Revisiting the Notion of Conflicting Belief Functions

Sébastien Destercke and Thomas Burger

Abstract. The problem of conflict measurement between information sources knows a regain of interest. In most works related to this issue, Dempter's rule plays a central role. In this paper, we propose to revisit conflict from a different perspective. We do not make a priori assumption about dependencies and start from the definition of conflicting sets, studying its possible extensions to the framework of belief functions.

Keywords: Consistency, Fusion, Contour Function, Dependence.

1 Introduction

In this paper, we revisit the notion of conflict and its quantification in Dempster-Shafer theory (DST), in which it plays an essential role. In particular, its uses in merging rules is the matter of lively debates [1]. Recently, some researchers have questioned the validity of the usual conflict measure (i.e., the mass attributed to the empty set after combination) [2,3]. To solve the issue, they have mostly proposed to complement the usual measure with others. In this work, we take a rather different approach. Two main ideas have motivated this study:

1. First, the idea that conflict between belief functions should be an extension of conflict between sets: when belief functions reduce to sets, the conflict measure should be a binary value that is maximum in case of disjoint sets, minimum otherwise.
2. Second, the idea that conflict between sources should not *a priori* depend on a specific independence assumption between the sources. This is coherent with the *least commitment* principle.

Sebastien Destercke
CNRS, UMR Heudiasyc, Centre de recherche de Royallieu, 60205 COMPIEGNE
e-mail: sebastien.destercke@hds.utc.fr

Thomas Burger
CNRS (FR3425), CEA (iRTSV/BGE), INSERM (EDyP, U1038), Grenoble, France
e-mail: thomas.burger@cea.fr

T. Denœux & M.-H. Masson (Eds.): Belief Functions: Theory & Appl., AISC 164, pp. 153–160.
springerlink.com © Springer-Verlag Berlin Heidelberg 2012

After recalling some basics (Section 2), Section 3 investigates how consistency degree of a single mass assignment can be defined. Then, in Sections 4 and 5, we investigate the case of conflict between sets, and the case of conflict between mass functions. This study leads us to two different propositions of conflict measures, whose differences are briefly discussed in Section 6.

2 Preliminaries

We assume the reader to be familiar with DST [4, 5], and we only present notations and unusual definitions. A *mass assignment* m over Ω is a mapping $m : \wp(\Omega) \to [0,1]$, with $\wp(\Omega)$ the power set of Ω and s.t. $\sum_{A \in \wp(\Omega)} m(A) = 1$. $\mathcal{M}_\Omega$ denote the set of all mass assignments over Ω. A subset $A \subseteq \Omega$ is a *focal element* of m if $m(A) \neq 0$. The set of focal elements of m is noted $\mathcal{F}$. m is *normalised* if $m(\emptyset) = 0$. From m, in addition to the classical *belief*, *plausibility* and *commonality* functions [4], respectively denoted Bel, Pl and Q we use the *contour function* $pl : \Omega \to [0,1]$ of a mass assignment that corresponds to its plausibility on singletons. Recall that m can be associated to a probability set $\mathcal{P}_m := \{Pr(.) \mid \forall A \subseteq \Omega, Bel(A) \leq Pr(A)\}$.

Among the existing interpretations of belief functions, we focus on Shafer's view [4], extensively taken over by Smets in his Transferable Belief Model [5]. In this view, $m(A)$ is the mass of belief exactly committed to the hypothesis $\{\omega_0 \in A\}$, where ω_0 is the true value of an ill-known variable $\mathcal{W}$. A difference between Shafer's view and the TBM is that the latter allows $m(\emptyset) \neq 0$. Note that in the TBM original exposure, $m(\emptyset)$ is not related to conflict itself, but to the open-world assumption in which $m(\emptyset)$ quantifies the belief that the true value does not lie in Ω.

