FPT Results for Signed Domination*

Ying Zheng[1], Jianxin Wang[1], Qilong Feng[1], and Jianer Chen[1,2]

[1] School of Information Science and Engineering
Central South University
Changsha 410083, P.R. China
[2] Department of Computer Science and Engineering
Texas A&M University
College Station, Texas 77843-3112, USA

Abstract. A function $f : v \rightarrow \{-1, +1\}$ defined on the vertices of a graph G is a signed dominating function if the sum of its function values over any closed neighborhood is at least one. The weight of a signed dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The signed domination number of a graph G, denoted by $\gamma_s(G)$, equals the minimum weight of a signed dominating function of G. The decision problem corresponding to the problem of computing γ_s is an important NP-complete problem derived from social network. A signed dominating set is a set of vertices assigned the value $+1$ under the function f in the graph. In this paper, we give some fixed parameter tractable results for signed dominating set problem, specifically the kernels for signed dominating set problem on general and special graphs. These results generalize the parameterized algorithm for this problem. Furthermore we propose a parameterized algorithm for signed dominating set problem on planar graphs.

1 Introduction

Signed domination is a variation of dominating set problem, there is a variety of applications for this variation. By assigning the values -1 or $+1$ to the vertices of a graph, which can be modeled as networks of positive and negative electrical charges, networks of positive and negative spins of electrons, and networks of people or organizations in which global decisions must be made(e.g. yes-no, agree-disagree, like-dislike, etc.). In such a context, the signed domination number represents the minimum number of people whose positive votes can assure that all local groups of voters(represented by closed neighborhoods in graphs) have more positive than negative voters, even though the entire network may have far more people whose vote negative than positive. Hence this variation of domination studies situations in which, in spite of the presence of negative vertices, the closed neighborhoods of all vertices are required to maintain a positive sum.

* This work is supported by the National Natural Science Foundation of China under Grant (61103033,61128006), the Doctoral Discipline Foundation of Higher Education Institution of China under Grant (20090162110056).

M. Agrawal, S.B. Cooper, and A. Li (Eds.): TAMC 2012, LNCS 7287, pp. 572–583, 2012.

Given a graph $G = (V, E)$, for each vertex $v \in V$, let $N(v)$ be all neighbors of v, and $N[v] = N(v) \cup \{v\}$, $N(v)$ and $N[v]$ are called the *open* and the *closed* neighborhood of v. Similarly, for a set S of vertices, define the open neighborhood $N(S) = \cup N(v)$ over all v in S and the closed neighborhood $N[S] = N(S) \cup S$. A set S of vertices is a *dominating set* if $N[S] = V$. For an integer function $f : V \to N$, the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, therefore, $w(f) = f(V)$. For convenience, we use $f(N[v])$ to denote $\sum_{u \in N[v]} f(u)$.

A Dominating Set of a graph $G = (V, E)$ is a vertex set $D \subseteq V$ such that each $v \in V$ is contained in D or v is a neighbor of at least one vertex in D. In other words, let $f : V \to \{0, 1\}$ be a function which assigns to each vertex of a graph an element in the set $\{0, 1\}$. Then, f is called *dominating function* if for every $v \in V, f(N[v]) \geq 1$. The *domination number*, denoted by $\gamma(G)$, $\gamma(G)$ =min$\{f(V) : f$ is a dominating function of $G\}$.

Let $f : V \to \{-1, 1\}$ be a function which assigns each vertex of a graph an element in the set $\{-1, 1\}$. Then, f is called *signed dominating function* if for every $v \in V, f(N[v]) \geq 1$. The *signed domination number*, denoted by $\gamma_s(G)$, of G is the minimum weight of the $\sum_{v \in V} f(v)$ over all such functions, $\gamma_s(G)$ =min$\{f(V) : f$ is a signed dominating function of $G\}$. We define $P \subseteq V$ the *signed dominating set* which is the set of vertices with value $+1$ assigned by f.

Definition 1. *(Parameterized Signed Dominating Set) Given a graph $G = (V, E)$ and a non-negative integer k, does there exist a signed dominating set P of size at most k such that for each $v \in V$, $\sum_{u \in N[v]} f(u) > 0$.*

The concept of signed domination in graphs was introduced by Zelinka[4] and studied in [1][3][5][7]. The decision problem corresponding to the problem of computing γ_s is NP-complete, even when the graph restricted to chordal graph or bipartite graph. For a fixed k, the problem of determining if a graph has a signed dominating function of weight at most k is also NP-complete. A linear time algorithm for finding a minimum signed dominating function in an arbitrary tree was presented in [2]. The research dealing with signed domination has many focused on computing better upper and lower bounds on the signed domination number γ_s for graphs. Dunbar et. al[1] investigated the properties of signed domination number and established upper and lower bounds for γ_s. For r-regular n-vertex graphs, $\gamma_s \geq \frac{n}{r+1}$ when r is even, Henning and Slater[11] pointed out that $\gamma_s \geq \frac{2n}{r+1}$ when r is odd. The upper bounds are given by Henning[12] and Favaron[8], when r is odd, $\gamma_s \leq \frac{(r+1)^2}{r^2+4r-1} \cdot n$, and when r is even, $\gamma_s \leq \frac{r+1}{r+3} \cdot n$.

