8 The Rigid Finite Element Method

Actual kinematic chains commonly contain links whose flexibility greatly exceeds that of other links. It may then be necessary to take that flexibility into account. Booms of cranes and certain links of manipulators count among those. A large number of approaches in analysis of multibody systems can be found in literature with with one and more flexible links [Zienkiewicz O. C., 1972], [Wittbrodt E., 1983], [Wojciech S., 1984], [Huston R. L., Wanga Y., 1994], [Arteaga M. A., 1998], [Zienkiewicz O. C., Taylor R. L., 2000], [Berzeri M., et al., 2001], [Adamiec-Wójcik I., 2003], [Wittbrodt E., et al., 2006]. Chapter 9 introduces models of offshore cranes (a column one and an A-frame) which enable taking into account the flexibility of the supporting structure.

Let us consider a flexible link numbered p of a sample mechanism depicted in Fig. 8.1. Let $\{p,0\}$ be the coordinate system attached to the link p as if it were rigid. Its position relative to the preceding link s is given by the coordinates of the following vector:

$$\tilde{\mathbf{q}}^{(p,0)} = \begin{bmatrix} \tilde{q}_1^{(p,0)} & \cdots & \tilde{q}_{\tilde{n}_{p,0}}^{(p,0)} \end{bmatrix}^T. \tag{8.1}$$

The number $\tilde{n}_{p,0}$ of coordinates of the vector $\tilde{\mathbf{q}}^{(p,0)}$ is less than 6 and depends on the class of the kinematic joint connecting the links s and p. These coordinates will henceforth be called rigid (configuration) coordinates of the link p.

In order to fully describe the relative motion of a flexible link, the vector (8.1) needs to be supplemented with a vector whose elements are called elastic coordinates. Their choice depends on the discretisation method used for the flexible link. Regardless of the method, the vector of generalized coordinates of the flexible link p describing its motion in the kinematic chain may be written as:

$$\tilde{\mathbf{q}}^{(p)} = \begin{bmatrix} \tilde{\mathbf{q}}^{(p,0)} \\ \tilde{\mathbf{q}}^{(p,f)} \end{bmatrix}, \tag{8.2}$$

where $\tilde{\mathbf{q}}^{(p,0)}$ – vector of generalized configuration (rigid) coordinates of the link p,

$\tilde{\mathbf{q}}^{(p,f)} = \begin{bmatrix} \tilde{q}_1^{(p,f)} & \ldots & \tilde{q}_{\tilde{n}_{p,f}}^{(p,f)} \end{bmatrix}^T$ – vector of generalized elastic (flexible) coordinates of the link p,

$\tilde{n}_{p,f}$ – number of elastic coordinates of the link p.

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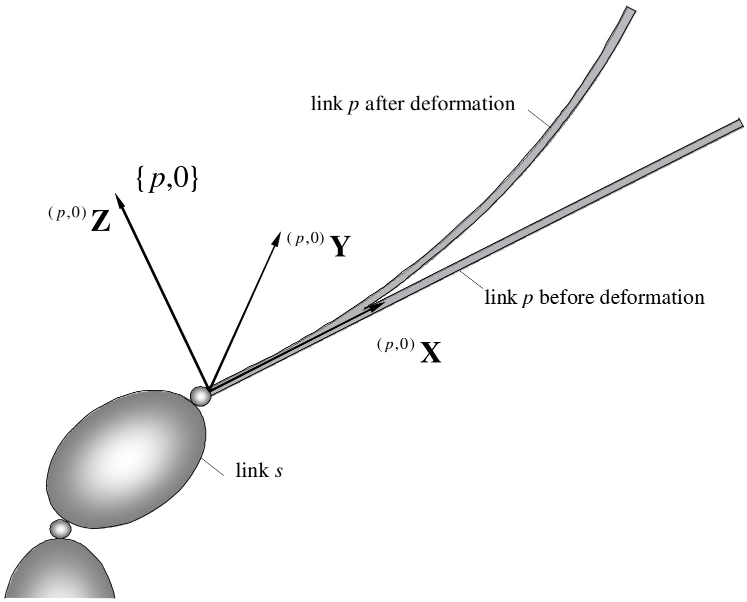

Fig. 8.1. A flexible link *p*

Let us also assume that the transformation of coordinates from the local coordinate system $\{p,0\}$ to the preceding coordinate system (with index s) is given by the matrix:

$$ {}_{(p,0)}^{\ \ s}\mathbf{T} = {}_{(p,0)}^{\ \ s}\mathbf{T}\left(\tilde{\mathbf{q}}^{(p,0)}\right). \tag{8.3}$$

One of many discretisation methods of flexible links will be presented below. This is the rigid finite element (RFE) method. It has two variants: classical and modified.

8.1 The RFE Method: Classical Formulation

The rigid finite element method has for many years been applied at the Gdańsk University of Technology, initially by Prof. Kruszewski, then by Prof. Wittbrodt, and their co-workers, to model multibody systems. The formulation of the method presented in [Kruszewski J., et al., 1975], in which each finite element is assumed to possess six degrees of freedom in its relative motion, is called classical. The description of the method expounded herein deviates from that which is found in papers by professor Kruszewski and his co-authors. Namely, joint coordinates and homogeneous transformations are used to derive the equations of motion, following [Adamiec-Wójcik I., 2003] and [Wittbrodt E., et al., 2006].

8.1.1 Generalized Coordinates: Transformation Matrices

Let p be a flexible beam link in a kinematic chain. That link is replaced with a series of rigid finite elements connected with spring-damping elements using discretisation which is detailed by Kruszewski and co-authors in [Kruszewski J., et al., 1975], [Kruszewski J., et al., 1999]. In the case of a beam with constant section, the procedure is as follows: first, this is the so-called primary division, the beam of length L_p is divided into m_p equally long segments (Fig. 8.2a).

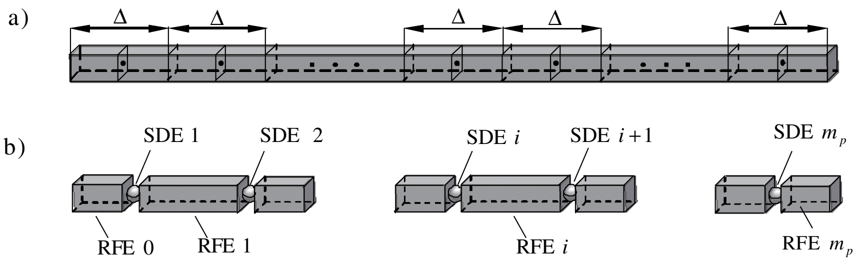

Fig. 8.2. Division of a flexible link: a) primary division, b) secondary division

Flexibility traits of the elements are inherited by the spring-damping elements (SDE) placed at the centre of each segment of length Δ. In this way, one obtains a secondary division of the flexible link into m_p+1 rigid finite elements (RFEs) connected by m_p massless and dimensionless spring-damping elements (Fig. 8.2b).

Division of beam links with variable sections and a method of determining characteristic parameters of RFEs and SDE are expounded, among other things, in the work [Wittbrodt E., et al., 2006]. Since each RFE (except RFE 0) has a coordinate system attached with origin in its centre of mass and axes coinciding with the principal axes of inertia (Fig. 8.3), the position of the element in undeformed state can be determined unambiguously relative to the system $\{p,0\}$ of RFE 0, provided that the transformation matrices are known:

$$\tilde{\mathbf{T}}^{(p,i')} = \text{const} . \tag{8.4}$$

In the general case, the transformation matrices with constant coefficientstake the form:

$$\tilde{\mathbf{T}}^{(p,i')} = \begin{bmatrix} \tilde{\mathbf{R}}^{(p,i')} & \tilde{\mathbf{r}}^{(p,i')} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{8.5}$$

where $\tilde{\mathbf{R}}^{(p,i')}$ – direction cosine matrix of the axes of the system $\{p,i'\}$ relative to the system $\{p,0\}$,

$\tilde{\mathbf{r}}^{(p,i')}$ – vector of coordinates of the origin system of the system $\{p,i'\}$ in $\{p,0\}$.

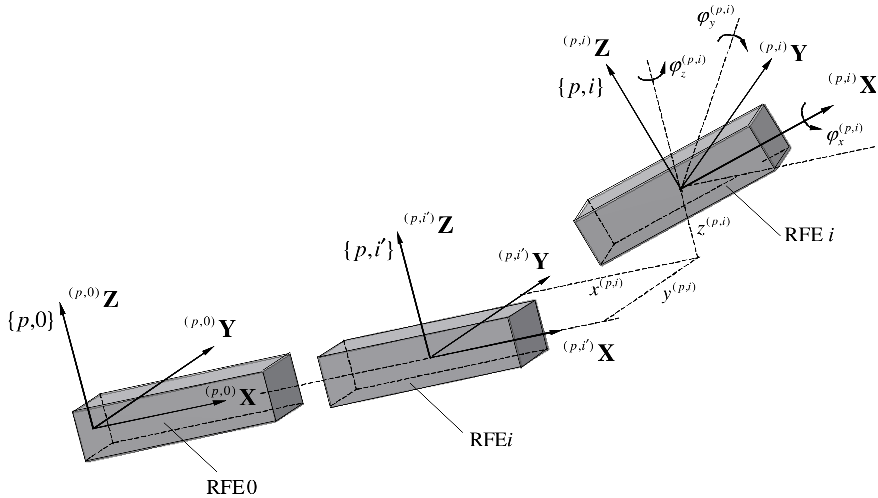

Fig. 8.3. Coordinate systems related to a flexible link: { } – the inertial system, $\{p,0\}$ – the system attached to RFE 0, $\{p,i'\}$ – the system attached to RFE i in undeformed state of the beam, $\{p,i\}$ – the system attached in a fixed way to RFE i whose axes coincide with the principal central axes of inertia of the element, $x^{(p,i)}, y^{(p,i)}, z^{(p,i)}$ – coordinates of the origin of the coordinate system $\{p,i\}$ in $\{p,i'\}$, $\varphi_x^{(p,i)}, \varphi_y^{(p,i)}, \varphi_z^{(p,i)}$ – ZYX Euler angles described in chapter 4

If the system $\{p,i'\}$ has axes parallel to the axes of the system $\{p,0\}$, the rotation matrix $\tilde{\mathbf{R}}^{(p,i')}$ is the identity matrix. Due to the lifting motion and external loads, individual RFEs are subjected to displacements. The generalized coordinates being the components of the vector:

$$\tilde{\mathbf{q}}^{(p,i)} = \begin{bmatrix} x^{(p,i)} & y^{(p,i)} & z^{(p,i)} & \varphi_x^{(p,i)} & \varphi_y^{(p,i)} & \varphi_z^{(p,i)} \end{bmatrix}^T , \tag{8.6}$$

decribe the motion of the i-th RFE ($i = 1,...,m_p$) of the link p relative to the system $\{p,i'\}$ attached to the RFE i in undeformed state. The transformation matrix ${}_i^{i'}\tilde{\mathbf{T}}^{(p)}$ from the system $\{p,i\}$ to the system $\{p,i'\}$ in the nonlinear model, allowing the rotation angles $\varphi_x^{(p,i)}, \varphi_y^{(p,i)}, \varphi_z^{(p,i)}$ to be large, takes the following form:

$${}_i^{i'}\tilde{\mathbf{T}}^{(p)} = \begin{bmatrix} c_z^{(p,i)}c_y^{(p,i)} & c_z^{(p,i)}s_y^{(p,i)}s_x^{(p,i)} - s_z^{(p,i)}c_x^{(p,i)} & c_z^{(p,i)}s_y^{(p,i)}c_x^{(p,i)} + s_z^{(p,i)}s_x^{(p,i)} & x^{(p,i)} \\ s_z^{(p,i)}c_y^{(p,i)} & s_z^{(p,i)}s_y^{(p,i)}s_x^{(p,i)} + c_z^{(p,i)}c_x^{(p,i)} & s_z^{(p,i)}s_y^{(p,i)}c_x^{(p,i)} - c_z^{(p,i)}s_x^{(p,i)} & y^{(p,i)} \\ -s_y^{(p,i)} & c_y^{(p,i)}s_x^{(p,i)} & c_y^{(p,i)}c_x^{(p,i)} & z^{(p,i)} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8.7}$$

where $c_\alpha^{(p,i)} = \cos\varphi_\alpha^{(p,i)}, \quad s_\alpha^{(p,i)} = \sin\varphi_\alpha^{(p,i)} \qquad \text{for} \quad \alpha \in \{x, y, z\}.$.

When small rotation angles of RFEs are assumed, leading to omission of higher rank small terms from the approximations of trigonometric functions of the angles $\varphi_x^{(p,i)}$, $\varphi_y^{(p,i)}$, $\varphi_z^{(p,i)}$ (the linear model), the matrix ${}_i^{i'}\tilde{\mathbf{T}}^{(p)}$ may be written [Adamiec-Wójcik I., 2003] as:

$$ {}_i^{i'}\tilde{\mathbf{T}}^{(p)} = \begin{bmatrix} 1 & -\varphi_z^{(p,i)} & \varphi_y^{(p,i)} & x^{(p,i)} \\ \varphi_z^{(p,i)} & 1 & -\varphi_x^{(p,i)} & y^{(p,i)} \\ -\varphi_y^{(p,i)} & \varphi_x^{(p,i)} & 1 & z^{(p,i)} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.8} $$

The transformation matrix from the system $\{p,i\}$ to the system $\{p,0\}$, whether the model is linear or nonlinear, has this form:

$$ \tilde{\mathbf{T}}^{(p,i)} = \tilde{\mathbf{T}}^{(p,i)}\left(\tilde{\mathbf{q}}^{(p,i)}\right) = \tilde{\mathbf{T}}^{(p,i')}\,{}_i^{i'}\tilde{\mathbf{T}}^{(p)}. \tag{8.9} $$

8.1.2 Kinetic Energy of a Flexible Link

Let us assume, as in chapter 5, that the concerned multibody system is situated on a movable base $\{A\}$ (Fig. 5.1) whose motion relative to the inertial (global) system $\{0\} = \{\,\}$ is known.

Rigid finite elements of the link p may be treated as m_p+1 consecutive bodies appended to the link s of the kinematic chain. In further considerations, the first rigid finite element in the chain (RFE 0) is treated separately, because the generalized coordinates describing the relative motion of this RFE depend on the type of the kinematic joint connecting the link p with its preceding link s and their number is less than 6. The coordinate system $\{p,0\}$ plays the role of the configuration system of the link p.

Let the vector of generalized coordinates $\overline{\mathbf{q}}^{(p)}$ contain the coordinates of RFE 0 of the link p and the coordinates of the link s which precedes the link p. Let also the transformation matrix $\mathbf{T}^{(p,0)}$ define the transformation from the system $\{p,0\}$ attached to RFE 0 of the flexible link p to the inertial system. The following notation is introduced:

$$ \overline{\mathbf{q}}^{(p)} = \begin{bmatrix} \mathbf{q}^{(s)} \\ \tilde{\mathbf{q}}^{(p,0)} \end{bmatrix}, \tag{8.10.1} $$

$$ \mathbf{T}^{(p,0)} = {}_A^0\mathbf{T}(t)\,{}_s^A\mathbf{T}\left(\mathbf{q}^{(s)}\right){}_{(p,0)}^{\;s}\mathbf{T}\left(\tilde{\mathbf{q}}^{(p,0)}\right) = {}_A^0\mathbf{T}(t)\,\overline{\mathbf{T}}^{(p,0)}\left(\overline{\mathbf{q}}^{(p)}\right), \tag{8.10.2} $$

where $\overline{\mathbf{T}}^{(p,0)} = {}_s^A\mathbf{T}\left(\mathbf{q}^{(s)}\right){}_{(p,0)}^{\;s}\mathbf{T}\left(\tilde{\mathbf{q}}^{(p,0)}\right)$.

