# 5 Equations of Motion of Systems with Rigid Links

In the current chapter the main steps of determining the components of the equation of motion for open kinematic chains consisting of rigid links are presented [Wittbrodt E., et al., 2006]. The method is based on the Lagrange equations of the second order, homogeneous transformations and joint coordinates.

The Lagrange equations of the second order may be written as:

$$\boldsymbol{\varepsilon}_{\mathbf{q}}(E)+\frac{\partial V}{\partial \mathbf{q}}+\frac{\partial D}{\partial \dot{\mathbf{q}}}=\mathbf{Q}, \tag{5.1}$$

where $\boldsymbol{\varepsilon}_{\mathbf{q}}(E)=\left(\frac{d}{dt}\frac{\partial E}{\partial \dot{q}_k}-\frac{\partial E}{\partial q_k}\right)_{k=1,\dots,n}$

$$\frac{\partial V}{\partial \mathbf{q}}=\left(\frac{\partial V}{\partial q_k}\right)_{k=1,\dots,n}, \qquad \frac{\partial D}{\partial \dot{\mathbf{q}}}=\left(\frac{\partial D}{\partial \dot{q}_k}\right)_{k=1,\dots,n}, \qquad \mathbf{Q}=\left(Q_k\right)_{k=1,\dots,n}.$$

$\mathbf{q}=\begin{bmatrix}q_1 & \dots & q_k & \dots & q_n\end{bmatrix}^T$ – vector of generalized coordinates,

$\dot{\mathbf{q}}=\begin{bmatrix}\dot{q}_1 & \dots & \dot{q}_k & \dots & \dot{q}_n\end{bmatrix}^T$ – vector of generalized velocities,

$E$ – kinetic energy,

$V$ – potential energy,

$D$ – function of dissipation energy,

$Q_k$ – non-potential generalized force corresponding to the $k$-th generalized coordinate,

$n$ – number of generalized.

In the following reasoning, the dissipation of energy is omitted ($D$=0) and the multibody system is assumed to be situated on a movable base $\{A\}$ (Fig. 5.1) whose motion relative to the inertial (global) system $\{0\} = \{\}$ is known.

E. Wittbrodt et al.: Rigid Finite Element Method, OEO 1, pp. 59–73.
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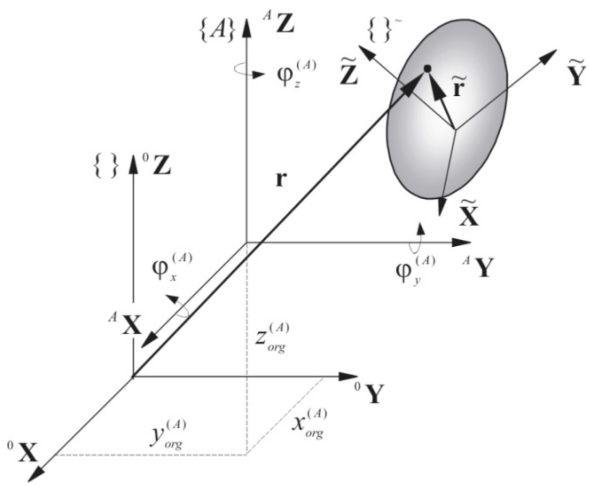


**Fig. 5.1.** Coordinate systems: { } – stationary (inertial) global one, {$A$} – that of the movable base, { }$^\sim$ – local one attached to the considered link

For the sake of notation's clarity, the coordinate system {0} will for the remaining part be identified with the inertial system { }. Additionally, the following notation will be assumed:

$$ {}_p^0\mathbf{T} = \mathbf{T}^{(p)}, \tag{5.2}$$

where $p$ – number of the link in the kinematic chain.

Let us introduce the following denotations:

$$ x_{org}^{(p)},\ y_{org}^{(p)},\ z_{org}^{(p)}, \tag{5.3}$$

for the origin of the system {$p$} in the coordinate system of the preceding link and:

$$ \varphi_x^{(p)}, \varphi_y^{(p)}, \varphi_z^{(p)}, \tag{5.4}$$

for ZYX Euler angles determining the orientation of the axes of the system {$p$} relative to the axes of the preceding system.

Matrix of the homogeneous transformation ${}_A^0\mathbf{T}$ taking into account the motion of the system {$A$} relative to the system { } may be represented as a product of six matrices, each of which being a function of a single time-dependent variable only:

$$ {}_A^0\mathbf{T}(t) = {}_A^0\mathbf{T}_1\ {}_A^0\mathbf{T}_2\ {}_A^0\mathbf{T}_3\ {}_A^0\mathbf{T}_6\ {}_A^0\mathbf{T}_5\ {}_A^0\mathbf{T}_4\ , \tag{5.5}$$

where

$$ {}_A^0\mathbf{T}_1 = {}_A^0\mathbf{T}_1\left(x_{org}^{(A)}\right) = \begin{bmatrix} 1 & 0 & 0 & x_{org}^{(A)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}_A^0\mathbf{T}_4 = {}_A^0\mathbf{T}_4\left(\varphi_x^{(A)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\varphi_x^{(A)} & -s\varphi_x^{(A)} & 0 \\ 0 & s\varphi_x^{(A)} & c\varphi_x^{(A)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}_A^0\mathbf{T}_2 = {}_A^0\mathbf{T}_2\left(y_{org}^{(A)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_{org}^{(A)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}_A^0\mathbf{T}_5 = {}_A^0\mathbf{T}_5\left(\varphi_y^{(A)}\right) = \begin{bmatrix} c\varphi_y^{(A)} & 0 & s\varphi_y^{(A)} & 0 \\ 0 & 1 & 0 & 0 \\ -s\varphi_y^{(A)} & 0 & c\varphi_y^{(A)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}_A^0\mathbf{T}_3 = {}_A^0\mathbf{T}_3\left(z_{org}^{(A)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z_{org}^{(A)} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}_A^0\mathbf{T}_6 = {}_A^0\mathbf{T}_6\left(\varphi_z^{(A)}\right) = \begin{bmatrix} c\varphi_z^{(A)} & -s\varphi_z^{(A)} & 0 & 0 \\ s\varphi_z^{(A)} & c\varphi_z^{(A)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ x_{org}^{(A)} = x_{org}^{(A)}(t), \qquad y_{org}^{(A)} = y_{org}^{(A)}(t), \qquad z_{org}^{(A)} = z_{org}^{(A)}(t), $$

$$ \varphi_x^{(A)} = \varphi_x^{(A)}(t), \qquad \varphi_y^{(A)} = \varphi_y^{(A)}(t), \qquad \varphi_z^{(A)} = \varphi_z^{(A)}(t). $$

The order of rotations included in the matrix ${}_A^0\mathbf{T}$ conforms to the convention for ZYX Euler angles presented in section 4.1.