Since the research dealing with signed domination has mainly focused on improving better upper and lower bounds on the signed domination number γ_s, therefore, in this paper, we study this problem from the point of algorithm complexity and present a variety of fixed parameter tractable(FPT) results for signed dominating set problem. We study signed dominating set problem in general graphs, particularly show that signed dominating set problem is NP-complete even restricted to bipartite or chordal graphs. We also present a linear

Table 1. FPT results for signed dominating set in general graphs and special graphs

Graph Class	Parameterized Complexity	Kernel
general	FPT	$O(k^2)$
planar	FPT	$O(k)$
bipartite	FPT	$O(k^2)$
r-regular	FPT	$O(k)$
$\Delta \leq 5$	FPT	$O(k)$

kernel $O(k)$ and an efficient FPT algorithm of time $O((6\sqrt{k})^{O(\sqrt{k})}|V|)$ for signed dominating set problem on planar graphs. Finally we give the kernels for signed dominating set on the following graph classes: bipartite graphs, $\Delta \leq 5$ graphs and r-regular graphs. FPT results for signed dominating set problem are given in Table 1.

2 Preliminaries

A signed dominating function is a labeling of the vertices by values $-1, +1$ such that the sum of labels in $N[v]$ is positive, for each v. For convenience, we will also say that each v is "dominated" if the sum of labels in $N[v]$ is positive.

Let P and M be the sets of vertices with labels $+1$ and -1, also called positive and negative vertices, respectively. Let P_i denote the set of those positive vertices having exactly i negative neighbors. Similarly M_i is defined to be the set of those negative vertices having exactly i positive neighbors. It is easy to see that $M_0 = M_1 = \emptyset$. Let D_i be the set of all vertices with degree i. Δ is the maximum degree of the graph. The symbols $|P|$, $|M|$, $|P_i|$, $|M_i|$ denote the cardinalities of the sets P, M, P_i, M_i respectively. If X, Y are disjoint sets of vertices of graph G, $K_{|X|,|Y|}$ denotes the bipartite subgraph of G consisting of the parts X and Y.

Definition 2. *[6] The pair (T, X) is a tree decomposition of a graph G if*
1. T is a tree,
2. $X = \{X_i | X_i \subseteq V(G), i \in V(T)\}$, and $\bigcup_{X_i \in X} = V(G)$, ($X_i$ is called a bag),
3. (Containment) $\forall u, v, (u, v) \in E(G), \exists i \in V(T)$ such that $u, v \in X_i$, and
4. (Connectivity) $\forall i, j, k \in V(T)$, if k is on the path from i to j in tree T, then $X_i \cap X_j \subseteq X_k$.
The width *of (T, X) is defined as $\max_{i \in V(T)}\{|X_i|\} - 1$.*

The treewidth of the graph G is the minimum width of all possible tree decompositions of the graph.

Definition 3. *[9] A* nice tree decomposition *is a tree decomposition (T, X) in which one node of T is considered to be the root, and each node i in T is of one of the four following types.*
-Leaf: node i is a leaf of T and $|X_i| = 1$.
-Join: node i has exactly two children, say j and k, and $X_i = X_j = X_k$.

-Introduce: node i has exactly one child, say j, and there is a vertex $v \in V(G)$ with $X_i = X_j \cup \{v\}$.
-Forget: node i has exactly one child, say j, and there is a vertex $v \in V(G)$ with $X_i = X_j - \{v\}$.

Every tree decomposition can be transformed into a nice tree decomposition[10].

Lemma 1. *Given a tree decomposition of width k with $O(n)$ nodes of a graph G, where n is the number of vertices of G, one can find a nice tree decomposition of G that has the same width k and $O(n)$ nodes in linear time.*

3 Signed Dominating Set in General Graph

The decision problem corresponding to the problem of computing γ_s is well-known NP-complete [2]. We show that signed dominating set problem is NP-complete even restricted to bipartite or chordal graphs.
Problem: **Dominating Set**
Instance: A graph $G = (V, E)$ and a positive integer k.
Question: Does G have a dominating set of cardinality k or less.