The kinetic energy of RFE 0 of the link p is given by the expression:

$$E_{p,0} = \frac{1}{2}\mathrm{tr}\left\{\dot{\mathbf{T}}^{(p,0)}\,\mathbf{H}^{(p,0)}\,\dot{\mathbf{T}}^{(p,0)T}\right\}, \tag{8.11}$$

where $\mathbf{H}^{(p,0)}$ – matrix of inertia of RFE 0 of the link p.

A derivation similar to that in chapter 5 yields:

$$\boldsymbol{\varepsilon}_{\overline{q}_k^{(p)}}(E_{p,0}) = \sum_{i=1}^{n_{p,0}} a_{k,i}^{(p,0)}\ddot{q}_i^{(p,0)} + e_k^{(p,0)}, \tag{8.12}$$

where $a_{k,i}^{(p,0)} = \mathrm{tr}\left\{\mathbf{T}_k^{(p,0)}\,\mathbf{H}^{(p,0)}\,\mathbf{T}_i^{(p,0)T}\right\}$,

$$e_k^{(p,0)} = \sum_{i=1}^{n_{p,0}}\sum_{j=1}^{n_{p,0}} \mathrm{tr}\left\{\mathbf{T}_k^{(p,0)}\mathbf{H}^{(p,0)}\,\mathbf{T}_{i,j}^{(p,0)}\right\}\dot{q}_i^{(p,0)}\,\dot{q}_j^{(p,0)} + $$
$$+\,\mathrm{tr}\left\{\mathbf{T}_k^{(p,0)}\,\mathbf{H}^{(p,0)}\left[{}_A^0\ddot{\mathbf{T}}\,\overline{\mathbf{T}}^{(p,0)} + 2\,{}_A^0\dot{\mathbf{T}}\,\dot{\overline{\mathbf{T}}}^{(p,0)}\right]^T\right\},$$

$n_{p,0} = n_s + \tilde{n}_{p,0}$.

The equation (8.12) may be put in a matrix form:

$$\boldsymbol{\varepsilon}_{\overline{q}_k^{(p)}}(E_{p,0}) = \begin{bmatrix}\mathbf{A}_{s,s}^{(p,0)} & \mathbf{A}_{s,0}^{(p,0)}\\ \mathbf{A}_{0,s}^{(p,0)} & \mathbf{A}_{0,0}^{(p,0)}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}^{(s)}\\ \ddot{\tilde{\mathbf{q}}}^{(p,0)}\end{bmatrix} + \begin{bmatrix}\mathbf{e}_s^{(p,0)}\\ \mathbf{e}_0^{(p,0)}\end{bmatrix}. \tag{8.13}$$

The remaining RFEs of the flexible link are treated as elements of the kinematic chain appended to RFE 0. Hence, the coordinates of an arbitrary point in the local system $\{p,i\}$ of RFE i of the link p ($i = 1,...,m_p$) may be transformed, following the procedure presented in chapter 5, to the inertial system. The following equality is used:

$$\mathbf{r}^{(p,i)} = \mathbf{T}^{(p,i)}\,\tilde{\mathbf{r}}^{(p,i)}, \tag{8.14}$$

where $\mathbf{T}^{(p,i)} = \mathbf{T}^{(p,0)}\,\tilde{\mathbf{T}}^{(p,i)}$

$\mathbf{r}^{(p,i)}$ – vector of coordinates in the inertial system { },

$\tilde{\mathbf{r}}^{(p,i)}$ – vector of local coordinates in the system $\{p,i\}$.

The kinetic energy of REF i of the link p equals:

$$E_{p,i} = \frac{1}{2}\mathrm{tr}\left\{\dot{\mathbf{T}}^{(p,i)}\,\mathbf{H}^{(p,i)}\,\dot{\mathbf{T}}^{(p,i)T}\right\}, \tag{8.15}$$

where $\mathbf{H}^{(p,i)}$ – matrix of inertia of RFE i of the link p.

Defining a vector with $n_{p,i} = n_s + \tilde{n}_{p,0} + 6 = n_{p,0} + 6$ components:

$$\mathbf{q}^{(p,i)} = \begin{bmatrix}\overline{\mathbf{q}}^{(p)}\\ \tilde{\mathbf{q}}^{(p,i)}\end{bmatrix}, \tag{8.16}$$

the following may be written:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p,i)}}(E_{p,i}) = \mathbf{A}^{(p,i)}\ddot{\mathbf{q}}^{(p,i)} + \mathbf{e}^{(p,i)}, \tag{8.17}$$

where $\mathbf{A}^{(p,i)} = \left(\tilde{a}_{l,s}^{(p,i)}\right)_{l,s=1,\dots,n_{p,i}} = \mathrm{tr}\left\{\mathbf{T}_l^{(p,i)}\,\mathbf{H}^{(p,i)}\,\mathbf{T}_s^{(p,i)T}\right\}$,

$$\mathbf{e}^{(p,i)} = \left(\tilde{e}_l^{(p,i)}\right)_{l=1,\dots,n_{p,i}} = \sum_{s=1}^{n_{p,i}}\sum_{j=1}^{n_{p,i}} \mathrm{tr}\left\{\mathbf{T}_l^{(p,i)}\,\mathbf{H}^{(p,i)}\,\mathbf{T}_{s,j}^{(p,i)T}\right\}\dot{q}_s^{(p,i)}\,\dot{q}_j^{(p,i)} + \\ + \mathrm{tr}\left\{\mathbf{T}_l^{(p,i)}\,\mathbf{H}^{(p,i)}\left[{}_A^0\ddot{\mathbf{T}}\overline{\mathbf{T}}^{(p,i)} + {}_A^0\dot{\mathbf{T}}\dot{\overline{\mathbf{T}}}^{(p,i)}\right]\right\}.$$

The same may be expressed in the block form, thus:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p,i)}}\left(E_{p,i}\right) = \begin{bmatrix} \mathbf{A}_{s,s}^{(p,i)} & \mathbf{A}_{s,0}^{(p,i)} & \mathbf{A}_{s,i}^{(p,i)} \\ \mathbf{A}_{0,s}^{(p,i)} & \mathbf{A}_{0,0}^{(p,i)} & \mathbf{A}_{0,i}^{(p,i)} \\ \mathbf{A}_{i,s}^{(p,i)} & \mathbf{A}_{i,0}^{(p,i)} & \mathbf{A}_{i,i}^{(p,i)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(s)} \\ \ddot{\tilde{\mathbf{q}}}^{(p,0)} \\ \ddot{\tilde{\mathbf{q}}}^{(p,i)} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_s^{(p,i)} \\ \mathbf{e}_0^{(p,i)} \\ \mathbf{e}_i^{(p,i)} \end{bmatrix}. \tag{8.18}$$

8.1.3 Potential Energy of Gravity Forces and Deformations of a Flexible Link p

The potential energy of gravity forces of the RFE i is given by:

$$V_{p,i}^{g} = m^{(p,i)}\, g\, \boldsymbol{\theta}_3 \mathbf{T}^{(p,i)}\tilde{\mathbf{r}}_C^{(p,i)}, \tag{8.19}$$

where $\tilde{\mathbf{r}}_C^{(p,i)}$ – vector determining the position of the centre of mass of the RFE i in the local coordinate system $\{p,i\}$,

$m^{(p,i)}$ – mass of REF i.

Hence, after ironing out the differences in the definitions of matrices $\mathbf{T}^{(p,0)}$ and $\mathbf{T}^{(p,i)}$, for $i = 1,\dots,m_p$ the following holds:

$$\mathbf{G}^{(p,0)} = \frac{\partial V_{p,0}^{g}}{\partial \overline{\mathbf{q}}^{(p)}} = \begin{bmatrix} \mathbf{G}_s^{(p,0)} \\ \mathbf{G}_0^{(p,0)} \end{bmatrix}, \tag{8.20.1}$$

$$\mathbf{G}^{(p,i)} = \frac{\partial V_{p,i}^{g}}{\partial \mathbf{q}^{(p,i)}} = \begin{bmatrix} \mathbf{G}_s^{(p,i)} \\ \mathbf{G}_0^{(p,i)} \\ \mathbf{G}_i^{(p,i)} \end{bmatrix}, \tag{8.20.2}$$

where $\mathbf{G}^{(p,i)} = \left(g_k^{(p,i)}\right)_{k=1,\ldots,n_{p,i}}$, $g_k^{(p,i)} = m^{(p,i)} g\, \mathbf{T}_k^{(p,i)} \tilde{\mathbf{r}}_C^{(p,i)}$ for $i = 0,1,\ldots,m_p$,

$\mathbf{G}_s^{(p,0)}, \mathbf{G}_0^{(p,0)}, \mathbf{G}_s^{(p,i)}, \mathbf{G}_0^{(p,i)}, \mathbf{G}_i^{(p,i)}$ – appropriate blocks of vectors $\mathbf{G}^{(p,0)}$ and $\mathbf{G}^{(p,i)}$ corresponding to the coordinates $\mathbf{q}^{(s)}, \tilde{\mathbf{q}}^{(p,0)}, \tilde{\mathbf{q}}^{(p,i)}$.

Since the considered link is flexible, before formulating its equations of motion the expressions resulting from the energy of elastic deformation of SDE must be determined. Their derivations in the case of linear physical dependencies describing the properties of the material are presented below. The way with nonlinear physical dependencies will be discussed later. In the considerations pertaining to the deformation of spring-damping elements the reference coordinate system is assumed to be $\{p,0\}$, which is attached to RFE 0, and the matrices $\tilde{\mathbf{R}}^{(p,i')}$, which occur in (8.5), to be identity matrices. A consequence of this is the proposition that in the undeformed state of the link p the axes of all the coordinate systems attached to RFEs from 0 to m_p are parallel. A general algorithm omitting this assumption is presented in [Wittbrodt E., et al., 2006]. A numerically efficient modification of the algorithm will also be described later in this chapter.

Let SDE e connect the RFEs l and r of a flexible link p (Fig. 8.4).

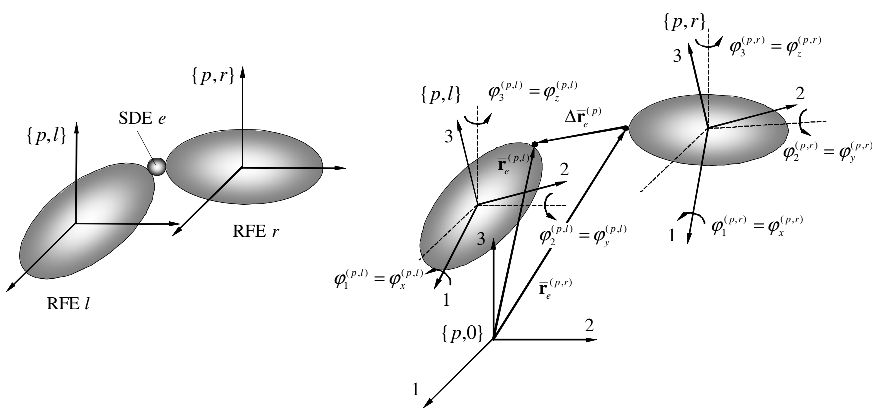

Fig. 8.4. A model of a spring-damping element: a) connection of RFEs l and r by SDE e, b) notation assumed

The energy of elastic deformation of this element is given by the formula:

$$V_{p,e}^{s} = \frac{1}{2}\sum_{j=1}^{3} c_{e,j}^{(p)}\left[\bar{x}_{r,j} - \bar{x}_{l,j}\right]^2 + \frac{1}{2}\sum_{j=1}^{3} c_{e,j+3}^{(p)}\left[\varphi_j^{(p,r)} - \varphi_j^{(p,l)}\right]^2, \tag{8.21}$$

where $c_{e,j}^{(p)}$ for j=1,2,3 – coefficients of translational stiffness of the SDE e of the link p,

$c_{e,j}^{(p)}$ for j=4,5,6 – coefficients of rotational stiffness of the SDE e of the link p,

$\overline{\mathbf{r}}_e^{(p,l)} = \begin{bmatrix} \overline{x}_{l,1} & \overline{x}_{l,2} & \overline{x}_{l,3} & 1 \end{bmatrix}^T$, $\overline{\mathbf{r}}_e^{(p,r)} = \begin{bmatrix} \overline{x}_{r,1} & \overline{x}_{r,2} & \overline{x}_{r,3} & 1 \end{bmatrix}^T$ – vectors of coordinates of the SDE e (treated first as a point of the RFE l, and next as a point of the RFE r) expressed in the system $\{p,0\}$,

$\varphi_j^{(p,r)}, \varphi_j^{(p,l)}$ – rotation angles of the RFEs r and l of the link p.

In Fig. 8.4 and formula (8.21), the axes of the coordinate system are denoted with (1, 2, 3) instead of (**X** ,**Y**, **Z**) used hitherto. This shortens the formulas considerably.

The coordinates of the SDE e in the systems attached to the RFEs l and r are assumed to be represented by vectors $\tilde{\mathbf{r}}_e^{(p,l)}$ and $\tilde{\mathbf{r}}_e^{(p,r)}$ in Fig. 8.4, respectively. Consequently, the coordinates of this spring-damping element in the reference coordinate system $\{p,0\}$ are expressed by:

$$\overline{\mathbf{r}}_e^{(p,i)} = \tilde{\mathbf{T}}^{(p,i)} \tilde{\mathbf{r}}_e^{(p,i)}, \tag{8.22}$$

where $i \in \{r,l\}$.

The vector $\Delta\overline{\mathbf{r}}_e^{(p)}$ (Fig. 8.4b) is given as:

$$\Delta\overline{\mathbf{r}}_e^{(p)} = \overline{\mathbf{r}}_e^{(p,r)} - \overline{\mathbf{r}}_e^{(p,l)} = \tilde{\mathbf{T}}^{(p,r)}\tilde{\mathbf{r}}_e^{(p,r)} - \tilde{\mathbf{T}}^{(p,l)}\tilde{\mathbf{r}}_e^{(p,l)}, \tag{8.23}$$

and the potential energy of elastic deformation of the SDE e may be put in the following form:

$$V_{p,e}^s = \frac{1}{2}\Delta\overline{\mathbf{r}}_e^{(p)T}\mathbf{C}_T^{(p,e)}\Delta\overline{\mathbf{r}}_e^{(p)} + \frac{1}{2}\left[\tilde{\mathbf{q}}^{(p,r)} - \tilde{\mathbf{q}}^{(p,l)}\right]^T \mathbf{C}_R^{(p,e)}\left[\tilde{\mathbf{q}}^{(p,r)} - \tilde{\mathbf{q}}^{(p,l)}\right], \tag{8.24}$$

where $\mathbf{C}_T^{(p,e)} = \begin{bmatrix} c_{e,1}^{(p)} & 0 & 0 & 0 \\ 0 & c_{e,2}^{(p)} & 0 & 0 \\ 0 & 0 & c_{e,3}^{(p)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{C}_R^{(p,e)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{e,4}^{(p)} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{e,5}^{(p)} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{e,6}^{(p)} \end{bmatrix}$.

