Finitely Supported Measures on $SL_2(\mathbb{R})$ Which are Absolutely Continuous at Infinity

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Abstract We construct finitely supported symmetric probability measures on $SL_2(\mathbb{R})$ for which the Furstenberg measure on $\mathbb{P}_1(\mathbb{R})$ has a smooth density.

1 Introduction

In this note, we give explicit examples of finitely supported symmetric probability measures ν on $SL_2(\mathbb{R})$ for which the corresponding Furstenberg measure μ on $\mathbb{P}_1(\mathbb{R})$ is absolutely continuous wrt to Haar measure $d\theta$, and moreover $\frac{d\mu}{d\theta}$ is of class C^r, with r any given positive integer. Probabilistic constructions of finitely supported (non-symmetric measures ν on $SL_2(\mathbb{R})$ with absolutely continuous Furstenberg measure appear in the paper [1], setting (in the negative) a conjecture from [4]. The construction in [1] may be viewed as a non-commutative analogue of the theory of random Bernoulli convolutions and uses methods from [5, 6].

It is not clear if this technique may produce Furstenberg measures with say C^1-density. Our method also addresses the issue of obtaining a symmetric ν (raised in [4]), which seems problematic with the [1] technique.

Our starting point is a construction from [2] of certain Hecke operators on $SL_2(\mathbb{R})$ whose projective action exhibits a spectral gap. The mathematics underlying [2] is closely related to the paper [3] and makes essential use of results and techniques from arithmetic combinatorics. In particular, it should be pointed out that

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the spectral gap is not achieved by exploiting hyperbolicity, at least not in the usual way. Our measure ν has in fact a Lyapounov exponent that can be made arbitrary small, while the spectral gap (in an appropriate restricted sense) remains uniformly controlled (the size of supp ν becomes larger of course).

We believe that similar constructions are possible also in the $SL_d(\mathbb{R})$-setting, for $d > 2$ (cf. [4]). In fact, such Hecke operators can be produced using the construction from Lemmas 1 and 2 below in $SL_2(\mathbb{R})$ and considering a suitable family of $SL_2(\mathbb{R})$-embeddings in SL_d. We do not present the details here.

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2 Preliminaries

We recall Lemmas 2.1 and 2.2 from [2].

Lemma 1. *Given $\varepsilon > 0$, there is $Q \in \mathbb{Z}_+$ and $\mathcal{G} \subset SL_2(\mathbb{R}) \cap \left(\frac{1}{Q} Mat_2(\mathbb{Z})\right)$ with the following properties*

$$\frac{1}{\varepsilon} < Q < \left(\frac{1}{\varepsilon}\right)^{c_1} \tag{1}$$

$$|\mathcal{G}| > Q^{c_2} \tag{2}$$

$$\text{The elements of } \mathcal{G} \text{ are free generators of a free group} \tag{3}$$

$$\|g - 1\| < \varepsilon \text{ for } g \in \mathcal{G} \tag{4}$$

Here c_1, c_2 are constants independent of ε.

Define the probability measure ν on $SL_2(\mathbb{R})$ as

$$\nu = \frac{1}{2|\mathcal{G}|} \sum_{g \in \mathcal{G}} (\delta_g + \delta_{g^{-1}}). \tag{5}$$

Denote also $P_\delta, \delta > 0$, an approximate identity on $SL_2(\mathbb{R})$. For instance, one may take $P_\delta = \frac{1_{B_\delta(1)}}{|B_\delta(1)|}$ where $B_\delta(1)$ is the ball of radius δ around 1 in $SL_2(\mathbb{R})$.

Lemma 2. *Fix $\tau > 0$. Then we have*

$$\|\nu^{(\ell)} * P_\delta\|_\infty < \delta^{-\tau} \tag{6}$$

provided

$$\ell > c_3(\tau) \frac{\log 1/\delta}{\log 1/\varepsilon} \tag{7}$$

and assuming δ small enough (depending on Q and τ).