Problem: **Signed Dominating Set**
Instance: A graph $H = (V, E)$ and a positive integer j.
Question: Does H have a signed dominating set P of cardinality at most j.

Theorem 1. *Signed dominating set problem is NP-complete, even restricted to bipartite or chordal graphs.*

Proof. It is obvious that signed dominating set problem is a member of NP since we can in polynomial time verify that H has a signed dominating set of size at most j for a function $f : V \to \{-1, +1\}$. To show that signed dominating set problem is NP-complete even restricted to bipartite or chordal graphs, we establish a polynomial reduction from the NP-complete problem dominating set. Let (G, k) be an instance of Dominating Set consisting of the dominating set of size k. We construct an instance (H, j) of Signed Dominating Set as follows.

Given a graph $G = (V, E)$ and a positive integer k, construct the graph H by adding for each vertex v of G a set of $deg_G v$ paths P_2 on two vertices. Let $m = |E(G)|$ and $n = |V(G)|$. Then $|V(H)| = n + 2\sum_{v \in V} deg_G v = n + 4m$ and $|E(H)| = m + 2\sum_{v \in V} deg_G v = 5m$. It is easy to see that graph H can be constructed in polynomial time.

Next, we show the equivalence between the instances, that is, (G, k) is a yes-instance of Dominating Set if and only if (H, j) is a yes-instance of Signed Dominating Set.

Let D be a dominating set of graph G of size k. Let $f : V(H) \to \{-1, +1\}$ be the function defined by $f(v) = +1$ if $v \in (V(H) - V(G)) \cup D$ and $f(v) = -1$ if $v \in V(G) - D$. Then f is a signed dominating function of graph H, and $|P| \leq j = |V(H) - V(G)| + |D| = k + 4m$.

Let f be a signed dominating function of graph H. If x is a degree-1 vertex and y is its neighbor, by the definition of the signed dominating function, then $f(x) = +1$ and $f(y) = +1$. It follows that $f(w) = +1$ for every $w \in V(H) - V(G)$. In other words, for the signed dominating function f, if $f(v) = -1$, then $v \in V(G) \subseteq V(H)$. Furthermore, $f : V(H) \to \{-1, +1\}$, $v \in V(G)$, v has even degree in graph H and $f(N[v]) \geq 1$. Since exactly half the neighbors of v belong to $V(H) - V(G)$ and all those vertices are assigned the value $+1$, it follows that at least one neighbor of v in G is assigned the value $+1$ under f. That is, if f is a signed dominating function of H, then $f(v) = +1$ for $v \in V(H) - V(G)$ and the set of vertices D in graph G which are assigned the value $+1$ under f form a dominating set in graph G. Since the signed dominating set P in graph H with size at most $j = k + 4m$ and $|V(H) - V(G)| = 2\sum_{v \in V} deg_G v = 4m$, therefore, $|D| \leq k$.

It is also easy to verify the reduction preserves the properties "bipartite", "chordal". □

Kernelization can be seen as the strategy of analyzing preprocessing or data reduction heuristics from a parameterized complexity perspective. Given a graph $G = (V, E)$, we propose to develop data reduction rules as follows.

Rule: If there exists a vertex v of degree larger than $2k$ in graph G, then remove vertex v from graph G.

Theorem 2. *Signed dominating set problem in general graphs admits a $O(k^2)$ kernel.*

Proof. It is easy to verify the correctness of this rule. If there is a vertex of degree larger than $2k$, then at least half of its neighbors should be assigned the value $+1$ under the signed dominating function, it contradicts with the signed dominating set of size bounded by k.

The left graph has all vertices of degree smaller than $2k$. Since each vertex with value $+1$ has at most half neighbors assigned the value -1, now the graph has at most k vertices with value $+1$, then the number of vertices with value -1 is less than k^2. Then we can get $|V| \leq k(k+1)$, therefore, we can conclude that $|V| \leq k^2$, the kernel is $O(k^2)$. □

4 Signed Dominating Set on Planar Graph

4.1 Linear Kernel for Signed Dominating Set

It is observed that for a fixed parameter tractable problem on planar graphs, if some kernelization rules can be developed to bound the size of "lower degree" vertices, then we can get a kernel of this problem soon. Next, we will analyze the kernel for signed dominating set on planar graphs.

Lemma 2. *Let v be a degree-1 vertex in G with $N(v) = \{u\}$, then G has a signed dominating set of size bounded by k if and only if $G \setminus v$ has a signed dominating set bounded by k that contains v and u.*

Lemma 3. *Let v be a degree-2 vertex in G with $N(v) = \{u, w\}$ and $(u, w) \in E(G)$. Then G has a signed dominating set of size bounded by k if and only if $G \setminus v$ has a signed dominating set of size bounded by k that contains u and w.*

Based on Lemma 2 and lemma 3, we can get the following reduction rules.