A main source of conflict comes from the conjunctive combination of information coming from not fully agreeing sources. The most classical conjunctive combination is the conjunctive rule [5], or Dempster's [6] unnormalised rule, that assumes that the sources of information are independent. In this paper, we consider a more general framework [7] where other dependency structures are considered. Given two mass assignments m_1 and m_2 defined on Ω, we consider that a conjunctive combination is achieved in two steps:

1. A joint mass assignment $\mathbf{m} : \wp(\Omega) \times \wp(\Omega) \to [0,1]$ is built s.t.

$$\sum_{B \subseteq \Omega} \mathbf{m}(A \times B) = m_1(A) \ ; \ \sum_{A \subseteq \Omega} \mathbf{m}(A \times B) = m_2(B) \ \ \forall A, B \in \wp(\Omega). \quad (1)$$

2. A mass $m_\cap : \wp(\Omega) \to [0,1]$ such that $m_\cap(C) = \sum_{A \cap B = C} \mathbf{m}(A \times B)$.

The joint mass $\mathbf{m}$ encodes the dependence structure between the two sources m_1, m_2. The conjunctive rule, whose result is denoted $m_\oplus$, corresponds to choose $\mathbf{m}(A \times B) = m_1(A)m_2(B)$ in step 1. We denote by $\mathcal{M}_{12}$ the set of all mass $m_\cap$ obtainable by a conjunctive combination of m_1 and m_2. Note that all mass assignments in $\mathcal{M}_{12}$ are specialisations of both m_1 and m_2. Recall that a mass m with $\mathcal{F} = \{E_1, \ldots, E_q\}$ is a specialisation of m' with $\mathcal{F}' = \{E'_1, \ldots, E'_p\}$ if and only if there exists a non-negative matrix $G = [g_{ij}]$ such that for $j = 1, \ldots, p, \quad \sum_{i=1}^q g_{ij} = 1, g_{ij} > 0 \Rightarrow E_i \subseteq E'_j$, and for $i = 1, \ldots, q, \quad \sum_{j=1}^p m'(E'_j) g_{ij} = m(E_i)$, where

g_{ij} is the proportion of E'_j that "flows down" to E_i. In other words, m_1 is s-included in m_2 ($m_1 \sqsubseteq_s m_2$) if the mass of any focal element E_j of m_2 can be redistributed among subsets of E_j in m_1. In fact, s-inclusion is a direct extension of the relation of inclusion between sets. As for set inclusion, s-inclusion can therefore be used to compare informative contents, $m_1 \sqsubseteq_s m_2$ meaning that m_1 is less informative than m_2.

3 Consistent Mass Assignments

We first define the notion of consistent set, before extending it to mass assignment. When information is provided as a single set $\omega_0 \in A$, this information is consistent if and only if $A \neq \emptyset$. A can be seen, for instance, as the set of models of a logic base that could be inconsistent. In this case, either a set is consistent (i.e. non-empty) or it is not, and a degree of consistency ϕ can only takes two values. Moreover, it should obey the following properties:

Property 1 (Bounded). *ϕ should be bounded.*

Property 2 (Extreme consistency). *ϕ should be maximal iff information is totally consistent, and minimal iff information is totally inconsistent.*

For simplicity, we assume that the bounds are $[0, 1]$. In the case of sets, we define the consistency degree as $\phi : \wp(\Omega) \to \{0, 1\}$ such that

$$\phi(A) = 1 \text{ if } A \neq \emptyset, 0 \text{ otherwise} \tag{2}$$

which satisfies Properties 1 and 2. We now extend it to generic mass functions. We consider first extreme cases of totally consistent and totally inconsistent mass functions: It is natural to associate totally inconsistent information with the mass $m(\emptyset) = 1$. On the other hand, the totally consistent information on sets can be extended in two main different ways. A first definition of consistent belief functions (see [7, 8]) is the following:

Definition 1. A mass assignment m is said to be *logically consistent* if and only if $\bigcap_{E \in \mathcal{F}} E \neq \emptyset$.

That is, a (normalized) mass m whose focal elements have a non-empty intersection. Next lemma characterizes these masses in terms of contour function.

Lemma 1. $\bigcap_{E \in \mathcal{F}} E \neq \emptyset \quad \Leftrightarrow \quad \exists \omega \in \Omega$ *s.t.* $pl(\omega) = 1$

m is *logically consistent* iff its contour function is normalized. This form of consistency is in accordance with the TBM interpretation, as a source is logically consistent if it considers at least one state of the world to be totally plausible. Among logically consistent mass assignments, *consonant* ones play a particular role, displaying an even stronger form of consistency: the intersection of any two focal sets is still a focal set of this mass assignment (since if $A \subset B$, $A \cap B = A$), which is not the case for general logically consistent mass assignments. The next definition provides a weaker form of consistency:

Definition 2. A mass assignment m is said to be *probabilistically consistent* if and only if $m(\emptyset) = 0$.