The above considerations pertain to the general case in which the transformation matrices $\tilde{\mathbf{T}}^{(p,i)}$ dla $i \in \{l,r\}$ are nonlinear. When small oscillations are considered, i.e. when the transformation matrices ${}_i^{i'}\tilde{\mathbf{T}}^{(p)}$ conform to the formula (8.8), the transformation formula taking the system $\{p,i\}$ to the system $\{p,0\}$ may be represented thusly:

$$\bar{\mathbf{r}}_e^{(p,i)} = \mathbf{r}_e'^{(p,i)} + \mathbf{D}_e'^{(p,i)}\tilde{\mathbf{q}}^{(p,i)}, \tag{8.25}$$

where $\mathbf{r}_e'^{(p,i)} = \begin{bmatrix} \tilde{x}_{i,1} + a_1^{(p,i')} \\ \tilde{x}_{i,2} + a_2^{(p,i')} \\ \tilde{x}_{i,3} + a_3^{(p,i')} \\ 1 \end{bmatrix}$ – vector with constant coefficients,

$\mathbf{D}_e'^{(p,i)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \tilde{x}_{i,3} & -\tilde{x}_{i,2} \\ 0 & 1 & 0 & -\tilde{x}_{i,3} & 0 & \tilde{x}_{i,1} \\ 0 & 0 & 1 & \tilde{x}_{i,2} & -\tilde{x}_{i,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ – matrix with constant coefficients,

$a_j^{(p,i')}$ – components of the vector $\tilde{\mathbf{r}}^{(p,i')}$ from the formula (8.5) for $j = 1, 2, 3$,

$\tilde{x}_{i,1}, \tilde{x}_{i,2}, \tilde{x}_{i,3}$ – coordinates of considered point in $\{p,i\}$.

The formula (8.21) for small deformations takes the form:

$$\begin{aligned} V_{p,e}^s = &\frac{1}{2}\left[\Delta\mathbf{r}_e'^{(p)} + \mathbf{D}_e'^{(p,r)}\tilde{\mathbf{q}}^{(p,r)} - \mathbf{D}_e'^{(p,l)}\tilde{\mathbf{q}}^{(p,l)}\right]^T \mathbf{C}_T^{(p,e)}\left[\Delta\mathbf{r}_e'^{(p)} + \mathbf{D}_e'^{(p,r)}\tilde{\mathbf{q}}^{(p,r)} - \mathbf{D}_e'^{(p,l)}\tilde{\mathbf{q}}^{(p,l)}\right] + \\ &+ \frac{1}{2}\left[\tilde{\mathbf{q}}^{(p,r)} - \tilde{\mathbf{q}}^{(p,l)}\right]^T \mathbf{C}_R^{(p,e)}\left[\tilde{\mathbf{q}}^{(p,r)} - \tilde{\mathbf{q}}^{(p,l)}\right] \end{aligned}, \tag{8.26}$$

where $\Delta\mathbf{r}_e'^{(p)} = \tilde{\mathbf{r}}_e'^{(p,r)} - \tilde{\mathbf{r}}_e'^{(p,l)}$.

In the case of beam links, the SDE i connects the RFE i–1 with the RFE i, therefore $\tilde{\mathbf{q}}^{(p,l)} = \tilde{\mathbf{q}}^{(p,i-1)}$ and $\tilde{\mathbf{q}}^{(p,r)} = \tilde{\mathbf{q}}^{(p,i)}$.

The potential energy of elastic deformation of the link p equals the sum of energies of all the SDE:

$$V_p^s = \sum_{e=1}^{m_p} V_{p,e}^s. \tag{8.27}$$

One should take into account that the formula expressing the elastic energy of SDE 1 of the link p is a variant of the formulas (8.21) and (8.24), and it takes the form:

$$V_{(p,1)}^s = \frac{1}{2}\Delta\bar{\mathbf{r}}_1^{(p)T}\mathbf{C}_T^{(p,e)}\Delta\bar{\mathbf{r}}_1^{(p)} + \frac{1}{2}\tilde{\mathbf{q}}^{(p,1)T}\mathbf{C}_R^{(p,e)}\tilde{\mathbf{q}}^{(p,1)}, \tag{8.28}$$

where $\Delta\bar{\mathbf{r}}_1^{(p)} = \tilde{\mathbf{T}}^{(p,1)}\tilde{\mathbf{r}}_1^{(p,1)} - \tilde{\mathbf{r}}_1^{(p,0)}$.

Taking (8.27) into account leads to:

$$\frac{\partial V_p^s}{\partial \tilde{\mathbf{q}}^{(p,i)}} = -\mathbf{C}_R^{(p,i)}\tilde{\mathbf{q}}^{(p,i-1)} + \left(\mathbf{C}_R^{(p,i)} + \mathbf{C}_R^{(p,i+1)}\right)\tilde{\mathbf{q}}^{(p,i)} - \mathbf{C}_R^{(p,i+1)}\tilde{\mathbf{q}}^{(p,i+1)} + \tilde{\mathbf{S}}^{(p,i)}, \quad (8.29)$$

where $\tilde{\mathbf{S}}^{(p,i)} = \frac{\partial \Delta\overline{\mathbf{r}}_i^{(p)T}}{\partial \tilde{\mathbf{q}}^{(p,i)}}\mathbf{C}_T^{(p,i)}\Delta\overline{\mathbf{r}}_i^{(p)} + \frac{\partial \Delta\overline{\mathbf{r}}_{i+1}^{(p)T}}{\partial \tilde{\mathbf{q}}^{(p,i)}}\mathbf{C}_T^{(p,i+1)}\Delta\overline{\mathbf{r}}_{i+1}^{(p)}$.

Let us remark that for $i = 0$ and $i = m_p$ the following should be assumed, respectively: $\mathbf{C}_R^{(p,0)} = \mathbf{0}$, $\mathbf{C}_R^{(p,i_p+1)} = \mathbf{C}_T^{(p,i_p+1)} = \mathbf{0}$. The formula (8.29) is valid both for linear and nonlinear oscillations. The form of the vectors $\tilde{\mathbf{S}}^{(p,i)}$ in the linear case may be determined easily by means of the formula (8.26). Problems related to the choice of stiffness coefficients when analysing large deflections are discussed in the following papers: [Adamiec-Wójcik I., 1992], [Wojciech S., Adamiec-Wójcik I., 1993], [Wojciech S., Adamiec-Wójcik I., 1994] and [Wittbrodt E., et al., 2006].

8.1.4 Generalized Forces: Equations of Motion

Let us assume that the following act upon the RFE *i*: a force $\tilde{\mathbf{F}}^{(p,i)}$ and a pair of forces whose moment $\tilde{\mathbf{M}}^{(p,i)}$ has the components:

$$\tilde{\mathbf{F}}^{(p,i)} = \left[\tilde{F}_x^{(p,i)} \quad \tilde{F}_y^{(p,i)} \quad \tilde{F}_z^{(p,i)} \quad 0\right]^T, \quad (8.30.1)$$

$$\tilde{\mathbf{M}}^{(p,i)} = \left[\tilde{M}_x^{(p,i)} \quad \tilde{M}_y^{(p,i)} \quad \tilde{M}_z^{(p,i)} \quad 0\right]^T. \quad (8.30.2)$$

Applying the formulas (5.40) and (5.42) along with the procedure presented in [Adamiec-Wójcik I., et al., 2008] yields these forms of generalized forces due to their presence:

$$\begin{aligned} Q_k^{(p,i)}\left(\tilde{\mathbf{F}}^{(p,i)}, \tilde{\mathbf{M}}^{(p,i)}\right) &= \tilde{\mathbf{F}}^{(p,i)T}\mathbf{T}^{(p,i)T}\mathbf{T}_k^{(p,i)}\tilde{\mathbf{r}}^{(p,i)} + \tilde{M}_1^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,3}\left(\mathbf{T}_k^{(p,i)}\right)_{j,2} + \\ &+ \tilde{M}_2^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,1}\left(\mathbf{T}_k^{(p,i)}\right)_{j,3} + \tilde{M}_3^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,2}\left(\mathbf{T}_k^{(p,i)}\right)_{j,1}. \end{aligned} \quad (8.31)$$

When forces acting on RFE 0 are considered, it may be written:

$$\mathbf{Q}^{(p,0)} = \begin{bmatrix} \mathbf{Q}_s^{(p,0)} \\ \mathbf{Q}_0^{(p,0)} \end{bmatrix}, \quad (8.32.1)$$

whereas for a force $\tilde{\mathbf{F}}^{(p,i)}$ and a pair of forces with moment $\tilde{\mathbf{M}}^{(p,i)}$ acting on RFEs from 1 to m_p the following holds:

$$\mathbf{Q}^{(p,i)} = \begin{bmatrix} \mathbf{Q}_s^{(p,i)} \\ \mathbf{Q}_0^{(p,i)} \\ \mathbf{Q}_i^{(p,i)} \end{bmatrix}. \tag{8.32.2}$$

In the case of a flexible link decomposed into m_p+1 rigid finite elements, the following vector of generalized coordinates of the link and expressions giving the kinetic energy and the potential energy of the gravity forces may be defined:

$$\mathbf{q}^{(p)} = \begin{bmatrix} \mathbf{q}^{(s)} \\ \tilde{\mathbf{q}}^{(p,0)} \\ \tilde{\mathbf{q}}^{(p,1)} \\ \vdots \\ \tilde{\mathbf{q}}^{(p,m_p)} \end{bmatrix}, \tag{8.33.1}$$

$$E_p = \sum_{i=1}^{m_p} E_{p,i}\ , \tag{8.33.2}$$

$$V_p^g = \sum_{i=1}^{m_p} V_{p,i}^g\ . \tag{8.33.3}$$

From the equations (8.13), (8.18), (8.20), (8.29) and (8.32) it follows that the equations of motion of the link p, including the term due to the energy of elastic deformation, take the form:

$$\mathbf{A}^{(p)}\ddot{\mathbf{q}}^{(p)} + \mathbf{K}_R^{(p)}\mathbf{q}^{(p)} = -\mathbf{e}^{(p)} - \mathbf{G}^{(p)} - \mathbf{S}^{(p)} + \mathbf{Q}^{(p)}, \tag{8.34}$$

where $$\mathbf{A}^{(p)} = \begin{bmatrix} \sum_{i=0}^{m_p} \mathbf{A}_{s,s}^{(p,i)} & \sum_{i=0}^{m_p} \mathbf{A}_{s,0}^{(p,i)} & \mathbf{A}_{s,1}^{(p,1)} & \cdots & \mathbf{A}_{s,m_p}^{(p,m_p)} \\ \sum_{i=0}^{m_p} \mathbf{A}_{0,s}^{(p,i)} & \sum_{i=0}^{m_p} \mathbf{A}_{0,0}^{(p,i)} & \sum_{i=0}^{m_p} \mathbf{A}_{0,1}^{(p,i)} & \cdots & \mathbf{A}_{0,m_p}^{(p,m_p)} \\ \mathbf{A}_{1,s}^{(p,1)} & \mathbf{A}_{1,0}^{(p,1)} & \mathbf{A}_{1,1}^{(p,1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{m_p,s}^{(p,m_p)} & \mathbf{A}_{m_p,0}^{(p,m_p)} & 0 & \cdots & \mathbf{A}_{m_p,m_p}^{(p,m_p)} \end{bmatrix},$$

$$\mathbf{K}_R^{(p)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_R^{(p,1)} & -\mathbf{C}_R^{(p,1)} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_R^{(p,1)} & \mathbf{C}_R^{(p,1)} + \mathbf{C}_R^{(p,2)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_R^{(p,m_p)} \end{bmatrix},$$

$$\mathbf{G}^{(p)} = \begin{bmatrix} \sum_{i=0}^{m_p} \mathbf{G}_s^{(p,i)} \\ \sum_{i=0}^{m_p} \mathbf{G}_0^{(p,i)} \\ \mathbf{G}_1^{(p,1)} \\ \vdots \\ \mathbf{G}_{m_p}^{(p,m_p)} \end{bmatrix}, \mathbf{e}^{(p)} = \begin{bmatrix} \sum_{i=0}^{m_p} \mathbf{e}_s^{(p,i)} \\ \sum_{i=0}^{m_p} \mathbf{e}_0^{(p,i)} \\ \mathbf{e}_1^{(p,1)} \\ \vdots \\ \mathbf{e}_{m_p}^{(p,m_p)} \end{bmatrix}, \mathbf{S}^{(p)} = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{S}}^{(p,0)} \\ \tilde{\mathbf{S}}^{(p,1)} \\ \vdots \\ \tilde{\mathbf{S}}^{(p,m_p)} \end{bmatrix}, \ \mathbf{Q}^{(p)} = \begin{bmatrix} \sum_{i=0}^{m_p} \mathbf{Q}_s^{(p,i)} \\ \sum_{i=0}^{m_p} \mathbf{Q}_0^{(p,i)} \\ \mathbf{Q}_1^{(p,1)} \\ \vdots \\ \mathbf{Q}_i^{(p,m_p)} \end{bmatrix}.$$

A remark is due that the matrices $\mathbf{A}^{(p)}$ and $\mathbf{K}_R^{(p)}$ contain many zeroes. This fact may be leveraged in an implementation of the algorithm on a computer. The equations of motion of the system's links from 1 to p, forming a kinematic chain, may be generated in the way described in section 5.4. The equations for a rigid link may be obtained as a special case of a flexible link taking $m_p = 0$. The rigid link may then be treated as RFE 0.

When a link p follows a flexible link in a kinematic chain, the model includes a connection between the last RFE of the flexible link s and the next link (namely, with RFE 0 of the next link). If linear oscillations are considered, i.e. the transformation matrix for the RFE i of the flexible link takes the form (8.18), the matrix of masses $\mathbf{A}^{(p)}$ is a diagonal matrix in the fragment from RFE 1 to RFE m_p of the link p. Calculations are considerably simples when this fact is used in the integration of the equations (8.34). Additionally, the stiffness matrix $\mathbf{K}_R^{(p)}$ is a block-tridiagonal matrix, which is also helpful in solving the equations of motion. A product of matrices with constant coefficients may be distinguished in the vector $\mathbf{S}^{(p,f)} = \left[\tilde{\mathbf{S}}^{(p,1)} \quad \ldots \quad \tilde{\mathbf{S}}^{(p,m_p)}\right]^T$ in the linear case [Wojnarowski J., Adamiec-Wójcik I., 2005], thus assuming:

$$\mathbf{S}^{(p,f)} = \mathbf{K}_T^{(p,f)} \tilde{\mathbf{q}}^{(p,f)} + \mathbf{S}_c^{(p,f)}, \tag{8.35}$$

where $\mathbf{K}_T^{(p,f)}, \mathbf{S}_c^{(p,f)}$ – a matrix and a vector with constant coefficients,

$$\tilde{\mathbf{q}}^{(p,f)} = \left[\tilde{\mathbf{q}}^{(p,1)T} \quad \cdots \quad \tilde{\mathbf{q}}^{(p,\tilde{n}_p)T}\right]^T.$$

The presented model includes all possible displacements of the RFEs into which a flexible link is divided. If just one type of flexibility (e.g. to bending in one plane or torsion) is dominant in the link, models with fewer degrees of freedom of the RFEs may be easily obtained as a special case of the given formulas by appropriately fixing the vector of generalized coordinates of the rigid element.

8.2 Modification of the Rigid Finite Element Method

The classical rigid finite element method enables taking into account arbitrary displacements of finite elements and therefore analysis of the following deformations: lateral, longitudinal, rotational and shear. The displacements of each element are considered relative to the reference coordinate system attached to RFE 0. In this section**,** a modification of the rigid finite element method is presented which also has applications to discretisation of flexible beam links. In the modification only lateral and rotational deformations are assumed**,** and displacements of each RFE are defined relative to its preceding RFE. The method is presented in [Wojciech S., 1984] for planar systems and in the papers [Wojciech S., 1990], [Adamiec-Wójcik I., 1992], [Adamiec-Wójcik I., 1993] and [Adamiec-Wójcik I., 2003] as well as in [Wittbrodt E., et al., 2006] for spatial systems. The modification allows large deflections of flexible links to be analysed.

8.2.1 Generalized Coordinates: Transformation Matrices

Discretisation of a flexible beam link is performed in the same way as in the classical rigid finite element method, i.e. with primary and secondary divisions (Fig. 8.4). To each rigid finite element, a coordinate system is attached whose origin is located in its preceding spring-damping element (Fig. 8.5).