If $\tilde{\mathbf{r}} = \begin{bmatrix}\tilde{x} & \tilde{y} & \tilde{z} & 1\end{bmatrix}^T$ is a vector determining the coordinates of a mass *dm* in the local system $\{\}^{\sim}$ attached to given link of the system, then the coordinates of this mass in the system $\{\}$ can be given with this formula:

$$ \mathbf{r} = {}_A^0\mathbf{T}(t)\,\overline{\mathbf{T}}(\mathbf{q})\,\tilde{\mathbf{r}} = \mathbf{T}\,\tilde{\mathbf{r}}\,, \tag{5.6} $$

where $\overline{\mathbf{T}}(\mathbf{q}) = {}_{\{\}^{\sim}}^{A}\mathbf{T}(q_1,\ldots,q_n)$ – matrix of coordinate transformation from the local system $\{\}^{\sim}$ to the system $\{A\}$, dependent on the generalized coordinates of the link,

$\mathbf{T} = {}_A^0\mathbf{T}(t)\overline{\mathbf{T}}(\mathbf{q})$ .

In a particular case whereby the base $\{A\}$ of a multibody system is motionless, the following may be assumed:

$$ {}_A^0\mathbf{T}(t) = \mathbf{I}\,, \tag{5.7.1} $$

where $\mathbf{I}$ is the identity matrix.

Then:

$$ \mathbf{T} = \overline{\mathbf{T}}(\mathbf{q})\,. \tag{5.7.2} $$

## 5.1 Kinetic Energy of a Link

Kinetic energy $E$ of a link with mass $m$ can be calculated using the trace of a matrix [Paul R. P., 1981], [Jurewič E. I., 1984]. The kinetic energy of an elementary mass $dm$ with coordinates $(x, y, z)$ can then be represented as:

$$dE = \frac{1}{2}\mathrm{tr}\{\dot{\mathbf{r}}\,\dot{\mathbf{r}}^T\}dm = \frac{1}{2}\mathrm{tr}\left\{\begin{bmatrix} \dot{x}\dot{x} & \dot{x}\dot{y} & \dot{x}\dot{z} & 0 \\ \dot{y}\dot{x} & \dot{y}\dot{y} & \dot{y}\dot{z} & 0 \\ \dot{z}\dot{x} & \dot{z}\dot{y} & \dot{z}\dot{z} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}dm = \tag{5.8}$$

$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dm = \frac{1}{2}v^2 dm\,,$$

where $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{u} a_{ii}$ – trace of the matrix $\mathbf{A}_{u\times u} = (a_{ij})_{i,j=1,\ldots,u}$,

$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

Since the vector $\tilde{\mathbf{r}}$ which determines the position of the elementary mass $dm$ in the local coordinate system has constant coordinates (in time), then:

$$\dot{\mathbf{r}} = \dot{\mathbf{T}}\,\tilde{\mathbf{r}}\,, \tag{5.9}$$

and the expression giving the kinetic energy of the considered link takes the form:

$$E = \frac{1}{2}\int_m \mathrm{tr}\{\dot{\mathbf{r}}\,\dot{\mathbf{r}}^T\}dm = \frac{1}{2}\int_m \mathrm{tr}\{\dot{\mathbf{T}}\,\dot{\tilde{\mathbf{r}}}\,\dot{\tilde{\mathbf{r}}}^T\,\dot{\mathbf{T}}^T\}dm =$$
$$= \frac{1}{2}\mathrm{tr}\left\{\dot{\mathbf{T}}\left[\int_m \dot{\tilde{\mathbf{r}}}\,\dot{\tilde{\mathbf{r}}}^T dm\right]\dot{\mathbf{T}}^T\right\} = \frac{1}{2}\mathrm{tr}\{\dot{\mathbf{T}}\,\mathbf{H}\,\dot{\mathbf{T}}^T\}\,. \tag{5.10}$$

The matrix $\mathbf{H}$ occurring in the above formula is the matrix of inertia of the link whose elements may be calculated thus:

$$\mathbf{H} = \int_m \dot{\tilde{\mathbf{r}}}\,\dot{\tilde{\mathbf{r}}}^T dm = \begin{bmatrix} J_{(\tilde{X}\tilde{X})} & J_{\tilde{X}\tilde{Y}} & J_{\tilde{X}\tilde{Z}} & J_{\tilde{X}} \\ J_{\tilde{X}\tilde{Y}} & J_{(\tilde{Y}\tilde{Y})} & J_{\tilde{Y}\tilde{Z}} & J_{\tilde{Y}} \\ J_{\tilde{X}\tilde{Z}} & J_{\tilde{Y}\tilde{Z}} & J_{(\tilde{Z}\tilde{Z})} & J_{\tilde{Z}} \\ J_{\tilde{X}} & J_{\tilde{Y}} & J_{\tilde{Z}} & m \end{bmatrix}, \tag{5.11}$$

where $J_{\tilde{X}\tilde{X}} = \int_m \tilde{x}^2 dm, \quad J_{\tilde{Y}\tilde{Y}} = \int_m \tilde{y}^2 dm, \quad J_{\tilde{Z}\tilde{Z}} = \int_m \tilde{z}^2 dm$ – planar moments of inertia in the coordinate system $\{\}^{\sim}$,

$J_{\tilde{X}\tilde{Y}} = \int_m \tilde{x}\tilde{y} dm, \quad J_{\tilde{X}\tilde{Z}} = \int_m \tilde{x}\tilde{z} dm, \quad J_{\tilde{Y}\tilde{Z}} = \int_m \tilde{y}\tilde{z} dm$ – centrifugal (deviatoric) moments of inertia in the coordinate system $\{\}^{\sim}$,

$J_{\tilde{X}} = \int_m \tilde{x} dm, \quad J_{\tilde{Y}} = \int_m \tilde{y} dm, \quad J_{\tilde{Z}} = \int_m \tilde{z} dm$ – static moments of inertia of the link in the coordinate system $\{\}^{\sim}$,

$m$ – mass of the link.

