2)}$ couples a pair of opposite nodes $(x_{1,k}, 0)$ and $(x_{1,k}, \varepsilon)$ of the horizontal edges. For the second preconditioner (66), the 2×2 matrix $\Lambda_i^{(0)}$ is diagonal with two equal nonzero entries. Due to the pointed out property the system with the matrix $\mathscr{Y}^{\mathrm{E}}$

can be solved in $\mathscr{O}((n_1+n_2)\log\overline{n})$ arithmetical operations. The Schur complement $\mathbf{Y}_E$ and the preconditioners $\mathscr{Y}^{\mathrm{E}}$ can be calculated in the trigonometric basis for $n_1 \times n_2$ arithmetical operations, whereas matrix vector multiplications by $\mathscr{Y}^{\mathrm{E}}$ require $\mathscr{O}((n_1+n_2)\log\overline{n})$ arithmetical operations, see, e.g., Korneev [31] and Rytov [46]. Costs of some of these operations can be considerably reduced, if some up to date techniques are applied, such as $\mathscr{H}$-matrices and tensor-train decompositions Hackbusch et al. [25], Khoromskij & Wittum [28], Dolgov et al. [16]. For instance, the $\mathscr{H}$-matrix approximation technique provides the cost $\mathscr{O}(n_{\partial\Omega}\log^g n_{\partial\Omega})$ for the computation of the matrix $\mathbf{Y}_E$, $n_{\partial\Omega} = 2(n_1+n_2)$, as well as for storage operations and matrix vector multiplications.

5 Piecewise Orthotropic Discretizations on Domains Composed of Rectangles with Arbitrary Aspect Ratios

5.1 Schur Complement and Domain Decomposition Algorithms

We turn now to the piecewise orthotropic problem (4) and its piecewise orthotropic FE discretization (6) by means of decomposition and discretization meshes (2)-(3). The bilinear form $\alpha_\Omega(u,v)$, $\forall u,v \in \mathring{\mathscr{V}}(\Omega)$, induces the FE stiffness matrix $\mathbf{K}$ and its inter-subdomain Schur complement $\mathbf{S} = \mathbf{K}_B - \mathbf{K}_{B,I}\mathbf{K}_I^{-1}\mathbf{K}_{I,B}$, which, obviously, may be viewed as assembled from the stiffness matrices $\mathbf{K}_j$ and the corresponding Schur complements $\mathbf{S}_j$ for subdomains Ω_j. Preconditioners for subdomain matrices $\mathbf{K}_j$, $\mathbf{S}_j$, studied in the preceding sections, will be used in the fast solvers for the systems (6) and

$$\mathbf{S}\mathbf{u}_B = \mathbf{F}_B \tag{76}$$

and we start from the Schur complement solver. It is based on the use of the two preconditioners $\mathscr{S}_k$, $k = 1,2$, for the matrix $\mathbf{S}$ with different properties. It is implied that $\mathscr{S}_1$ is as close as possible to $\mathbf{S}$ in the spectrum and is cheap, at least much cheaper than $\mathbf{S}$, for matrix-vector multiplications. The preconditioner $\mathscr{S}_2$ is allowed to be less close to $\mathbf{S}$ in the spectrum, but is cheap for operations $\mathscr{S}_2^{-1}\mathbf{y}$ and at least much cheaper than $\mathbf{S}$ and $\mathscr{S}_1$ for operations $\mathbf{S}^{-1}\mathbf{y}$, $\mathscr{S}_1^{-1}\mathbf{y}$. Under these assumptions, we solve the system (76) by a PCG with the preconditioner $\mathscr{S}_1$, whereas systems $\mathscr{S}_1\mathbf{x} = \mathbf{y}$, arising at each PCG iteration, are solved inexactly by means of the iterative processes