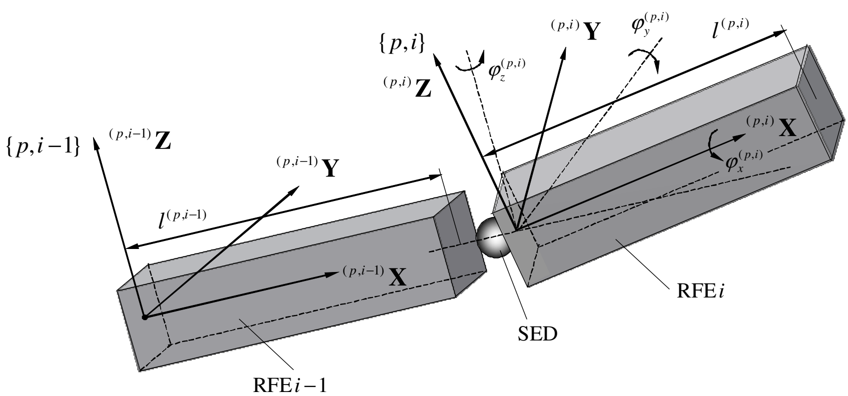

Fig. 8.5. Generalized coordinates of the *i*-th RFE and local coordinate systems

The generalized coordinates describing the position of the *i*-th RFE relative to the preceding *i*-1-th RFE of the flexible link *p* are the angles $\varphi_x^{(p,i)}$, $\varphi_y^{(p,i)}$, $\varphi_z^{(p,i)}$, of which the latter two correspond to bending and the first one to torsion of the element. Upon discretisation, the flexible link may be viewed as a system of rigid links connected by joints of the 3rd class. Similarly to the model formed using classical finite elements, a rigid link is a special case of a flexible link ($m_p = 0$). The transformation matrix $\tilde{\mathbf{T}}^{(p,i)}$ from the system $\{p,i\}$ attached to RFE i ($i = 1,...,m_p$) to the system $\{p,i-1\}$ in the nonlinear model, i.e. allowing the angles $\varphi_\alpha^{(p,i)}$ for $\alpha \in \{x, y, z\}$ to be large, takes the form:

$$\tilde{\mathbf{T}}^{(p,i)} = \begin{bmatrix} c_z^{(p,i)}c_y^{(p,i)} & c_z^{(p,i)}s_y^{(p,i)}s_x^{(p,i)} - s_z^{(p,i)}c_x^{(p,i)} & c_z^{(p,i)}s_y^{(p,i)}c_x^{(p,i)} + s_z^{(p,i)}s_x^{(p,i)} & l^{(p,i-1)} \\ s_z^{(p,i)}c_y^{(p,i)} & s_z^{(p,i)}s_y^{(p,i)}s_x^{(p,i)} + c_z^{(p,i)}c_x^{(p,i)} & s_z^{(p,i)}s_y^{(p,i)}c_x^{(p,i)} - c_z^{(p,i)}s_x^{(p,i)} & 0 \\ -s_y^{(p,i)} & c_y^{(p,i)}s_x^{(p,i)} & c_y^{(p,i)}c_x^{(p,i)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (8.36.1)$$

where $c_\alpha^{(p,i)} = \cos \varphi_\alpha^{(p,i)}$, $s_\alpha^{(p,i)} = \sin \varphi_\alpha^{(p,i)}$ for $\alpha \in \{x, y, z\}$,
$l^{(p,i-1)}$ – length of RFE i–1 of the link p.

When the angles $\varphi_\alpha^{(p,i)}$ are small, the following may be assumed:

$$\tilde{\mathbf{T}}^{(p,i)} = \begin{bmatrix} 1 & -\varphi_z^{(p,i)} & \varphi_y^{(p,i)} & l^{(p,i-1)} \\ \varphi_z^{(p,i)} & 1 & -\varphi_x^{(p,i)} & 0 \\ -\varphi_y^{(p,i)} & \varphi_x^{(p,i)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (8.36.2)$$

When all three types of oscillations are considered (rotational and lateral in two planes), the generalized coordinates describing the motion of the i-th RFE of the link p relative to its predecessor may be written as components of the following vector:

$$\tilde{\mathbf{q}}^{(p,i)} = \begin{bmatrix} \phi_x^{(p,i)} & \phi_y^{(p,i)} & \phi_z^{(p,i)} \end{bmatrix}^T \quad \text{for } i = 1,\ldots,m_p \,. \quad (8.37)$$

RFE 0 is treated like in the classical rigid finite element method, its generalized coordinates being given by the vector $\tilde{\mathbf{q}}^{(p,0)}$. A series of intermediate transformations yields the transformation matrix from the local system $\{p,i\}$ (i=1,...,m_p) to the global system:

$$\mathbf{T}^{(p,i)} = \mathbf{T}^{(p,i-1)}\tilde{\mathbf{T}}^{(p,i)}, \quad (8.38)$$

where $\mathbf{T}^{(p,i-1)} = \mathbf{T}^{(p,0)}\tilde{\mathbf{T}}^{(p,1)}\ldots\tilde{\mathbf{T}}^{(p,i-1)}$

$\mathbf{T}^{(p,0)}$ – matrix given by (8.10.2),

$\tilde{\mathbf{T}}^{(p,i)}$ – matrix defined by the formula (8.36) for i=1,...,m_p.

The kinetic energy, the potential energy of gravity forces and the generalized forces caused by external forces and moments thereof acting on the flexible link are calculated as in section 5.3.

An important property of formula (8.38) is that the matrix $\mathbf{T}^{(p,i)}$ depends not only on the vector $\mathbf{q}^{(s)}$ of generalized coordinates of the link which precedes the flexible link, but also on all the RFEs preceding the RFE i. Defining the vectors:

$$\mathbf{q}^{(p,i)} = \begin{bmatrix} \mathbf{q}^{(s)} \\ \tilde{\mathbf{q}}^{(p,0)} \\ \tilde{\mathbf{q}}^{(p,1)} \\ \vdots \\ \tilde{\mathbf{q}}^{(p,i-1)} \\ \tilde{\mathbf{q}}^{(p,i)} \end{bmatrix}, \tag{8.39}$$

and taking (8.2) into account allows us to write:

$$\mathbf{q}^{(p)} = \begin{bmatrix} \mathbf{q}^{(s)} \\ \tilde{\mathbf{q}}^{(p,0)} \\ \tilde{\mathbf{q}}^{(p,f)} \end{bmatrix}, \tag{8.40}$$

where $\tilde{\mathbf{q}}^{(p,f)} = \begin{bmatrix} \tilde{\mathbf{q}}^{(p,1)T} & \cdots & \tilde{\mathbf{q}}^{(p,m_p)T} \end{bmatrix}^T$.

8.2.2 Kinetic Energy: Lagrange Operators

From (8.38) it follows:

$$\mathbf{T}^{(p,i)} = \mathbf{T}^{(p,0)}\,\hat{\mathbf{T}}^{(p,i)}\left(\tilde{\mathbf{q}}^{(p,1)},\ldots,\tilde{\mathbf{q}}^{(p,i)}\right), \tag{8.41}$$

where $\hat{\mathbf{T}}^{(p,i)} = \prod_{j=1}^{i} \tilde{\mathbf{T}}^{(p,j)}\left(\tilde{\mathbf{q}}^{(p,j)}\right)$.

Since the kinetic energy of the link p may be written as:

$$E_p = \sum_{i=0}^{m_p} E_{p,i}\,, \tag{8.42}$$

where $E_{p,i} = \mathrm{tr}\left\{\dot{\mathbf{T}}^{(p,i)}\mathbf{H}^{(p,i)}\dot{\mathbf{T}}^{(p,i)T}\right\}$, calculations analogous to those presented in chapter 5 give:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p,i)}}\left(E_{p,i}\right)=\begin{bmatrix}\mathbf{A}_{s,s}^{(p,i)} & \mathbf{A}_{s,0}^{(p,i)} & \mathbf{A}_{s,1}^{(p,i)} & \cdots & \mathbf{A}_{s,j}^{(p,i)} & \cdots & \mathbf{A}_{s,i}^{(p,i)}\\ \mathbf{A}_{0,s}^{(p,i)} & \mathbf{A}_{0,0}^{(p,i)} & \mathbf{A}_{0,1}^{(p,i)} & \cdots & \mathbf{A}_{0,j}^{(p,i)} & \cdots & \mathbf{A}_{0,i}^{(p,i)}\\ \mathbf{A}_{1,s}^{(p,i)} & \mathbf{A}_{1,0}^{(p,i)} & \mathbf{A}_{1,1}^{(p,i)} & \cdots & \mathbf{A}_{1,j}^{(p,i)} & \cdots & \mathbf{A}_{1,i}^{(p,i)}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ \mathbf{A}_{i,s}^{(p,i)} & \mathbf{A}_{i,0}^{(p,i)} & \mathbf{A}_{i,1}^{(p,i)} & \cdots & \mathbf{A}_{i,j}^{(p,i)} & \cdots & \mathbf{A}_{i,i}^{(p,i)}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}^{(s)}\\ \ddot{\tilde{\mathbf{q}}}^{(p,0)}\\ \ddot{\tilde{\mathbf{q}}}^{(p,1)}\\ \vdots\\ \ddot{\tilde{\mathbf{q}}}^{(p,j)}\\ \vdots\\ \ddot{\tilde{\mathbf{q}}}^{(p,i)}\end{bmatrix}+\begin{bmatrix}\mathbf{e}_{s}^{(p,i)}\\ \mathbf{e}_{0}^{(p,i)}\\ \mathbf{e}_{1}^{(p,i)}\\ \vdots\\ \mathbf{e}_{j}^{(p,i)}\\ \vdots\\ \mathbf{e}_{i}^{(p,i)}\end{bmatrix}= \tag{8.43}$$

$$=\mathbf{A}^{(p,i)}\ddot{\mathbf{q}}^{(p,i)}+\mathbf{e}^{(p,i)},$$

where $\mathbf{A}_{\alpha,\beta}^{(p,i)}$ – appropriate blocks of the matrix $\mathbf{A}^{(p,i)}$,

$$\mathbf{A}^{(p,i)}=\left(a_{k,j}^{(p,i)}\right)_{k,j=1,\dots,n_{p,i}}=\operatorname{tr}\left\{\mathbf{T}_{k}^{(p,i)}\,\mathbf{H}^{(p,i)}\,\mathbf{T}_{j}^{(p,i)T}\right\}$$

$$\mathbf{e}^{(p,i)}=\left(e_{k}^{(p,i)}\right)_{k=1,\dots,n_{p,i}}=\sum_{j=1}^{n_{p,i}}\sum_{l=1}^{n_{p,i}}\operatorname{tr}\left\{\mathbf{T}_{k}^{(p,i)}\,\mathbf{H}^{(p,i)}\mathbf{T}_{j,l}^{(p,i)}\right\}\dot{q}_{j}^{(p,i)}\,\dot{q}_{l}^{(p,i)}+$$

$$+\operatorname{tr}\left\{\mathbf{T}_{k}^{(p,i)}\mathbf{H}^{(p,i)}\left[{}_{A}^{0}\ddot{\mathbf{T}}\,\overline{\mathbf{T}}^{(p,i)}+2\,{}_{A}^{0}\dot{\mathbf{T}}\,\dot{\overline{\mathbf{T}}}^{(p,i)}\right]^{T}\right\},$$

$$\overline{\mathbf{T}}^{(p,i)}={}_{s}^{A}\mathbf{T}\left(\mathbf{q}^{(s)}\right)\prod_{j=0}^{i}\tilde{\mathbf{T}}^{(p,j)},$$

$$n_{p,i}=n_{s}+\tilde{n}_{p,0}+3i.$$

As before, the gravity forces of the RFEs and their derivatives may be put in the form:

$$V_{p,i}^{g}=m^{(p,i)}g\,\boldsymbol{\theta}_{3}\mathbf{T}^{(p,i)}\tilde{\mathbf{r}}_{C}^{(p,i)}, \tag{8.44}$$

and further:

$$\frac{\partial V_{p,i}^{g}}{\partial\mathbf{q}^{(p,i)}}=\begin{bmatrix}\mathbf{G}_{s}^{(p,i)}\\ \mathbf{G}_{0}^{(p,i)}\\ \vdots\\ \mathbf{G}_{i}^{(p,i)}\end{bmatrix}, \tag{8.45}$$

where $\mathbf{G}_{\alpha}^{(p,i)}$ – appropriate blocks of the vector $\mathbf{G}^{(p,i)}$,

$$\mathbf{G}^{(p,i)}=\left(g_{k}^{(p,i)}\right)_{k=1,\dots,n_{p,i}},$$

$$g_{k}^{(p,i)}=m^{(p,i)}g\,\boldsymbol{\theta}_{3}\mathbf{T}_{k}^{(p,i)}\tilde{\mathbf{r}}_{C}^{(p,k)}.$$

The generalized forces may be similarly presented. If $\tilde{\mathbf{F}}^{(p,i)}$ and $\tilde{\mathbf{M}}^{(p,i)}$ specified in (8.30) act on RFE i of the link, then:

$$\mathbf{Q}^{(p,i)} = \begin{bmatrix} \mathbf{Q}_s^{(p,i)} \\ \mathbf{Q}_0^{(p,i)} \\ \vdots \\ \mathbf{Q}_i^{(p,i)} \end{bmatrix}, \tag{8.46}$$

where $\mathbf{Q}_\alpha^{(p,i)}$ – appropriate blocks of the vector $\mathbf{Q}^{(p,i)}$,

$$\mathbf{Q}^{(p,i)} = \left(\mathbf{Q}_k^{(p,i)}\right)_{k=1,\ldots,n_{p,i}} = \tilde{\mathbf{F}}^{(p,i)T}\mathbf{T}^{(p,i)T}\,\mathbf{T}_k^{(p,i)}\,\tilde{\mathbf{r}}^{(p,i)} + \tilde{M}_1^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,3}\left(\mathbf{T}_k^{(p,i)}\right)_{j,2} +$$
$$+ \tilde{M}_2^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,1}\left(\mathbf{T}_k^{(p,i)}\right)_{j,3} + \tilde{M}_3^{(p,i)}\sum_{j=1}^{3}\left(\mathbf{T}^{(p,i)}\right)_{j,2}\left(\mathbf{T}_k^{(p,i)}\right)_{j,1},$$

$\tilde{\mathbf{r}}^{(p,i)}$ – vector giving the coordinates of the point to which the force in the system $\{p,i\}$.

Formulation of the equations of motion further requires the determination of the elastic energy and its derivatives. The reasoning below pertains to linear physical dependencies.

8.2.3 Energy of Elastic Deformation

The potential energy of elastic deformation of an SDE of a flexible link is calculated based on the fact that the generalized coordinates specify relative angles. For the spring-damping element connecting the RFEs i–1 and i it is given by the formula:

$$V_{p,i}^{s} = \frac{1}{2}\sum_{j=1}^{3} c_{i,3+j}^{(p)}\left[\varphi_j^{(p,i)}\right]^2, \tag{8.47}$$

where $c_{i,3+j}^{(p)}$ are the appropriate coefficients of rotational stiffness defined in (8.21).