The following relations hold:

$$J_{(\tilde{X}\tilde{X})} = \frac{1}{2}\left(\bar{J}_{\tilde{Y}} + \bar{J}_{\tilde{Z}} - \bar{J}_{\tilde{X}}\right), \tag{5.12.1}$$

$$J_{(\tilde{Y}\tilde{Y})} = \frac{1}{2}\left(\bar{J}_{\tilde{X}} + \bar{J}_{\tilde{Z}} - \bar{J}_{\tilde{Y}}\right), \tag{5.12.2}$$

$$J_{(\tilde{Z}\tilde{Z})} = \frac{1}{2}\left(\bar{J}_{\tilde{X}} + \bar{J}_{\tilde{Y}} - \bar{J}_{\tilde{Z}}\right), \tag{5.12.3}$$

where $\bar{J}_{\tilde{X}} = \int_m \left(\tilde{y}^2 + \tilde{z}^2\right) dm, \ \bar{J}_{\tilde{Y}} = \int_m \left(\tilde{x}^2 + \tilde{z}^2\right) dm, \ \bar{J}_{\tilde{Z}} = \int_m \left(\tilde{x}^2 + \tilde{y}^2\right) dm$ are mass moments of inertia of the link relative to the axes $\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{Z}}$, respectively.

Taking (5.6) into account, we may write the matrix $\dot{\mathbf{T}}$ as:

$$\dot{\mathbf{T}} = \frac{d\mathbf{T}}{dt} = \frac{d}{dt}\left[{}^0_A\mathbf{T}(t)\,\overline{\mathbf{T}}(\mathbf{q})\right] = {}^0_A\dot{\mathbf{T}}\,\overline{\mathbf{T}} + {}^0_A\mathbf{T}\,\dot{\overline{\mathbf{T}}}. \tag{5.13}$$

Since:

$$\dot{\overline{\mathbf{T}}} = \frac{d\overline{\mathbf{T}}}{dt} = \sum_{i=1}^{n} \frac{\partial \overline{\mathbf{T}}}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} \overline{\mathbf{T}}_i \, \dot{q}_i, \tag{5.14}$$

where $\overline{\mathbf{T}}_i = \dfrac{\partial \overline{\mathbf{T}}}{\partial q_i}$, the following is obtained:

$$\dot{\mathbf{T}} = {}_A^0\dot{\mathbf{T}}\overline{\mathbf{T}} + \sum_{i=1}^{n} \mathbf{T}_i\,\dot{q}_i\,, \tag{5.15}$$

where $\mathbf{T}_i = {}_A^0\mathbf{T}\,\overline{\mathbf{T}}_i$.
A reasoning analogous to that in [Wittbrodt E., et al., 2006] leads to:

$$\varepsilon_i(E) = \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_i} - \frac{\partial E}{\partial q_i} = \mathrm{tr}\left\{\mathbf{T}_i\mathbf{H}\ddot{\mathbf{T}}^T\right\} \quad i=1,\ldots,n. \tag{5.16}$$

The matrix $\ddot{\mathbf{T}}$ may be calculated by differentiating (5.13):

$$\ddot{\mathbf{T}} = {}_A^0\ddot{\mathbf{T}}\overline{\mathbf{T}} + 2\,{}_A^0\dot{\mathbf{T}}\dot{\overline{\mathbf{T}}} + {}_A^0\mathbf{T}\ddot{\overline{\mathbf{T}}}\,, \tag{5.17.1}$$

where $\dot{\overline{\mathbf{T}}}$ is defined in (5.14).

The matrix $\ddot{\overline{\mathbf{T}}}$ is obtained by differentiating by time the formula (5.14), giving:

$$\ddot{\overline{\mathbf{T}}} = \sum_{i=1}^{n}\left(\frac{d\overline{\mathbf{T}}_i}{dt}\dot{q}_i + \overline{\mathbf{T}}_i\,\ddot{q}_i\right) = \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\frac{\partial\overline{\mathbf{T}}_i}{\partial q_j}\dot{q}_j\right)\dot{q}_i + \overline{\mathbf{T}}_i\,\ddot{q}_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\mathbf{T}}_{i,j}\,\dot{q}_i\,\dot{q}_j + \sum_{i=1}^{n}\overline{\mathbf{T}}_i\,\ddot{q}_i\,, \tag{5.17.2}$$

where $\overline{\mathbf{T}}_{i,j} = \dfrac{\partial\overline{\mathbf{T}}_i}{\partial q_j} = \dfrac{\partial^2\overline{\mathbf{T}}}{\partial q_i \partial q_j}$.

Taking (5.17) into account, we may rewrite the relation (5.16) as:

$$\begin{aligned}\varepsilon_i(E) &= \mathrm{tr}\left\{\mathbf{T}_i\,\mathbf{H}\left[{}_A^0\ddot{\mathbf{T}}\overline{\mathbf{T}} + 2\,{}_A^0\dot{\mathbf{T}}\dot{\overline{\mathbf{T}}} + \sum_{l=1}^{n}\sum_{j=1}^{n}\mathbf{T}_{l,j}\,\dot{q}_l\,\dot{q}_j + \sum_{l=1}^{n}\mathbf{T}_l\,\ddot{q}_l\right]^T\right\} = \\ &= \sum_{l=1}^{n} a_{i,l}(\mathbf{q})\,\ddot{q}_l + e_i(\mathbf{q}) \qquad \text{for} \quad i=1,2,\ldots,n,\end{aligned} \tag{5.18}$$

where $a_{i,l}(\mathbf{q}) = \mathrm{tr}\left\{\mathbf{T}_i\ \mathbf{H}\ \mathbf{T}_l{}^T\right\}$,

$$\begin{aligned}e_i(\mathbf{q}) &= \mathrm{tr}\left\{\mathbf{T}_i\mathbf{H}\left[{}_A^0\ddot{\mathbf{T}}\overline{\mathbf{T}} + 2\,{}_A^0\dot{\mathbf{T}}\dot{\overline{\mathbf{T}}} + \sum_{l=1}^{n}\sum_{j=1}^{n}\mathbf{T}_{l,j}\,\dot{q}_l\,\dot{q}_j\right]^T\right\} = \\ &= \mathrm{tr}\left\{\mathbf{T}_i\mathbf{H}\left[{}_A^0\ddot{\mathbf{T}}\overline{\mathbf{T}} + 2\,{}_A^0\dot{\mathbf{T}}\sum_{j=1}^{n}\overline{\mathbf{T}}_j\,\dot{q}_j + \sum_{l=1}^{n}\sum_{j=l}^{n}\delta_{l,j}\,\mathbf{T}_{l,j}\,\dot{q}_l\,\dot{q}_j\right]^T\right\},\end{aligned}$$

$$\delta_{l,j} = \begin{cases}1 & \text{when} \quad l=j\\ 2 & \text{when} \quad l\neq j\end{cases}.$$