3 Furstenberg Measure

Denote for $g \in SL_2(\mathbb{R})$ by τ_g the action on $P_1(\mathbb{R})$ that we identify with the circle $\mathbb{R}/\mathbb{Z} = \mathbb{T}$. Thus if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, then

$$e^{i\tau_g(\theta)} = \frac{(a\cos\theta + b\sin\theta) + i(c\cos\theta + d\sin\theta)}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{\frac{1}{2}}}. \tag{8}$$

Assume μ on $P_1(\mathbb{R})$ is ν-stationary, i.e.

$$\mu = \sum \nu(g) g_*[\mu]. \tag{9}$$

4 A Restricted Spectral Gap

Take $\mathcal{G}$ as in Lemma 1 and $\nu = \frac{1}{2r}\sum_{g\in\mathcal{G}}(\delta_g + \delta_{g-1})$ with $r = |\mathcal{G}|$.

Lemma 3. *There is some constant $K > 0$ (depending on ν), such that if $f \in L^2(\mathbb{T})$ satisfies*

$$\|f\|_2 \le 1 \text{ and } \hat{f}(n) = 0 \text{ for } |n| < K \tag{10}$$

then

$$\left\| \int (f \circ \tau_g) d\nu \right\|_2 < \frac{1}{2}. \tag{11}$$

Proof. Define $\rho_g f = (\tau_g')^{1/2}(f \circ \tau_g)$, hence ρ is the projective representation. Since $\|1 - g\| < \varepsilon$, $|\tau_g' - 1| \lesssim \varepsilon$ and (11) will follow from

$$\left\| \int (\rho_g f)\nu(dg) \right\|_2 < \frac{1}{3}. \tag{12}$$

Assume (12) fails. By almost orthogonality, there is $f \in L^2(\mathbb{T})$ such that

$$\operatorname{supp} \hat{f} \subset [2^k, 2^{k+1}] \tag{13}$$

$$\|f\|_2 = 1 \tag{14}$$

$$\left\| \int (\rho_g f)\nu(dg) \right\|_2 > c (\text{for some } c > 0). \tag{15}$$

Let $\ell < k$ to be specified. From (15), since ν is symmetric,

$$\left\| \int (\rho_g f)\nu^{(\ell)}(dg) \right\|_2 > c^\ell \tag{16}$$

and hence

$$\int |\langle \rho_g f, f\rangle| \nu^{(2\ell)}(dg) = \iint |\langle \rho_g f, \rho_h f\rangle| \nu^{(\ell)}(dg)\nu^{(\ell)}(dh) > c^{2\ell}. \tag{17}$$

Take $\delta = 10^{-k}$. Recalling (13), straightforward approximation permits us to replace in (16) the discrete measure $\nu^{(\ell)}$ by $\nu^{(\ell)} * P_\delta$, where $P_\delta(\delta > 0)$ denotes the approximate identity on $SL_2(\mathbb{R})$. Hence (17) becomes

$$\int_{SL_2(\mathbb{R})} |\langle \rho_g f, f\rangle|(\nu^{2\ell} * P_\delta)(g)dg + 2^{-k} > c^{2\ell}. \tag{18}$$

Fix a small constant $\tau > 0$ and apply Lemma 2. This gives

$$\ell \sim C(\tau)\frac{\log \frac{1}{\delta}}{\log \frac{1}{\varepsilon}} \tag{19}$$

such that

$$\|\nu^{(\ell)} * P_\delta\|_\infty < \delta^{-\tau}. \tag{20}$$

Note that supp $\nu^{(\ell)}$ is contained in a ball of radius at most $(1+\varepsilon)^\ell$, by (4).

Introduce a smooth function $0 \le \omega \le 1$ on $\mathbb{R}$, $\omega = 1$ on $[-(1+\varepsilon)^{4\ell}, (1+\varepsilon)^{4\ell}]$ and $\omega = 0$ outside $[-2(1+\varepsilon)^{4\ell}, 2(1+\varepsilon)^{4\ell}]$.