The formula (8.47) may be rewritten as:

$$V_{p,i}^{s} = \frac{1}{2}\tilde{\mathbf{q}}^{(p,i)T}\mathbf{C}^{(p,i)}\,\tilde{\mathbf{q}}^{(p,i)}, \tag{8.48}$$

where $\mathbf{C}^{(p,i)} = \begin{bmatrix} c_{i,4}^{(p)} & 0 & 0 \\ 0 & c_{i,5}^{(p)} & 0 \\ 0 & 0 & c_{i,6}^{(p)} \end{bmatrix}.$

The derivatives of the potential energy of elastic deformation relative to the generalized coordinates have the form:

$$\frac{\partial V^{s}_{(p,i)}}{\partial \tilde{\mathbf{q}}^{(p,i)}} = \mathbf{C}^{(p,i)}\tilde{\mathbf{q}}^{(p,i)} . \tag{8.49}$$

8.2.4 Equations of Motion

Whereas the kinetic energy and the potential energy of gravity forces of the link p are given by the formulas:

$$E_p = \sum_{i=0}^{m_p} E_{p,i} , \tag{8.50.1}$$

$$V_p^g = \sum_{i=0}^{m_p} V_{p,i}^g , \tag{8.50.2}$$

and taking (8.43), (8.45), (8.46) and (8.49) into account, equations of motion of the link p may be written as:

$$\mathbf{A}^{(p)}\ddot{\mathbf{q}}^{(p)} = \mathbf{f}^{(p)}, \tag{8.51.1}$$

or decomposed with blocks:

$$\begin{bmatrix} \mathbf{A}_{s,s}^{(p)} & \mathbf{A}_{s,0}^{(p)} & \mathbf{A}_{s,1}^{(p)} & \cdots & \mathbf{A}_{s,j}^{(p)} & \cdots & \mathbf{A}_{s,m_p}^{(p)} \\ \mathbf{A}_{0,s}^{(p)} & \mathbf{A}_{0,0}^{(p)} & \mathbf{A}_{0,1}^{(p)} & \cdots & \mathbf{A}_{0,j}^{(p)} & \cdots & \mathbf{A}_{0,m_p}^{(p)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \mathbf{A}_{i,s}^{(p)} & \mathbf{A}_{i,0}^{(p)} & \mathbf{A}_{i,1}^{(p)} & \cdots & \mathbf{A}_{i,j}^{(p)} & \cdots & \mathbf{A}_{i,m_p}^{(p)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \mathbf{A}_{m_p,s}^{(p)} & \mathbf{A}_{m_p,0}^{(p)} & \mathbf{A}_{m_p,1}^{(p)} & \cdots & \mathbf{A}_{m_p,j}^{(p)} & \cdots & \mathbf{A}_{m_p,m_p}^{(p)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(s)} \\ \ddot{\tilde{\mathbf{q}}}^{(p,0)} \\ \vdots \\ \ddot{\tilde{\mathbf{q}}}^{(p,j)} \\ \vdots \\ \ddot{\tilde{\mathbf{q}}}^{(p,m_p)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s^{(p)} \\ \mathbf{f}_0^{(p)} \\ \vdots \\ \mathbf{f}_i^{(p)} \\ \vdots \\ \mathbf{f}_{m_p}^{(p)} \end{bmatrix}, \tag{8.51.2}$$

where $\mathbf{A}_{s,s}^{(p)} = \sum_{i=0}^{m_p} \mathbf{A}_{s,s}^{(p,i)}$, $\mathbf{A}_{s,j}^{(p)} = \sum_{i=j}^{m_p} \mathbf{A}_{s,j}^{(p,i)}$, $\mathbf{A}_{i,s}^{(p)} = \sum_{j=i}^{m_p} \mathbf{A}_{i,s}^{(p,j)}$,

$\mathbf{A}_{i,j}^{(p)} = \sum_{l=\max\{i,j\}}^{m_p} \mathbf{A}_{i,j}^{(p,l)}$ for $i, j = 0,1,\ldots,m_p$,

$$\mathbf{f}_s^{(p)} = \sum_{i=0}^{m_p} \left[-\mathbf{e}_s^{(p,i)} - \mathbf{G}_s^{(p,i)} + \mathbf{Q}_s^{(p,i)} \right],$$

$$\mathbf{f}_0^{(p)} = \sum_{i=0}^{m_p} \left[-\mathbf{e}_0^{(p,i)} - \mathbf{G}_0^{(p,i)} + \mathbf{Q}_0^{(p,i)} \right],$$

$$\mathbf{f}_i^{(p)} = \sum_{j=i}^{m_p} \left[-\mathbf{e}_i^{(p,j)} - \mathbf{G}_i^{(p,j)} + \mathbf{Q}_i^{(p,j)} \right] - \mathbf{C}^{(p,i)} \tilde{\mathbf{q}}^{(p,i)} \qquad \text{for } i = 1, \ldots, m_p .$$

8.3 Modelling of Planar System

By means of the rigid finite element method, an arbitrary description of the geometry of a system may be given. The traditional approach may be used instead of homogeneous transformations and joint coordinates proposed in earlier chapters. In the present chapter an example is given of modelling a planar system using the rigid finite element method in its modified form and a classical description of the system's geometry.

8.3.1 Determination of Generalized Coordinates

In Fig. 8.6, a sample decomposition of a k-th flexible links into n_k+1 rigid finite elements connected at points $A_1^{(k)}, \ldots, A_{n_k}^{(k)}$ by n_k massless spring-damping elements is presented. Since the problem considered is plane, in the relative motion each RFE enjoys one degree of freedom which is the inclination angle of the axis ${}^{(k,i)}\mathbf{X}$ of the RFE i to the axis **X** of the global system (Fig. 8.7). Further analysis assumes the angles to be measured relative to the global system.

The position of the link k being discredited is therefore described by n_k+3 coordinates. Two of them, (x_k, y_k), are the coordinates of the point $A^{(k)}$ which equals the point $A_0^{(k)}$ of the first RFE (usually being one of the nodes of the whole mechanism). The remaining coordinates are the angles already mentioned which will be denoted $\varphi^{(k,0)}, \ldots, \varphi^{(k,n_k)}$. As a noteworthy observation, these angles correspond to those from (8.37) – $\varphi_y^{(p,i)}$. Thus, the vector of coordinates of the link k may be defined:

$$\mathbf{q}^{(k)} = \left[x_k, y_k, \varphi^{(k,0)}, \varphi^{(k,1)}, \ldots, \varphi^{(k,n_k)} \right]^T . \tag{8.52}$$

Following [Wojciech S., 1984], [Szczotka M., 2011b], when introducing denotations for coordinates of the point $A_i^{(k)}\left(a_{i,i}^{(k)}, b_{i,i}^{(k)}\right)$ and $A_{i+1}^{(k)}\left(a_{i,i+1}^{(k)}, b_{i,i+1}^{(k)}\right)$ in the local coordinate system $0_i^{(k)} \xi_i^{(k)} \eta_i^{(k)}$ attached to the centre of mass of the

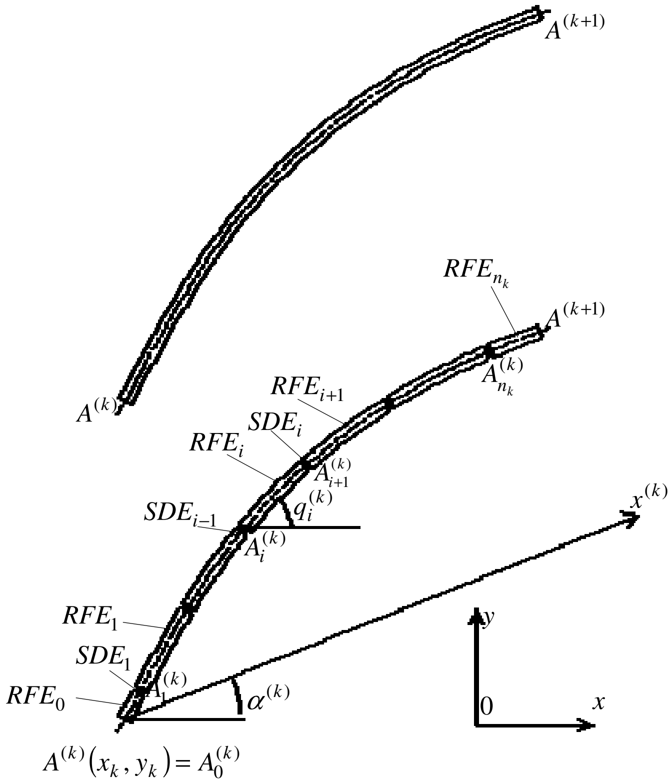

Fig. 8.6. Decomposition of a flexible link into rigid finite elements

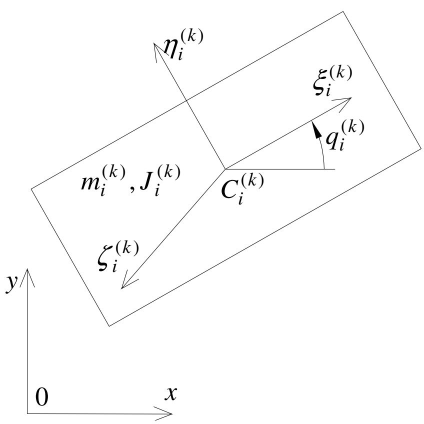

Fig. 8.7. Inclination angles of an RFE to the axes of a stationary coordinate system

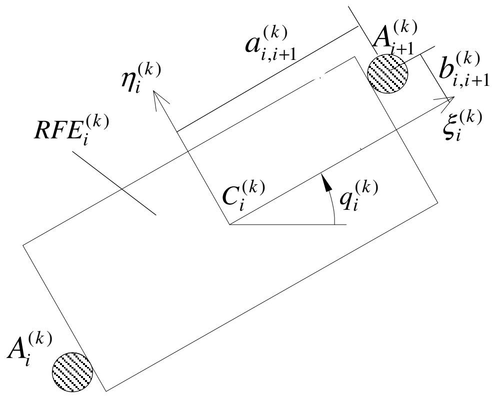

Fig. 8.8. Coordinates of a point in the local coordinate system

RFE i (Fig. 8.8) we may write the coordinates of the centre of mass of the RFE i in the coordinate system $\{A\}$ as follows:

$$\begin{aligned} x_{c_i}^{(k)} &= x_k + \sum_{j=0}^{i} h_{i,j}^{(k)} \cos\left(\varphi^{(k,j)} + \varphi_{i,j}^{(k)}\right) \\ y_{c_i}^{(k)} &= y_k + \sum_{j=0}^{i} h_{i,j}^{(k)} \sin\left(\varphi^{(k,j)} + \varphi_{i,j}^{(k)}\right) \end{aligned}, \tag{8.53}$$

where $$h_{i,j}^{(k)} = \begin{cases} \sqrt{\left(-a_{j,j}^{(k)} + a_{j,j+1}^{(k)}\right)^2 + \left(-b_{j,j}^{(k)} + b_{j,j+1}^{(k)}\right)^2} & \text{when} \quad j < i \\ \sqrt{\left(a_{i,i}^{(k)}\right)^2 + \left(b_{i,i}^{(k)}\right)^2} & \text{when} \quad j = i \end{cases},$$

$$\varphi_{i,j}^{(k)} = \begin{cases} \operatorname{arctg} \dfrac{b_{j,j+1}^{(k)} - b_{j,j}^{(k)}}{a_{j,j+1}^{(k)} - a_{j,j}^{(k)}} & \text{when} \quad j < i \\ \operatorname{arctg} \dfrac{b_{i,i}^{(k)}}{a_{i,i}^{(k)}} + \pi & \text{when} \quad j = i \end{cases}.$$

8.3.2 *Equations of Motion of a Link*

The kinetic energy of the i-th RFE equals:

$$E_{k,i} = \frac{1}{2} m_i^{(k)} \left\{ \left[\dot{x}_{c_i}^{(k)}\right]^2 + \left[\dot{y}_{c_i}^{(k)}\right]^2 \right\} + \frac{1}{2} J_i^{(k)} \left[\varphi^{(k,j)}\right]^2, \tag{8.54}$$

where $m_i^{(k)}$ – mass of the i-th RFE,

$J_i^{(k)}$ – moment of inertia of the i-th RFE relative to the central axis perpendicular to the plane **XY**,

and then the following sum gives the energy of the entire link k:

$$E_k = \sum_{i=0}^{n_k} E_{k,i} \; . \tag{8.55}$$

Using (8.53), (8.54) and the following identities, which may be proved by induction:

$$\sum_{i=0}^{n_k} m_i^{(k)} \sum_{j=0}^{i} \dot{\varphi}^{(k,j)} b_{i,j}^{(k)} = \sum_{i=0}^{n_k} \dot{\varphi}^{(k,i)} \sum_{j=0}^{n_k} m_j^{(k)} b_{j,i}^{(k)} \, , \tag{8.56.1}$$

$$\sum_{i=0}^{n_k} m_i^{(k)} \left[\sum_{j=0}^{i} \dot{\varphi}^{(k,j)} b_{i,j}^{(k)} \right]^2 = \sum_{i=0}^{n_k} \dot{\varphi}^{(k,i)} \sum_{j=0}^{n_k} \dot{\varphi}^{(k,j)} \sum_{l=\max\{i,j\}}^{n_k} m_l^{(k)} b_{l,i}^{(k)} b_{l,j}^{(k)} \, , \tag{8.56.2}$$

we may express the kinetic energy of the link k as:

$$E_k = \frac{1}{2} \left[\dot{\mathbf{q}}^{(k)} \right]^T \mathbf{A}^{(k)} \dot{\mathbf{q}}^{(k)} , \tag{8.57}$$

where $\mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{M}_1^{(k)} & \mathbf{M}_2^{(k)} \\ \left[\mathbf{M}_2^{(k)}\right]^T & \mathbf{M}^{(k)} \end{bmatrix}$,

$$\mathbf{M}_1^{(k)} = \begin{bmatrix} m^{(k)} & 0 \\ 0 & m^{(k)} \end{bmatrix},$$

$$\mathbf{M}_2^{(k)} = \begin{bmatrix} -\overline{A}_0^{(k)} \sin\left(\varphi^{(k,0)} + \alpha_0^{(k)}\right) \ldots - \overline{A}_i^{(k)} \sin\left(\varphi^{(k,i)} + \alpha_i^{(k)}\right) \ldots - \overline{A}_{n_k}^{(k)} \sin\left(\varphi^{(k,n_k)} + \alpha_{n_k}^{(k)}\right) \\ \overline{A}_0^{(k)} \cos\left(\varphi^{(k,0)} + \alpha_0^{(k)}\right) \ldots + \overline{A}_i^{(k)} \cos\left(\varphi^{(k,i)} + \alpha_i^{(k)}\right) \ldots + \overline{A}_{n_k}^{(k)} \cos\left(\varphi^{(k,n_k)} + \alpha_{n_k}^{(k)}\right) \end{bmatrix},$$

$$\mathbf{M}^{(k)} = \left(m_{i,j}^{(k)} \right)_{i,j=0}^{n_k}, \qquad m_{i,j}^{(k)} = \overline{A}_{i,j}^{(k)} \cos\left(\varphi^{(k,i)} - \varphi^{(k,j)} - \alpha_{i,j}^{(k)}\right),$$

$$\overline{A}_i^{(k)} = \sqrt{\left[\overline{a}_i^{(k)}\right]^2 + \left[\overline{b}_i^{(k)}\right]^2} \, , \qquad \alpha_i^{(k)} = \operatorname{arctg} \frac{\overline{b}_i^{(k)}}{\overline{a}_i^{(k)}} \, ,$$

$$\overline{a}_i^{(k)} = \sum_{j=0}^{n_k} m_j^{(k)} h_{j,i} \cos \varphi_{j,i}^{(k)} \, , \qquad \overline{b}_i^{(k)} = \sum_{j=0}^{n_k} m_j^{(k)} h_{j,i} \sin \varphi_{j,i}^{(k)} \, ,$$

$$\overline{A}_{i,j}^{(k)} = \sqrt{\left[\overline{a}_{i,j}^{(k)}\right]^2 + \left[\overline{b}_{i,j}^{(k)}\right]^2} \, , \qquad \alpha_{i,j}^{(k)} = \operatorname{arctg} \frac{\overline{b}_{i,j}^{(k)}}{\overline{a}_{i,j}^{(k)}} \, ,$$

$$\overline{a}_{i,j}^{(k)} = \sum_{l=\max\{i,j\}}^{n_k} m_l^{(k)} h_{l,i} h_{l,j} \cos\left(\varphi_{l,i}^{(k)} - \varphi_{l,j}^{(k)}\right) + \delta_{i,j} J_i^{(k)},$$

$$\overline{b}_{i,j}^{(k)} = \sum_{l=\max\{i,j\}}^{n_k} m_l^{(k)} h_{l,i} h_{l,j} \sin\left(\varphi_{l,i}^{(k)} - \varphi_{l,j}^{(k)}\right),$$

$\delta_{i,j}$ – Kronecker delta,

$$m^{(k)} = \sum_{i=0}^{n_k} m_i^{(k)} .$$

This enables transforming the Lagrange equation of the link k to:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(k)}} + \frac{\partial D^{(k)}}{\partial \dot{\mathbf{q}}^{(k)}} + \frac{\partial V_k^g}{\partial \mathbf{q}^{(k)}} + \frac{\partial V_k^s}{\partial \mathbf{q}^{(k)}} = \mathbf{Q}^{(k)}, \tag{8.58}$$

where $\boldsymbol{\varepsilon}_{\mathbf{q}^{(k)}} = \begin{bmatrix} \varepsilon_{x_k} & \varepsilon_{y_k} & \varepsilon_{\varphi^{(k,1)}} & \dots & \varepsilon_{\varphi^{(k,n_k)}} \end{bmatrix}^T$,

$D^{(k)}$ – dissipation function of the k link's energy,

V_k^g, V_k^s – potential energy of deformation and gravity forces of the link k,

$\mathbf{Q}^{(k)}$ – vector of generalized forces.