The above equation may be presented in a matrix form:

$$\boldsymbol{\varepsilon}(E) = \mathbf{A}\ddot{\mathbf{q}} + \mathbf{e}\,, \tag{5.19}$$

where $\mathbf{A} = \mathbf{A}(t,\mathbf{q}) = \left(a_{i,l}(t,\mathbf{q})\right)_{i,l=1,\ldots,n}$,

$$\mathbf{e} = \mathbf{e}(t,\mathbf{q},\dot{\mathbf{q}}) = \left(e_i(t,\mathbf{q},\ddot{\mathbf{q}})\right)_{i=1,\ldots,n}.$$

An important property of the matrix **A** is its symmetry. The dependencies (5.18) and (5.19) will be used to formulate the equations of motion of analysed multibody systems.

## 5.2 Potential Energy of Gravity Forces of a Link

Let the coordinates of the centre of mass of a given link in its local coordinate system $\{\}^{\sim}$ be specified by the vector:

$$\tilde{\mathbf{r}}_C = \begin{bmatrix}\tilde{x}_C & \tilde{y}_C & \tilde{z}_C & 1\end{bmatrix}^T. \tag{5.20}$$

Assuming the axis ${}^0\mathbf{Z}$ of the global (inertial) coordinate system $\{\}$ to be perpendicular to the Earth's surface, we obtain the following formula that gives the potential energy of the gravity forces of the link:

$$V^g = m\,g\,z_C\,, \tag{5.21}$$

where $g$ – acceleration due to gravity,
$z_C$ – component in the direction of the axis ${}^0\mathbf{Z}$ of the vector $\mathbf{r}_C = \begin{bmatrix}x_C & y_C & z_C & 1\end{bmatrix}^T$ specifying the position of the centre of mass of the link in the inertial system.

By knowing the transformation matrix **T** from the local coordinate system $\{\}^{\sim}$ to the global one $\{\}$ the following may be obtained from equation (5.21):

$$\frac{\partial V^g}{\partial \mathbf{q}} = \mathbf{G}\,, \tag{5.22}$$

where $\mathbf{G} = \mathbf{G}(\mathbf{q}) = (g_i)_{i=1,\ldots,n}$,

$$g_i = \frac{\partial V}{\partial q_i} = m\,g\,\boldsymbol{\theta}_3\,\mathbf{T}_i\,\mathbf{r}'_C\,,$$

$$\boldsymbol{\theta}_3 = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}.$$

Elements of the vector **G** depend therefore on the matrix $\mathbf{T}_i$ and hence on time $t$ and the vector of generalized coordinates **q**.

## 5.3 Generalized Forces: Equations of Motion of a Link

If non-potential forces or moments thereof act on a given link, they must be taken into account in the equations of motion as generalized forces. When the convention of homogeneous transformations and coordinates is applied, vectors of forces and their moments, unlike those of positions, have zero as their fourth coordinate:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \tilde{F}_x & \tilde{F}_y & \tilde{F}_z & 0 \end{bmatrix}^T, \tag{5.23}$$

$$\tilde{\mathbf{M}} = \begin{bmatrix} \tilde{M}_x & \tilde{M}_y & \tilde{M}_z & 0 \end{bmatrix}^T. \tag{5.24}$$

Let us assume that a force $\tilde{\mathbf{F}}$ is applied to the link at the point $N$ (Fig. 5.2).

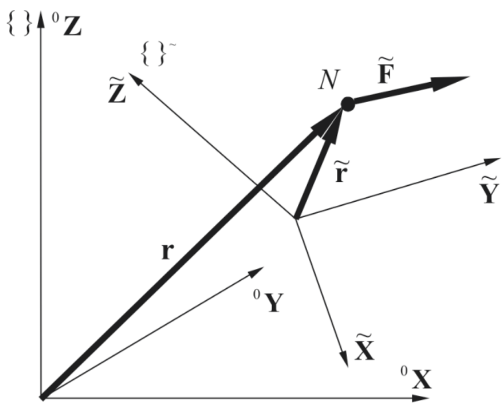


**Fig. 5.2.** Force acting in the local coordinate system

The force $\tilde{\mathbf{F}}$ is described in the inertial system by:

$$\mathbf{F} = \mathbf{T}\tilde{\mathbf{F}}. \tag{5.25}$$

The generalized force corresponding to the $i$-th generalized coordinate [Leyko J., 1996] may be written thus:

$$Q_i(\mathbf{F}) = \mathbf{F}^T \frac{\partial \mathbf{r}}{\partial q_i} = \mathbf{F}^T \mathbf{T}_i \tilde{\mathbf{r}} \quad \text{for } i = 1,...,n. \tag{5.26}$$

Using (5.25), we may transform the formula (5.26) to obtain:

$$Q_i(\tilde{\mathbf{F}}) = \tilde{\mathbf{F}}^T \, \mathbf{T}^T \, \mathbf{T}_i \, \tilde{\mathbf{r}} \, . \tag{5.27}$$

If an external moment of force $\tilde{\mathbf{M}}$ specified by (5.24) is applied to a given link, it is possible, by representing its components as pairs of forces [Grzegożek W., et al., 2003] and performing appropriate transformations, obtain the formula for the generalized force corresponding to the *i*-th generalized coordinate which is due to the moment $\tilde{\mathbf{M}}$:

$$Q_i(\tilde{\mathbf{M}}) = \tilde{M}_x \sum_{l=1}^{3} t_{l,3}\, t_{i,l,2} + \tilde{M}_y \sum_{l=1}^{3} t_{l,1}\, t_{i,l,3} + \tilde{M}_z \sum_{l=1}^{3} t_{l,2}\, t_{i,l,1}\,, \tag{5.28}$$

where $\left(t_{m,l}\right)_{m,l=1\ldots3}$, $\left(t_{i,m,l}\right)_{\substack{i=1,\ldots,n \\ m,l=1,\ldots,3}}$ are the corresponding elements of matrices $\mathbf{T}$ and $\mathbf{T}_i$, respectively.