Let $\omega_1(g) = \omega(a^2+b^2+c^2+d^2)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

From (20), the first term of (18) is bounded by

$$\delta^{-\tau}\int_{SL_2(\mathbb{R})} |\langle \rho_g f, f\rangle|\omega_1(g)dg. \tag{21}$$

Note also that by assuming ε a sufficiently small constant, we can ensure that $\ell \ll k$ and $2^{-k} < c^{2\ell}$. Thus

$$\int_{SL_2(\mathbb{R})} |\langle \rho_g f, f\rangle|\omega_1(g)dg > \frac{1}{2}\delta^\tau c^{2\ell} \tag{22}$$

and applying Cauchy-Schwarz

$$\begin{aligned} c^{4\ell}\delta^{2\tau}(1+\varepsilon)^{-6\ell} &\le \int_{SL_2(\mathbb{R})} |\langle \rho_g f, f\rangle|^2 \omega_1(g)dg \\ &= \left|\int_{SL_2(\mathbb{R})}\int_{\mathbb{T}}\int_{\mathbb{T}} f(x)\overline{f(y)}\,\overline{f(\tau_g x)}f(\tau_g y)(\tau_g'(x))^{1/2}(\tau_g'(y))^{1/2}\omega_1(g)dgdxdy\right| \\ &\le \int_{\mathbb{T}}\int_{\mathbb{T}} |f(x)|\,|f(y)|\left|\int_{SL_2(\mathbb{R})} f(\tau_g x)\overline{f(\tau_g y)}(\tau_g'(x))^{1/2}\big(\tau_g'(y)\big)^{\frac{1}{2}}\omega_1(g)dg\right|dxdy. \end{aligned} \tag{23}$$

Fix $x \neq y$ and consider the inner integral. If we restrict $g \in SL_2(\mathbb{R})$ s.t. $\tau_g x = \theta$ (fixed), there is still an averaging in $\psi = \tau_g y$ that can be exploited together with (13). By rotations, we may assume $x = \theta = 0$. Write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $dg = \frac{dadbdc}{a}$ on the chart $a \neq 0$. Since

$$e^{i\tau_g x} = \frac{(a\cos x + b\sin x) + i(c\cos x + d\sin x)}{[(a\cos x + b\sin x)^2 + (c\cos x + d\sin x)^2]^{1/2}}$$

the condition $\tau_g 0 = 0$ means $c = 0$ and thus

$$e^{i\psi} = e^{i\tau_g y} = \frac{(a\cos y + b\sin y) + \frac{i}{a}\sin y}{[(a\cos y + b\sin y)^2 + \frac{1}{a^2}\sin^2 y]^{\frac{1}{2}}}.$$

Hence, fixing a

$$\frac{\partial\psi}{\partial b} = -a\sin^2\psi. \tag{24}$$

Also

$$\tau_g'(z) = \frac{\cos^2\tau_g(z)}{(a\cos z + b\sin z)^2} = a^2\,\frac{\sin^2\tau_g(z)}{\sin^2 z} \tag{25}$$

implying

$$\tau_g'(0) = \frac{1}{a^2} \text{ and } \tau_g'(y) = \frac{a^2\sin^2\psi}{\sin^2 y}. \tag{26}$$

Substituting (24), (26) in (23) gives for the inner integral the bound

$$\frac{1}{|\sin(x-y)|}\iint d\theta\frac{da}{a^2}|f(\theta)|$$
$$\cdot\left|\int f(\psi)\frac{1}{|\sin(\theta-\psi)|}\omega\left(a^2+\frac{1}{a^2}+\left(\frac{1}{a}\operatorname{cotg}(\psi-\theta)-a\operatorname{cotg}(y-x)\right)^2\right)d\psi\right|. \tag{27}$$

The weight function restricts a to $(1+\varepsilon)^{-2\ell} \lesssim |a| \lesssim (1+\varepsilon)^{2\ell}$ and clearly

$$|\sin(\theta-\psi)| \gtrsim (1+\varepsilon)^{-4\ell}|\sin(x-y)|. \tag{28}$$

If we restrict $|\sin(x-y)| > 2^{-\frac{k}{10}}$, Assumption (13) gives a bound at most $2^{-k}\|f\|_1$ for the ψ-integral in (27). Indeed, if β is a smooth function vanishing on a neighborhood of 0 and $|n| \sim 2^k$, partial integration implies that for any given $A > 0$

$$\int e^{-in\psi}\frac{1}{\sin(\theta-\psi)}\,\beta\big(2^{\frac{k}{10}}(\theta-\psi)\big)d\psi \lesssim 2^{-Ak}.$$

Thus

$$(27) < 2^{k/10}(1+\varepsilon)^{2\ell}\, 2^{-k}\|f\|_1^2. \tag{29}$$

The contribution to (23) is at most

$$2^{-k/2}(1+\varepsilon)^{2\ell}\|f\|_1^4. \tag{30}$$

Next we consider, the contribution of $|\sin(x-y)| \le 2^{-\frac{k}{10}}$ to (23).