By taking into account (8.57), the following is obtained:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(k)}} = \mathbf{A}^{(k)} \ddot{\mathbf{q}}^{(k)} + \overline{\mathbf{B}}^{(k)} \dot{\mathbf{q}}^{(k)}, \tag{8.59}$$

where $\mathbf{A}^{(k)}$ – defined in (8.57),

$\overline{\mathbf{B}}^{(k)}$ – matrix with the following elements:

$$\overline{b}_{1,i+3}^{(k)} = -\dot{\varphi}^{(k,i)} \overline{A}_i^{(k)} \cos\left(\varphi^{(k,i)} + \alpha_i^{(k)}\right), \qquad \overline{b}_{i+3,1}^{(k)} = 0 ,$$

$$\overline{b}_{2,i+3}^{(k)} = -\dot{\varphi}^{(k,i)} \overline{A}_i^{(k)} \sin\left(\varphi^{(k,i)} + \alpha_i^{(k)}\right), \qquad \overline{b}_{i+3,2}^{(k)} = 0 ,$$

$$\overline{b}_{3+j,i+3}^{(k)} = -\dot{\varphi}^{(k,i)} \overline{A}_{i,j}^{(k)} \sin\left(\varphi^{(k,i)} - \varphi^{(k,j)} - \alpha_{i,j}^{(k)}\right) \qquad i, j = 0,1,\dots,n_k ,$$

$$\overline{b}_{1,1}^{(k)} = \overline{b}_{1,2}^{(k)} = \overline{b}_{2,1}^{(k)} = \overline{b}_{2,2}^{(k)} = 0 .$$

The following formula gives the potential energy of gravity forces:

$$V_k^g = \sum_{i=0}^{n_k} m_i^{(k)} \left[y_{c_i}^{(k)} - y_{c_i,0}^{(k)} \right], \tag{8.60}$$

whereby $y_{c_i,0}^{(k)} = \text{const}$.

It then follows from (8.53):

$$\frac{\partial V_k^g}{\partial \mathbf{q}^{(k)}} = \begin{bmatrix} 0 \\ m^{(k)} \\ m_0^{(k)} \sum_{i=0}^{n_k} h_{i,0}^{(k)} \cos\left(\varphi^{(k,0)} + \varphi_{i,0}^{(k)}\right) \\ \vdots \\ m_j^{(k)} \sum_{i=j}^{n_k} h_{i,1}^{(k)} \cos\left(\varphi^{(k,j)} + \varphi_{i,j}^{(k)}\right) \\ \vdots \\ m_{n_k}^{(k)} h_{n_k,n_k}^{(k)} \cos\left(\varphi^{(k,n_k)} + \varphi_{n_k,n_k}^{(k)}\right) \end{bmatrix}. \tag{8.61}$$

The expression giving the energy V_k^s of elastic deformation and the dissipation function $D^{(k)}$ of the k link's energy depend on the form of assumed physical dependencies between the deformations and stresses characteristic to the spring-damping elements. In the case of linear Kelvin-Voigt model, counting V_k^s and $D^{(k)}$ as components due to deformation of the spring-damping elements (Fig. 8.6), it may be written:

$$\frac{\partial V_k^s}{\partial \mathbf{q}^{(k)}} = \mathbf{C}^{(k)} \mathbf{q}^{(k)}, \tag{8.62}$$

$$\frac{\partial D^{(k)}}{\partial \mathbf{q}^{(k)}} = \mathbf{D}^{(k)} \dot{\mathbf{q}}^{(k)}, \tag{8.63}$$

where $\mathbf{C}^{(k)}$ and $\mathbf{D}^{(k)}$ are the stiffness and damping matrix, respectively, whose coefficients are constant and dependent on the geometry of the link and the constants determining the stiffness and the damping of the SDE.

The vector of generalized forces $\mathbf{Q}^{(k)}$ is formed by the values of forces caused by external loads and reactions in the joints. If load shown in Fig. 8.9 is applied to the i-th RFE, the components of the generalized forces due to the loads take the form:

$$\begin{aligned} & Q_{i,1}^{(k)} = P_{ix}^{(k)} \\ & Q_{i,2}^{(k)} = P_{iy}^{(k)} \\ & Q_{i,j+2}^{(k)} = -\left[h_{i,j}^{(k)} \sin\left(\phi^{(k,j)} + \phi_{i,j}^{(k)}\right) + \delta_{i,j} r_i^{(k)} \sin\left(\phi^{(k,j)} + \gamma_i^{(k)}\right)\right] P_{ix}^{(k)} + \\ & \qquad + \left[h_{i,j}^{(k)} \cos\left(\phi^{(k,j)} + \phi_{i,j}^{(k)}\right) + \delta_{i,j} r_i^{(k)} \cos\left(\phi^{(k,j)} + \gamma_i^{(k)}\right)\right] P_{iy}^{(k)} + \delta_{i,j} M_i^{(k)} \\ & \qquad\qquad \text{for } j = 0,1,\ldots i \\ & Q_{i,j+2}^{(k)} = 0 \qquad\qquad \text{for } j > i \end{aligned} \tag{8.64}$$

where $h_{i,j}^{(k)}, \varphi_{i,j}^{(k)}$ – defined as in formula (8.53),

$r_i^{(k)}, \gamma_i^{(k)}$ – polar coordinates (relative to the middle $C_i^{(k)}$ of the i-th RFE of the link) of the point to which the load is applied.

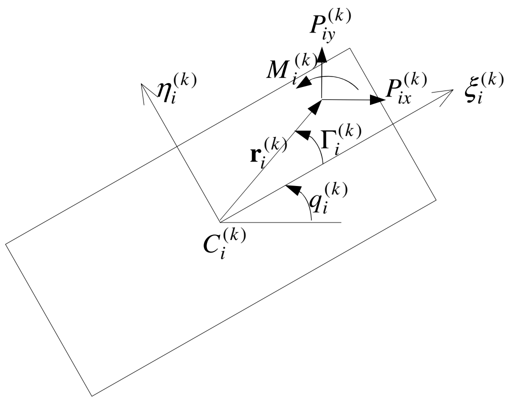

Fig. 8.9. Coordinates of the point of application of external load in the local coordinate system of the i-th RFE of the k-th link

Summing:

$$Q_i^{(k)} = \sum_{j=0}^{n_k} Q_{i,j}^{(k)}, \tag{8.65}$$

yields the components of the vector of generalized forces. Furthermore, with (8.58), (8.59), (8.61), (8.62), (8.63) the equations of motion may be rewritten:

$$\mathbf{A}^{(k)}\ddot{\mathbf{q}}^{(k)} + \mathbf{B}^{(k)}\dot{\mathbf{q}}^{(k)} + \mathbf{C}^{(k)}\mathbf{q}^{(k)} = \mathbf{Q}^{(k)} - \frac{\partial V_k^g}{\partial \mathbf{q}^{(k)}}, \tag{8.66}$$

where $\mathbf{B}^{(k)} = \overline{\mathbf{B}}^{(k)} + \mathbf{D}^{(k)}$.

In the vector of generalized forces $\mathbf{Q}^{(k)}$ both the reactions of constraints and known loads are included. For some operations, it is convenient to have this vector written as:

$$\mathbf{Q}^{(k)} = -\mathbf{K}^{(k)}\mathbf{R}^{(k)} + \mathbf{Q}_P^{(k)}, \tag{8.67}$$

where $\mathbf{K}^{(k)}$ – matrix with n_k+3 rows whose elements depend on $\mathbf{q}^{(k)}$,

$\mathbf{R}^{(k)}$ – vector of reaction (its elements are the components of reaction in revolute and translational connections and forces occurring therein as well as moments of undeveloped friction),

$\mathbf{Q}_P^{(k)}$ – vector of generalized forces due to known external loads, reactions in flexible connections and forces of developed dry friction and viscous friction.

Given the form of the vector (8.67), the equations of motion of the link k may be written as follows:

$$\mathbf{A}^{(k)}\ddot{\mathbf{q}}^{(k)} + \mathbf{B}^{(k)}\dot{\mathbf{q}}^{(k)} + \mathbf{C}^{(k)}\mathbf{q}^{(k)} + \mathbf{K}^{(k)}\mathbf{R}^{(k)} = \mathbf{F}^{(k)}, \tag{8.68}$$

where $\mathbf{F}^{(k)} = -\dfrac{\partial V_k^g}{\partial \mathbf{q}^{(k)}} + \mathbf{Q}_P^{(k)}$.

The elements of the matrices $\mathbf{A}^{(k)}$, $\mathbf{C}^{(k)}, \mathbf{K}^{(k)}$ depend on $\mathbf{q}^{(k)}$ and the elements of the matrix $\mathbf{B}^{(k)}$ and the vector $\mathbf{F}^{(k)}$ depend on $\mathbf{q}^{(k)}$ and $\dot{\mathbf{q}}^{(k)}$. In the special case of n_k=0, the concerned link is modelled as rigid.

Motion of the base $\{A\}$ may be taken into consideration by assuming it to be the RFE 0 whose motion is described by:

$$\begin{aligned} x_0 &= x_{org}^{(A)}(t) \\ y_0 &= y_{org}^{(A)}(t) \\ \varphi_{0,0}^{(0)} &= \psi^{(A)}(t) \end{aligned} \quad . \tag{8.69}$$

The vector of reaction in the connection is thence defined by the vector:

$$\mathbf{R}^{(0)} = \begin{bmatrix} F_x^{(0)} & F_y^{(0)} & M_z^{(0)} \end{bmatrix}^T, \tag{8.70}$$

whose components describe the forces and moment which realize the excitation (8.69).

A detailed description of the algorithm of combining the equations of subsystems for revolute and translational connections is presented in [Wojciech S., 1984].

8.4 Modelling Large Deflections and Inclusion of Nonlinear Physical Dependencies

Most of the applications already discussed in which the RFE method is used pertain to systems containing beam links. Some of the considerations in this book are for pipelines which may be subjected to deflections much larger than typical beam systems. Although the RFE method enables analysis involving large deflections, the specific dynamic behaviour of offshore pipelines and cableswhen laid on the bottom of a sea, calls for considerable modifications in the formulation of the equations of motion according to this method [Szczotka M., 2011b]. They will be later applied in some of the examples presented.

When the deflections of the link are large, the length of the chord *AB'* may differ (be smaller) from its primary length *AB=l* (Fig. 8.10). Let us remind that, according to (5.5), the motion of the base (the vessel's hull) is known to be given by the vector:

$$\mathbf{q}^{(A)} = \begin{bmatrix} x_{org}^{(A)} & y_{org}^{(A)} & z_{org}^{(A)} & \varphi_z^{(A)} & \varphi_y^{(A)} & \varphi_x^{(A)} \end{bmatrix}^T. \tag{8.71}$$

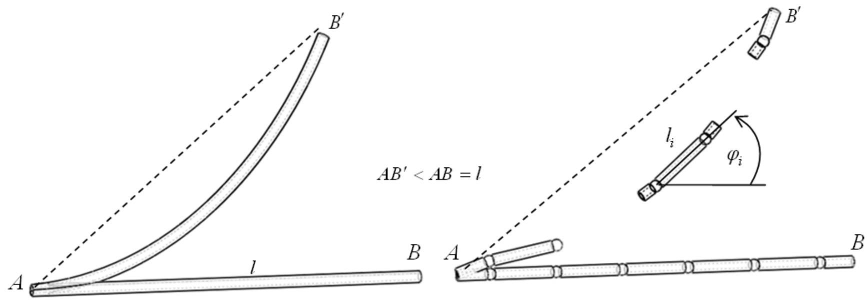

Fig. 8.10. Division of a beam with length *l* into RFEs and SDE: a) primary beam, b) equivalent system of RFEs and SDE

Let the components of the following vector determine the displacements and orientation of the RFE *i* in the system $\{A\}$:

$$\tilde{\mathbf{q}}^{(i)} = \begin{bmatrix} \tilde{\mathbf{r}}^{(i)T} & \tilde{\mathbf{\Phi}}^{(i)T} \end{bmatrix}^T, \tag{8.72}$$

where $\tilde{\mathbf{r}}^{(i)} = \begin{bmatrix} x^{(i)} & y^{(i)} & z^{(i)} \end{bmatrix}^T$ – coordinates of the origin of the system $\{i\}$ attached to the RFE *i* in the system $\{A\}$,

$\tilde{\mathbf{\Phi}}^{(i)} = \begin{bmatrix} \varphi_z^{(i)} & \varphi_y^{(i)} & \varphi_x^{(i)} \end{bmatrix}^T$ – ZYX Euler angles determining the orientation of the axes of the system $\{i\}$ relative to $\{A\}$.

Based on the information from previous chapters, let us write the transformation matrices from the system $\{i\}$ to the base system $\{A\}$ in the form:

$$\tilde{\mathbf{T}}^{(i)} = \begin{bmatrix} \tilde{\mathbf{R}}^{(i)} & \tilde{\mathbf{r}}^{(i)} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{8.73}$$

where $$\tilde{\mathbf{R}}^{(i)} = \begin{bmatrix} c\varphi_z^{(i)} & -s\varphi_z^{(i)} & 0 \\ s\varphi_z^{(i)} & c\varphi_z^{(i)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\varphi_y^{(i)} & 0 & s\varphi_y^{(i)} \\ 0 & 1 & 0 \\ -s\varphi_y^{(i)} & 0 & c\varphi_y^{(i)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi_x^{(i)} & -s\varphi_x^{(i)} \\ 0 & s\varphi_x^{(i)} & c\varphi_x^{(i)} \end{bmatrix},$$

and the transformation matrices from the system $\{i\}$ to the global system, according to (5.4.1) and (5.6), are as follows:

$$\mathbf{T}^{(i)} = \mathbf{T}^{(i)}\left(\tilde{\mathbf{q}}^{(i)}, t\right) = {}_A^0\mathbf{T}(t)\tilde{\mathbf{T}}^{(i)}\left(\tilde{\mathbf{q}}^{(i)}\right). \tag{8.74}$$

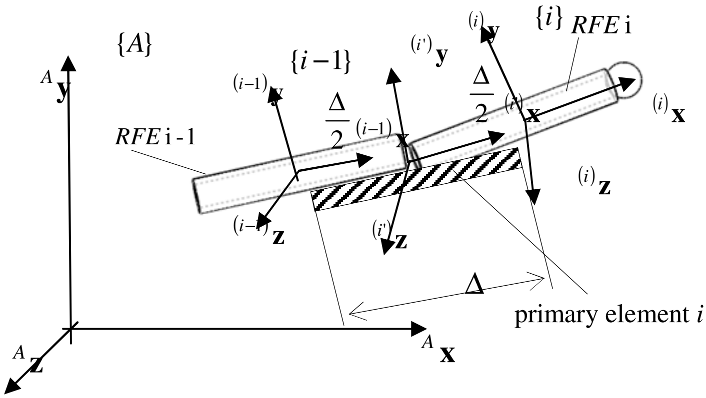

Fig. 8.11. The primary element *i*, RFEs *i*-1 and *i* having load applied to the beam

Large displacements of the links cause the primary element as well as the RFEs *i* and *i*-1 created in the secondary division to be in the configuration depicted in Fig. 8.11.