Finally, using the Lagrange equations of the second kind, the equations of motion of the link concerned are put in this form:

$$\mathbf{A}\,\ddot{\mathbf{q}} = \mathbf{Q} - \mathbf{G} - \mathbf{e}\,, \tag{5.29}$$

where $\mathbf{A}$ – matrix of inertia defined in (5.19),
$\mathbf{G}$ – vector of gravity forces defined in (5.22),
$\mathbf{e}$ – vector of nonlinear forces defined in (5.19),
$\mathbf{Q}$ – vector of non-potential forces,
$\mathbf{Q} = \left(Q_i\right)_{i=1,\ldots,n}$,
$Q_i = Q_i\left(\tilde{\mathbf{F}}\right) + Q_i\left(\tilde{\mathbf{M}}\right)$.

## 5.4 Generalization of the Procedure

The equations of motion for a single link having been determined, the equations of motion of an arbitrary open kinematic chain (Fig. 5.3) can be formulated.

Since joint coordinates are used to describe motion, the motion of a link *p* depends on its generalized coordinates, of which there are $\tilde{n}_p$, and on the generalized coordinates of its predecessor *s* in the chain. The total number of generalized coordinates for a link *p* (including all the generalized coordinates of preceding links) will be denoted by $n_p$. The vector of generalized coordinates of a link *p* may therefore be written:

$$\mathbf{q}^{(p)} = \begin{bmatrix} \mathbf{q}^{(s)} \\ \tilde{\mathbf{q}}^{(p)} \end{bmatrix}, \tag{5.30}$$

where $\mathbf{q}^{(s)}$ – vector of generalized coordinates describing the motion of the link *s* preceding the link *p*,

$\tilde{\mathbf{q}}^{(p)} = [\tilde{q}_1^{(p)} \quad \ldots \quad \tilde{q}_{\tilde{n}_p}^{(p)}]^T$ – vector of generalized coordinates of the link $p$ describing its motion relative to the link $s$,

$\tilde{q}_i^{(p)} \in \left\{x^{(p)} \quad y^{(p)} \quad z^{(p)} \quad \varphi_x^{(p)} \quad \varphi_y^{(p)} \quad \varphi_z^{(p)}\right\}$,

$\mathbf{q}^{(p)} = [q_1^{(p)} \quad \ldots \quad q_{n_p}^{(p)}]^T$,

$n_p = n_s + \tilde{n}_p$ .

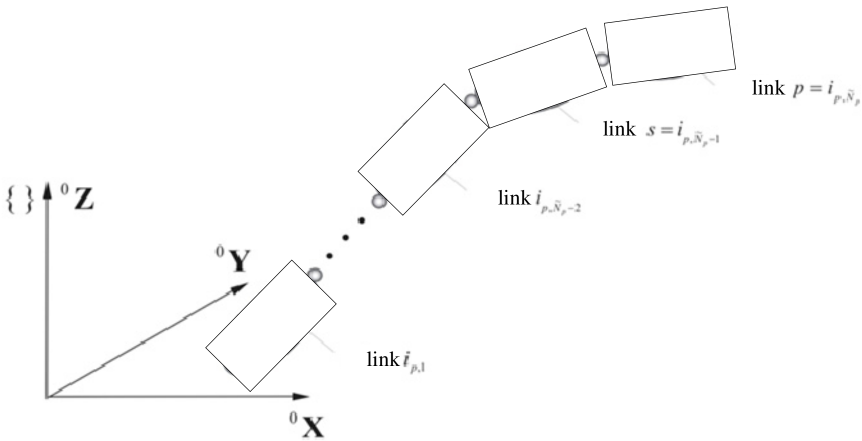


**Fig. 5.3.** A link $p$ and links preceding it in a kinematic chain

The presented procedure takes the tree structure of kinematic chains into consideration. Therefore, consecutive links in a chain need not be assigned consecutive ordinal numbers. In such cases, one needs to define an ordered set of indices of the preceding links in the kinematic chain along with the index of the concerned link $p$:

$$N_p = \{i_{p,1},\ldots,i_{p,l},\ldots,i_{p,\tilde{N}_p}\}, \tag{5.31}$$

whereas:

$$\begin{aligned} &\tilde{N}_p \quad \text{– number of elements of the set } N_p, \\ &i_{p,\tilde{N}_p} = p, \\ &i_{p,\tilde{N}_p-1} = s, \\ &N_p = N_s \cup \{p\}, \\ &\tilde{N}_p = \tilde{N}_s + 1. \end{aligned} \tag{5.32}$$

The vector $\mathbf{q}^{(p)}$ from the formula (5.30) may now be written as:

$$\mathbf{q}^{(p)} = \begin{bmatrix} \tilde{\mathbf{q}}^{(i_{p,1})^T} & \ldots & \tilde{\mathbf{q}}^{(i_{p,l})^T} & \ldots & \tilde{\mathbf{q}}^{(i_{p,\tilde{N}_p})^T} \end{bmatrix}^T , \tag{5.33}$$

and the transformation from the coordinate system of the point $p$ of the link to the inertial system { } may be expressed as follows:

$$\mathbf{r}^{(p)} = \mathbf{T}^{(p)}\,\tilde{\mathbf{r}}^{(p)} = \mathbf{T}^{(s)}\,\tilde{\mathbf{T}}^{(p)}\,\tilde{\mathbf{r}}^{(p)} = {}^{0}_{A}\mathbf{T}\,\overline{\mathbf{T}}^{(p)}\,\tilde{\mathbf{r}}^{(p)} , \tag{5.34}$$

where $\mathbf{T}^{(s)} = {}^{0}_{A}\mathbf{T}(t)\,\overline{\mathbf{T}}^{(s)}\left(\mathbf{q}^{(s)}\right)$,

$\mathbf{T}^{(p)} = {}^{0}_{A}\mathbf{T}(t)\,\overline{\mathbf{T}}^{(s)}\left(\mathbf{q}^{(s)}\right)\tilde{\mathbf{T}}^{(p)}\left(\tilde{\mathbf{q}}^{(p)}\right) = {}^{0}_{A}\mathbf{T}(t)\,\overline{\mathbf{T}}^{(p)}\left(\mathbf{q}^{(p)}\right)$,

$\overline{\mathbf{T}}^{(p)} = \overline{\mathbf{T}}^{(s)}\,\tilde{\mathbf{T}}^{(p)}$,

$\tilde{\mathbf{r}}^{(p)}$ – vector of coordinates of the point in the local coordinate system $\{p\}$.