Rule 1. If a vertex v is a degree-1 vertex, then remove v and decrease the parameter k by two.

Rule 2. If a vertex v has two neighbors u and w and $(u, w) \in E(G)$, remove v and decrease the parameter k by two.

It is easy to verify that any rule can be applied at most polynomial times. Following is a useful property of planar graphs.

Lemma 4. *For a planar graph G, let S be a subset of V with at least 2 vertices, and let $J = \{v|v \in V(G) \setminus S, |N(v) \cap S| \geq 3\}$, then $|J| \leq 2|S| - 4$.*

Proof. It is not hard to see that this lemma is true for $|S| = 2$. Then we suppose $|S| \geq 3$. Let $B = G[S \cup J] \setminus (E(G[S]) \cup E(G[J]))$. Since there is no K_3 in B, by Euler's formula, $|E(B)| \leq 2(|S| + |J| - 2)$. Then $|E(B)| \geq 3|J|$. Thus, we have $|J| \leq 2|S| - 4$. □

Lemma 5. *[14] For a planar graph $G = (V, E)$ with at least three vertices, then $|E| \leq 3|V| - 6$.*

Lemma 6. *Signed dominating set on planar graphs admit a $6k - 10$ kernel.*

Proof. For any instance (G, k) of signed domination set, we reduce the instance by rule 1- rule 2. Let (G', k') is the reduced instance with $k' \leq k$. Suppose G' has a signed dominating set P of size k'. Let $J_i = \{v|v \in V(G') \setminus V(P), |N(v) \cap V(P)| = i\}$, where $i = 0, 1, 2$, and let $J_3^+ = \{v|v \in V(G') \setminus V(P), |N(v) \cap V(P)| \geq 3\}$. It is clearly that $|J_0| = 0$. Since G' has already been reduced by rule 1- rule 2, there exists no degree-1 vertices and degree-2 vertices whose neighbors are adjacent in $V(G') \setminus V(P)$. Otherwise, assume that there is a degree-2 vertex v in $V(G') \setminus V(P)$ with $N(v) = \{u, w\}$. If $(u, w) \in E(G')$, then rule 2 can be used, contradicting that (G', k') is reduced. Therefore, $|J_1| = 0$. It is easy to see that J_2 induced an independent set. P is an induced graph of planar graph G', according to lemma 5 we obtain $|E(P)| \leq 3|V(P)| - 6$. Each vertex in J_2 has exactly two neighbors in P and these two neighbors are not connected. Therefore, $|J_2| \leq 3|V(P)| - 6$. Since G' is a planar graph, by lemma 4, $|J_3^+| \leq 2|V(P)| - 4$. Thus $|G'| = |V(P)| + |J_0| + |J_1| + |J_2| + |J_3^+| \leq 6|V(P)| - 10$. Since $|V(P)| \leq k$, the size of the kernel is $6k - 10$.

The procedure of reduction just takes the operation of deleting vertices and edges, therefore, the kernelization takes polynomial times. □

4.2 FPT Algorithm for Signed Dominating Set

In this section we will solve signed dominating set problem by dynamic programming on the tree-decomposition. What we show is a FPT algorithm with respect to the parameter treewidth.

Given a graph $G = (V, E)$ and $V = \{x_1, ..., x_n\}$, assume that the vertices in the bags are given in increasing order when used as indices of the dynamic programming tables, that is $X_i = \{x_{i1}, ..., x_{in_i}\}$ with $i1 \leq ... \leq in_i$, $1 \leq i \leq |V(T)|$. We use eight different "colors" that will be assigned to the vertices in bag.

- "blue": represented by 1, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) > 1$ at the current stage of the algorithm.
- "white": represented by $1_{[}$, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) = 1$ at the current stage of the algorithm.
- "grey": represented by $1_{]}$, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) = 0$ at the current stage of the algorithm.
- "pink": represented by 1_*, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) < 0$ at the current stage of the algorithm.
- "black": represented by -1, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) > 1$ at the current stage of the algorithm.
- "red": represented by $-1_{[}$, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) = 1$ at the current stage of the algorithm.
- "green": represented by $-1_{]}$, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) = 0$ at the current stage of the algorithm.
- "brown": represented by -1_*, meaning that the vertex x_{it} satisfies $f(N[x_{it}]) < 0$ at the current stage of the algorithm.