The coordinate systems $\{i\text{-}1\}$ and $\{i\}$ are attached to RFEs *i*-1 and *i*. On the other hand, to the primary element *i* the coordinate system $\{i'\}$ is attached. The coordinate system $\{A\}$ may in further considerations be the global system or one attached to the deck of a vessel or platform.

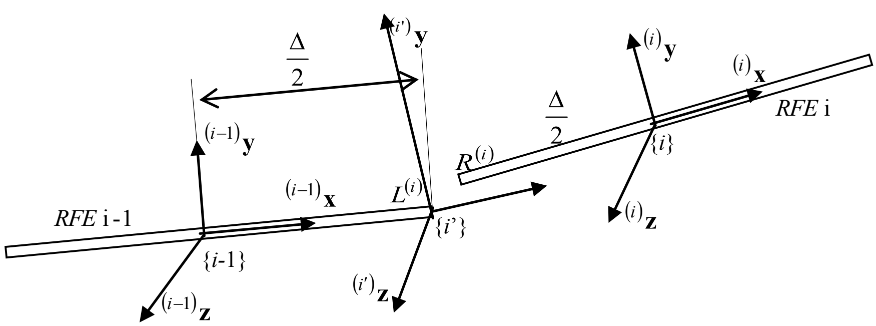

Fig. 8.12. Position and orientation of the system $\{i'\}$

If division of a beam into rigid finite elements is fine enough, differences between the angles $\varphi_z^{(i)} - \varphi_z^{(i-1)}$, $\varphi_y^{(i)} - \varphi_y^{(i-1)}$, $\varphi_x^{(i)} - \varphi_x^{(i-1)}$ which are components of the vector:

$$\Delta\tilde{\mathbf{\Phi}}^{(i)} = \tilde{\mathbf{\Phi}}^{(i)} - \tilde{\mathbf{\Phi}}^{(i-)} = \begin{bmatrix} \varphi_z^{(i)} - \varphi_z^{(i-1)} \\ \varphi_y^{(i)} - \varphi_y^{(i-1)} \\ \varphi_x^{(i)} - \varphi_x^{(i-1)} \end{bmatrix} \tag{8.75}$$

may be assumed to be small. Let us assume that the origin of the coordinate system {*i*'} (of the primary element) coincides with the right end of the RFE *i*-1 and its orientation to is determined by ZYX Euler angles being the arithmetic means of the Euler angles of the RFEs *i*-1 and *i* – Fig. 8.12. Therefore:

$$\mathbf{r}^{(i')} = \tilde{\mathbf{r}}^{(i-1)} + \tilde{\mathbf{R}}^{(i-1)}\tilde{\mathbf{r}}_R^{(i-1')}, \tag{8.76}$$

where $\tilde{\mathbf{r}}_R^{(i-1')} = \left[\Delta/2 \quad 0 \quad 0\right]^T$,

and:

$$\tilde{\mathbf{\Phi}}^{(i')} = \begin{bmatrix} \varphi_z^{(i')} \\ \varphi_y^{(i')} \\ \varphi_x^{(i')} \end{bmatrix} = \frac{1}{2}\left[\tilde{\mathbf{\Phi}}^{(i-1)} + \tilde{\mathbf{\Phi}}^{(i)}\right]. \tag{8.77}$$

The coordinates of the right end of the RFE *i*-1 (point $L^{(i)}$) and the left RFE *i* (point $R^{(i)}$) in the base system $\{A\}$ are determined thus:

$$\tilde{\mathbf{r}}_L^{(i)} = \tilde{\mathbf{r}}^{(i-1)} + \tilde{\mathbf{R}}^{(i-1)}\tilde{\mathbf{r}}_R^{(i-1')}, \tag{8.78}$$

$$\tilde{\mathbf{r}}_R^{(i)} = \tilde{\mathbf{r}}^{(i1)} + \tilde{\mathbf{R}}^{(i1)}\tilde{\mathbf{r}}_L^{(i-1')}, \tag{8.79}$$

where $\tilde{\mathbf{r}}_R^{(i-1')}$ – defined in (8.76),

$$\tilde{\mathbf{r}}_L^{(i-1')} = \begin{bmatrix} -\frac{\Delta}{2} & 0 & 0 \end{bmatrix}^T.$$

These vectors may be represented in the system {*i*'}:

$$\mathbf{r}_L^{(i')} = \mathbf{R}^{(i')T}\left(\tilde{\mathbf{r}}_L^{(i)} - \mathbf{r}^{(i')}\right), \tag{8.80}$$

$$\mathbf{r}_R^{(i')} = \mathbf{R}^{(i')T}\left(\tilde{\mathbf{r}}_R^{(i)} - \mathbf{r}^{(i')}\right), \tag{8.81}$$

where $\mathbf{R}^{(i')}$ – rotation matrix corresponding to the angles $\varphi_x^{(i')}, \varphi_y^{(i')}, \varphi_z^{(i')}$, whereas, considering (8.77), the vectors $\mathbf{\Phi}_L^{(i')}$ and $\mathbf{\Phi}_R^{(i')}$ are:

$$\mathbf{\Phi}_L^{(i')} = \tilde{\mathbf{\Phi}}^{(i-1)} - \tilde{\mathbf{\Phi}}^{(i')} = -\frac{1}{2}\mathbf{\Delta}\tilde{\mathbf{\Phi}}^{(i)}, \tag{8.82}$$

$$\mathbf{\Phi}_R^{(i')} = \tilde{\mathbf{\Phi}}^{(i)} - \tilde{\mathbf{\Phi}}^{(i')} = \frac{1}{2}\mathbf{\Delta}\tilde{\mathbf{\Phi}}^{(i)}. \tag{8.83}$$

The vector of deformation of the SDE *i* takes the form:

$$\Delta\mathbf{q}^{(i')} = \begin{bmatrix} \Delta\mathbf{r}^{(i')} \\ \Delta\mathbf{\Phi}^{(i')} \end{bmatrix}, \tag{8.84}$$

where $\Delta\mathbf{r}^{(i')} = \mathbf{r}_R^{(i')} - \mathbf{r}_L^{(i')}$,

$\Delta\mathbf{\Phi}^{(i')} = \mathbf{\Phi}_R^{(i')} - \mathbf{\Phi}_L^{(i')}$.

Taking (8.80) – (8.83) into account:

$$\Delta\mathbf{q}^{(i')} = \begin{bmatrix} \mathbf{R}^{(i')T}\left[\tilde{\mathbf{r}}_R^{(i)} - \mathbf{r}^{(i')} - \tilde{\mathbf{r}}_L^{(i)} + \mathbf{r}^{(i')}\right] \\ \frac{1}{2}\Delta\tilde{\mathbf{\Phi}} - \left(-\frac{1}{2}\Delta\tilde{\mathbf{\Phi}}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(i')T}\left[\tilde{\mathbf{r}}_R^{(i)} - \tilde{\mathbf{r}}_L^{(i)}\right] \\ \Delta\tilde{\mathbf{\Phi}} \end{bmatrix}. \tag{8.85}$$

The axes of the SDE are the principal deformation axes, hence the following formulas for the forces and moments caused by the deformation of the SDE *i*:

$$\tilde{\mathbf{F}}^{(i)} = \mathbf{C}_r^{(i)}\Delta\mathbf{r}^{(i')}, \tag{8.86}$$

$$\tilde{\mathbf{M}}^{(i)} = \mathbf{C}_\Phi^{(i)}\Delta\mathbf{\Phi}^{(i')}, \tag{8.87}$$

where $\tilde{\mathbf{F}}^{(i)} = \left[\tilde{F}_x^{(i)} \quad \tilde{F}_y^{(i)} \quad \tilde{F}_z^{(i)}\right]^T$,

$\tilde{\mathbf{M}}^{(i)} = \left[\tilde{M}_x^{(i)} \quad \tilde{M}_y^{(i)} \quad \tilde{M}_z^{(i)}\right]^T$,

$\mathbf{C}_r^{(i)} = diag\left\{c_x^{(i)}, c_y^{(i)}, c_z^{(i)}\right\}$,

$\mathbf{C}_\Phi^{(i)} = diag\left\{c_\psi^{(i)}, c_\theta^{(i)}, c_\varphi^{(i)}\right\}$,

$c_x^{(i)}, \ldots, c_\varphi^{(i)}$ – stiffness coefficients.

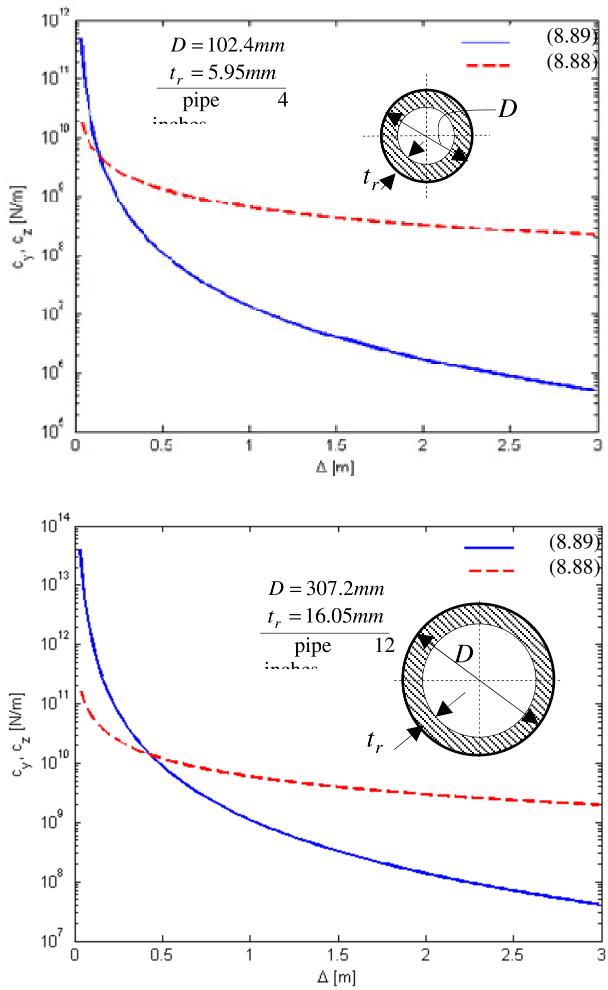

Fig. 8.13. Values of shear coefficients according to (8.88) and (8.89)

In [Kruszewski J., et al., 1999] the following formulas for stiffness coefficients of the elements are given:

$$c_x = \frac{EA}{\Delta}, \qquad c_y = \frac{GA}{\kappa\Delta}, \qquad c_z = \frac{GA}{\kappa\Delta},$$
$$c_\varphi = \frac{GJ_x}{\Delta}, \qquad c_\theta = \frac{GJ_y}{\Delta}, \qquad c_\psi = \frac{GJ_z}{\Delta}. \tag{8.88}$$

The work [Szczotka M., 2011b] takes another approach to defining the shear stiffness coefficients c_y and c_z by assuming:

$$c_y = \frac{12EJ_z}{\Delta^3}, \qquad c_z = \frac{12EJ_y}{\Delta^3}, \tag{8.89}$$

and maintaining the conformance of the values of remaining coefficients to (8.88). The above modification of the coefficients c_y and c_z enables the same expressions to give the potential energy of elastic deformation of the primary element obtained with the RFE method and the energy of elastic deformation of the deformable element considered in FEM.

The values of shear stiffness coefficients determined by formulas (8.88) and (8.89) are shown in Fig. 8.13 for different lengths of the element. Calculations were performed for two different sections of pipes which are analysed in later chapters of this volume. Appropriate division into finite elements enables both coefficients to share the same value. The stiffness coefficients c_y and c_z in the formulas (8.89) have smaller values when the elements resulting from the division are longer.

Forces $+\tilde{\mathbf{F}}_{est}^{(i)}$ applied to the point whose coordinates are given by (8.78) and pairs of forces $+\tilde{\mathbf{M}}_{est}^{(i)}$ act on the RFE i-1. Forces $-\tilde{\mathbf{F}}_{est}^{(i)}$ applied to the point whose coordinates are given by (8.79) and pairs of forces $-\tilde{\mathbf{M}}_{est}^{(i)}$ act on the RFE i. (Fig. 8.14).

The forces (8.86) and the moments (8.87) are given in the coordinate system $\{i'\}$. Their transformation to the global system is done as follows:

$$\mathbf{F}^{(i)} = \mathbf{R}^{(i')}\tilde{\mathbf{F}}^{(i)}, \tag{8.90.1}$$

$$\mathbf{M}^{(i)} = \mathbf{R}^{(i')}\tilde{\mathbf{M}}^{(i)}, \tag{8.90.2}$$

and in the following way to the coordinate systems of the RFEs i and i-1:

$$\tilde{\mathbf{F}}_{est}^{(i-1)} = \mathbf{R}^{(i-1)^T}\mathbf{F}^{(i)}, \tag{8.91.1}$$

$$\tilde{\mathbf{F}}_{est}^{(i)} = \mathbf{R}^{(i)^T}\mathbf{F}^{(i)}, \tag{8.91.2}$$

$$\tilde{\mathbf{M}}_{est}^{(i-1)} = \mathbf{R}^{(i-1)^T}\mathbf{M}^{(i)}, \tag{8.91.3}$$

$$\tilde{\mathbf{M}}_{est}^{(i)} = \mathbf{R}^{(i)^T}\mathbf{M}^{(i)}. \tag{8.91.4}$$

The presented discussion shows that the crucial change introduced with respect to the original formulation of the RFE method (section 8.1) is having the system

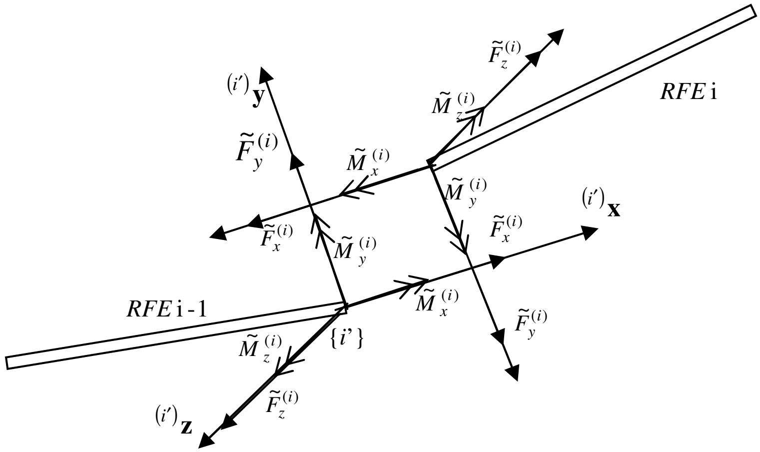

Fig. 8.14. Forces and moments acting on the RFEs *i*-1 and *i* caused by deformation of the SDE *i*

of principal deformation of the SDE {*i*'} "follow" large displacements of the finite elements. A similar approach to planar systems with variable configuration is presented in [Wittbrodt E., 1983]. Furthermore, the modification of shear stiffness coefficients enables the energy of an element's deformation to be expressed in the same form as in the method of deformable finite elements. This conclusion holds for linear physical dependencies. Also of importance is an observation that since the presented proposal assumes the vectors of generalized coordinates of the RFEs take forms described by (8.84), no distinction is made among the variables to configuration (describing the motion of the beam as a rigid body) and flexible ones, as in section 8.1.