The name probabilistic consistency comes from the fact that requiring $m(\emptyset) = 0$ is equivalent to requiring that the probability set $\mathcal{P}_m$ induced by m is non-empty. It is also in accordance with logic-based interpretation of belief functions [9].

Definitions 1 and 2 each suggests a different measure of consistency. The following measures ϕ_{pl}, ϕ_m from $\mathcal{M}_\Omega$ to $[0,1]$, such that:

$$\phi_{pl}(m) = \max_{\omega \in \Omega} pl(\omega), \tag{3}$$

$$\phi_m(m) = 1 - m(\emptyset) \tag{4}$$

do satisfy Property 2 for totally inconsistent information and for Definitions 1 and 2 of totally consistent information, respectively. When $\exists A \in \Omega / m(A) = 1$, then both ϕ_m and ϕ_{pl} reduce to Eq. (2).

Although Definition 2 and Eq. (4) appear less adapted to the TBM interpretation than Definition 1, we will see in further sections that Eq. (4) can be useful in the TBM interpretation as well. Also, let us note that the inequality $\phi_{pl} \leq \phi_m$ always holds, and $\phi_{pl} = \phi_m$ if and only if $\bigcap_{E \in \mathcal{F} \setminus \emptyset} E \neq \emptyset$. Moreover, for consonant masses ϕ_{pl}, ϕ_m are the consistency degree of possibility theory [10].

4 Conflict between Sets

We can now study conflict between sources, starting with sets. Similar to possibility theory [10], we measure conflict as the inconsistency (inconsistency being the inverse of consistency) resulting from the conjunctive merging of information. Considering two sources of information (extension $N > 2$ is straightforward), we define the conflict of sets as $\kappa : \wp(\Omega) \times \wp(\Omega) \to \{0, 1\}$ embedding the combination step.

In the case of sources assessing that $\omega_0 \in A$ and $\omega_0 \in B$, two extreme cases may occur: they are conflicting ($A \cap B = \emptyset$) or not ($A \cap B \neq \emptyset$). As for the consistency measure, a (bounded) measure of conflict κ should take its maximal / minimal values in such cases, giving

Property 3 (Extreme conflict). *A conflict measure should be maximal value iff sources are totally conflicting, and minimal iff sources are non-conflicting.*

In other words, conflict κ for sets should be such that

$$\kappa(A, B) = 1 - \phi(A \cap B) = 1 \text{ if } A \cap B = \emptyset, \ 0 \text{ otherwise} \tag{5}$$

Other desirable properties may be formulated by observing sets. A first property should be symmetry, as we consider the two sources of equal importance.

Property 4 (Symmetry). *A measure of conflict should be symmetric.*

This translates into $\kappa(A, B) = \kappa(B, A)$. The other properties concern the behaviour of the measure with respect to some changes in the information.

Property 5 (Imprecision monotonicity). *A measure of conflict should be non-increasing if a source becomes less informative.*

If $A \cap B \neq \emptyset$, then considering $A' \supseteq A$ implies $A' \cap B \neq \emptyset$, hence κ should not increase. In contrast, we may have $A \cap B = \emptyset$ but $A' \cap B \neq \emptyset$, in which case κ should decrease. This translates by the constraint $\kappa(A', B) \leq \kappa(A, B)$.

Property 6 (Ignorance is bliss). *A measure of conflict should be insensitive to combination with ignorance.*

If $B = \Omega$, then $A \cap B \neq \emptyset$ unless $A = \emptyset$, and a state of ignorance should not conflict with any information, unless the latter is inconsistent. This translates by the constraint $\kappa(A, \Omega) = 1 - \phi(A)$.

5 Conflict between Mass Assignments

In the case of mass assignments m_1, m_2, the conjunctive combination is no longer unique (Eq. (1)), unless a specific (in)dependence structure is given. In our opinion, conflict measurement should reflect our knowledge of dependence. In particular, $m_{\oplus}$ **should not be used** to measure conflict, unless independence assumption between sources holds. This results in the following property.