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \sigma_k\mathscr{S}_2^{-1}(\mathscr{S}_1\mathbf{x}^k - \mathbf{y}), \quad k = 1,2,\ldots,k_s, \tag{77}$$

with Chebyshev iteration parameters σ_k for some fixed numbers k_s of iterations. In other words, the system $\mathbf{S}\mathbf{u}_B = \mathbf{F}_B$ is solved by a PCG with the preconditioner $\mathscr{S}_{1,\mathrm{it}}$ which inverse is

$$\mathscr{S}_{1,\mathrm{it}}^{-1} = \Big[\mathbf{I} - \prod_{k=1}^{k_s}(\mathbf{I} - \sigma_k\mathscr{S}_2^{-1}\mathscr{S}_1)\Big]\mathscr{S}_1^{-1}. \tag{78}$$

Proposition 1. *Suppose that*

ι) preconditioners $\mathscr{S}_k$ satisfy

$$\underline{\gamma}_1 \mathscr{S}_1 \prec \mathbf{S} \prec \overline{\gamma}_1 \mathscr{S}_1, \quad \underline{\gamma}_2 \mathscr{S}_2 \prec \mathscr{S}_1 \prec \overline{\gamma}_2 \mathscr{S}_2,$$

ιι) matrix-vector multiplications by $\mathbf{S}$ *and* $\mathscr{S}_1$ *spend* $\mathscr{N}_S$ *and* $\mathscr{N}_{\mathscr{S}_1}$ *arithmetical operations, respectively, and*
ιιι) solving the system $\mathscr{S}_2 \mathbf{v}_B = \mathscr{F}_B$, $\forall \mathscr{F}_B$, *requires* $\mathscr{N}_{\mathscr{S}_2}$ *operations.*

Then solving the system (76) with the prescribed accuracy $\varepsilon \in (0,1)$ *in the norm* $\|\cdot\|_{\mathbf{S}}$ *requires not more than*

$$c\sqrt{\overline{\gamma}_1/\underline{\gamma}_1}\left[\mathscr{N}_S + \sqrt{\overline{\gamma}_2/\underline{\gamma}_2}(\mathscr{N}_{\mathscr{S}_1} + \mathscr{N}_{\mathscr{S}_2})\right]\log\varepsilon^{-1}$$

arithmetical operations, $c = \text{const.}$

Proof. The proof of these statements can be found in Nepomnyaschikh [40] and many other places, see, e.g., Korneev & Langer [32]. □

Preconditioner $\mathscr{S}_1$

We transform each subdomain Ω_j to $\Omega_{j,\varepsilon}$ and by means of the condensed transformed mesh we define the preconditioner (28), denoted now $\mathbf{C}_{\mathrm{ff,j}}$. Setting $\mathscr{S}_{1,j} = \sqrt{\rho_{1,j}\rho_{2,j}}\,\mathbf{C}_{\mathrm{ff,j}}$, we assemble $\mathscr{S}_1$ from these subdomain matrices. As it follows from these definitions and Corollary 3,

$$\mathscr{S}_1 \prec \mathbf{S} \prec \mathscr{S}_1 \tag{79}$$

and

$$\text{ops}\,[\mathscr{S}_1 \mathbf{v}] \prec (\log \overline{n}) \sum_j (n_{1,j} + n_{2,j}) \prec (J_1 N_2 + J_2 N_1) \log \overline{n}, \quad \forall \mathbf{v}.$$

Preconditioner $\mathscr{S}_2$

For each subdomain Ω_j, we consider the preconditioner $\mathscr{S}_{2,j} = \mathscr{Y}_j$, where $\mathscr{Y}_j = \text{diag}\,[\mathscr{Y}_j^{\mathrm{E}}, \sqrt{\rho_{1,j}\rho_{2,j}}\,\mathbf{B}_{0,j}]$ is defined in the same way as $\mathscr{Y}$ in (67) for the domain $\Omega = \Omega_j$. Then $\mathscr{S}_2$ is assembled from subdomain preconditioners $\mathscr{S}_{2,j}$ and has the block diagonal form $\mathscr{S}_2 = \text{diag}\,[\mathscr{S}_E, \mathbf{K}_V]$, where $\mathbf{K}_V$ is the block of the FE matrix $\mathbf{K}$ for vertices. Obviously, $\mathscr{S}_E$ is assembled from the matrices $\mathscr{S}_{E,j} = \mathscr{Y}_j^{\mathrm{E}}$, and according to Theorem 4

$$\min_j \underline{\mu}_j \mathscr{S}_2 \prec \mathbf{S} \prec \mathscr{S}_2, \tag{80}$$

where $\underline{\mu}_j$ is the value of $\underline{\mu}$ in (69) for a particular subdomain Ω_j.