8.4.1 Equations of Motion When Using the Classical RFE Method

The equations of motion of a system taking into account the dependencies from previous chapters may be put in the form:

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}), \tag{8.92}$$

where $\mathbf{A} = diag\left[\tilde{\mathbf{A}}_0, ..., \tilde{\mathbf{A}}_n\right]$,

$$\mathbf{q} = \left[\tilde{\mathbf{q}}_0^T, ..., \tilde{\mathbf{q}}_n^T\right]^T,$$

$$\mathbf{f} = \mathbf{Q} - \mathbf{e} - \mathbf{G} = \left[\tilde{\mathbf{f}}_0^T, ..., \tilde{\mathbf{f}}_n^T\right]^T,$$

$$\mathbf{Q} = \left[\left(\mathbf{Q}_0^{(est)} + \mathbf{Q}_i^{(S)}\right)^T, ..., \left(\mathbf{Q}_n^{(est)} + \mathbf{Q}_n^{(S)}\right)^T\right]^T,$$

$$\mathbf{e} = \left[\tilde{\mathbf{e}}_0^T, \ldots, \tilde{\mathbf{e}}_n^T\right]^T,$$

$$\mathbf{G} = \left[\left(\frac{\partial V_0}{\partial \tilde{\mathbf{q}}_0}\right)^T, \ldots, \left(\frac{\partial V_n}{\partial \tilde{\mathbf{q}}_n}\right)^T\right]^T,$$

n – number of RFEs in the concerned model.

Note that the matrix **A** is diagonal, which is of great importance when integrating the system's equations of motion. Such form is characteristic of systems modelled with the classical RFE method.

Let us assume that at the point whose coordinates are given by the vector $\tilde{\mathbf{r}}_i'$ in the local system of the RFE i there act: an external force and a pair of forces given by:

$$\begin{aligned}\tilde{\mathbf{F}}_i &= \left[\tilde{F}_i^{(x)} \quad \tilde{F}_i^{(y)} \quad \tilde{F}_i^{(z)} \quad 0\right]^T,\\ \tilde{\mathbf{M}}_i &= \left[\tilde{M}_i^{(x)} \quad \tilde{M}_i^{(y)} \quad \tilde{M}_i^{(z)} \quad 0\right]^T.\end{aligned} \tag{8.93}$$

Their corresponding generalized forces may then be determined from the formulas [Wittbrodt E., et al., 2006]:

$$\begin{aligned}\left(\mathbf{Q}_i\left(\tilde{\mathbf{F}}_i\right)\right)_{k=1,\ldots,6} &= \tilde{\mathbf{F}}_i^T \mathbf{T}_i^T \mathbf{T}_{i,k} \tilde{\mathbf{r}}_i',\\ \left(\mathbf{Q}_i\left(\tilde{\mathbf{M}}_i\right)\right)_{k=1,\ldots,6} &= \tilde{M}_i^{(x)} \sum_{j=1}^{3} (\mathbf{T}_i)_{j,3} \left(\mathbf{T}_{i,k}\right)_{j,2} + \tilde{M}_i^{(y)} \sum_{j=1}^{3} (\mathbf{T}_i)_{j,1} \left(\mathbf{T}_{i,k}\right)_{j,3} + \tilde{M}_i^{(z)} \sum_{j=1}^{3} (\mathbf{T}_i)_{j,2} \left(\mathbf{T}_{i,k}\right)_{j,1}.\end{aligned} \tag{8.94}$$

The relations (8.94) allow us to determine the generalized forces and moments pertaining to the impact of the sea environment:

$$\mathbf{Q}_i^{(S)} = \mathbf{Q}_i^{(h)}\left(\tilde{\mathbf{F}}_i^h, \tilde{\mathbf{M}}_i^h\right) + \mathbf{Q}_i^{(b)}\left(\tilde{\mathbf{F}}_i^b, \tilde{\mathbf{M}}_i^b\right) + \mathbf{Q}_i^{(w)}\left(\tilde{\mathbf{F}}_i^t, \tilde{\mathbf{M}}_i^t\right) + \mathbf{Q}_i^{(d)}\left(\tilde{\mathbf{F}}_i^d, \tilde{\mathbf{M}}_i^d\right), \tag{8.95}$$

where $\tilde{\mathbf{F}}_i^h, \tilde{\mathbf{M}}_i^h$ – vectors of forces and moments due to interaction of the element i with the liquid (including the influence of waves, sea currents and hydrodynamics),

$\tilde{\mathbf{F}}_i^b, \tilde{\mathbf{M}}_i^b$ – vectors of forces and hydrostatic buoyancy moments (hydrostatic buoyancy of the pipeline and additional buoyant modules),

$\tilde{\mathbf{F}}_i^t, \tilde{\mathbf{M}}_i^t$ – vectors of forces and moments of the action of guiding structures (e.g. reel, guiding ramp, mechanisms),

$\tilde{\mathbf{F}}_i^d, \tilde{\mathbf{M}}_i^d$ – vectors of forces and moments due to the action of the seabed.

The forces and moments caused by the deformation of the SDE may be included similarly. According to (8.86) and (8.87), forces and moments caused by the deformation of the SDE i (left end of the RFE i) and the SDE i+1 (right end of the RFE i) act upon the RFE i. Hence:

$$\mathbf{Q}_i^{(est)} = \mathbf{Q}_i^{(est)}\left(-\mathbf{F}_i^{(est)}, -\widetilde{\mathbf{M}}_i^{(est)}, \mathbf{F}_{i+1}^{(est)}, \widetilde{\mathbf{M}}_{i+1}^{(est)}\right), \tag{8.96}$$

where $\mathbf{F}^{(i)}, \widetilde{\mathbf{M}}^{(i)}$ – defined in (8.86) and (8.87),

$$\begin{aligned} \mathbf{Q}_{i,k}^{(est)} = {} & \mathbf{F}_{i+1}^{(est)^T}\mathbf{T}_{i,k}\widetilde{\mathbf{r}}_{R,i}^{'} - \mathbf{F}_i^{(est)^T}\mathbf{T}_{i,k}\widetilde{\mathbf{r}}_{L,i}^{'} + \\ & + \left[\widetilde{M}_{i+1,x}^{(est)} - \widetilde{M}_{i,x}^{(est)}\right] \cdot \sum_{j=1}^{3} \left(\mathbf{T}_i\right)_{j,3}\left(\mathbf{T}_{i,k}\right)_{j,2} + \\ & + \left[\widetilde{M}_{i+1,y}^{(est)} - \widetilde{M}_{i,y}^{(est)}\right] \cdot \sum_{j=1}^{3} \left(\mathbf{T}_i\right)_{j,1}\left(\mathbf{T}_{i,k}\right)_{j,3} + \\ & + \left[\widetilde{M}_{i+1,z}^{(est)} - \widetilde{M}_{i,z}^{(est)}\right] \cdot \sum_{j=1}^{3} \left(\mathbf{T}_i\right)_{j,2}\left(\mathbf{T}_{i,k}\right)_{j,1}. \end{aligned}$$

8.4.2 Inclusion of Nonlinear Physical Dependencies

In the rigid finite element method, flexibility is described in an approximate manner (displacements are realized in the SDE only). Therefore, the tensor $\sigma_{jk}^{(i)}$ present in (7.1), defined for each SDE i in the plane normal to the beam's axis, is given as [Szczotka M., 2011b]:

$$\sigma_{jk}^{(i)} = \begin{bmatrix} \sigma_x^{(i)} & \tau_{yx}^{(i)} & \tau_{zx}^{(i)} \\ \tau_{xy}^{(i)} & 0 & 0 \\ \tau_{xz}^{(i)} & 0 & 0 \end{bmatrix}, \tag{8.97}$$

where $\sigma_x = \dfrac{\widetilde{F}_x^{(i)}}{A}$,

$$\tau_{xy}^{(i)} = \frac{\widetilde{F}_y^{(y)}}{A} + \frac{1}{2}\tau_S^{(i)},$$

$$\tau_{xz}^{(i)} = \frac{\widetilde{F}_z^{(y)}}{A} + \frac{1}{2}\tau_S^{(i)},$$

$$\tau_{yx}^{(i)} = \tau_{xy}^{(i)}, \qquad \tau_{zx}^{(i)} = \tau_{xz}^{(i)},$$

$\tau_S^{(i)} = \dfrac{\widetilde{M}_x^{(i)}}{2At_{\min}^{(i)}}$ – stress tangent to the torque,

$t_{\min}^{(i)}$ – minimal thickness of the section's side.

The form of the stress tensor in a flexible beam modelled with the RFE method is similar to the tensor obtained in the Saint-Venant problem for torsion and bending of beams [Nowacki W., 1970]. The stresses $\sigma_{g,y}^{(i)}$, $\sigma_{g,z}^{(i)}$ due to the bending moments at which the transition from elasticity to plasticity occurs, may be determined from the following equation taking (7.1) into account:

$$\sqrt{3J_2^{(i)} + \left(\tilde{\sigma}_g^{(i)}\right)^2} - \sigma_0 = 0, \tag{8.98}$$

where $\tilde{\sigma}_g^{(i)}$ – equivalent bending stress,

$J_2^{(i)}$ – specified as in (7.1).

Some models of pipelines presented later use the given dependencies to construct a module allowing us to determine the forces and moments in the SDE when occurrence of elasto-plastic deformations is possible. Fig. 8.15 schematically shows a flowchart of actions comprising the procedure of determining the bending moment acting on the RFE on the assumption that $\vartheta_i \in \{\varphi_{yi}, \varphi_{zi}\}$.

The diagram uses the following notation:

F_i – mark specifying the state of the material,

X_s – maximal deformation which causes the phase to change from elastic to plastic,

$\tilde{\vartheta}_i^0$ – neutral value of displacement (at which $m_\vartheta^{(i)} = 0$),

$C_e^{(i)}$ – stiffness coefficient of the SDE within the elastic region,

$C_p^{(i)}$ – stiffness coefficient of the SDE within the plastic region, $C_p^{(i)} = \mu C_e^{(i)}$, $\mu = 0.01, 0.1, \ldots$,

$f_{mat}(\ )$ – function describing the shape of the characteristic $\sigma = f(\varepsilon_p)$ within the plastic region,

$M_B^{(i)}$ – value $m_\vartheta^{(i)}$ determined in the previous step $t-h$,

$\vartheta_B^{(i)}$ – plastic deformation in the previous step $t-h$.

A control procedure for the mark F_i and for calculating the values $\vartheta_B^{(i)}$ and $M_B^{(i)}$ (Fig. 8.16) is also necessary.

Approximation of the characteristic of a material may be performed for arbitrary data obtained e.g. from measurements. The linear segments (elastic region, linear reinforcement in the plastic region) may be interspersed with nonlinear ones, thus leading to significantly greater stability of the calculations. An example of such characteristic can be found in [Szczotka M., 2010].

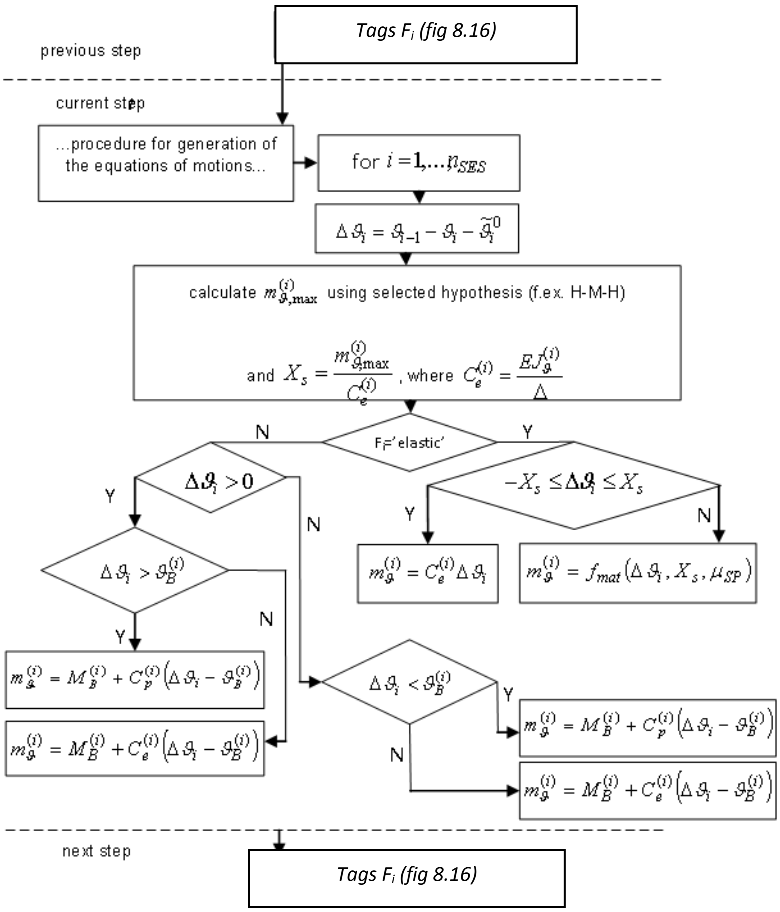

Fig. 8.15. Flow diagram of the algorithm determining the value $m_\vartheta^{(i)}$

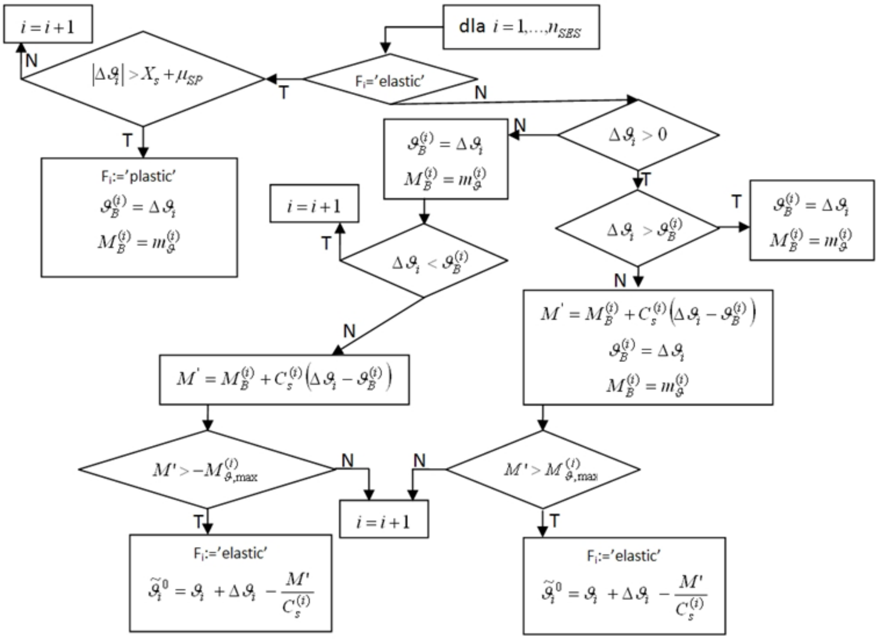

Fig. 8.16. Flow diagram of state markers control, F_i and calculations of the values $\vartheta_B^{(i)}$ and $M_B^{(i)}$