Following the reasoning in section 5.1, thus is the kinetic energy of the link $p$:

$$E_p = \frac{1}{2}\mathrm{tr}\{\dot{\mathbf{T}}^{(p)}\,\mathbf{H}^{(p)}\,\dot{\mathbf{T}}^{(p)^T}\} , \tag{5.35}$$

where $\mathbf{H}^{(p)}$ – defined as in (5.11),

and the Lagrange operator for the link $p$ takes the form:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p)}}(E_p) = \tilde{\mathbf{A}}^{(p)}\,\ddot{\mathbf{q}}^{(p)} + \tilde{\mathbf{e}}^{(p)} , \tag{5.36}$$

where $\tilde{\mathbf{A}}^{(p)} = (\tilde{a}^{(p)}_{k,j})_{k,j=1,\ldots,n_p}$,

$\tilde{\mathbf{e}}^{(p)} = (\tilde{e}^{(p)}_{k})_{k=1,\ldots,n_p}$,

$\tilde{a}^{(p)}_{k,j} = \mathrm{tr}\left\{\mathbf{T}^{(p)}_k\,\mathbf{H}^{(p)}\,\mathbf{T}^{(p)^T}_j\right\}$,

$$\tilde{e}^{(p)}_k = \mathrm{tr}\left\{\mathbf{T}^{(p)}_k\,\mathbf{H}^{(p)}\left[{}^{0}_{A}\ddot{\mathbf{T}}\,\overline{\mathbf{T}}^{(p)} + 2\,{}^{0}_{A}\dot{\mathbf{T}}\dot{\overline{\mathbf{T}}}^{(p)} + \sum_{l=1}^{n_p}\sum_{j=1}^{n_p}\mathbf{T}^{(p)}_{l,j}\,\dot{q}^{(p)}_l\,\dot{q}^{(p)}_j\right]^T\right\},$$

$$\mathbf{T}^{(p)}_k = \frac{\partial\mathbf{T}^{(p)}}{\partial q^{(p)}_k} = {}^{0}_{A}\mathbf{T}(t)\frac{\partial\overline{\mathbf{T}}^{(p)}}{\partial q^{(p)}_k} = \begin{cases} {}^{0}_{A}\mathbf{T}(t)\overline{\mathbf{T}}^{(s)}_k\tilde{\mathbf{T}}^{(p)} & \text{for} \quad k = 1,\ldots,n_s, \\ {}^{0}_{A}\mathbf{T}(t)\overline{\mathbf{T}}^{(s)}\tilde{\mathbf{T}}^{(p)}_{k-n_s} & \text{for} \quad k = n_s+1,\ldots,n_p, \end{cases}$$

$$\mathbf{T}^{(p)}_{l,j} = \frac{\partial\mathbf{T}^{(p)}_l}{\partial q^{(p)}_j} = {}^{0}_{A}\mathbf{T}(t)\frac{\partial^2\overline{\mathbf{T}}^{(p)}}{\partial q^{(p)}_l\partial q^{(p)}_j} = \begin{cases} {}^{0}_{A}\mathbf{T}\overline{\mathbf{T}}^{(s)}_{l,j}\tilde{\mathbf{T}}^{(p)} & \text{for} \quad l,j = 1,\ldots,n_s, \\ {}^{0}_{A}\mathbf{T}\overline{\mathbf{T}}^{(s)}_l\tilde{\mathbf{T}}^{(p)}_{j-n_s} & \text{for} \quad \begin{cases} l = 1,\ldots,n_s, \\ j = n_s+1,\ldots,n_p, \end{cases} \\ {}^{0}_{A}\mathbf{T}\overline{\mathbf{T}}^{(s)}\tilde{\mathbf{T}}^{(p)}_{l-n_s,j-n_s} & \text{for} \quad l,j = n_s+1,\ldots,n_p. \end{cases}$$

This relation may also be written with a matrix and vector blocks:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p)}}(E^{(p)}) = \begin{bmatrix} \tilde{\mathbf{A}}^{(p)}_{i_{p,1},i_{p,1}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,1},i_{p,k}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,1},i_{p,\tilde{N}_p}} \\ \vdots & & \vdots & & \vdots \\ \tilde{\mathbf{A}}^{(p)}_{i_{p,j},i_{p,1}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,j},i_{p,k}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,j},i_{p,\tilde{N}_p}} \\ \vdots & & \vdots & & \vdots \\ \tilde{\mathbf{A}}^{(p)}_{i_{p,\tilde{N}_p},i_{p,1}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,\tilde{N}_p},i_{p,k}} & \cdots & \tilde{\mathbf{A}}^{(p)}_{i_{p,\tilde{N}_p},i_{p,\tilde{N}_p}} \end{bmatrix} \begin{bmatrix} \ddot{\tilde{\mathbf{q}}}^{(i_{p,1})} \\ \vdots \\ \ddot{\tilde{\mathbf{q}}}^{(i_{p,j})} \\ \vdots \\ \ddot{\tilde{\mathbf{q}}}^{(i_{p,\tilde{N}_p})} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{e}}^{(p)}_{i_{p,1}} \\ \vdots \\ \tilde{\mathbf{e}}^{(p)}_{i_{p,j}} \\ \vdots \\ \tilde{\mathbf{e}}^{(p)}_{i_{p,\tilde{N}_p}} \end{bmatrix}, \tag{5.37}$$

where $\widetilde{\mathbf{A}}^{(p)}_{i_{p,j},i_{p,k}} = \left(\widetilde{a}^{(p)}_{n_{\alpha_j}+l,n_{\alpha_k}+m}\right)_{\substack{l=1,\ldots,\widetilde{n}_{i_{p,j}} \\ m=1,\ldots,\widetilde{n}_{i_{p,k}}}}$,