Property 7 (Independence to dependence). *A conflict measure should not depend on a dependence assumption not supported by evidence.*

5.1 Characterising Total Conflict and Conflict Absence

It is natural to say that two sources are totally conflicting if none of their focal elements intersect (i.e., only $\emptyset$ can have positive mass after merging). Let $\mathcal{D}_i = \cup_{A \in \mathcal{F}_i} A$, then

Definition 3. m_1 and m_2 are *totally conflicting* when $D_1 \cap D_2 = \emptyset$.

If $m_1(A) = 1$ and $m_2(B) = 1$, we retrieve the set definition. To extend the notion of non-conflicting sets, we see two main ways fitting the TBM interpretation, given here from the most to the least constraining.

Definition 4. m_1, m_2 are strongly non-conflicting iff $\bigcap_{A \in \mathcal{F}_{m_1} \cup \mathcal{F}_{m_2}} A \neq \emptyset$.

Definition 5. m_1, m_2 are non-conflicting iff $\forall (A, B)$ such that $A \in \mathcal{F}_{m_1}$, $B \in \mathcal{F}_{m_2}$, we have $A \cap B \neq \emptyset$.

Definition 4 requires all focal elements to have a non-empty intersection, and is stronger than requiring that all pairs of focal elements from m_1 and m_2 have a non-empty intersection (Definition 5). If $m_1(A) = 1$ and $m_2(B) = 1$, the two definitions reduce to non-empty intersecting sets. The next proposition shows that strongly non-conflicting masses are related to plausibility measures, hence to consistency given by Eq. (3).

Proposition 1. $\bigcap_{A\in\{\mathcal{F}_{m_1}\cup\mathcal{F}_{m_2}\}} A\neq\emptyset$ *iff* $\forall m_\cap \in \mathcal{M}_{12}, \exists\omega\in\Omega$ *s.t.* $pl_{m_\cap}(\omega)=1$

This suggests to use the contour function to evaluate the conflict when conflict absence corresponds to Definition 4 (Strong non-conflict). Proposition 1 says that two sources are strongly non-conflicting iff there is at least one state of the world ω that they both consider "normal" or totally plausible. This is in agreement with the TBM interpretation and similar to Daniel [3] proposal. Definition 5, on the other hand, is related to the consistency measure given by Eq. (4) and we have

Proposition 2. $A\cap B\neq\emptyset\ \ \forall A\in\mathcal{F}_{m_1},\ \forall B\in\mathcal{F}_{m_2}$ *iff* $m_\cap(\emptyset)=0\ \forall m_\cap\in\mathcal{M}_{12}$

This suggests to use $m_\cap(\emptyset)$ to measure conflict under Definition 5 (Non-conflict). It is by far the most common value used to estimate conflict between information sources in Dempster-Shafer theory.

5.2 Measuring Conflict between Mass Assignments

We now propose different measure of conflicts corresponding to each notion of conflict absence, some of them being imprecise (reflecting a possible lack of knowledge about source dependencies). First, we reformulate some properties of conflict measurement κ in the vocabulary of mass assignments:

- **Prop. 3 (Extreme conflict):** $\kappa(m_1,m_2)=0$ if and only if m_1 and m_2 are non-conflicting (according to the considered definition);
- **Prop. 4 (Symmetry):** $\kappa(m_1,m_2)=\kappa(m_2,m_1)$;
- **Prop. 5 (Imprecision monotonicity):** if $m_1 \sqsubset_s m_1'$, then $\kappa(m_1',m_2)\leq\kappa(m_1,m_2)$;
- **Prop. 6 (Ignorance is bliss):** if $m_2(\Omega)=1$, then $\kappa(m_1,m_2)=1-\phi(m_1)$;

Measures for strong non-conflict: Given Proposition 1, it is natural to use ϕ_{pl} (Eq. (3)) to measure conflict from strong non-conflict. We propose to distinguish three cases:

- the case where dependence is unknown, and where one accepts imprecise conflict. In this case, if $\mathcal{I}([0,1])$ denote intervals of $[0,1]$, the measure of conflict is an application $\kappa^1_{pl}:\mathcal{M}_\Omega\times\mathcal{M}_\Omega\to\mathcal{I}([0,1])$ such that

$$\begin{aligned}\kappa^1_{pl}(m_1,m_2) &= [\min_{m_\cap\in\mathcal{M}_{12}} 1-\phi_{pl}(m_\cap), \max_{m_\cap\in\mathcal{M}_{12}} 1-\phi_{pl}(m_\cap)] \\ &= [\min_{m_\cap\in\mathcal{M}_{12}} 1-\max_{\omega\in\Omega} pl_\cap(\omega), \max_{m_\cap\in\mathcal{M}_{12}} 1-\max_{\omega\in\Omega} pl_\cap(\omega)];\end{aligned} \quad (6)$$

- the case where dependence is unknown, but the least commitment principle is followed to get a unique conflict value. In this case, we propose to select the minimal conflicting situation and $\kappa^2_{pl}:\mathcal{M}_\Omega\times\mathcal{M}_\Omega\to[0,1]$ is such that

$$\kappa^2_{pl}(m_1,m_2)=\min_{m_\cap\in\mathcal{M}_{12}} 1-\phi_{pl}(m_\cap)=\min_{m_\cap\in\mathcal{M}_{12}} 1-\max_{\omega\in\Omega} pl_\cap(\omega) \quad (7)$$

- the case where dependence is known (i.e., a joint mass $\mathbf{m}$ is specified) and where the result of conjunction is a single $m_\cap$: We propose to simply use

$$\kappa_{pl}^3(m_1, m_2) = 1 - \phi_{pl}(m_\cap) = 1 - \max_{\omega \in \Omega} pl_\cap(\omega) \quad (8)$$

They all satisfy properties 3- 6, and can deal with unknown dependence. Note that both κ_{pl}^3 and κ_{pl}^2 are straightforward to compute (the latter using results from [7]), and only the upper bound of κ_{pl}^1 requires the use of linear programming techniques.

Measures for non-conflict: As Proposition 2 is linked to Definition 2, we use ϕ_m (Eq.(4)) to derive three measures under non-conflict:

$$\kappa_m^1(m_1, m_2) = [\min_{m_\cap \in \mathcal{M}_{12}} 1 - \phi_m(m_\cap), \max_{m_\cap \in \mathcal{M}_{12}} 1 - \phi_m(m_\cap)] \quad (9)$$

$$\kappa_m^2(m_1, m_2) = \min_{m_\cap \in \mathcal{M}_{12}} 1 - \phi_m(m_\cap) = \min_{m_\cap \in \mathcal{M}_{12}} m_\cap(\emptyset) \quad (10)$$

$$\kappa_m^3(m_1, m_2) = 1 - \phi_m(m_\cap) = m_\cap(\emptyset) \quad (11)$$

$\kappa_m^1(m_1, m_2)$, $\kappa_m^2(m_1, m_2)$ corresponding to unknown dependence (without and with least commitment principle, respectively) and $\kappa_m^3(m_1, m_2)$ corresponding to known dependence. They all satisfy properties 3- 6 and can deal with unknown dependence. Classical conflict measure $m_\oplus(\emptyset)$ is captured by $\kappa_m^3(m_1, m_2)$ when independence between sources can be assumed. Computing the two bounds of κ_m^1 require the use of linear programs, while κ_m^3 remains straightforward to evaluate.

6 Short Exemplified Discussion

Let us take two different examples, showing that the proposed measures of conflict behave differently, and each have their own interest.

First, let us consider m_1, m_2 on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ such that $m_1(\{\omega_1, \omega_2\}) = 0.6$, $m_1(\{\omega_1, \omega_3\}) = 0.4$ and $m_2(\{\omega_2, \omega_3\}) = 0.5$, $m_2(\Omega) = 0.5$. Both are logically and probabilistically consistent, and we have $\kappa_{pl}^1(m_1, m_2) = [0.4, 0.4] = 0.4$ while $\kappa_m^1(m_1, m_2) = [0, 0] = 0$. According to the measure based on the contour functions, there is some conflict, whereas according to the one based on $m(\emptyset)$ there is not. While each source is consistent, they disagree on which state of the world is the most plausible (ω_1 for m_1 and ω_2 or ω_3 for m_2). Hence, in some sense (meaningful in a TBM interpretation), the two sources can be considered as conflicting. Clearly, only the measure based on contour functions is able to detect it.