First, from (25), we have that

$$|\tau_g'| \lesssim a^2 + \frac{1}{a^2} + b^2 \lesssim \|g\|^2 < (1+\varepsilon)^{4\ell}.$$

By Cauchy-Schwarz, the inner integral in (23) is at most

$$(1+\varepsilon)^{4\ell}\left(\int |f(\tau_g x)|^2 \omega_1(g) dg\right) < (1+\varepsilon)^{10\ell}\|f\|_2^2.$$

Hence, we obtain

$$\left[\int_{|x-y|<2^{-k/10}} |f(x)|\,|f(y)| dx dy\right](1+\varepsilon)^{10\ell}\|f\|_2^2$$
$$< 2^{-k/20}(1+\varepsilon)^{10\ell}\|f\|_2^4. \tag{31}$$

From (30), (31),

$$(23) \le 2^{-k/20}(1+\varepsilon)^{10\ell}$$

and hence, by (19)

$$2^{k/10} < 100^{k\tau}.C^{C(\tau)(\log\frac{1}{\varepsilon})^{-1}k}. \tag{32}$$

Taking (in order) τ and ε small enough, a contradiction follows.

This proves Lemma 3.

5 Absolute Continuity of the Furstenberg Measure and Smoothness of the Density

Our aim is to establish the following.

Theorem. *Let μ be the stationary measure introduced in (9). Given $r \in \mathbb{Z}_+$ and taking ε in Lemma 1 small enough will ensure that $\frac{d\mu}{d\theta} \in C^r$.*

This will be an immediate consequence of

Lemma 4. *Let $k > k(\varepsilon)$ be sufficiently large and $f \in L^\infty(\mathbb{T}), |f| \le 1$ such that* supp $\hat{f} \subset [2^{k-1}, 2^k]$. *Then*

$$|\langle f, \mu\rangle| < C_\varepsilon^{-k} \tag{33}$$

where $C_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \infty$.

Proof. Clearly, for any $\ell \in \mathbb{Z}_+$

$$|\langle f, \mu\rangle| \leq \Big\| \sum_g \nu^{(\ell)}(g)(f\circ\tau_g)\Big\|_\infty. \tag{34}$$

We will iterate Lemma 3 and let $K = K(\varepsilon)$ satisfy (10), (11).

We assume $2^k > 10K^{10}$. For $m < \ell$ and $|n| < K$, we evaluate $|\widehat{F_m}(n)|$, denoting

$$F_m = \sum_g \nu^{(m)}(g)(f\circ\tau_g). \tag{35}$$

Clearly $|\widehat{F_m}(n)| \leq \max_{g\in \operatorname{supp}\nu^{(m)}} |(f\circ\tau_g)^\wedge(n)|$ and by assumption on supp $\hat{f}$

$$\begin{aligned}|(f\circ\tau_g)^\wedge(n)| &= \left|\int f\big(\tau_g(x)\big)e^{-2\pi i n x}dx\right| \\ &\leq 2^{k/2}\|f\|_2 \max_{n'\in[2^{k-1},2^k]}\left|\int e^{2\pi i(n'\tau_g(x)-nx)}dx\right|.\end{aligned}$$

Performing a change of variables gives

$$\begin{aligned}\left|\int e^{2\pi i(n'\tau_g(x)-nx)}dx\right| &= \left|\int e^{2\pi i(n'y-n\tau_{g^{-1}}(y))}\tau'_{g^{-1}}(y)dy\right| \\ &\ll_r \|e^{-2\pi i n\tau_{g-1}}\tau'_{g-1}\|_{C^r}|n'|^{-r} \\ &\ll_r \frac{K^r}{|n'|^r}(1+\varepsilon)^{2m(r+1)} \ll_r 2^{-\frac{3}{4}kr}(1+\varepsilon)^{2\ell(r+1)}\end{aligned} \tag{36}$$

by partial integration and our assumptions. It follows from (36) that if ℓ satisfies

$$\ell < \frac{k}{100\varepsilon} \tag{37}$$

then for $m < \ell$ and $k > k(r)$

$$\max_{|n|<K}|\widehat{F_m}(n)| < 2^{-\frac{kr}{2}} \tag{38}$$

(with r a fixed large integer).