$$\widetilde{\mathbf{e}}^{(p)}_{i_{p,j}} = \left(\widetilde{e}^{(p)}_{n_{\alpha_j}+l}\right)_{l=1,\ldots,\widetilde{n}_{i_{p,j}}},$$

$$n_{\alpha_j} = \sum_{v=1}^{j-1} \widetilde{n}_{i_{p,v}}$$

or:

$$\boldsymbol{\varepsilon}_{\mathbf{q}^{(p)}}(E^{(p)}) = \begin{bmatrix} \hat{\mathbf{A}}^{(p)}_{s,s} & \hat{\mathbf{A}}^{(p)}_{s,p} \\ \hat{\mathbf{A}}^{(p)}_{p,s} & \hat{\mathbf{A}}^{(p)}_{p,p} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(s)} \\ \ddot{\tilde{\mathbf{q}}}^{(p)} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{e}}^{(p)}_s \\ \hat{\mathbf{e}}^{(p)}_p \end{bmatrix}, \tag{5.38}$$

where $\hat{\mathbf{A}}^{(p)}_{s,s} = \begin{bmatrix} \widetilde{\mathbf{A}}^{(p)}_{i_{p,1},i_{p,1}} & \cdots & \widetilde{\mathbf{A}}^{(p)}_{i_{p,1},s} \\ \vdots & \vdots & \vdots \\ \widetilde{\mathbf{A}}^{(p)}_{i_{p,1},s} & \cdots & \widetilde{\mathbf{A}}^{(p)}_{s,s} \end{bmatrix}$, $\hat{\mathbf{A}}^{(p)}_{s,p} = \begin{bmatrix} \widetilde{\mathbf{A}}^{(p)}_{i_{p,1},i_{p,\widetilde{N}^{(p)}}} \\ \vdots \\ \widetilde{\mathbf{A}}^{(p)}_{s,i_{p,\widetilde{N}^{(p)}}} \end{bmatrix}$,

$$\hat{\mathbf{A}}^{(p)}_{p,s} = \begin{bmatrix} \widetilde{\mathbf{A}}^{(p)}_{p,i_{p,1}} & \cdots & \widetilde{\mathbf{A}}^{(p)}_{p,s} \end{bmatrix}, \qquad \hat{\mathbf{A}}^{(p)}_{p,p} = \widetilde{\mathbf{A}}^{(p)}_{p,p},$$

$$\hat{\mathbf{e}}^{(p)}_s = \begin{bmatrix} \widetilde{\mathbf{e}}^{(p)}_{i_{p,1}} \\ \vdots \\ \widetilde{\mathbf{e}}^{(p)}_s \end{bmatrix}, \qquad \hat{\mathbf{e}}^{(p)}_p = \left[\widetilde{\mathbf{e}}^{(p)}_p\right],$$

$s, p$ – defined in (5.32).

Making use of the dependency (5.22) obtained in section 5.2, we may express the derivative of potential energy of gravity forces of the link $p$ with respect to generalized coordinates as:

$$\frac{\partial V^g_p}{\partial \mathbf{q}^{(p)}} = \left(\widetilde{g}^{(p)}_l\right)_{l=1,\ldots,n_p}, \tag{5.39}$$

where $\widetilde{g}^{(p)}_l = m^{(p)}\, g\, \boldsymbol{\theta}_3\, \mathbf{T}^{(p)}_l\, \widetilde{\mathbf{r}}^{(p)}_c$.

The dependency (5.39) may thus be rewritten:

$$\frac{\partial V_p^g}{\partial \mathbf{q}^{(p)}} = \tilde{\mathbf{G}}^{(p)} = \begin{bmatrix} \tilde{\mathbf{G}}_1^{(p)} \\ \vdots \\ \tilde{\mathbf{G}}_{i_{p,j}}^{(p)} \\ \vdots \\ \tilde{\mathbf{G}}_{i_{p,\tilde{N}_p}}^{(p)} \end{bmatrix}, \tag{5.40}$$

where $\tilde{\mathbf{G}}_j^{(p)} = \left(\tilde{g}_{n_{\alpha_j}+l}^{(p)}\right)_{l=1,\ldots,\tilde{n}_{i_{p,j}}}$,

equivalently:

$$\frac{\partial V_p^g}{\partial \mathbf{q}^{(p)}} = \tilde{\mathbf{G}}^{(p)} = \begin{bmatrix} \hat{\mathbf{G}}_s^{(p)} \\ \hat{\mathbf{G}}_p^{(p)} \end{bmatrix}, \tag{5.41}$$

where $\hat{\mathbf{G}}_s^{(p)} = \begin{bmatrix} \tilde{\mathbf{G}}_{i_{p,1}}^{(p)} \\ \vdots \\ \tilde{\mathbf{G}}_s^{(p)} \end{bmatrix}$, $\hat{\mathbf{G}}_p^{(p)} = \tilde{\mathbf{G}}_p^{(p)}$.

The generalized forces due to external forces and moments thereof are calculated like in section 5.3 giving:

$$\tilde{\mathbf{Q}}^{(p)} = \begin{bmatrix} \tilde{\mathbf{Q}}_{i_{p,1}}^{(p)} \\ \vdots \\ \tilde{\mathbf{Q}}_{i_{p,j}}^{(p)} \\ \vdots \\ \tilde{\mathbf{Q}}_{i_{p,\tilde{N}_p}}^{(p)} \end{bmatrix}, \tag{5.42}$$

where $\tilde{\mathbf{Q}}_{i_{p,j}}^{(p)} = \left(\tilde{Q}_{n_{\alpha_j}+l}^{(p)}(\tilde{\mathbf{F}}^{(p)}) + \tilde{Q}_{n_{\alpha_j}+l}^{(p)}(\tilde{\mathbf{M}}^{(p)})\right)_{l=1,\ldots,\tilde{n}_{i_{p,j}}}$,

$n_{\alpha_j}$ – defined in (5.37),

or:

$$\tilde{\mathbf{Q}}^{(p)} = \begin{bmatrix} \hat{\mathbf{Q}}_s^{(p)} \\ \hat{\mathbf{Q}}_p^{(p)} \end{bmatrix}, \tag{5.43}$$

where $\hat{\mathbf{Q}}_s^{(p)} = \begin{bmatrix} \tilde{\mathbf{Q}}_{i_{p,1}}^{(p)} \\ \vdots \\ \tilde{\mathbf{Q}}_s^{(p)} \end{bmatrix}$, $\hat{\mathbf{Q}}_p^{(p)} = \tilde{\mathbf{Q}}_p^{(p)}$.