Taking into account the representation of the FE stiffness matrix in the block form

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix}, \tag{81}$$

we define the inverse to the DD preconditioner $\mathscr{K}_{\rm DD}$ by the expression

$$\mathscr{K}_{\rm DD}^{-1} = \mathscr{K}_I^{+} + \mathbb{P}\mathscr{S}_{1,\rm it}^{-1}\mathbb{P}^{\top}. \tag{82}$$

Here the matrix $\mathscr{K}_I$ is related to the interior degrees of freedom for each subdomain and, like the block $\mathbf{K}_I = \mathrm{diag}\,[\mathbf{K}_{I,j}]_{j_1,j_2=1}^{J_1,J_2}$, has the block diagonal structure $\mathscr{K}_I = \mathrm{diag}\,[\mathscr{K}_{I,j}]_{j_1,j_2=1}^{J_1,J_2}$, $\mathbb{P}$ is the prolongation matrix $\mathbb{P}^{\top} = (\mathbb{P}_I^{\top}, \mathbf{I})^{\top}$. It is assumed that

$$\mathscr{K}_{I,j} \prec \mathbf{K}_{I,j} \prec \mathscr{K}_{I,j}, \quad \|\mathbb{P}_{B_j}\mathbf{v}_{B_j}\|_{\mathbf{K}_j} \prec \|\mathbf{v}_{B_j}\|_{\mathbf{S}_j}, \tag{83}$$

where $\mathbb{P}_{B_j}$ is the restriction of the prolongation operator $\mathbb{P}$ to the subdomain $\overline{\Omega}$. It is assumed additionally that

$$\mathrm{ops}\,[\mathscr{K}_I^{-1}\mathbf{f}_I] = N_\Omega,\ \forall \mathbf{f}, \quad \mathrm{ops}\,[\mathbb{P}\mathbf{v}_B] \prec N_\Omega,\ \forall \mathbf{v}_B. \tag{84}$$

There is a variety of such preconditioners and prolongation operators in the literature, and we refer only to papers by Oswald [44], Griebel & Oswald [22] and Nepomnyaschikh [41] for examples.

We restrict our considerations to the problem with subdomain-wise constant coefficients $\wp = \rho$. Clearly, the numerical complexity of DD preconditioners $\mathscr{K}_{\rm DD}$ for (1)-(5) will differ only by a constant depending on μ_1, μ_2. However, an implementation of the DD Schur complement solver will in general differ noticeably, since it requires the calculation of $\mathbf{S}$ and multiplications by it, which can be expensive. For instance, an implementation of $\mathscr{H}$-matrix techniques for the calculation of $\mathbf{S}$ gives the complexity $\mathscr{O}(N_\Omega \log N_\Omega)$, $N_\Omega = N_1N_2$. At the same time, the complexity of the same operation and of the matrix vector multiplication by $\mathbf{S}$ is $\mathscr{O}(N_\Gamma \log N_\Gamma)$, $N_\Gamma = J_1N_1 + J_2N_2$, if $\wp = \rho$, cf. Hackbusch [24] and Hackbusch et al. [25]. The last estimates are assumed to hold in what follows.

Theorem 5. *Let $H_{k,j_k} \in (0,1)$, $n_{k,j_k} \ge 1$, $\rho_{k,j} > 0$ be arbitrary in the pointed out ranges. Then the total arithmetical costs $Q_{\mathbf{K}}$, $Q_{\mathbf{S}}$ of the DD and Schur complement algorithms satisfy the bounds*

$$\begin{aligned} Q_{\mathbf{K}} &\prec N_1N_2 + [N_\Gamma(1+\log\overline{N}) + \Upsilon(J_1J_2)]\sqrt{\overline{N}}(1+\log\overline{N})^{1/2}, \\ Q_{\mathbf{S}} &\prec N_\Gamma(1+\log N_\Gamma) + [N_\Gamma(1+\log\overline{N}) + \Upsilon(J_1J_2)]\sqrt{\overline{N}}(1+\log\overline{N})^{1/2}, \end{aligned} \tag{85}$$

where $\overline{N} = \max_k N_k$ and $\Upsilon(J_1,J_2)$ stands for the cost of the solution of the subsystem with the matrix $\mathbf{K}_V$.