As a second example, consider two identical masses on $\Omega = \{\omega_1, \omega_2\}$ such that $m_1(\{\omega_1\}) = m_2(\{\omega_1\}) = 0.5$ and $m_1(\{\omega_2\}) = m_2(\{\omega_1\}) = 0.5$. First, note that $\phi_{pl}(m_i) = 0.5$ for $i = 1, 2$, a rather low score indicating some internal inconsistency for each source. Also, the conflict measures are $\kappa_{pl}^1(m_1, m_2) = [0.5, 1]$ and $\kappa_m^1(m_1, m_2) = [0, 1]$. The highest and lowest conflict value being obtained for the combination $\mathbf{m}(\omega_1 \times \omega_2) = 0.5$ and $\mathbf{m}(\omega_2 \times \omega_1) = 0.5$ and for the combination $\mathbf{m}(\omega_1 \times \omega_1) = 0.5$ and $\mathbf{m}(\omega_2 \times \omega_2) = 0.5$ (idempotent merging), respectively. Note that every possible dependency between these extremes may be considered. This example shows that some conflict is generated from the combination, but that contour-function based measures tend to mix it with some initial inconsistency, while κ_m does detect that sources can totally agree in case of dependence. Hence, contrarily to the first example, here, measures based on $m(\emptyset)$ provide some interesting information which are not captured by measures based on contour functions. This short discussion shows that the measures have different behaviors, and that an extended discussion would be interesting. A first quick conclusion is that $m(\emptyset)$ based measures identify conflict arising from combination only, while contour-function based measures also capture some internal inconsistency. Hence, $m(\emptyset)$ seems better fitted to measure conflict **between** sources.

7 Conclusion

We have considered conflict as the inconsistency resulting from conjunctive combination. Starting from sets, we have derived a number of results regarding consistency and conflict on mass assignments. Then, we have proposed several conflict measurements not relying on Dempster's rule and able to cope with unknown (or partially known) dependencies. Our findings show that using the contour function may be a better conflict measure within the TBM interpretation, but that using $m(\emptyset)$ may be useful to characterise conflict between mass assignments.

The next step is to relate this study with other works. For instance, how it can be used to differentiate between internal and external conflict [3]. Our approach should also be compared to conflict measurements based on distances [2, 11], however we can already notice that dissimilarities based on distances do not generally satisfied the properties required here (e.g., Prop. 3 and 5), hence the two approaches are likely to give different conclusions in some cases.

References

1. Smets, P.: Analyzing the combination of conflicting belief functions. Information Fusion 8, 387–412 (2006)
2. Liu, W.: Analyzing the degree of conflict among belief functions. Artif. Intell. 170(11), 909–924 (2006)
3. Daniel, M.: Conflicts within and between Belief Functions. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) IPMU 2010. LNCS, vol. 6178, pp. 696–705. Springer, Heidelberg (2010)
4. Shafer, G.: A mathematical Theory of Evidence. Princeton University Press, New Jersey (1976)
5. Smets, P., Kennes, R.: The transferable belief model. Artificial Intelligence 66, 191–234 (1994)
6. Dempster, A.P.: Upper and lower probabilities induced by a multivalued mapping. Annals of Mathematical Statistics 38, 325–339 (1967)
7. Destercke, S., Dubois, D.: Idempotent conjunctive combination of belief functions: Extending the minimum rule of possibility theory. Information Sciences 181(18), 3925 (2011), doi:10.1016/j.ins.2011.05.007
8. Cuzzolin, F.: On Consistent Approximations of Belief Functions in the Mass Space. In: Liu, W. (ed.) ECSQARU 2011. LNCS, vol. 6717, pp. 287–298. Springer, Heidelberg (2011)
9. Cattaneo, M.: Combining belief functions issued from dependent sources. In: Proc. Third International Symposium on Imprecise Probabilities and Their Application (ISIPTA 2003), Lugano, Switzerland, pp. 133–147 (2003)
10. Dubois, D., Prade, H.: Possibility theory and data fusion in poorly informed environments. Control Eng. Practice 2, 811–823 (1994)
11. Martin, A., Jousselme, A.-L., Osswald, C.: Conflict measure for the discounting operation on belief functions. In: The 11th International Conference on Information Fusion, pp. 1003–1010 (2008)