It is worthy of note that there are $|X_i| - 1$ number of states for $f(N[x_{it}]) > 1$, those states $f(N[x_{it}]) = 2, ..., f(N[x_{it}]) = |X_i|$ can be denoted by $1_2, ..., 1_{|X_i|}$ and $-1_2, ..., -1_{|X_i|}$ respectively. Moreover, there are $|X_i|$ number of states for $f(N[x_{it}]) < 0$. Similarly we use $1_{-1}, ..., 1_{-|X_i|}$ and $-1_{-1}, ..., -1_{-|X_i|}$ to denote those states $f(N[x_{it}]) = -1, ..., f(N[x_{it}]) = -|X_i|$ respectively. In order to express the dynamic programming algorithm easily, we still use 1, 1_*, -1 and -1_* to denote those states. Therefore, mapping

$$C_i : \{x_{i1}, ..., x_{in_i}\} \rightarrow \{1, 1_{[}, 1_{]}, 1_*, -1, -1_{[}, -1_{]}, -1_*\}$$

is called a coloring for the bag $X_i = \{x_{i1}, ..., x_{in_i}\}$, and the color assigned to vertex x_{it} by C_i is given by $C_i(x_{it})$. The colors in the bag can be represented as $(C(x_{i1}), ..., C(x_{in_i}))$. For each bag X_i with $X_i = n_i$, the algorithm use a mapping

$$m_i : \{1, 1_{[}, 1_{]}, 1_*, -1, -1_{[}, -1_{]}, -1_*\}^{n_i} \rightarrow \mathbb{N} \cup +\infty$$

For a coloring C_i, the value $m_i(C_i)$ stores how many vertices are needed for a minimum signed dominating set of the graph visited up to the current stage of the algorithm. A color is locally invalid if there is some vertex in the bag that is colored -1 or $-1_{[}$ but this vertex is not dominated by the vertices within the bag. Note that a locally invalid coloring may still be a correct coloring if this vertex is not dominated within the bag but dominated by some vertices from bags that have been considered earlier. For a coloring $c = (c_1, ..., c_m) \in \{1, 1_{[}, 1_{]}, 1_*, -1, -1_{[}, -1_{]}, -1_*\}^m$ and a color $d \in \{1, 1_{[}, 1_{]}, 1_*, -1, -1_{[}, -1_{]}, -1_*\}$, let

$$\sharp_d(c) = |\{t \in \{1, ..., m\} | c_t = d\}|$$

Theorem 3. *Given a graph $G = (V, E)$ with tree decomposition (T, X), a minimum signed dominating set problem can be computed in $O((6tw)^{tw} \cdot |V|)$ time, where tw is the treewidth of the tree decomposition.*

Proof. In order to describe the algorithm clearly, assume the dynamic programming algorithm is based on the nice tree decomposition computing the minimum signed dominating set.

Step 1: Table initialization.

For all tables X_i and each coloring $c \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_i}$ let

$$m_i(c) = \begin{cases} +\infty & \text{if } c \text{ is locally invalid for } X_i \\ \sharp_{1,1_[,1_],1_*}(c) & \text{otherwise} \end{cases}$$

Since the check for local invalidity takes $O(n_i)$ time, this step take time $O((4n_i)^{n_i} \cdot n_i)$.

Step 2: Dynamic programming.

After the initialization, the algorithm visits the bags of the tree decomposition from the leaves to the root, there are three kinds of nodes during the dynamic programming procedure should be considered. We evaluate the corresponding mappings in each node according to the following rules.

Forget Nodes: Assume i is a forget node with child j and $X_i = \{x_{i1}, ..., x_{in_i}\}$, $X_j = \{x_{i1}, ..., x_{in_i}, x\}$.

For all colorings $c \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_i}$, let

$$m_i(c) = \min_{d \in \{1, 1_[, -1, -1_[\}} \{m_j(c \times \{d\})\}$$

Note that for X_j the vertex x is assigned color $1_]$, $-1_]$, 1_* and -1_*, that is, x is not dominated by a graph vertex. But by the consistency property of tree decompositions, the vertex x would never appear in a bag for the rest of the algorithm, a coloring will not lead to a signed dominating set because x cannot be dominated. That is why in the above equation x just takes colors $1, 1_[, -1, -1_[$ only.

Introduce Nodes: Assume i is an introduce node with child node j, let $X_j = \{x_{j1}, ..., x_{jn_j}\}$, $X_i = \{x_{i1}, ..., x_{in_i}, x\}$. Suppose $N(x) \cap X_j = \{x_{jp_1}, ..., x_{jp_s}\}$ be the neighbors of the vertex x which are contained in the bag X_i. Now define a function on the set of colorings of X_j.

$$\phi : \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_j} \to \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_j}$$

For $c = (c_1, ..., c_{n_j}) \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_j}$, define $\phi(c) = (c'_1, ..., c'_{n_j})$ such that

$$c'_t = \begin{cases} 1_](-1_]) & \text{if } t \in \{p_1, ..., p_s\} \text{ and} c_t = 1_[(-1_[) \\ c_t & \text{otherwise} \end{cases}$$

Compute the mapping m_i of X_i as follows: for all colorings $c = (c_1, ..., c_{n_j}) \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_j}$, if we assign color 1 to vertex x, then the vertices in $\{x_{jp_1}, ..., x_{jp_s}\}$ with colors $1_]$ or $-1_]$ can be assigned colors $1_[$ or $-1_[$.