Next, decompose

$$F_m = F_m^{(1)} + F_m^{(2)} \text{ where } F_m^{(1)}(x) = \sum_{|n|<K}\widehat{F_m}(n)e^{2\pi i n x}.$$

Hence, by (38)

$$\|F_m^{(1)}\|_\infty < 2K2^{-\frac{kr}{2}}. \tag{39}$$

Estimate using (39) and Lemma 3

$$\begin{aligned}\|F_{m+1}\|_2 &\le \left\|\int (F_m^{(1)} \circ \tau_g) d\nu\right\|_\infty + \left\|\int (F_m^{(2)} \circ \tau_g) d\nu\right\|_2 \\ &\le \|F_m^{(1)}\|_\infty + \frac{1}{2}\|F_m^{(2)}\|_2 \\ &\le 3K2^{-\frac{kr}{2}} + \frac{1}{2}\|F_m\|_2. \end{aligned} \tag{40}$$

Iteration of (40) implies by (37)

$$\|F_\ell\|_2 \le 4K2^{-\frac{kr}{2}} + 2^{-\ell} \lesssim 2^{-\frac{kr}{2}} + 2^{-\frac{k}{100\varepsilon}}. \tag{41}$$

Also

$$|F_\ell'| \le \max_{g \in \operatorname{supp} \nu^{(\ell)}} \|(f \circ \tau_g)'\|_\infty \le \|f'\|_\infty (1+\varepsilon)^{2\ell} \lesssim 5^k \tag{42}$$

and interpolation between (41), (42) implies for r (resp. ε) large (resp. small) enough

$$\|F_\ell\|_\infty \lesssim (41)^{1/2}.(42)^{1/2} < 2^{-\frac{kr}{5}} + 2^{-\frac{k}{300\varepsilon}} \tag{43}$$

provided $k > k(\varepsilon, r)$.

In view of (34), this proves (33).

Remark. For ν finitely supported (with positive Lyapounov exponent), one cannot obtain a Furstenberg measure μ that equals Haar measure on $\mathbb{P}_1(\mathbb{R}) \simeq \mathbb{T}$. Indeed, otherwise for any f on $\mathbb{T}$, we would have

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{T}} f d\mu = \int \nu(dg)\Big[\int (f \circ \tau_g) d\mu\Big] \\ &= \int \nu(dg)\Big[\int f(x)(\tau_{g^{-1}})'(x) dx\Big]. \end{aligned} \tag{44}$$

For $g \in SL_2(\mathbb{R})$,

$$\begin{aligned}\int f(x)(\tau_{g^{-1}})'(x)dx &= \int f(\theta) P_z(2\theta) d\theta \\ &= \hat{f}(0) + \sum_{n \neq 0} |z|^{|n|} e^{2\pi i n (Arg z)} \hat{f}(-2n) \end{aligned} \tag{45}$$

for some $z \in D = \{z \in \mathbb{C}; |z| < 1\}$, with $P_z(\theta) = \frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2}$ the Poisson kernel.

From (44), (45), taking $\nu = \sum_{j=1}^{r} c_j \delta_{g_j}, c_j > 0$ and $\sum c_j = 1$ and $\{z_j\}$ the corresponding points in D, we get

$$\sum_{1}^{r} c_j |z_j|^n e^{2\pi i n (Arg z_j)} = 0 \text{ for all } n \neq 0. \tag{46}$$

This easily implies that $z_1 = \cdots = z_r = 0$. But then each g_j has unimodular spectrum and ν vanishing Lyapounov exponent.

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