Proof. Let us list the main factors contributing to the complexity of the Schur complement algorithm:

– the number of external PCG iterations $k_{\rm PCG} = \text{const}$,
– the cost $\mathcal{N}_S \prec N_\Gamma \log N_\Gamma$ arithmetical operations of one matrix-vector multiplication by $\mathbf{S}$,
– the number of secondary iterations (77)

$$k_s \prec 1/\sqrt{\underline{\mu}_{12}} \prec \sqrt{\max_j \overline{n}_j (1+\log \underline{n}_j)} \prec \sqrt{\overline{N}(1+\log \overline{N})},$$

  * the cost of the matrix-vector multiplication by $\mathscr{S}_1$ at each secondary iteration

$$\mathcal{N}_{\mathscr{S}_1} \prec N_\Gamma (1+\log \overline{N}),$$

  * the cost of solving the system with the preconditioner $\mathscr{S}_2$ at each secondary iteration

$$\mathcal{N}_{\mathscr{S}_2} \prec N_\Gamma (1+\log \overline{N}) + \Upsilon(J_1, J_2).$$

Taking into account the proof of Theorem 4, we conclude that

$$\underline{\mu}_{12} \mathscr{S}_2 \prec \mathscr{S}_1 \prec \mathscr{S}_2, \quad \underline{\mu}_{12} = \min_j \left[\frac{1}{(1+\log \underline{m}_j)} \min\left(\frac{1}{\theta_j}, \frac{1}{(1+\log \underline{m}_j)} \right) \right], \tag{86}$$

and, therefore, the given number k_s of secondary iterations provides the spectral equivalence $\mathscr{S}_{1,\rm it} \asymp \mathscr{S}_1$.

Now, the last relationship and (79) guarantee that $k_{\rm PCG} \log \varepsilon^{-1}$ PCG iterations provide the relative error in the norm $\|\cdot\|_{\mathbf{S}}$ bounded by the prescribed $\varepsilon > 0$. The first term in the expression for $\mathcal{N}_{\mathscr{S}_2}$ bounds the arithmetical cost of solving the system with the matrix $\mathscr{S}_E$. Implementing above bounds according to Proposition 1, we come to the bound (85) for $Q_{\mathbf{S}}$.

The DD peconditioner $\mathscr{K}_{\rm DD}$ is spectrally equivalent to the FE matrix $\mathbf{K}$, what follows from Corollary 3 and (83). The estimate of the DD solver cost is obtained by taking additionally into account the above costs of operations, related to the interface, and (84). □

Suppose that $N_1 = N_2 = N$, $J_1 = J_2 = J$ and the decomposition mesh is fixed. Then

$$Q_{\mathbf{K}} \prec N^2. \tag{87}$$

If the number of subdomains grows with the growth of the numbers N_k of the source mesh lines, the contribution $\Upsilon(J_1 J_2)$ of the solver for the vertex subproblem can compromise this bound. Assuming that a direct solver for systems with the matrix $\mathbf{K}_V$ is sufficiently fast, the bound (87) is retained under the condition

$$J_k \le N^{1/2}/(1+\log N)^{3/2}.$$

It is worth emphasizing that the above estimates are relatively crude, since we practically made no restrictions on H_{k,j_k}, n_{k,j_k}, and $\rho_{k,j} > 0$ and their change from subdomain to subdomain. It can help to improve the bounds, if the variation of these values can be characterized by some functions.

6 Concluding Remarks

The properties of the interface Schur complement preconditioners in relation to the discretization meshes were discussed in Subsect. 2.1, and, clearly, the Schur complement exhibits the same properties. Therefore, DD algorithms with the same type of Schur complement preconditioning can be efficiently used for a much wider class of discretizations.

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