The vertices in $\{x_{jp_1}, ..., x_{jp_s}\}$ with colors 1_* or -1_* can be pushed to the "upper" colors, for example, 1_{-1} is changed into $1_]$. In the same way, if we assign color -1 to vertex x, the vertices in $\{x_{jp_1}, ..., x_{jp_s}\}$ with colors 1 or -1 can be pulled to the "lower" colors, for example, 1_2 is changed into $1_[$.

$$m_i(c \times \{-1_], -1_*\}) = m_j(c)$$

$$m_i(c \times \{-1, -1_[\}) = m_j(c) \text{if } x \text{ has neighbors in } X_j \text{ with colors } 1, 1_[$$

$$m_i(c \times \{1, 1_[, 1_], 1_*\}) = m_j(\phi(c)) + 1$$

Since it needs $O(n_i)$ time to check a coloring is locally invalid, the computation of m_i can be carried out in $O((4n_i)^{n_i} \cdot n_i)$ time.
Join Nodes: Assume i is a join node with children j and k, let $X_i = X_j = X_k = \{x_{i1}, ..., x_{in_i}\}$. Let $c = (c_1, ..., c_{n_i}) \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_i}$ be a coloring for X_i. $c' = (c'_1, ..., c'_{n_i})$, $c'' = (c''_1, ..., c''_{n_i}) \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_i}$. Since for a join node, we have to consider a child of colors $1, 1_[$ combined with colors $1_], 1_*$ of another child. Similar with $-1, -1_[$ combined with $-1_], -1_*$. Then
$c_t = 1 \Rightarrow (c'_t, c''_t \in \{1, 1_*\}) \wedge (c'_t = 1 \vee c''_t = 1)$
$c_t = 1 \Rightarrow (c'_t, c''_t \in \{1, 1_]\}) \wedge (c'_t = 1 \vee c''_t = 1)$
$c_t = 1_[\Rightarrow (c'_t, c''_t \in \{1_[, 1_*\}) \wedge (c'_t = 1_[\vee c''_t = 1_[)$
$c_t = 1_[\Rightarrow (c'_t, c''_t \in \{1_[, 1_]\}) \wedge (c'_t = 1_[\vee c''_t = 1_[)$
$c_t = -1 \Rightarrow (c'_t, c''_t \in \{-1, -1_*\}) \wedge (c'_t = -1 \vee c''_t = -1)$
$c_t = -1 \Rightarrow (c'_t, c''_t \in \{-1, -1_]\}) \wedge (c'_t = -1 \vee c''_t = -1)$
$c_t = -1_[\Rightarrow (c'_t, c''_t \in \{-1_[, -1_*\}) \wedge (c'_t = -1_[\vee c''_t = -1_[)$
$c_t = -1_[\Rightarrow (c'_t, c''_t \in \{-1_[, -1_]\}) \wedge (c'_t = -1_[\vee c''_t = -1_[)$

Then, the computation of the mapping m_i of X_i as follows: for all colorings $c \in \{1, 1_[, 1_], 1_*, -1, -1_[, -1_], -1_*\}^{n_i}$, let

$$m_i(c) = \min\{m_j(c') + m_k(c'') - \sharp_{1,1_[,1_],1_*}(c)\}$$

Computing the value m_i, we should look up the corresponding values for coloring c in m_j and in m_k, add the corresponding values and subtract the number of color $1, 1_[$ in c. If color c of node i assigns the color $1_]$, 1_*, $-1_]$ or -1_* to a vertex x from X_i, then in color c' of X_j and color c'' of X_k, we should assign the same color to x. However, if c assigns color 1, $1_[$, -1 or $-1_[$ to x, it is necessary to justify this color by only one of the colorings c' or c''. Combine the states of c' and c'', therefore, computing m_i can be done in $O((6n_i)^{n_i} \cdot n_i)$ time.
Step 3 Let r be the root of T, the signed dominating set number is given by

$$\min\{m_r(c) | c \in \{1, 1_[, -1, -1_[\}^{n_r}\}$$

The minimum number of signed dominating set is taken only over colorings containing colors $1, 1_[, -1, -1_[$ since the colors $1_]$, 1_*, $-1_]$ and -1_* mean that the corresponding vertex still needs to be dominated.