Finally, the equations of motion of the link $p$ may be written thus:

$$\tilde{\mathbf{A}}^{(p)}\ddot{\mathbf{q}}^{(p)} = \tilde{\mathbf{f}}^{(p)}, \tag{5.44}$$

where $\tilde{\mathbf{f}}^{(p)} = \tilde{\mathbf{Q}}^{(p)} - \tilde{\mathbf{e}}^{(p)} - \tilde{\mathbf{G}}^{(p)}$,

$\tilde{\mathbf{A}}^{(p)}, \tilde{\mathbf{e}}^{(p)}, \tilde{\mathbf{G}}^{(p)}, \tilde{\mathbf{Q}}^{(p)}$ – defined by formulas (5.36), (5.40) and (5.42), respectively.

The equations (5.44) describing the motion of the link $p$ also indicate how its motion depends on the generalized coordinates of the preceding links (i.e. the coordinates $\tilde{\mathbf{q}}^{(i_{p,1})},\ldots,\tilde{\mathbf{q}}^{(s)}$ of the vector $\mathbf{q}^{(s)}$) and its own coordinates, i.e. $\tilde{\mathbf{q}}^{(p)}$. Therefore, those equations may be written in the form:

$$\begin{bmatrix} \hat{\mathbf{A}}_{s,s}^{(p)} & \hat{\mathbf{A}}_{s,p}^{(p)} \\ \hat{\mathbf{A}}_{p,s}^{(p)} & \hat{\mathbf{A}}_{p,p}^{(p)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(s)} \\ \ddot{\tilde{\mathbf{q}}}^{(p)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{f}}_s^{(p)} \\ \hat{\mathbf{f}}_p^{(p)} \end{bmatrix}, \tag{5.45}$$

where $\hat{\mathbf{f}}_s^{(p)} = \hat{\mathbf{Q}}_s^{(p)} - \hat{\mathbf{e}}_s^{(p)} - \hat{\mathbf{G}}_s^{(p)}$, $\hat{\mathbf{f}}_p^{(p)} = \hat{\mathbf{Q}}_p^{(p)} - \hat{\mathbf{e}}_p^{(p)} - \hat{\mathbf{G}}_p^{(p)}$.

The above equations of motion are obtained taking into account the kinetic energy and the potential energy of gravity forces of a single link $p$ as well as the force $\tilde{\mathbf{F}}^{(p)}$ and the moment of force $\tilde{\mathbf{M}}^{(p)}$ acting upon this link.

If a kinematic chain has links numbered 1 to $p$, the energies: kinetic and potential of gravity forces of the system are given by the expressions:

$$E = \sum_{i=1}^{p} E_i , \tag{5.46}$$

$$V^g = \sum_{i=1}^{p} V_i^g . \tag{5.47}$$

Let us assume that the equations of motion of the links 1 to $p$-1, which take into account the kinetic energy $E_i$, the potential energy $V_i^g$, the forces $\tilde{\mathbf{F}}^{(i)}$ and the moments of forces $\tilde{\mathbf{M}}^{(i)}$ ($i = 1,\ldots, p-1$), have the form:

$$\mathbf{A}^{(p-1)}\ddot{\mathbf{q}}^{(p-1)} = \mathbf{f}^{(p-1)} , \tag{5.48}$$

where $\mathbf{A}^{(p-1)}$ is a matrix of dimension $n_{p-1} \times n_{p-1}$, and $\mathbf{q}$ and $\mathbf{f}$ are $n_{p-1} = \sum_{i=1}^{p-1} \tilde{n}_i$ -element vectors.

Adding a link $p$ so that it connects with the link $s \le p-1$ belonging to the considered kinematic chain makes the equations of motion of the entire system of links 1 to $p$ expressible as:

$$\mathbf{A}^{(p)}\ddot{\mathbf{q}}^{(p)} = \mathbf{f}^{(p)}, \tag{5.49.1}$$

or:

$$\left[\begin{array}{c|c} \mathbf{A}^{(p-1)} + \breve{\mathbf{A}}^{(p)}_{s,s} & \breve{\mathbf{A}}^{(p)}_{s,p} \\ \hline \breve{\mathbf{A}}^{(p)}_{p,s} & \hat{\mathbf{A}}^{(p)}_{p,p} \end{array}\right] \begin{bmatrix} \ddot{\mathbf{q}}^{(p-1)} \\ \ddot{\tilde{\mathbf{q}}}^{(p)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(p-1)} + \breve{\mathbf{f}}^{(p)}_{s} \\ \hat{\mathbf{f}}^{(p)}_{p} \end{bmatrix}, \tag{5.49.2}$$

where $\breve{\mathbf{A}}^{(p)}_{s,s}, \breve{\mathbf{A}}^{(p)}_{s,p}, \breve{\mathbf{A}}^{(p)}_{p,s}, \breve{\mathbf{f}}^{(p)}_{s}$ – matrices of dimensions $n_{p-1} \times n_{p-1}$, $n_{p-1} \times \tilde{n}_p$, $\tilde{n}_p \times n_{p-1}$ in which the appropriate submatrices with indices $i, j \in N_p$ are calculated according to:

$$\left(\breve{\mathbf{A}}^{(p)}_{s,s}\right)_{k,l} = \begin{cases} \tilde{\mathbf{A}}^{(p)}_{k,l} & \text{when } k,l \in N_p, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{for } k,l = 1,\ldots,n_{p-1},$$

$$\left(\breve{\mathbf{A}}^{(p)}_{s,p}\right)_{k} = \begin{cases} \tilde{\mathbf{A}}^{(p)}_{k,p} & \text{when } k \in N_p, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{for } k = 1,\ldots,n_{p-1},$$

$$\left(\breve{\mathbf{A}}^{(p)}_{p,s}\right)_{k} = \begin{cases} \tilde{\mathbf{A}}^{(p)}_{p,k} & \text{when } k \in N_p, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{for } k = 1,\ldots,n_{p-1},$$

$$\left(\breve{\mathbf{f}}^{(p)}_{s}\right)_{k} = \begin{cases} \tilde{\mathbf{f}}^{(p)}_{k} & \text{when } k \in N_p, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{for } k = 1,\ldots,n_{p-1}.$$