Since $|X_i| = n_i$, if the given graph has treewidth tw, then computing the minimum signed dominating number can be done in $O((6tw)^{tw} \cdot |V|)$ time. □

Theorem 4. *[13] If a planar graph has a dominating set of size at most k, then its treewidth is bounded by $O(\sqrt{k})$.*

Lemma 7. *Given a graph $G = (V, E)$, if there exist a signed dominating set of size at most k, then there must exist a dominating set of size at most k.*

Proof. Assume there is a signed dominating set P with $|P| \leq k$ in graph G, then for each $v \in V$, the signed dominating function makes each v satisfy $f(N[v]) \geq 1$. If we use 0 to replace all the -1 in the graph, it is easy to see that P is also a dominating set of graph G. Each vertex with value -1 needs at least two vertices with value $+1$ to dominate, but each vertex with value 0 just needs at least one vertex with value $+1$ to dominate, therefore, a dominating set of graph G is of size at most k. Conversely, it does not hold. □

By Theorem 3, Theorem 4 and Lemma 7, it is easy to see that signed dominating set problem on a planar graph has fixed parameter tractable algorithm.

Corollary 1. *Signed dominating set on planar graphs is solved in $O((6\sqrt{k})^{O(\sqrt{k})} \cdot |V|)$ time.*

5 Signed Dominating Set in Special Graph

5.1 Polynomial Kernel in Bipartite Graph

For $p \geq 1, q \geq 1$, let $K_{p,q}$ be a bipartite graph with vertices $\{x_1, \ldots, x_i, y_1, \ldots, y_j\}$ by connecting all edges of the type $(x_i y_j)$, where $1 \leq i \leq p, 1 \leq j \leq q$.

Theorem 5. *In a bipartite graph $K_{p,q}$, the kernel for signed dominating set is $O(k^2)$.*

Proof. Let the X^+ and X^- be two sets of vertices in X which are assigned with $+1$ and -1, respectively. Similarly define the Y^+ and Y^-. Then $|P| = |X^+| + |Y^+|$, $|M| = |X^-| + |Y^-|$. Every vertex in X^- has to be connected to at least two vertices in Y^+, according to the pigeonhole principle, at least one vertex in Y^+ has to be adjacent to at least $|X^-|/|Y^+|$ vertices in X^-. Since the vertex y_i in Y^+ should satisfy $f(N[y_i]) \geq 1$, it follows that $|X^+| - |X^-|/|Y^+| \geq 1$, then $|X^-| \leq |Y^+|(|X^+| - 1)$. Every vertex in Y^- has to be connected to at least two vertices in X^+, then at least one vertex in X^+ has to be adjacent to at least $|Y^-|/|X^+|$ vertices in Y^-, therefore $|Y^-| \leq |X^+|(|Y^+| - 1)$.

For the whole graph, $|V| = |X^+| + |X^-| + |Y^+| + |Y^-|$, $|V| \leq |X^+| + |Y^+|(|X^+|-1) + |Y^+| + |X^+|(|Y^+|-1) = 2|X^+||Y^+|$, $|X^+||Y^+| \geq \frac{|V|}{2}$. $|P| - |M| = 2(|X^+| + |Y^+|) - |V|$, $2k - |V| \geq 4\sqrt{|X^+| \cdot |Y^+|} - |V|$, $k \geq \sqrt{2|V|}$, therefore, the kernel is $O(k^2)$. □

5.2 Linear Kernel in $\Delta \leq d$ Graph

Lemma 8. *Any signed dominating function in a $\Delta \leq d$ graph satisfies*

$$|P| - |M| = |P_0| + |P_1|/2 + (\frac{d}{4} - \frac{1}{2})|M_{\frac{d}{2}+1}| + ... + (\frac{d}{2} - 1)|M_d| - \frac{1}{2}(|P_3| - |M_3|) \\ - ... - (\frac{d}{4} - 1)(|P_{\frac{d}{2}}| - |M_{\frac{d}{2}}|)$$

where d is a constant.

Proof. If d is even, $|P| = |P_0| + |P_1| + ... + |P_{\frac{d}{2}}|$, and if d is odd, $|P| = |P_0| + |P_1| + ... + |P_{\frac{d-1}{2}}|$. $|M| = |M_2| + |M_3| + ... + |M_d|$. For d even, the edge number of $K_{|P|,|M|}$ is $|P_1| + 2|P_2| + ... + \frac{d}{2}|P_{\frac{d}{2}}| = 2|M_2| + 3|M_3| + ... + d|M_d|$. Then

$$|P_1| + 2(|P_2| - |M_2|) + ... + \frac{d}{2}(|P_{\frac{d}{2}}| - |M_{\frac{d}{2}}|) = (\frac{d}{2} + 1)|M_{\frac{d}{2}+1}| + ... + d|M_d|$$

Then, in $\Delta \leq d$ graph,

$$|P| - |M| = |P_0| + |P_1|/2 + (\frac{d}{4} - \frac{1}{2})|M_{\frac{d}{2}+1}| + ... + (\frac{d}{2} - 1)|M_d| - \frac{1}{2}(|P_3| - |M_3|) \\ - ... - (\frac{d}{4} - 1)(|P_{\frac{d}{2}}| - |M_{\frac{d}{2}}|)$$

The same with the case d is odd. □

Theorem 6. *In a $\Delta \leq 5$ graph, the kernel for signed dominating set is $O(k)$.*

Proof. In a $\Delta \leq 5$ graph, according to lemma 8, $|P| - |M| = |P_0| + |P_1|/2 + |M_3|/2 + |M_4| + \frac{3}{2}|M_5|$. From the edge number $K_{|P_1|,|M_2|}$, we can see that $2|M_2| \leq |P_1|$ and $|P_2| = |M_2| + \frac{3}{2}|M_3| + 2|M_4| + \frac{5}{2}|M_5| - |P_1|/2$. Since $|V| = |P_0| + |P_1| + |P_2| + |M_2| + |M_3| + |M_4| + |M_5|$, therefore, $|V| = |P_0| + |P_1|/2 + 2|M_2| + \frac{5}{2}|M_3| + 3|M_4| + \frac{7}{2}|M_5| \leq 5(|P| - |M|)$. Since $|P| \leq k$ and $|M| \geq |V| - k$, then we obtain $|V| \leq \frac{5}{3}k$. □

5.3 Linear Kernel in r-Regular Graph

Theorem 7. *In a r-regular graph, the kernel for signed dominating set is $O(k)$.*

Proof. According to the lower bound of γ_s in [11], for every r-regular graph, $\gamma_s \geq \frac{1}{r+1}|V|$ for r is even, and $\gamma_s \geq \frac{2}{r+1}|V|$ for r is odd. Then $2k - |V| \geq \gamma_s \geq \frac{1}{r+1}|V|$, we get the kernel $O(k)$.

6 Conclusion

In this paper, we study signed dominating set problem from the parameterized perspective. There are still some problems deserved for further research. Firstly can we improve the kernel of the signed dominating set problem in general graphs, if it can do, the new fixed parameter tractable algorithm for this problem also follows naturally. Since for a fixed k, the problem of determining whether a graph has a signed dominating function of weight at most k is NP-complete, if signed domination problem is parameterized with the weight of the signed dominating function, is this problem still fixed parameter tractable?

References

1. Dunbar, J., Hedetniemi, S., Henning, M., Slater, P.: Signed domination in graphs. Graph Theory, Combinatorics and Applications, 311–322 (1995)
2. Hattingh, J., Henning, M., Slater, P.: The algorithmic complexity of signed domination in graphs. Australasian Journal of Combinatorics 12, 101–112 (1995)
3. Hass, R., Wexler, T.: Signed domination numbers of a graph and its complement. Discrete Math. 283, 87–92 (2004)
4. Zelinka, B.: Signed total domination number of a graph. Czechoslovak Math. 51, 225–229 (2001)
5. Matousek, J.: On the signed domination in graphs. Combinatorica 20, 103–108 (2000)
6. Robertson, N., Seymour, P.D.: Graph minors X. Obstructions to tree decomposition. J. of Combinatorial Theory, Series B 52, 153–190 (1991)
7. Zhang, Z., Xu, B., Li, Y., Liu, L.: A note on the lower bounds of signed domination number of a graph. Discrete Math. 195, 295–298 (1999)
8. Favaron, O.: Signed domination in regular graphs. Discrete Math. 158, 287–293 (1996)
9. Kloks, T. (ed.): Treewidth: Computations and Approximations. LNCS, vol. 842. Springer, Heidelberg (1994)
10. Bodlaender, H.L.: Treewidth: Algorithmic Techniques and Results. In: Privara, I., Ružička, P. (eds.) MFCS 1997. LNCS, vol. 1295, pp. 19–36. Springer, Heidelberg (1997)
11. Henning, M.A., Slater, P.J.: Irregularities relating domination parameters in cubic graphs. Discrete Mathematics 158, 87–98 (1996)
12. Henning, M.A.: Domination in regular graphs. Ars Combinatoria 43, 263–271 (1996)
13. Demaine, E.D., Fomin, F.V., Thilikos, D.M.: Bidimensional parameters and local treewidth. SIAM J. Disc. Math. 18(3), 501–511 (2004)
14. Alber, J., Fellows, M., Niedermeier, R.: Polynomial time data reduction for dominating set. J. ACM 51, 363–384 (2004)